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GLOBAL EXISTENCE OF SOLUTIONS FOR A REACTION-DIFFUSION SYSTEM

Yutaka Aoyagi†, Kimitoshi Tsutaya, and Yusuke Yamauchi
Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan

(Submitted by: Tohru Ozawa)

Abstract. We show the global existence of solutions of a reaction-diffusion system with the nonlinear terms $|x|^{\sigma_j}u^{p_j}v^{q_j}$. The proof is based on the existence of supersolutions and the comparison principle. We also prove that uniqueness of the global solutions holds in the superlinear case by contraction argument. Our conditions for the global existence are optimal in view of the nonexistence results proved by Yamauchi [12].

1. Introduction

We consider the Cauchy problem for the reaction-diffusion system:

$$\begin{cases} u_t - \Delta u = |x|^{\sigma_1}u^{p_1}v^{q_1}, & x \in \boldsymbol{R}^N, \ \ t > 0, \\ v_t - \Delta v = |x|^{\sigma_2}u^{p_2}v^{q_2}, & x \in \boldsymbol{R}^N, \ \ t > 0, \\ u(x,0) = u_0(x) \geq 0, \not\equiv 0, & x \in \boldsymbol{R}^N, \\ v(x,0) = v_0(x) \geq 0, \not\equiv 0, & x \in \boldsymbol{R}^N, \end{cases} \tag{1.1}$$

where $p_j, q_j \geq 0$, $\sigma_j \geq 0$ $(j = 1, 2)$, and $p_1, q_2 \neq 1$.

The combustion process of a single solid chemical material is expressed as a reaction-diffusion system (see, e.g., [2]). The two unknown functions in this original system represent the temperature and mass of the material. The nonlinearity in the original system consists of powers and exponential forms of the unknown functions. Our problem (1.1) describes a simplified model to investigate the effect of this type of nonlinearity.

Before stating our main results, we first recall known results for the single equation, $u_t - \Delta u = u^p$. Let N be the space dimension. Fujita [6] proved the existence of global solutions to the equation if $p > 1 + 2/N$ for exponentially

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†Current address: Fujitsu Nagano Systems Engineering Limited, Nagano, Japan 380-0813.

decaying small initial data. He also proved the nonexistence of global solutions if $p < 1 + 2/N$. In the critical case, $p = 1 + 2/N$, the nonexistence is proved by Hayakawa [7]; Kobayashi, Sirao and Tanaka [8]; and Weissler [11]. On the other hand, in the sublinear case, i.e., $0 < p < 1$, it was shown by Aguirre and Escobedo [1] that every solution to the equation exists globally in time.

For nonlinear terms with variable coefficients, Pinsky [10] showed the existence and nonexistence of global solutions to the equation $u_t - \Delta u = a(x)u^p$, where $p > 1$ and $a(x)$ behaves like $|x|^m$ with $m > -2$ for large $|x|$. He proved in [10] that there exist global solutions to the equation if $p > 1 + (2 + m)/N$, and that there exist no global solutions to the equation if $1 < p \leq 1 + (2 + m)/N$.

We next consider a system

$$\begin{cases} u_t - \Delta u = F_1(x, u, v), \\ v_t - \Delta v = F_2(x, u, v). \end{cases}$$

Escobedo and Herrero [4] studied the system with the nonlinear terms $F_1 = v^p$ and $F_2 = u^q$ with $p, q > 0$ for nonnegative, and continuous and bounded initial data. Their results are divided into three cases:

(i) $pq > 1$ and $(\max(p, q) + 1)/(pq - 1) < N/2$,
(ii) $pq > 1$ and $(\max(p, q) + 1)/(pq - 1) \geq N/2$,
(iii) $pq < 1$.

When $pq > 1$ and $(\max(p, q) + 1)/(pq - 1) < N/2$, there exist global solutions for small initial data. For large initial data, some solutions blow up in finite time. When $pq > 1$ and $(\max(p, q) + 1)/(pq - 1) \geq N/2$, there exist no global solutions. When $pq < 1$, every solution exists globally in time.

