LARGE TIME ASYMPTOTICS FOR THE OTT-SUDAN-OSTROVSKIY TYPE EQUATIONS ON A SEGMENT

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Abstract. We study the initial-boundary value problems for the nonlinear nonlocal equation on a segment $(0,a)$

$$\begin{cases} u_t + \lambda \left|u\right| \ u + C_1 \int_0^x \frac{u_{ss}(s,t)}{\sqrt{x-s}} ds = 0, \ t > 0, \\ u(x,0) = u_0(x), \\ u(a,t) = h_1(t), u_x(0,t) = h_2(t), t > 0, \end{cases} \tag{0.1}$$

where $\lambda \in \mathbf{R}$ and the constant C_1 is chosen by the condition of the dissipation, such that $\operatorname{Re} C_1 p^{\frac{3}{2}} > 0$ for $\operatorname{Re} p = 0$. The aim of this paper is to prove the global existence of solutions to the initial-boundary value problem and to find the main term of the asymptotic representation of solutions.

1. Introduction

We study global existence and asymptotic behavior of solutions to the initial-boundary value problem for the nonlinear nonlocal Ott-Sudan-Ostrovskiy type equations on a segment

$$\begin{cases} u_t + \lambda \left|u\right| \ u + C_1 \int_0^x \frac{u_{ss}(s,t)}{\sqrt{x-s}} ds = 0, \ t > 0, x \in (0,a), \\ u(x,0) = u_0(x), \ x \in (0,a), \\ u(a,t) = h_1(t), \ u_x(0,t) = h_2(t), t > 0, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbf{R}$ and the constant $C_1 < 0$ is chosen by the condition of the dissipation, such that

$$\operatorname{Re} C_1 p^{\frac{3}{2}} > 0 \text{ for } \operatorname{Re} p = 0, p \neq 0.$$

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Here and below p^α is the main branch of the complex analytic function in the half-complex plane $\Re p \geq 0$, so that $1^\alpha = 1$ (we make a cut along the negative real axis $(-\infty, 0)$).

The asymptotic theory of the initial-boundary value problems for the nonlinear pseudodifferential equations is relatively new, and traditional questions of a general theory are far from their conclusion. A description of the large time asymptotic behavior of solutions for the initial-boundary value problems requires new approaches and the reorientation of the points of view compared to the Cauchy problem. It is interesting to study the influence of the boundary data in the asymptotic behavior of solutions.

Note that a general theory of the initial-boundary value problem for the nonlinear pseudodifferential equations is difficult even in the case of linear evolutionary equations. The difficulty is explained by the fact that we need not only global existence of solutions, but also a number of additional a priori estimates of the Green function . Every type of the nonlinearity and boundary data should be studied individually.

A great number of publications have dealt with asymptotic representations of solutions to the Cauchy problem for nonlinear evolution equations in the last twenty years. While not attempting to provide a complete review of this publications, we do list some known results [1]-[8], [10], [11], [13], [14], [15]-[18], [21], [22], [25]-[35], where there were obtained local and global existence , optimal time decay estimates and asymptotic formulas of solutions to different nonlinear local and nonlocal dissipative equations. In the case of the Cauchy problem the nonlinear equations were divided into three general types: of asymptotically weak nonlinearity, of critical nonlinearity and of strong (or subcritical) nonlinearity. Also the critical and subcritical nonlinearities were divided on the convective and nonconvective types (for example, the Burgers equation $u_t - u_{xx} + u\ u_x = 0$ and the nonlinear heat equation $u_t - \Delta u + u^\rho = 0$ have convective and nonconvective types of the nonlinearity correspondingly). It was found that convective and nonconvective types of the nonlinearity define different behavior in the critical and subcritical cases. For example, in the critical convective case, the large time asymptotics is defined by the self-similar solution. In the critical nonconvective case, solutions have some additional logarithmic time decay rate compared to the corresponding linear equation. For the general theory of nonlinear pseudodifferential equations on a half-line, we refer to the book [20].

Up to now, the theory of nonlinear nonlocal initial-boundary value problems on a segment, is not developed well due to its difficulty. There are many open natural questions which we need to study. First of them is how many boundary data should be posed in the initial-boundary value problems for its correct solvability. There are some results in the case of nonlinear differential equations [9], [12]. However, as far as we know, there are few results in the case of nonlinear pseudodifferential equations. We give some review of these works. In paper [24], the authors considered the case of pseudodifferential operator $\mathbb{K}$ of order $\alpha \in (0,1)$ taking as an example of the nonlocal Schrödinger equation on a segment

$$\begin{cases} u_t + i\left|u\right|^2 u \; + \mathbb{K}u = 0,\; t > 0, x \in (0,a)\,, \\ u(x,0) = u_0(x),\; x \in (0,a)\,, \end{cases} \tag{1.2}$$

where the pseudodifferential operator on a segment with symbol $K(p) = Cp^{\alpha}$ was defined as follows

$$\mathbb{K}u = \theta_a(x)\mathcal{L}^{-1}\Big\{K(p)\Big(\mathcal{L}\{u\}(p,t) - \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1}u(0,t) - e^{-pa}\partial_x^{j-1}u(a,t)}{p^j}\Big)\Big\}.$$

Here by $[\alpha]$ we denote the integer part of the real number α and we define by $\theta_a(x)$ the step function

$$\theta_a(x) = \begin{cases} 1, & x \in (0,a) \\ 0, & x \notin (0,a)\,. \end{cases}$$

Also, the Laplace transform and the inverse Laplace transform we denote by symbols $\mathcal{L}$ and $\mathcal{L}^{-1}$ respectively. We denote by Lebesgue space $\mathbf{L}^p\left(0,a\right) = \{\varphi \in \mathcal{S}'; \|\varphi\|_{\mathbf{L}^p} < \infty\}$, where

$$\|\varphi\|_{\mathbf{L}^p} = \Big(\int_0^a \left|\varphi\left(x\right)\right|^p dx\Big)^{\frac{1}{p}}$$

for $1 \le p < \infty$ and

$$\|\varphi\|_{\mathbf{L}^\infty} = \text{ess}\sup_{x\in(0,a)} \left|\varphi\left(x\right),\right|$$

for $p = \infty$. In the [24] it was proved that if the initial data $u_0 \in \mathbf{L}^\infty\left(0,a\right)$ and the norm $\|u_0\|_{\mathbf{L}^\infty}$ is small, then there exists a unique solution $u \in \mathbf{C}\left(\left[0,\infty\right);\mathbf{L}^\infty\left(0,a\right)\right)$ of the initial-boundary value problem (1.2). Moreover, there exists a constant A such that the solution has the following large time asymptotics

$$u(x,t) = At^{-\frac{1}{\alpha}}\Lambda\Big(\frac{x}{t^{\frac{1}{\alpha}}}\Big) + O\Big(t^{-\frac{1+\delta}{\alpha}}\Big),$$

where $\Lambda(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-z^{\alpha}+zx}dz$.

In the paper [23], we studied the Whitham equation on a segment in the case of $\alpha \in (\frac{3}{2}, 2)$

$$\begin{cases} u_t + u\ u_x + \mathbb{K}u = 0, & t > 0, x \in (0,a)\,, \\ u(x,0) = u_0(x), & x \in (0,a)\,. \end{cases} \tag{1.3}$$

We proved that there exists a unique solution $u \in \mathbf{C}\left([0,\infty)\,;\mathbf{L}^2(0,a)\right) \cap \mathbf{C}\left([0,\infty)\,;\mathbf{H}^1(0,a)\right)$ of problem (1.3) for small initial data $u_0 \in \mathbf{L}^\infty(0,a)$. Here we denote by

$$\mathbf{H}^1\left(0,a\right) = \{f, \partial_x f \in \mathbf{L}^2\left(0,a\right); \|f\|^2_{\mathbf{H}^1} = \|f\|^2_{\mathbf{L}^2} + \|\partial_x f\|^2_{\mathbf{L}^2} < +\infty\},$$

the Sobolev space. Note that in both cases $\alpha \in (0,1)$ and $\alpha \in (\frac{3}{2}, 2)$, we did not put any boundary data in the problems (1.3) and (1.2).

This paper is the first attempt to give a systematic approach for obtaining the large time asymptotic representations for solutions to the nonlinear nonlocal equations on a segment with nonhomogeneous boundary data. We restrict our attention to the investigation of the typical well-known nonlinear nonlocal Ott-Sudan-Ostrovskiy type equation, which has many applications such as in the theory of ion-acoustic waves in plasma with the Landau damping (see [28], [29]). Some other applications can be found in [27]. Note that the nonlocal operator of this equation

$$L_2 u = \int_0^x \frac{u_{ss}(s,t)}{\sqrt{x-s}} ds,$$

has the order $\alpha = \frac{3}{2}$. Here we adopt the approach based on the estimates of the Green function.

Another difficulty for nonlocal equations on a segment is that the symbol $K(p)$ is not analytic in the left half-complex plane. Therefore, we cannot apply the Laplace theory directly.

To state the results of the present paper precisely we give some notations.

Let $\mathbf{B}$ be a Banach space; we then denote

$$\mathbf{C}([0,T]\,,\mathbf{B}) = \left\{ f(t) \in \mathbf{B} : \lim_{t_1 \to t, t_1 \in [0,T]} \|f(t_1) - f(t)\|_{\mathbf{B}} = 0, \forall t \in [0,T] \right\}.$$

We denote the metric spaces $\mathbf{Z}_\alpha = \left\{ h \in \mathbf{C}^1\left(0,\infty\right), \|h\|_{\mathbf{Z}_\alpha} < \infty \right\}$, with the norm

$$\|h\|_{\mathbf{Z}_\alpha} = \left\| \langle t \rangle^\alpha \sum_{n=0}^{1} t^{-n} \partial_t^n h \right\|_{\mathbf{L}^\infty},$$

Here and below $\langle t \rangle = 1 + t, \{t\} = t \langle t \rangle^{-1}$.

Now we state our results.

Theorem 1. *Let* $\alpha > \beta > 0$, $\alpha \in \left(0, \frac{3}{2}\right)$. *Suppose that* $u_0 \in \mathbf{C}^1$, $h_1 \in \mathbf{Z}_{\alpha}$,, $h_2 \in \mathbf{Z}_{\beta}$ *are such that the norm* $\|u_0\|_{\mathbf{C}^1} + \|h_1\|_{\mathbf{Z}_\alpha} + \|h_2\|_{\mathbf{Z}_\beta}$ *is sufficiently small. Then there exists a unique global solution*

$$u \in \mathbf{C}([0,\infty); \mathbf{L}^{\infty}(0,a)) \cap \mathbf{C}((0,\infty); \mathbf{C}^1([0,a])),$$

of problem (1.1). *Furthermore, the asymptotic formula with some* $\gamma > 0$

$$u(t) = h_1(t) + (x-a)t^{\frac{1}{3}}\Lambda \int_0^1 (1-z)^{-\frac{2}{3}} \mathcal{N}(h_1(tz))dz + O(t^{-\frac{1}{\alpha}-\gamma}(a-x)),$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in [0,a]$, *where*

$$\Lambda = \frac{\sqrt{3}}{3\pi} |C_1|^{\frac{2}{3}} \int_0^{\infty} e^{-q} q^{-\frac{1}{3}} dq.$$

2. Preliminaries

Now, we consider the following linear initial-boundary value problem

$$\begin{cases} u_t + C_1 \int_0^x \frac{u_{ss}(s,t)}{\sqrt{x-s}} ds = 0, \ t > 0, x \in (0,a), \\ u(x,0) = u_0(x), \ x \in (0,a), \\ u(a,t) = h_1(t), u_x(0,t) = h_2(t), t > 0. \end{cases} \tag{2.1}$$

Denote

$$\mathcal{G}\phi = \int_0^a G(x,y,t)\phi(y)dy,$$

where the function $G(x,y,t)$ is defined by

$$G(x,y,t) = \theta_a(x)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\Big[(1-e^{k(\xi)(x-a)})\Theta(\xi,y) \tag{2.2}$$

$$+ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp \frac{(e^{px} - e^{k(\xi)(x-a)}e^{pa})}{K(p)+\xi}\Big(e^{-py} - \frac{\xi}{p}\Theta(\xi,y)\Big)\Big].$$

Here

$$K(p) = C_1 p^{\frac{3}{2}},$$

$$\Theta(\xi,y) = \Psi(\xi)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{-qy}}{K(q)+\xi}\left(1 - e^{(q-k(\xi))a}\right) dq, \tag{2.3}$$

$$\Psi(\xi) = \Big(\frac{2}{3} + e^{-k(\xi)a} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{(q-k(\xi))a}}{K(q)+\xi}\frac{\xi}{q}dq\Big)^{-1}, \tag{2.4}$$

and the function $k(\xi) = |C_1|^{\frac{2}{3}} |\xi|^{\frac{2}{3}} e^{\frac{2}{3} i \arg \xi}$ is defined such that

$$K(k(\xi)) = -\xi, \quad \operatorname{Re} k(\xi) > 0 \text{ for all } \operatorname{Re} \xi > 0.$$

Also, we introduce

$$\mathcal{G}_{\frac{1}{2}} h = C_1 \int_0^t \lim_{y \to a-} \overline{\partial_y^{\frac{1}{2}}} G(x, y, t-\tau) h(\tau) d\tau, \tag{2.5}$$

$$\mathcal{G}_{-\frac{1}{2}} h = C_1 \int_0^t \lim_{y \to 0+} \overline{\partial_y^{-\frac{1}{2}}} G(x, y, t-\tau) h(\tau) d\tau, \tag{2.6}$$

where for $w \in (-1, 1)$

$$\overline{\partial_y^w} \phi = -\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-py} p^w \widehat{\phi}(-p) dp.$$

Proposition 1. *Let the initial data* $u_0 \in C^1(0, a)$ *and boundary data* $h_1(t) \in \mathbf{L}^1_{loc}(0, \infty), h_2(t) \in \mathbf{L}^1_{loc}(0, \infty)$. *Then there exists a unique semiclassical solution*

$$u(x, t) \in \mathbf{C}([0, \infty); \mathbf{L}^\infty(0, a)) \cap \mathbf{C}((0, \infty); \mathbf{C}^1([0, a])),$$

of the initial-boundary value problem (2.1), *which has representation*

$$u(x, t) = \mathcal{G} u_0 + \mathcal{G}_{\frac{1}{2}} h_1 + \mathcal{G}_{-\frac{1}{2}} h_2. \tag{2.7}$$

Proof. To derive an integral representation for solutions of the problem, (2.1) we suppose that there exists a solution $u(x, t)$ of problem (2.1), which is continued by zero outside of the interval $(0, a)$:

$$u(x, t) = 0 \quad \text{for all } x \notin [0, a].$$

However, everywhere below, we denote by

$$\begin{aligned} \partial_x^k u(0, t) &= \lim_{x \to +0} \partial_x^k u(x, t) \\ \partial_x^k u(a, t) &= \lim_{x \to a-0} \partial_x^k u(x, t), k = 0, 1. \end{aligned}$$

We denote operator

$$\mathbb{P}[\phi(p, t)] = \frac{1}{2\pi i} VP \int_{-i\infty}^{i\infty} \frac{e^{(q-p)a} - 1}{q - p} \phi(q, t) dq.$$

Since by Cauchy Theorem

$$\mathbb{P}\left[e^{-pa} \frac{1}{\sqrt{p}}\right] = \frac{1}{2\pi i} VP \int_{-i\infty}^{i\infty} \frac{e^{(q-p)a} - 1}{q - p} e^{-qa} \frac{1}{\sqrt{q}} dq$$

$$= \frac{1}{2\pi i} VP \int_{-i\infty}^{i\infty} \frac{e^{-pa} - e^{-qa}}{q-p} \frac{1}{\sqrt{q}} dq = 0,$$

we have for the Laplace transform

$$\begin{aligned} & \mathbb{P}\mathcal{L}\left\{ \int_0^x \frac{u_{ss}(s,t)}{\sqrt{x-s}} ds \right\} \\ = \ & \mathbb{P}\left[\frac{p^2}{\sqrt{p}} \left(\mathcal{L}\{u\} - \frac{u(0,t) - e^{-pa}u(a,t)}{p} - \frac{\partial_x u(0,t) - e^{-pa}u_x(a,t)}{p^2} \right) \right] \\ = \ & \mathbb{P}\left[\frac{p^2}{\sqrt{p}} \left(\mathcal{L}\{u\} - \frac{u(0,t) - e^{-pa}u(a,t)}{p} - \frac{\partial_x u(0,t)}{p^2} \right) \right]. \end{aligned}$$

Due to $\mathcal{L}\{u\}$ is analytic for all $p \in \mathbb{C}$ we get

$$\mathcal{L}\{u\} = \widehat{u}(p,t) = \mathbb{P}[\widehat{u}(p,t)].$$

Applying the Laplace transform with respect to x to problem (2.1) we obtain

$$\left\{ \begin{array}{c} \mathbb{P}\left[\widehat{u}_t + K(p)\left(\widehat{u}(p,t) - \frac{u(0,t) - e^{-pa}u(a,t)}{p} - \frac{\partial_x u(0,t)}{p^2} \right) \right] = 0, \\ t > 0, x \in (0,a), \\ \widehat{u}(p,0) = \widehat{u}_0(p), \\ u(a,t) = h_1(t), u_x(0,t) = h_2(t), t > 0. \end{array} \right. \tag{2.8}$$

We rewrite (2.8) in the form

$$\left\{ \begin{array}{c} \widehat{u}_t + K(p)\left(\widehat{u}(p,t) - \frac{u(0,t)}{p} + \frac{e^{-pa}u(a,t))}{p} - \frac{\partial_x u(0,t)}{p^2} \right) = \Phi(p,t), \\ \widehat{u}(p,0) = \widehat{u}_0(p), \\ u(a,t) = h_1(t), u_x(0,t) = h_2(t), t > 0, \end{array} \right. \tag{2.9}$$

with some function $\Phi(p,t)$, such that

$$\mathbb{P}[\Phi(p,t)] = 0,$$

and for all $p \in \mathbb{C}$, $|p| > 1$

$$|\Phi(p,t)| \leq C \frac{1 + |e^{-pa}|}{\sqrt{|p|}}.$$

Applying the Laplace transformation with respect to t to problem (2.9) we write $\mathcal{L}_t\{\widehat{u}(p,t)\} = \widehat{\widehat{u}}(p,\xi)$ as

$$\widehat{\widehat{u}}(p,\xi) = \frac{1}{K(p) + \xi} \left(\widehat{u}_0(p) + g_1(p,\xi) \right), \tag{2.10}$$

where

$$g_1(p,\xi) = K(p)\left(\frac{\widehat{u}(0,\xi)}{p} - \frac{e^{-pa}\widehat{u}(a,\xi)}{p} + \frac{\widehat{u}_x(0,\xi)}{p^2}\right) + \widehat{\Phi}(p,\xi).$$

Here, functions $\widehat{u}(0,\xi)$, $\widehat{u}(a,\xi)$, $\widehat{u}_x(0,\xi)$ and $\widehat{\Phi}(p,\xi)$, are the Laplace transforms of the boundary data and $\Phi(p,t)$ with respect to time. In order to get the integral formula for solution , we need to know the boundary values $u(0,t)$, $u(a,t)$, $u_x(0,t)$ and function $\Phi(p,t)$. We will find them using the analytic condition of function $\widehat{\widehat{u}}$ for all $p \in \mathbb{C}$. It is easy to prove that this condition is fulfilled in points $K(p)+\xi \neq 0, \operatorname{Re}\xi > 0$. In points, where $K(p)+\xi = 0$, we have to put the following necessary and sufficient condition

$$\hat{u}_0(p) + g_1(p,\xi) = 0. \tag{2.11}$$

We use the equation (2.11) to find the boundary values involved in formula (2.10).