In case $F_1 = |x|^{\sigma_1} v^p$ and $F_2 = |x|^{\sigma_2} u^q$ with $p, q > 1$, $0 \leq \sigma_j < N(p_j + q_j - 1)$ $(j = 1, 2)$, Mochizuki and Huang [9] showed the existence and nonexistence of global solutions and the asymptotic behavior of the global solution. Let $\alpha = \{(p(\sigma_2 + 2) + (\sigma_1 + 2)\}/2\{(pq - 1)\}$ and $\beta = \{(q(\sigma_1 + 2) + (\sigma_2 + 2)\}/\{2(pq - 1)\}$. They proved that if $0 < \max(\alpha, \beta) < N/2$, then a global solution exists for small initial data and does not exist for large initial data. On the other hand, if $\max(\alpha, \beta) \geq N/2$, then there exist no global solutions.

For $F_j = u^{p_j} v^{q_j}$ with $p_j, q_j \geq 0$ $(j = 1, 2)$, $0 < p_1 + q_1 \leq p_2 + q_2$, Escobedo and Levine [5] showed the following results. Let $\alpha = (q_1 - q_2 + 1)/\{p_2 q_1 - (1 - p_1)(1 - q_2)\}$ and $\beta = (p_1 - p_2 + 1)/\{p_2 q_1 - (1 - p_1)(1 - q_2)\}$.

(i) Let $p_1 \le 1$. If $0 \le \max(\alpha, \beta) < N/2$, then a global solution exists for small initial data and does not exist for large initial data. If $\max(\alpha, \beta) < 0$, then every solution exists globally in time. If $\max(\alpha, \beta) \ge N/2$, then there exist no global solutions.

(ii) Let $p_1 > 1$. If $p_1 + q_1 > 1 + 2/N$, then a global solution exists for small initial data and does not exist for large initial data. If $p_1 + q_1 \le 1 + 2/N$, then there exist no global solutions.

In this paper, we consider (1.1) as a generalization of these nonlinear terms. We prove the existence of global solutions to (1.1) under some conditions on N, σ_j, p_j, and q_j $(j = 1, 2)$, which are optimal in view of the nonexistence results proved by Yamauchi [12].

Let $F_j = |x|^{\sigma_j} u^{p_j} v^{q_j}$ with $p_j, q_j \ge 0$, $\sigma_j > \max(-2, -N)$ $(j = 1, 2)$, $0 < (p_1 + q_1 - 1)/(2 + \sigma_1) \le (p_2 + q_2 - 1)/(2 + \sigma_2)$, and put α and β as in (2.1).

(i) Let $p_1 < 1$. If $0 < \max(\alpha, \beta) < N/2$, then a global solution does not exist for large initial data. If $\max(\alpha, \beta) \ge N/2$, then there exist no global solutions.

(ii) Let $p_1 > 1$. If $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, then a global solution does not exist for large initial data. If $p_1 + q_1 \le 1 + (2 + \sigma_1)/N$, then there exist no global solutions.

We emphasize that if $\sigma_j = 0$ $(j = 1, 2)$ or $p_1 = q_2 = 0$, our conditions for the global existence coincide with those of [5] or [9], respectively.

In showing time local and global existence of the solutions, we have two essential differences between the proof in this paper and that in previous one. First, we improved the estimates with weighted norm in [9]. For our system we need to weaken the function spaces of the solutions in [9] because the nonlinearities are strongly coupled. Therefore, we prepared the estimates for the spaces again. Second, we reconstructed supersolutions corresponding with our nonlinearities. Since we can not apply a contraction-mapping argument in the sublinear case, we employ the proof with a supersolution and comparison argument. However, the super-solution in [5] does not seem applicable because of the variable coefficient $|x|^\sigma$ in the nonlinearities. Thus, especially in the sublinear case $p_1, q_2 < 1$, we reconstructed supersolutions, which contain the exponential function e^{kt} $(k > 0)$.

Our plan of this paper is as follows. In Section 2, we state some notation and our main results. In Section 3, we show the local existence of classical solutions of the system (1.1). In Sections 4 and 5, we show the existence of global solutions by comparison principle and contraction argument, respectively.