There exists only one root $k(\xi)$ of equation $K(p) = -\xi$, such that in right-half complex plane $\operatorname{Re}\xi > 0$, $\operatorname{Re} k > 0$. Taking this root $k(\xi)$, the condition (2.11) can be written as equation

$$\widehat{u}_0(k) - \xi\frac{\widehat{u}(0,\xi)}{k} + \frac{\xi}{k}e^{-ka}\widehat{u}(a,\xi) - \xi\frac{\widehat{u}_x(0,\xi)}{k^2} + \widehat{\Phi}(k,\xi) = 0, \tag{2.12}$$

for Re $\xi > 0$.Here

$$\widehat{u}_0(k) = \int_0^a e^{-ky}u_0(y)dy.$$

We introduce a contour Γ_ε for $\varepsilon > 0$

$$\Gamma_\varepsilon = \left\{p \in \mathbb{C}, p \in \left(\infty e^{i(-\pi-\varepsilon)}, 0e^{i(-\pi-\varepsilon)}\right) \cup \left(0e^{i(\pi+\varepsilon)}, \infty e^{i(\pi+\varepsilon)}\right)\right\}. \tag{2.13}$$

Since

$$\widehat{\widehat{u}}(p,\xi) = \frac{1}{K(p)+\xi}\Big(\widehat{u}_0(p) - \frac{\xi\widehat{u}(0,\xi)}{p} + \frac{\xi}{p}e^{-pa}\widehat{u}(a,\xi) - \frac{\xi\widehat{u}_x(0,\xi)}{p^2} + \widehat{\Phi}(p,\xi)\Big)$$
$$- \frac{e^{-pa}}{p}\widehat{u}(a,\xi) + \frac{1}{p}\widehat{u}(0,\xi) + \frac{1}{p^2}\widehat{u}_x(0,\xi)$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{e^{qa}}{q-p}\widehat{\widehat{u}}(q,\xi)dq = 0,$$

we obtain for all $p \notin \Gamma_\varepsilon$

$$\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{e^{qa}}{q-p}\widehat{\widehat{u}}(q,\xi)dq \tag{2.14}$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{e^{qa}}{q-p}\frac{1}{K(q)+\xi}\Big(\widehat{u}_0(q) - \frac{\xi\widehat{u}(0,\xi)}{q} - \frac{\xi\widehat{u}_x(0,\xi)}{q^2} + \frac{\xi}{q}e^{-qa}\widehat{u}(a,\xi)\Big)dq$$
$$+ \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{e^{qa}}{q-p}\frac{1}{K(q)+\xi}\widehat{\Phi}(q,\xi)dq + \frac{1}{p}\widehat{u}(a,\xi) - \frac{1}{p}\widehat{u}(0,\xi) - (\frac{a}{p}+\frac{1}{p^2})\widehat{u}_x(0,\xi)$$
$$= 0.$$

Using for $\varepsilon_1 > 0$

$$\Phi(p,t) = -e^{-pa}\frac{1}{2\pi i}\int_{\Gamma_{\varepsilon_1}} \frac{e^{qa}}{q-p}\Phi(q,t)dq,$$

by Cauchy Theorem, we get for $\varepsilon_1 < \varepsilon$

$$\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{e^{qa}}{q-p}\frac{1}{K(q)+\xi}\widehat{\Phi}(q,\xi)dq$$
$$= -\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{e^{qa}}{q-p}\frac{1}{K(q)+\xi}e^{-qa}\frac{1}{2\pi i}\int_{\Gamma_{\varepsilon_1}} \frac{e^{q_1 a}}{q_1-q}\Phi(q,t)dq$$
$$= \frac{1}{k-p}k\prime(\xi)e^{k(\xi)a}\widehat{\Phi}(k,\xi) - \frac{e^{pa}}{K(p)+\xi}\widehat{\Phi}(p,\xi).$$

Also, we have

$$\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{e^{qa}}{q-p}\frac{1}{K(q)+\xi}e^{-qa}\frac{\xi}{q}\widehat{u}(a,\xi)dq$$
$$= \frac{1}{k-p}k\prime(\xi)\frac{\xi}{k}\widehat{u}(a,\xi) - \frac{\xi}{(K(p)+\xi)p}\widehat{u}(a,\xi).$$

Therefore, from (2.14) we get

$$\frac{1}{K(p)+\xi}\widehat{\Phi}(p,\xi)$$
$$= e^{-pa}\Big(\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{e^{qa}}{q-p}\frac{1}{K(q)+\xi}\Big(\widehat{u}_0(q) - \frac{\xi\widehat{u}(0,\xi)}{q} - \frac{\xi\widehat{u}_x(0,\xi)}{q^2}\Big)dq \tag{2.15}$$
$$+ \frac{1}{k-p}k\prime(\xi)\Big(e^{k(\xi)a}\widehat{\Phi}(k,\xi) + \frac{\xi}{k}\widehat{u}(a,\xi)\Big)$$
$$+ \frac{K(p)}{p(K(p)+\xi)}\widehat{u}(a,\xi) - \frac{1}{p}\widehat{u}(0,\xi) - (\frac{a}{p}+\frac{1}{p^2})\widehat{u}_x(0,\xi)\Big).$$

Since $\frac{1}{q-p} = -\frac{1}{p} + \frac{q}{p(q-p)}$, $\frac{1}{k-p} = -\frac{1}{p} + \frac{k}{p(k-p)}$ from (2.15), we obtain

$$\widehat{\Phi}(p,\xi) = e^{-pa}(K(p)+\xi)(-\frac{1}{p})A_1(\xi) + O\Big(\Big|\frac{1+e^{-pa}}{\sqrt{p}}\Big|\Big), \tag{2.16}$$

where

$$A_1(\xi) = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} e^{qa}\frac{1}{K(q)+\xi}\Big(\widehat{u}_0(q) - \frac{\xi\widehat{u}(0,\xi)}{q} - \frac{\xi\widehat{u}_x(0,\xi)}{q^2}\Big)dq$$
$$+ k\prime(\xi)\Big(e^{k(\xi)a}\widehat{\Phi}(k,\xi) + \frac{\xi}{k}\widehat{u}(a,\xi)\Big) - \widehat{u}(a,\xi) + \widehat{u}(0,\xi) + a\widehat{u}_x(0,\xi).$$

By the definition of function $\widehat{\Phi}(p,\xi)$, we have for all $p \in \mathbb{C}, |p| > 1$

$$\left|\widehat{\Phi}(p,\xi)\right| \le C\frac{1+|e^{-pa}|}{\sqrt{|p|}},$$

and therefore from (2.16) we must put condition

$$A_1(\xi) = 0. \tag{2.17}$$

Thus, using (2.12), we obtain the system of two equations with unknown functions $\widehat{\Phi}(k,\xi)$ and $\widehat{u}(0,\xi), \widehat{u}_x(0,\xi)$, $\widehat{u}(a,\xi)$

$$\begin{cases} \widehat{u}_0(k) - \xi\frac{\widehat{u}(0,\xi)}{k} + \frac{\xi}{k}e^{-ka}\widehat{u}(a,\xi) - \xi\frac{\widehat{u}_x(0,\xi)}{k^2} + \widehat{\Phi}(k,\xi) = 0 \\ A_1(\xi) \\ = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} e^{qa}\frac{1}{K(q)+\xi}\left(\widehat{u}_0(q) - \frac{\xi\widehat{u}(0,\xi)}{q} - \frac{\xi\widehat{u}_x(0,\xi)}{q^2}\right)dq \\ +k\prime(\xi)\left(e^{k(\xi)a}\widehat{\Phi}(k,\xi) + \frac{\xi}{k}\widehat{u}(a,\xi)\right) - \widehat{u}(a,\xi) + \widehat{u}(0,\xi) + a\widehat{u}_x(0,\xi) \\ = 0. \end{cases} \tag{2.18}$$

So, we must include in the initial-boundary value problem (2.1) two boundary data. Taking for example, $\widehat{u}(a,\xi) = \widehat{h}_1(\xi), \widehat{u}_x(0,\xi) = \widehat{h}_2(\xi)$, we can find the rest of the Laplace transforms of unknown functions from the system (2.18).