2. MAIN RESULTS

We begin with stating some notation. Put

$$\begin{cases} \alpha = \dfrac{q_1(\sigma_2+2)+(1-q_2)(\sigma_1+2)}{2\{p_2q_1-(1-p_1)(1-q_2)\}}, \\ \beta = \dfrac{p_2(\sigma_1+2)+(1-p_1)(\sigma_2+2)}{2\{p_2q_1-(1-p_1)(1-q_2)\}}, \end{cases} \tag{2.1}$$

$$\begin{cases} \delta_1 = \dfrac{q_1\sigma_2+(1-q_2)\sigma_1}{p_2q_1-(1-p_1)(1-q_2)}, \\ \delta_2 = \dfrac{p_2\sigma_1+(1-p_1)\sigma_2}{p_2q_1-(1-p_1)(1-q_2)}. \end{cases} \tag{2.2}$$

We note that (α,β) are solutions to the system

$$\begin{pmatrix} -\alpha-1 \\ -\beta-1 \end{pmatrix} = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} + \begin{pmatrix} \sigma_1/2 \\ \sigma_2/2 \end{pmatrix}. \tag{2.3}$$

For $a \in \boldsymbol{R}$, we define the following function spaces:

$$I^a = \{w \in C(\boldsymbol{R}^N) : w(x) \geq 0,\ \limsup_{|x|\to\infty} |x|^a w(x) < \infty\},$$

$$L^\infty_a = \{w \text{ is measurable function on } \boldsymbol{R}^N : \\ w(x) \geq 0,\ \|w\|_{\infty,a} \equiv \|\langle x\rangle^a w(x)\|_\infty < \infty\},$$

where $\langle x\rangle = \left(1+|x|^2\right)^{1/2}$. We also define

$$E_T = \{(u,v) : [0,T] \to L^\infty_{\delta_1} \times L^\infty_{\delta_2},\ u,v \geq 0,\ \|(u,v)\|_{E_T} < \infty\},$$

where

$$\|(u,v)\|_{E_T} = \sup_{t\in[0,T]} (\|u(t)\|_{\infty,\delta_1} + \|v(t)\|_{\infty,\delta_2}).$$

We state our main results. Throughout this paper, we assume that $(u_0,v_0) \in I^{\delta_1} \times I^{\delta_2}$.

Theorem 2.1. *Let* $p_1 < 1$ *and* $q_2 < 1$.

(i) *If* $0 < \max(\alpha,\beta) < N/2$, *then there exist global classical solutions of* (1.1) *for small initial data.*

(ii) *If* $\max(\alpha,\beta) < 0$, *then every classical solution of* (1.1) *exists globally in time.*

Theorem 2.2. *Let* $p_1 > 1$ *and* $q_2 < 1$. *If* $\alpha < N/2$ *and* $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, *then there exist global classical solutions of* (1.1) *for small initial data.*

Theorem 2.3. *Let* $p_1 < 1$ *and* $q_2 > 1$. *If* $\beta < N/2$ *and* $p_2 + q_2 > 1 + (2 + \sigma_2)/N$, *then there exist global classical solutions of* (1.1) *for small initial data.*

Theorem 2.4. *Let* $p_1 > 1$ *and* $q_2 > 1$. *If* $p_1 + q_1 > 1 + (2 + \sigma_1)/N$ *and* $p_2 + q_2 > 1 + (2 + \sigma_2)/N$, *then there exist global classical solutions of* (1.1) *for small initial data.*

Tables 1–4 at the end of this paper summarize Theorems 2.1–2.4 and global nonexistence theorems in [12]. We abbreviate global existence and blowing-up to "G.E." and "B.L." in the tables, respectively.

For $p_1 < 1$, we can rewrite Theorems 2.1 and 2.3 in the manner of [5] as follows.

Corollary 2.5. *Assume that*

$$\frac{p_1 + q_1 - 1}{\sigma_1 + 2} \leq \frac{p_2 + q_2 - 1}{\sigma_2 + 2}, \tag{2.4}$$

and let $p_1 < 1$ *and* $q_2 \neq 1$.