Solving (2.18) we get

$$\widehat{u}(0,\xi) = \frac{1}{2\pi i}\Psi(\xi)\Big(\int_{-i\infty}^{i\infty}\frac{1-e^{(q-k(\xi))a}}{K(q)+\xi}\Big(\widehat{u}_0(q) + \frac{K(q)}{q^2}\widehat{h}_2(\xi)\Big)dq + e^{-k(\xi)a}\widehat{h}_1(\xi)\Big), \tag{2.19}$$

where

$$\Psi(\xi) = \Big(\frac{2}{3} + e^{-k(\xi)a} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{(q-k(\xi))a}}{K(q)+\xi}\frac{\xi}{q}dq\Big)^{-1}. \tag{2.20}$$

Also, we obtain

$$\frac{1}{K(p)+\xi}\widehat{\Phi}(p,\xi) \tag{2.21}$$

$$= \quad e^{-pa}\Big(\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{e^{qa}q}{(q-p)p}\frac{1}{K(q)+\xi}\Big(\widehat{u}_0(q)-\frac{\xi\widehat{u}(0,\xi)}{q}+\frac{K(q)}{q^2}\widehat{h}_2(\xi)\Big)dq$$
$$-\frac{k}{(k-p)p}\Big(\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{qa}}{K(q)+\xi}\Big(\widehat{u}_0(q)-\frac{\xi\widehat{u}(0,\xi)}{q}+\frac{K(q)\widehat{h}_2(\xi)}{q^2}\Big)dq$$
$$+\widehat{u}(0,\xi)-\widehat{h}_1(\xi)\Big)-\frac{\xi}{p(K(p)+\xi)}\widehat{h}_1(\xi)\Big).$$

By construction, we have

$$\widehat{u}(x,\xi)=\theta_a(x)\mathcal{L}^{-1}\{\widehat{\widehat{u}}(p,\xi)\}=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{px}\frac{1}{K(p)+\xi} \tag{2.22}$$
$$\times\Big(\widehat{u}_0(p)-\xi\frac{\widehat{u}(0,\xi)}{p}+\frac{K(p)\widehat{h}_2(\xi)}{p^2}+\xi e^{-pa}\frac{h_1(\xi)}{p}+\widehat{\Phi}(p,\xi)\Big)+\widehat{u}(0,\xi).$$

Using (2.21) we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{px}\frac{1}{K(p)+\xi}\widehat{\Phi}(p,\xi)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{p(x-a)} \tag{2.23}$$
$$\times\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{e^{qa}q}{(q-p)p}\frac{1}{K(q)+\xi}\Big(\widehat{u}_0(q)-\frac{\xi\widehat{u}(0,\xi)}{q}+\frac{K(q)}{q^2}\widehat{h}_2(\xi)\Big)dq$$
$$-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{p(x-a)}\frac{k}{(k-p)p}$$
$$\times\Big(\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{qa}}{K(q)+\xi}\Big(\widehat{u}_0(q)-\frac{\xi\widehat{u}(0,\xi)}{q}+\frac{K(q)\widehat{h}_2(\xi)}{q^2}\Big)dq$$
$$+\widehat{u}(0,\xi)-\widehat{h}_1(\xi)\Big)-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{p(x-a)}\frac{\xi}{p(K(p)+\xi)}\widehat{h}_1(\xi).$$

Therefore, from (2.22) and (2.23) we get

$$\widehat{u}(x,\xi)=\mathcal{L}^{-1}\{\widehat{\widehat{u}}(p,\xi)\}=\theta_a(x)\widehat{u}(0,\xi)(1-e^{k(\xi)(x-a)}) \tag{2.24}$$
$$+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{px}\frac{1}{K(p)+\xi}\Big(\widehat{u}_0(p)-\xi\frac{\widehat{u}(0,\xi)}{p}+\frac{K(p)\widehat{h}_2(\xi)}{p^2}\Big)$$
$$-e^{k(\xi)(x-a)}\Big(\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{qa}}{K(q)+\xi}\Big(\widehat{u}_0(q)$$
$$-\frac{\xi\widehat{u}(0,\xi)}{q}+\frac{K(q)\widehat{h}_2(\xi)}{q^2}\Big)dq-\widehat{h}_1(\xi)\Big),$$

where the Laplace transform $\widehat{u}(0,\xi)$ of unknown function $u(0,t)$ we find from (2.19).

Substituting (2.19) into (2.24) and taking inverse Laplace transform with respect to ξ, we get

$$u(x,t) = \mathcal{G}u_0 + \mathcal{G}_1 h_1 + \mathcal{G}_2 h_2, \tag{2.25}$$

where operators $\mathcal{G}$,$\mathcal{G}_{\frac{1}{2}}, \mathcal{G}_{-\frac{1}{2}}$ was defined by (2.2),(2.5) and (2.6).

Now, we prove that for $t > 0$

$$G_t + \mathbb{K}G = \delta(t)\delta(x-y). \tag{2.26}$$

Indeed, we have

$$\begin{aligned}\widehat{\widehat{G}}(p,y,\xi) &= \Theta(\xi,y)\frac{1}{p} + \frac{1}{K(p)+\xi}\Big(e^{-py} - \xi\frac{\Theta(\xi,y)}{p}\Big)\\ &+ \frac{1}{p}\int_{\Gamma_\varepsilon} e^{(q-p)a}\Big(\frac{q}{(q-p)} - \frac{k}{(k-p)}\Big)\Big(\frac{1}{K(q)+\xi}(e^{-qy} - \frac{\xi\Theta(\xi,y)}{q})\Big)dq\\ &- e^{-pa}\frac{k}{(k-p)p}\Theta(\xi,y).\end{aligned}$$

By definition of $\Theta(\xi,y)$, we have

$$-e^{-k(\xi)a}\Theta(\xi,y) + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp\frac{(1-e^{-k(\xi)a}e^{pa})}{K(p)+\xi}\Big(e^{-py} - \frac{\xi}{p}\Theta(\xi,y)\Big) = 0.$$

Therefore,

$$\begin{aligned}\widehat{G}(0,y,\xi) &= (1-e^{-k(\xi)a})\Theta(\xi,y)\\ &\quad + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp\frac{(1-e^{-k(\xi)a}e^{pa})}{K(p)+\xi}\Big(e^{-py} - \frac{\xi}{p}\Theta(\xi,y)\Big) = \Theta(\xi,y).\end{aligned}$$

Also, since for all $y > 0$

$$\begin{aligned}\lim_{x\to 0+}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp\frac{pe^{px}}{K(p)+\xi}e^{-py} &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp\frac{p}{K(p)+\xi}e^{-py}\\ = k'(\xi)e^{-k(\xi)y}k(\xi) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp\frac{k(\xi)}{K(p)+\xi}e^{-py},\end{aligned}$$

we get for all $y > 0$

$$\begin{aligned}&\widehat{G}_x(0,y,\xi) = -k(\xi)e^{-k(\xi)a}\Theta(\xi,y)\\ &-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp\frac{k(\xi)e^{-k(\xi)a}e^{pa}}{K(p)+\xi}\Big(e^{-py} - \frac{\xi}{p}\Theta(\xi,y)\Big)\end{aligned}$$

$$+\lim_{x\to 0+}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{pe^{px}}{K(p)+\xi}\Big(e^{-py}-\frac{\xi}{p}\Theta(\xi,y)\Big)$$
$$=k(\xi)\Big[-e^{-k(\xi)a}\Theta(\xi,y)+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{(1-e^{-k(\xi)a}e^{pa})}{K(p)+\xi}\Big(e^{-py}-\frac{\xi}{p}\Theta(\xi,y)\Big)\Big]$$
$$=0.$$

By direct calculation, we obtain $\widehat{G}(a,y,\xi)=0$. Since

$$\mathbb{K}G=C_1\int_0^x\frac{\partial_{ss}G(s,y,t)}{\sqrt{x-s}}ds=\Big[\frac{1}{2\pi i}\theta_a(x)\int_{-i\infty}^{i\infty}K(p)\Big(\widehat{G}(p,y,t)$$
$$-\frac{G(0,y,t)-e^{-pa}G(a,y,t)}{p}-\frac{\partial_xG(0,y,t)}{p^2}\Big)\Big],$$

we get

$$\mathbb{K}G=\frac{1}{2\pi i}\theta_a(x)\int_{-i\infty}^{i\infty}e^{\xi t}d\xi\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{px}\frac{(K(p)+\xi)-\xi}{K(p)+\xi}\Big(e^{-py}-\xi\frac{\Theta(\xi,y)}{p}\Big)$$
$$+e^{p(x-a)}\frac{K(p)}{p}\int_{\Gamma_\varepsilon}e^{qa}\Big(\frac{q}{(q-p)}-\frac{k}{(k-p)}\Big)\Big(\frac{1}{K(q)+\xi}(e^{-qy}-\frac{\xi\Theta(\xi,y)}{q})\Big)dq$$
$$-e^{p(x-a)}\frac{K(p)k}{(k-p)p}\Theta(\xi,y)\Big)$$
$$=\frac{1}{2\pi i}\theta_a(x)\int_{-i\infty}^{i\infty}e^{\xi t}d\xi\int_{-i\infty}^{i\infty}dpe^{px}\Big(e^{-py}-\xi\frac{\Theta(\xi,y)}{p}\Big)$$
$$-\int_{-i\infty}^{i\infty}e^{\xi t}\xi d\xi\int_{-i\infty}^{i\infty}dpe^{px}\frac{1}{K(p)+\xi}\Big(e^{-py}-\xi\frac{\Theta(\xi,y)}{p}\Big)$$
$$-e^{k(\xi)(x-a)}\xi\int_{\Gamma_\varepsilon}e^{qa}\Big(\frac{1}{K(q)+\xi}(e^{-qy}-\frac{\xi\Theta(\xi,y)}{q})\Big)dq+e^{k(\xi)(x-a)}\xi\Theta(\xi,y)$$
$$=-\theta_a(x)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\xi\Big[(1-e^{k(\xi)(x-a)})\Theta(\xi,y)$$
$$+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{(e^{px}-e^{k(\xi)(x-a)}e^{pa})}{K(p)+\xi}\Big(e^{-py}-\frac{\xi}{p}\Theta(\xi,y)\Big)\Big]$$
$$+\frac{1}{2\pi i}\theta_a(x)\int_{-i\infty}^{i\infty}e^{\xi t}d\xi\int_{-i\infty}^{i\infty}dpe^{p(x-y)}=-G_t+\delta(t)\delta(x-y).$$