(i) *If* $0 < \max(\alpha, \beta) < N/2$, *then there exist global classical solutions of* (1.1) *for small initial data.*

(ii) *If* $\max(\alpha, \beta) < 0$, *then every classical solution exists globally.*

Proof. (i) In the case $p_1, q_2 < 1$, it is clear from Theorem 2.1(1). We consider the case $p_1 < 1 < q_2$. From Theorem 2.3, it is enough to show that $p_2 + q_2 > 1 + (2 + \sigma_2)/N$ under the assumptions in Corollary 2.5. Using (2.4), we have

$$\begin{aligned}\beta &= \frac{p_2(\sigma_1 + 2) + (1 - p_1)(\sigma_2 + 2)}{2\{p_2 q_1 - (1 - p_1)(1 - q_2)\}} \\ &\geq \frac{p_2 \frac{(p_1 + q_1 - 1)(\sigma_2 + 2)}{(p_2 + q_2 - 1)} + (1 - p_1)(\sigma_2 + 2)}{2\{p_2 q_1 - (1 - p_1)(1 - q_2)\}} = \frac{\sigma_2 + 2}{p_2 + q_2 - 1}\ .\end{aligned}$$

Thus, $p_2 + q_2 > 1 + (2 + \sigma_2)/N$ holds by $\beta < N/2$.

(ii) Since $\beta < 0$ and $p_1 < 1$, we see that $2\{p_2 q_1 - (1 - p_1)(1 - q_2)\} < 0$ from the definition of β. Hence, we have $q_2 < 1$. In fact, if we assume $q_2 \geq 1$ then $2\{p_2 q_1 - (1 - p_1)(1 - q_2)\} \geq 0$. In the case $p_1, q_2 < 1$ we can apply Theorem 2.1(ii). □

Also for $p_1 > 1$, we can rewrite Theorems 2.2 and 2.4 in the way of [5].

Corollary 2.6. *Assume* (2.4), *and let* $p_1 > 1$ *and* $q_2 \neq 1$. *If* $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, *then there exist global classical solutions of* (1.1) *for small initial data.*

Proof. In the case $p_1, q_2 > 1$, it is clear from Theorem 2.4. We consider the case $p_1 > 1 > q_2$. From Theorem 2.2, it is enough to show that $\alpha < N/2$ under the assumptions in Corollary 2.6. Using (2.4), we have

$$\begin{aligned}\alpha &= \frac{q_1(\sigma_2+2)+(1-q_2)(\sigma_1+2)}{2\{p_2q_1-(1-p_1)(1-q_2)\}} \\ &\le \frac{q_1\frac{(p_2+q_2-1)(\sigma_1+2)}{p_1+q_1-1}+(1-q_2)(\sigma_1+2)}{2\{p_2q_1-(1-p_1)(1-q_2)\}} = \frac{\sigma_1+2}{2(p_1+q_1-1)}\ .\end{aligned}$$

Thus, $\alpha < N/2$ holds by $p_1 + q_1 > 1 + (2 + \sigma_1)/N$. □

Remark 2.7. We remark that the condition $p_1 + q_1 > 1 + (2 + \sigma_1)/N$ in Corollary 2.6 consists of only the exponents in one equation. This condition is the same as that for the global existence for the single equation $u_t - \Delta u = |x|^{\sigma_1} u^{p_1+q_1}$. See [10] for details.

In the strongly coupled system (1.1), interaction between the two equations and self-growth in one equation prevent the solutions from existing globally in time.

In the sublinear case $p_1, q_2 < 1$, interaction between the two equations makes the solutions blow up. Indeed, for the single sublinear equation $u_t - \Delta u = u^p$ $(p < 1)$ [1], the solution exists globally; that is, the self-growth of the solution does not work. We can also see the fact from the conditions for the blowing-up consisting of six exponents p_j, q_j, σ_j ($j = 1$ and 2).

On the other hand, in the superlinear case, self-growth of the solution is stronger than the interaction. We can also see the fact from the conditions for the blowing-up consisting of only three exponents p_j, q_j, σ_j ($j = 1$ or 2). Here, since both u and v have the same basic properties for the solutions of heat equation, the equation $u_t - \Delta u = |x|^{\sigma_1} u^{p_1+q_1}$ appears from (1.1) by identifying v with u.