Since,

$$\lim_{t\to 0+}G(x,y,t)=\theta_a(x)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi(1-e^{k(\xi)(x-a)})\Theta(\xi,y)$$

$$
\begin{aligned}
&- e^{k(\xi)(x-a)} \int_{-i\infty}^{i\infty} dp \frac{e^{pa}}{K(p)+\xi}\Big(e^{-py} - \frac{\xi}{p}\Theta(\xi,y)\Big) \\
&- \xi\Theta(\xi,y)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp \frac{e^{px}}{(K(p)+\xi)p} \\
&+ \lim_{t\to 0+} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp \frac{e^{p(x-y)}}{K(p)+\xi} = \delta(x-y),
\end{aligned}
$$

using (2.26) we easily can prove that (2.25) is the solution of problem (2.1). Proposition 1 is proved. □

Lemma 1. *The following estimates are true*

$$
\mathcal{G}(t_1)\phi(t_2) = (a-x)\Lambda t_1^{-\frac{2}{3}} \lim_{x\to a-} \phi(x,t_2) + O(t_1^{-\frac{2}{3}-\gamma}) \left\|\phi'\right\|_{\mathbf{L}^\infty},
$$

$$
\|\mathcal{G}(t_1)\phi(t_2)\|_{\mathbf{L}^\infty} \le C\left(\|\phi(t_2)\|_{L^\infty} + \left\|\phi'(t_2)\right\|_{\mathbf{L}^\infty}\right),
$$

$$
\sup_{t>0} \langle t\rangle^{\frac{1}{\alpha}} \left\|(1+\frac{\langle t\rangle^{\frac{2}{3}}}{(a-x)})\mathcal{G}_{\frac{1}{2}}h\right\|_{\mathbf{L}^\infty} \le C\|h\|_{\mathbf{Z}_\alpha},
$$

$$
\sup_{t>0} \langle t\rangle^{\frac{1}{\beta}-\gamma} \left\|\frac{1}{(a-x)}\mathcal{G}_{-\frac{1}{2}}h\right\|_{\mathbf{L}^\infty} \le C\|h\|_{\mathbf{Z}_\beta},
$$

for $\gamma > 0$ is small enough, and for $t\to\infty$ uniformly with respect to $x\in[0,a]$

$$
\mathcal{G}_{\frac{1}{2}}h = h(t) + (a-x)O(t^{-\frac{2}{3}-\frac{1}{\alpha}})\|h\|_{\mathbf{Z}_\alpha},
$$

provided that the right-hand sides are finite.

Proof. We denote by

$$
\begin{aligned}
\Gamma_1 &= \Big\{\xi \in [-t^{-1+\delta} - i\varepsilon_1, 0 - i\varepsilon_1) \cup \xi = \varepsilon_1 e^{i\phi}, \\
&\qquad \phi \in [-\tfrac{\pi}{2}; \tfrac{\pi}{2}] \cup (0 + i\varepsilon_1, -t^{-1+\delta} + i\varepsilon_1]\Big\}, \\
\Gamma_2 &= \Big\{\xi \in \Big[-t^{-1+\delta} - i\infty, -t^{-1+\delta} - i\varepsilon\Big) \cup \Big(-t^{-1+\delta} + i\varepsilon, -t^{-1+\delta} + i\infty\Big]\Big\}, \\
\Gamma_\varepsilon &= \Big\{p \in (-\infty - i\varepsilon; -\varepsilon i) \cup p = \varepsilon e^{i\phi}, \phi \in \left[-\tfrac{\pi}{2}; \tfrac{\pi}{2}\right] \cup p \in (\varepsilon i; -\infty + i\varepsilon)\Big\}, \\
\mathcal{C}_\varepsilon &= \Big\{p \in (-i\infty; -\varepsilon i) \cup p = \varepsilon e^{i\phi}, \phi \in \left[-\tfrac{\pi}{2}; \tfrac{\pi}{2}\right] \cup p \in (\varepsilon i; i\infty)\Big\},
\end{aligned}
$$

for $\varepsilon_1 > \varepsilon > 0, \delta > 0$. Using

$$
\Big[-e^{-k(\xi)a}\Theta(\xi,y) + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp \frac{(1-e^{-k(\xi)a}e^{pa})}{K(p)+\xi}\Big(e^{-py} - \frac{\xi}{p}\Theta(\xi,y)\Big)\Big] = 0,
$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_1\cup\Gamma_2} d\xi e^{\xi t}\frac{1}{2\pi i}\int_{\mathcal{C}_\varepsilon} dp\frac{(e^{px}-e^{pa})-(e^{k(\xi)x}-e^{k(\xi)a})}{K(p)+\xi}e^{-py}=0,$$

by Cauchy Theorem we can rewrite the Green function (see (2.2)) in the form

$$\begin{aligned} &G(x,y,t) \qquad (2.27)\\ &=-\theta_a(x)\frac{1}{2\pi i}\int_{\Gamma_1\cup\Gamma_2} d\xi e^{\xi t}\frac{1}{2\pi i}\int_{\mathcal{C}_\varepsilon} dp\frac{(e^{px}-e^{pa})-(e^{k(\xi)x}-e^{k(\xi)a})}{K(p)+\xi}\frac{\xi}{p}\Theta(\xi,y)\\ &=-\frac{1}{2\pi i}\int_{\Gamma_1\cup\Gamma_2} d\xi e^{\xi t}\Theta(\xi,y)\Big[\Big(e^{k(\xi)x}-e^{k(\xi)a}\Big)\\ &+\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} dp\frac{e^{px}-e^{pa}-e^{k(\xi)x}+e^{k(\xi)a}}{K(p)+\xi}\frac{K(p)}{p}\Big].\end{aligned}$$

Here, the function $\Theta(\xi,y)$ was introduced by formula (2.3).

Integrating by part we get

$$\mathcal{G}\phi=\phi(a)-\int_0^a\phi'(y)\widetilde{G}(x,y,t)dy,$$

where

$$\widetilde{G}(x,y,t)=M(x,y,t)+R(x,y,t), \qquad (2.28)$$

$$\begin{aligned} M(x,y,t) &= -\frac{1}{2\pi i}\int_{\Gamma_1} d\xi e^{\xi t}\widetilde{\Theta}(\xi,y)\Big[\Big(e^{k(\xi)x}-e^{k(\xi)a}\Big) \qquad (2.29)\\ &+\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} dp\frac{e^{px}-e^{pa}-e^{k(\xi)x}+e^{k(\xi)a}}{K(p)+\xi}\frac{K(p)}{p}\Big],\\ R(x,y,t) &= -\frac{1}{2\pi i}\int_{\Gamma_2} d\xi e^{\xi t}\widetilde{\Theta}(\xi,y)\Big[\Big(e^{k(\xi)x}-e^{k(\xi)a}\Big) \qquad (2.30)\\ &+\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} dp\frac{e^{px}-e^{pa}-e^{k(\xi)x}+e^{k(\xi)a}}{K(p)+\xi}\frac{K(p)}{p}\Big],\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Theta}(\xi,y) &=\Big(\frac{2}{3}+\frac{e^{-k(\xi)a}}{2\pi i}\int_{\Gamma_\varepsilon}\frac{e^{qa}}{K(q)+\xi}\frac{K(q)}{q}dq\Big)^{-1} \qquad (2.31)\\ &\times\xi^{-1}\Big(-\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{(1-e^{(q-k(\xi))a})K(q)e^{-qy}}{(K(q)+\xi)q}dq-e^{-k(\xi)a}\Big).\end{aligned}$$

First, we estimate function $\widetilde{\Theta}(\xi,y)$ for $\xi\in\Gamma_1$. Since

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{(1-e^{(q-k(\xi))a})K(q)}{(K(q)+\xi)q}dq \qquad (2.32)\\
&= -\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{(1-e^{(q-k(\xi))a})\xi}{q}\frac{2i\,|q|^{\frac{3}{2}}}{|q|^3+\xi^2}dq\\
&= \frac{1}{\pi}\int_{-\infty}^{0}\frac{(1-e^{(|k(\xi)|z-k(\xi))a})\xi}{|z|\,|k(\xi)|}\frac{|z|^{\frac{3}{2}}\,|\xi|\,|k(\xi)|}{|\xi|^2\left(|z|^3+\xi^2\,|\xi|^{-2}\right)}dz\\
&= -\frac{1}{\pi}\int_{0}^{\infty}\frac{(1-e^{(-|k(\xi)|z-k(\xi))a})}{z^{1-\frac{3}{2}}}\frac{1}{(z^3+1)}dz\\
&= \frac{1}{\pi}\,|k(\xi)|\,a\int_{0}^{\infty}\frac{z^{\frac{1}{2}}\left(z+k(\xi)\,|k(\xi)|^{-1}\right)}{(z^3+1)}dz+O(|k(\xi)|^{1+\gamma}),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{e^{(q-k(\xi))a}}{K(q)+\xi}\frac{K(q)}{q}dq=-\frac{1}{\pi}\int_{0}^{\infty}\frac{z^{\frac{1}{2}}e^{(-|k(\xi)|z-k(\xi))a}}{z^3+1}dz \qquad (2.33)\\
&=-\frac{1}{\pi}\int_{0}^{\infty}\frac{z^{\frac{1}{2}}(1-|k(\xi)|\,a(z+k(\xi)\,|k(\xi)|^{-1})}{z^3+1}dz+O(|k(\xi)|^{1+\gamma})\\
&=-\left(\frac{1}{3}(1-k(\xi)a)-\frac{2}{3}\,|k(\xi)|\,a\right)+O(|k(\xi)|^{1+\gamma}),
\end{aligned}$$

we get

$$\xi\widetilde{\Theta}(\xi,y)=-1+O(|k(\xi)|), \qquad (2.34)$$

Also, in the same way, we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}dp\frac{e^{px}-e^{k(\xi)x}}{K(p)+\xi}\frac{K(p)}{p}=-\frac{1}{\pi}\int_{0}^{\infty}\frac{z^{\frac{1}{2}}\left(e^{-|k(\xi)|zx}-e^{k(\xi)x}\right)}{z^3+1}dz\\
&=\frac{x\,|k(\xi)|}{\pi}\int_{0}^{\infty}\frac{z^{\frac{1}{2}}(z+k(\xi)\,|k(\xi)|^{-1})}{z^3+1}dz \qquad (2.35)\\
&=x\,|k(\xi)|\left(\frac{2}{3}+\frac{1}{3}k(\xi)\,|k(\xi)|^{-1}\right)+O(|k(\xi)|^{1+\gamma}).
\end{aligned}$$

Therefore, substituting (2.34) and (2.35) into (2.29) and using definition of $k(\xi)=|\xi|^{\frac{2}{3}}\,|C|^{\frac{2}{3}}\,e^{\frac{2}{3}i\arg\xi}$, and the estimate

$$e^{k(\xi)x}-e^{k(\xi)a}=(x-a)k(\xi)+O(|k(\xi)|^{1+\gamma}),$$

we get

$$M(x,y,t) = -(a-x)\frac{1}{2\pi i}\int_{\Gamma_1} d\xi e^{\xi t}\xi^{-1}\frac{2}{3}\left|k(\xi)\right|\left(1+2k(\xi)\left|k(\xi)\right|^{-1}\right) \quad (2.36)$$
$$+\int_{\Gamma_1} d\xi e^{\xi t}\xi^{-1}O(\left|k(\xi)\right|^{2+\gamma}) = \Lambda(a-x)t^{-\frac{2}{3}} + O(t^{-\frac{2+\gamma}{3}}),$$

where

$$\Lambda = -\frac{2}{3}\frac{1}{2\pi i}\int_{\Gamma_1} d\xi e^{q}q^{-1}|k(q)|\big(1+2k(q)|k(q)|^{-1}\big) = \frac{1}{\pi}\frac{\sqrt{3}}{3}\int_0^\infty e^{-q}q^{-1}k(q)dq.$$

Also, since

$$\Psi(\xi) = \Big(\frac{2}{3} + e^{-k(\xi)a} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{(q-k(\xi))a}}{K(q)+\xi}\frac{\xi}{q}dq\Big)^{-1} = O\left(1\right), \quad \xi\in\Gamma_2, \quad (2.37)$$

we get for $\xi\in\Gamma_2$

$$|\widetilde{\Theta}(\xi,y)| \le |\Psi(\xi)||\xi|^{-1}\int_{\Gamma_\varepsilon}\frac{1}{|K(q)+|\xi|^{-1}\xi|}dq \le C.$$

Using

$$\Big|\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} dp\frac{e^{px}-e^{k(\xi)x}}{K(p)+\xi}\frac{K(p)}{p}\Big|$$
$$\le \quad C\Big|\int_0^\infty \frac{z^{\frac{1}{2}}\left(e^{-|k(\xi)|zx}-e^{k(\xi)x}\right)}{z^3+1}dz\Big| \le Ce^{\operatorname{Re}k(\xi)x},$$

we attain

$$|R(x,y,t)| \le e^{-\delta\frac{t}{2}}C\int_{\Gamma_2} d\xi e^{\operatorname{Re}\xi\frac{t}{2}+\operatorname{Re}k(\xi)a}\left|\xi\right|^{-1}d\xi \le C\left\langle t\right\rangle^{-\frac{2}{3}-\gamma}. \quad (2.38)$$

Thus, from (2.36),(2.38) by (2.28) we get for some $\gamma>0$

$$\mathcal{G}(t_1)\phi(t_2) = (a-x)\Lambda t_1^{-\frac{2}{3}}\lim_{x\to a-}\phi(x,t_2) + O(t_1^{-\frac{2}{3}-\gamma})\left\|\phi'\right\|_{\mathbf{L}^\infty},$$

and

$$\left\|\mathcal{G}(t_1)\phi(t_2)\right\|_{\mathbf{L}^\infty} \le C\left(\left\|\phi\right\|_{L^\infty}+\left\|\phi'\right\|_{\mathbf{L}^\infty}\right).$$

Now we estimate operator $\mathcal{G}_{\frac{1}{2}}$. By direct calculation (see (2.5)) we have

$$G_{\frac{1}{2}}(x,t)$$
$$= \lim_{y\to a-}\partial_y^{\frac{1}{2}}G(x,y,t) = \theta_a(x)\frac{1}{2\pi i}\Bigg(\Sigma_{i=1,2}\int_{\Gamma_i} d\xi e^{\xi t}(1-e^{k(\xi)(x-a)})\Theta_{\frac{1}{2}}(\xi)$$

$$+ e^{k(\xi)(x-a)} + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp \frac{(e^{px} - e^{k(\xi)(x-a)}e^{pa})}{K(p)+\xi}\frac{K(p)}{p}\Theta_{\frac{1}{2}}(\xi)\Big) = I_1 + I_2,$$

where $\Theta_{\frac{1}{2}}(\xi) = e^{-k(\xi)a}\Psi(\xi)$. In the same way as (2.35) we get

$$\begin{aligned} I_1 &= \int_{\Gamma_1} e^{\xi t} d\xi \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp \frac{(e^{px} - e^{k(\xi)(x-a)}e^{pa})}{K(p)+\xi}\frac{K(p)}{p}\Theta_{\frac{1}{2}}(\xi) \\ &= -\frac{1}{\pi}\int_{\Gamma_1} e^{\xi t}\Theta_{\frac{1}{2}}(\xi)\frac{\xi\,|\xi|}{|\xi|^2}d\xi \int_0^\infty \frac{z^{\frac{1}{2}}\left(e^{-|k(\xi)|zx} - e^{k(\xi)(x-a)}e^{-|k(\xi)|za}\right)}{z^3 + |\xi|^{-2}\xi^2}dz \\ &= \frac{a-x}{\pi}\int_{\Gamma_1} e^{\xi t}\Theta_{\frac{1}{2}}(\xi)\,|k(\xi)|\Big(\int_0^\infty \frac{z^{\frac{1}{2}}(z + k(\xi)\,|k(\xi)|^{-1})}{z^3+1}dz \\ &+ O\Big(\int_0^\infty \frac{z^{\frac{1}{2}}(z^{1+\gamma} + |k(\xi)|^\gamma)}{|z^3+1|}dz\Big)\Big). \end{aligned}$$

Therefore, using $\Theta_{\frac{1}{2}}(\xi) = e^{-k(\xi)a}\Psi(\xi) = 1 + O(|k(\xi)|)$, we obtain

$$|I_1| \le C(1 + (a-x)\langle t\rangle^{-\frac{2}{3}-1}). \tag{2.39}$$

Since by (2.37) for $\xi \in \Gamma_2$, $|\Theta_{\frac{1}{2}}(\xi)| \le 1$, we attain

$$\begin{aligned} |I_2| &\le e^{-\delta\frac{t}{2}} C \int_{\Gamma_2} |d\xi|\, e^{\operatorname{Re}\xi\frac{t}{2}}\Big(1 + \int_0^{+\infty} dz \frac{z^{\frac{1}{2}}}{|K(z) + |\xi|^{-2}\xi|}\Big) \\ &\le Ct^{-1}(1 + (a-x)\langle t\rangle^{-\frac{2}{3}-\gamma}). \end{aligned} \tag{2.40}$$

Therefore, from (2.39) and (2.40)

$$|G_{\frac{1}{2}}(x,t)| \le Ct^{-1}(1 + (a-x)\langle t\rangle^{-\frac{2}{3}-\gamma}). \tag{2.41}$$

By (2.41), we have for $t > 1$

$$\begin{aligned} &\Big|\int_0^{\frac{t}{2}} G_{\frac{1}{2}}(x, t-\tau)h(\tau)d\tau\Big| \\ \le\ & C(a-x)t^{-\frac{2}{3}-1}(1+t)^{-\frac{1}{\alpha}+1}\|h\|_{\mathbf{Z}_\alpha} \le C(a-x)t^{-\frac{2}{3}-\frac{1}{\alpha}}\|h\|_{\mathbf{Z}_\alpha}, \end{aligned}$$

and

$$\begin{aligned} &\int_{\frac{t}{2}}^{t} G_{\frac{1}{2}}(x, t-\tau)h(\tau)d\tau \\ =\ & h(t)\int_0^{\frac{t}{2}} G_{\frac{1}{2}}(x,\tau)d\tau + \int_0^{\frac{t}{2}} G_{\frac{1}{2}}(x,\tau)(h(t-\tau) - h(\tau))d\tau \end{aligned}$$

$$= \quad h(t)\int_0^{+\infty} G_{\frac{1}{2}}(x,\tau)d\tau + R,$$

where

$$|R| \le C\Big|\,|h(t)|\int_{\frac{t}{2}}^{+\infty} G_{\frac{1}{2}}(x,\tau)d\tau + \int_0^{\frac{t}{2}} |G_{\frac{1}{2}}(x,\tau)|\,|h(t-\tau)-h(\tau)|\,d\tau\Big|$$
$$\le C(a-x)t^{-\frac{2}{3}-\frac{1}{\alpha}}\,\|h\|_{\mathbf{Z}_\alpha}.$$

Therefore, since

$$\int_0^{+\infty} G_{\frac{1}{2}}(x,\tau)d\tau = \widehat{G}_{\frac{1}{2}}(x,0) = 1,$$

we get for $t > 1$

$$\begin{aligned}\mathcal{G}_{\frac{1}{2}}h &= \int_0^{\frac{t}{2}} G(x,t-\tau)h(\tau)d\tau + \int_{\frac{t}{2}}^{t} G(x,t-\tau)h(\tau)d\tau \\ &= h(t) + (a-x)O(t^{-\frac{2}{3}-\frac{1}{\alpha}})\,\|h\|_{\mathbf{Z}_\alpha}.\end{aligned}$$

Using definition of $\Psi(\xi)$ by direct calculation we obtain

$$\begin{aligned}G_{\frac{1}{2}}(x,y,t) = -\theta_a(x)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}e^{-k(\xi)a}\Psi(\xi) \\ \times\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} dp\frac{(e^{px}-e^{k(\xi)x})}{K(p)+\xi}\frac{K(p)}{p}.\end{aligned}$$

Therefore, since $|\Psi(\xi)| \le C$, we get for $t < 1$

$$\begin{aligned}\left\|\mathcal{G}_{\frac{1}{2}}h\right\|_{\mathbf{L}^\infty} &\le C\int_{-i\infty}^{i\infty} d\,|\xi|\,e^{-\operatorname{Re}k(\xi)a}|\widehat{h}(\xi)| \\ &\quad\times\int_0^{+\infty} d\,|p|\,\frac{1+e^{\operatorname{Re}k(\xi)x}}{|K(p)+\xi|\xi|^{-1}|}\frac{|K(p)|}{|p|} \\ &\le C\int_{-i\infty}^{i\infty}|\widehat{h}(\xi)|d\xi \le C\,\|h\|_{\mathbf{Z}_\alpha}.\end{aligned} \tag{2.42}$$

Now, we estimate operator $\mathcal{G}_{-\frac{1}{2}}$.By definition (2.6) we have

$$G_{-\frac{1}{2}}(x,t)\lim_{y\to 0+}\partial_y^{-\frac{1}{2}}G(x,y,t) \tag{2.43}$$
$$= \theta_a(x)\frac{1}{2\pi i}\Big(\sum_{i=1}^{2}\int_{\Gamma_i} d\xi e^{\xi t}(1-e^{k(\xi)(x-a)})\Theta_{-\frac{1}{2}}(\xi)$$

$$+\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} dp\frac{(e^{px}-e^{k(\xi)(x-a)}e^{pa})}{K(p)+\xi}\Big(\frac{K(p)}{p^2}-\frac{\xi}{p}\Theta_{-\frac{1}{2}}(\xi)\Big)\Big)=M+R,$$

where

$$\Theta_{-\frac{1}{2}}(\xi)=-\Psi(\xi)\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{(1-e^{(q-k(\xi))a})K(q)}{(K(q)+\xi)q^2}dq.$$

As (2.32),(2.33) and (2.35), we obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{(1-e^{(q-k(\xi))a})K(q)}{(K(q)+\xi)q^2}dq\\ =\ &\frac{1}{\pi}a\int_0^\infty\frac{z^{-\frac{1}{2}}\left(z+k(\xi)\left|k(\xi)\right|^{-1}\right)}{(z^3+1)}dz+O(|k(\xi)|^\gamma),\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{(e^{px}-e^{k(\xi)(x-a)}e^{pa})}{K(p)+\xi}\frac{K(p)}{p^2}\\ =\ &\frac{1}{3}(a-x)(1+2k(\xi)\left|k(\xi)\right|^{-1})+O(|k(\xi)|^\gamma),\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{(e^{px}-e^{k(\xi)(x-a)}e^{pa})}{K(p)+\xi}\frac{K(p)}{p}\\ =\ &\frac{1}{3}(a-x)\left|k(\xi)\right|(1+2k(\xi)\left|k(\xi)\right|^{-1})+O(|k(\xi)|^\gamma),\end{aligned}$$

Therefore, we have

$$\begin{aligned}&|M(x,t)| \qquad (2.44)\\ =\ &\Big|\frac{1}{2\pi i}\int_{\Gamma_1}d\xi e^{\xi t}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}dp\frac{(e^{px}-e^{k(\xi)(x-a)}e^{pa})}{K(p)+\xi}\Big(\frac{K(p)}{p^2}+\frac{K(p)}{p}\Theta_{-\frac{1}{2}}(\xi)\Big)\Big|\\ \le\ &(a-x)C\Big|\int_{\Gamma_1}d\xi e^{\xi t}\left(1+2k(\xi)\left|k(\xi)\right|^{-1}\right)\Big|\le C(a-x)\left\langle t\right\rangle^{-1},\end{aligned}$$

Also, using (2.37) we get for $\xi\in\Gamma_2$

$$\begin{aligned}|\widetilde{\Theta}_{-\frac{1}{2}}(\xi)|\ &=\ \Big|\Psi(\xi)\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{1}{(K(q)+\xi)q^2}\left(1-e^{(q-k(\xi))a}\right)dq\Big| \qquad (2.45)\\ &\le\ |\Psi(\xi)|\left|k(\xi)\right|^{-1}\int_0^{+\infty}\frac{1}{|z^3+|\xi|^2\xi^{-2}||z|^{\frac{1}{2}}}dz\le C\left|k(\xi)\right|^{-1}.\end{aligned}$$

Therefore, we attain

$$\begin{aligned} &|R(x,y,t)| \qquad (2.46)\\ = &\Big|\frac{1}{2\pi i}\int_{\Gamma_2} d\xi e^{\xi t}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} dp\frac{(e^{px}-e^{k(\xi)(x-a)}e^{pa})}{K(p)+\xi}\Big(\frac{K(p)}{p^2}+\frac{K(p)}{p}\Theta_{-\frac{1}{2}}(\xi)\Big)\Big| \\ \le & e^{-\delta\frac{t}{2}}C\int_{\Gamma_2} d\xi e^{\mathrm{Re}\,\xi\frac{t}{2}}|k(\xi)|^{-1}\int_0^{+\infty}\frac{|z|^{\frac{1}{2}}+|z|^{-\frac{1}{2}}}{|z^3+|\xi|^2\xi^{-2}|}dz \\ \le & Ct^{-\frac{1}{3}}(1+(a-x)\langle t\rangle^{-\frac{2}{3}}). \end{aligned}$$

Thus, from (2.44),(2.46) by (2.43) we get

$$|G_{-\frac{1}{2}}(x,y,t)| \le Ct^{-\frac{1}{3}}(1+(a-x)\langle t\rangle^{-\frac{2}{3}}).$$

Substituting this estimate into the definition of operator $\mathcal{G}_{-\frac{1}{2}}$ and using

$$\int_0^{+\infty} G_{-\frac{1}{2}}(x,\tau)d\tau = \widehat{G}_{-\frac{1}{2}}(x,0)=0,$$

we find for $t>1$

$$\begin{aligned} |\mathcal{G}_{-\frac{1}{2}}h| &= (a-x)\Big|\int_0^{\frac{t}{2}}(t-\tau)^{-1}h(\tau)d\tau+\int_0^1\tau^{-1}(h(t-\tau)-h(t))d\tau \\ &\quad +h(t)\int_0^{+\infty}G_{-\frac{1}{2}}(x,\tau)d\tau+\int_1^{\frac{t}{2}}\tau^{-1}|(h(t-\tau)-h(t)|d\tau\Big| \\ &\le C(a-x)(t^{-\beta}+t^{-\beta+\gamma})\|h\|_{\mathbf{Z}_\beta}. \end{aligned}$$

As in the proof of, we get for $t<1$

$$\left\|\mathcal{G}_{-\frac{1}{2}}h\right\|_{\mathbf{L}^\infty} \le C\|h\|_{\mathbf{Z}_\beta}.$$

Lemma 1 is proved. □

In the same way we obtain the following lemma.

Lemma 2. *The following estimates are true*

$$\sup_{t>0} t^{\frac{2}{3}}\|\partial_x\mathcal{G}\phi\|_{\mathbf{L}^\infty(0,a)} \le C\|\phi\|_{\mathbf{L}^\infty(0,a)},$$

$$\sup_{t>0}\langle t\rangle^{\frac{1}{\alpha}}t^{\frac{2}{3}}\left\|(1+\frac{\langle t\rangle^{\frac{2}{3}}}{(a-x)})\partial_x\mathcal{G}_{\frac{1}{2}}h\right\|_{\mathbf{L}^\infty} \le C\|h\|_{\mathbf{Z}_\alpha},$$

$$\sup_{t>0}\langle t\rangle^{\frac{1}{\beta}-\gamma}t^{\frac{2}{3}}\left\|\frac{1}{(a-x)}\partial_x\mathcal{G}_{-\frac{1}{2}}h\right\|_{\mathbf{L}^\infty} \le C\|h\|_{\mathbf{Z}_\beta},$$

for $\gamma>0$ is small enough, provided that the right-hand sides are finite.

The local existence of the solutions for the initial boundary-value problem (1.1) can be obtained by the standard contraction mapping principle (for the proof see [20]).

Theorem 2. *Let the initial data* $u_0 \in \mathbf{C}^1(0,a)$. *Then for some* $T > 0$ *there exists a unique solution* $u(x,t) \in \mathbf{C}([0,T);\mathbf{L}^{\infty}(0,a)) \cap \mathbf{C}((0,T);\mathbf{C}^1([0,a]))$ *of the problem* (1.1).

3. Proof of Theorem1

From Proposition 1, we write the solution $u(x,t)$ of the problem (1.1) in the form

$$u(x,t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)\,d\tau + \mathcal{G}_{\frac{1}{2}}(t)h_1 + \mathcal{G}_{-\frac{1}{2}}(t)h_2, \tag{3.1}$$

where operators $\mathcal{G}, \mathcal{G}_{\frac{1}{2}}$ and $\mathcal{G}_{-\frac{1}{2}}$ were defined by (2.2),(2.5) and (2.6). From Lemma 1 we see that the operator

$$\mathcal{B}_0(t)h_1 = (1+t)^{-\frac{1}{\alpha}} \lim_{t\longrightarrow\infty}(1+t)^{\frac{1}{\alpha}}h_1(t)$$

is the asymptotic operator for the operator $\mathcal{G}_{\frac{1}{2}}$ in spaces $\mathbf{Y}$, $\mathbf{Z}$ such that the estimate is true for some $\gamma > 0$

$$\left\| t^{\gamma}(a-x)^{-1}\left(\mathcal{G}_{\frac{1}{2}}(t)h_1 - \mathcal{B}_0(t)h_1\right)\right\|_{\mathbf{Y}} \le C\,\|h_1\|_{\mathbf{Z}_\alpha}, \tag{3.2}$$

where

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0} t^{\frac{1}{\alpha}}\|\phi(\cdot,t)\|_{\mathbf{L}^{\infty}(0,a)}. \tag{3.3}$$

Let

$$\mathcal{G}_0\phi = (a-x)\Lambda t^{-\frac{2}{3}} \lim_{x\to a-}\phi. \tag{3.4}$$

From Lemma 1, we see that the operator $\mathcal{G}_0(t)\phi$ is the asymptotic operator for the Green operator $\mathcal{G}$ in spaces $\mathbf{Y}$, $\mathbf{C}^1(0,a)$ such that the estimate is true for any $\phi \in \mathbf{C}^1(0,a)$

$$\left\|\langle t\rangle^{\frac{2}{3}-\frac{1}{\alpha}}(a-x)^{-1}\left(\mathcal{G}(t)\phi - \mathcal{G}_0(t)\phi\right)\right\|_{\mathbf{Y}} \le C\,\|\phi'\|_{\mathbf{L}^{\infty}}. \tag{3.5}$$

After the change $\tau = tz$ we get for some constant θ

$$\int_0^t \mathcal{G}_0(t-\tau)\mathcal{N}(\mathcal{B}_0(\tau)h_1)d\tau = t^{-\frac{2}{\alpha}+\frac{1}{3}}\Lambda\theta^2\lambda(x-a)\int_0^1 (1-z)^{-\frac{2}{3}} z^{-\frac{2}{\alpha}}dz.$$

Since $\frac{1}{\alpha} \le \frac{2}{\alpha} - \frac{1}{3} < \frac{2}{3}$, we call the nonlinearity $\mathcal{N}$ in equation (1.1) critical convective.

We introduce a complete metric space $\mathbf{X}$ of functions defined on $[0,a]\times\mathbf{R}^+$

$$\mathbf{X} = \{\phi(x,t) \in \mathbf{C}([0,\infty);\ \mathbf{C}^1(0,a)); \|\phi\|_{\mathbf{X}} < +\infty\},$$

where

$$\|\phi\|_{\mathbf{X}} = \sum_{n=0}^{1} \sup_{t>0} \langle t\rangle^{\frac{1}{\alpha}} t^{\frac{2n}{3}} \|\partial_x^n \phi(t)\|_{\mathbf{L}^\infty}.$$

Now, we prove the global existence of the solution of the problem (1.1) in the space $\mathbf{X}$.Via Lemma 2.1 we have

$$\left\| \int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}(w) - \mathcal{N}(v)\right) d\tau \right\|_{\mathbf{X}} \le C\|w-v\|_{\mathbf{X}} (\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}). \quad (3.6)$$

Also, we have

$$\|\mathcal{G}\phi\|_{\mathbf{X}} \le C\sum_{n=0}^{1} \|\partial_x^n \phi\|_{\mathbf{L}^\infty}. \quad (3.7)$$

We apply the contraction mapping principle in the ball $\mathbf{X}_\rho = \{\phi \in \mathbf{X} : \|\phi\|_{\mathbf{X}} \le \rho\}$ of a radius

$$\rho = \frac{1}{2C}\Big(\sum_{n=0}^{1} \|\partial_x^n u_0\|_{\mathbf{L}^\infty} + \|h_1\|_{\mathbf{Z}_\alpha} + \|h_1\|_{\mathbf{Z}_\beta}\Big) > 0.$$

For $v \in \mathbf{X}_\rho$ we define the mapping $\mathcal{M}(v)$ by formula

$$\mathcal{M}(v) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)\,d\tau + \mathcal{G}_{\frac{1}{2}}(t)h_1 + \mathcal{G}_{-\frac{1}{2}}(t)h_2. \quad (3.8)$$

We first prove that $\|\mathcal{M}(v)\|_{\mathbf{X}} \le \rho$, where $\rho > 0$ is sufficiently small. From (3.6),(3.7) and the integral formula (3.8) we have

$$\begin{aligned}\|\mathcal{M}(v)\|_{\mathbf{X}} &\le \|\mathcal{G}u_0\|_{\mathbf{X}} + \left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))\,d\tau\right\|_{\mathbf{X}} \\ &+ \|\mathcal{G}_{\frac{1}{2}}(t)h_1\|_{\mathbf{X}} + \|\mathcal{G}_{-\frac{1}{2}}(t)h_2\|_{\mathbf{X}} \\ &\le C\Big(\sum_{n=0}^{1}\|\partial_x^n u_0\|_{\mathbf{L}^\infty} + \|h_1\|_{\mathbf{Z}_\alpha} + \|h_1\|_{\mathbf{Z}_\beta}\Big) + C\|v\|_{\mathbf{X}}^2 \\ &\le \frac{\rho}{2} + C\rho^2 < \rho,\end{aligned}$$

since $\rho > 0$ is sufficiently small. Hence the mapping $\mathcal{M}$ transforms the ball $\mathbf{X}_\rho$ into itself. In the same manner we estimate the difference

$$\|\mathcal{M}(w) - \mathcal{M}(v)\|_{\mathbf{X}} \le \frac{1}{2}\|w-v\|_{\mathbf{X}},$$

which shows that $\mathcal{M}$ is a contraction mapping. Therefore, there exists a unique solution $u \in \mathbf{X}$ to the problem (1.1) such that

$$\|u\|_{\mathbf{X}} \leq \varepsilon. \tag{3.9}$$

Because of this, a priory estimate and Theorem 2 by the standard continuation method, we get the global existence of a unique solution

$$u \in \mathbf{C}([0,\infty)\,;\mathbf{L}^{\infty}\,(0,a)) \cap \mathbf{C}((0,\infty);\mathbf{C}^{1}\,([0,a])).$$

We now compute the asymptotics of the solution. Denote by

$$V(t) = h_1(t) + (x-a)t^{\frac{1}{3}}\Lambda \int_0^1 (1-z)^{-\frac{2}{3}}\mathcal{N}(h_1(tz))dz,$$

By definition of operator $\mathcal{G}_0$ we have,

$$V = h_1 + \int_0^t \mathcal{G}_0(t-\tau)\mathcal{N}(V(\tau))d\tau,$$

and therefore,

$$\begin{aligned}
&\left\|\langle t\rangle^{\gamma}(a-x)^{-1}(V-u(t))\right\|_{\mathbf{Y}} \\
&\leq C\left\|\langle t\rangle^{\gamma}(a-x)^{-1}\left(t^{-\frac{1}{3}}V - \mathcal{G}(t)\,u_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)d\tau - \mathcal{G}_{\frac{1}{2}}h_1 - \mathcal{G}_{-\frac{1}{2}}h_2\right)\right\|_{\mathbf{Y}} \\
&\leq C\left\|\langle t\rangle^{\gamma}(a-x)^{-1}\mathcal{G}(t)\,u_0\right\|_{\mathbf{Y}} \\
&+ C\left\|\langle t\rangle^{\gamma}(a-x)^{-1}\int_0^t \mathcal{G}_0(t-\tau)\left(\mathcal{N}(u(t)) - \mathcal{N}(V)\right)d\tau\right\|_{\mathbf{Y}} \\
&+ C\left\|\langle t\rangle^{\gamma}(a-x)^{-1}\int_0^t \left(\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau)\right)\mathcal{N}(u(t))d\tau\right\|_{\mathbf{Y}} \\
&+ C\left(\left\|\langle t\rangle^{\gamma}(a-x)^{-1}\left(\mathcal{G}_{\frac{1}{2}}h_1 - h_1\right)\right\|_{\mathbf{Y}} + C\left\|\langle t\rangle^{\gamma}(a-x)^{-1}\mathcal{G}_{-\frac{1}{2}}h_2\right\|_{\mathbf{Y}}\right) \\
&\equiv I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

for some $\gamma > 0$. From Lemma 1 we obtain

$$I_1 \leq \left\|\langle t\rangle^{\gamma}(a-x)^{-1}\mathcal{G}(t)\,u_0\right\|_{\mathbf{Y}} \leq C\sum_{n=0}^{1}\|\partial_x^n u_0\|_{\mathbf{L}^{\infty}}.$$

Since,

$$\lim_{x \longrightarrow a-0} u(x,t) = h_1(t),$$

using (3.4) we gain

$$I_2 = C\left\|\langle t\rangle^{\gamma}(a-x)^{-1}\int_0^t \mathcal{G}_0(t-\tau)\left(\mathcal{N}(u(\tau)) - \mathcal{N}(V)\right)d\tau\right\|_{\mathbf{Y}} = 0.$$

Also, from estimate (3.5) we get

$$I_3 = C\left\| \langle t\rangle^{\gamma} (a-x)^{-1} \int_0^t \left(\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau)\right)\mathcal{N}(u(t))d\tau\right\|_{\mathbf{Y}} \le C.$$

Via Lemma 1 and the estimate (3.2) we obtain

$$I_4 = C\left(\left\|\langle t\rangle^{\gamma} (a-x)^{-1}\left(\mathcal{G}_{\frac{1}{2}}h_1 - h_1\right)\right\|_{\mathbf{Y}} + \left\|\langle t\rangle^{\gamma} (a-x)^{-1}\mathcal{G}_{\frac{1}{2}}h_2\right\|_{\mathbf{Y}}\right) \le C.$$

Hence,

$$\left\|\langle t\rangle^{\gamma} (a-x)^{-1} (u(t) - V)\right\|_{\mathbf{Y}} \le C;$$

this completes the proof of Theorem 1.

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