In the superlinear case, we have a uniqueness result.

Theorem 2.8. *Assume* (2.4) *and let* $p_j > 1$ *and* $q_j > 1$ $(j = 1, 2)$. *Assume that* $(u_0, v_0) \in I^a \times I^b$ $((\sigma_1 + 2)/(p_1 + q_1 - 1) < a, b < N)$. *If* $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, *then there exist unique global solutions of* (1.1) *for small initial data.*

We make a remark about the definition of (α, β).

Remark 2.9. We can see that the definition of (α, β) (2.1) is a generalization of that in known results. First, substituting $\sigma_1 = \sigma_2 = 0$ and $p_1 = q_2 = 0$ into (2.1), we see that (α, β) coincide with that in [5] and [9], respectively.

Moreover, substituting $\sigma_1 = \sigma_2 = 0$, $p_1 = q_2 = 0$, and $p_2 = q_1$ into (2.1), we also see that the condition coincides with that in the classical result for a single equation [6]. Second, if $(u(x,t), v(x,t))$ are solutions to (1.1), then $(\lambda^\alpha u(\sqrt{\lambda}x, \lambda t), \lambda^\beta v(\sqrt{\lambda}x, \lambda t))$ are solutions to (1.1) again. For the known results [4], [5] and [9], we can also obtain the exponents (α, β) by the same calculation as this.

3. Local existence theorem

In this section, we show the local existence of classical solutions of (1.1).

Theorem 3.1. *Let δ_1 and δ_2 be as defined in* (2.2). *Assume that* $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$. *Then there exist classical solutions* $(u(t), v(t)) \in E_T$ *for the system* (1.1) *for some* $T > 0$.

Proof. We first construct local solutions to the system of integral equations associated with (1.1):

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_1}u(s)^{p_1}v(s)^{q_1}ds, \tag{3.1}$$

$$v(t) = e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_2}u(s)^{p_2}v(s)^{q_2}ds, \tag{3.2}$$

where

$$e^{t\Delta}f(x) = (4\pi t)^{-\frac{N}{2}}\int_{\boldsymbol{R}^N}\exp\Big(-\frac{|x-y|^2}{4t}\Big)f(y)dy.$$

It is sufficient to show Propositions 3.2 and 3.4 below to prove Theorem 3.1. The local existence of solutions of (3.1) and (3.2) is given by the following proposition.

Proposition 3.2. *Let δ_1 and δ_2 be as defined in* (2.2). *Assume that* $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$. *Then there exist* $(u(t), v(t)) \in E_T$ *satisfying the integral equations* (3.1) *and* (3.2) *for some* $T > 0$.

To prove the proposition, we define $\{u_n(x,t)\}$ and $\{v_n(x,t)\}$ $(n = 1, 2, \dots)$ inductively by

$$u_{n+1}(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_1}u_n(s)^{p_1}v_n(s)^{q_1}ds,$$
$$v_{n+1}(t) = e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_2}u_n(s)^{p_2}v_n(s)^{q_2}ds,$$
$$u_1 = e^{t\Delta}u_0, \qquad v_1 = e^{t\Delta}v_0.$$

We use the following estimates by weighted norms and uniform estimates for the solutions.

Lemma 3.3. (1) *Let* $\delta, a \in \boldsymbol{R}$, $\sigma \geq 0$ *and* $\delta + a + \sigma = 0$. *Then we have*

$$\|e^{t\Delta}|x|^\sigma \langle x\rangle^a\|_{\infty,\delta} \leq \begin{cases} C(1+t)^{\max(-\delta,0,\delta-N)/2}, & (\delta \neq N), \\ C\log(2+t), & (\delta = N). \end{cases} \tag{3.3}$$

(2) *Suppose that* $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$. *Then there exist* $K > 0$ *and* $T > 0$ *such that*

$$\sup_{t\in[0,T]} \|u_n(t)\|_{\infty,\delta_1} < K, \qquad \sup_{t\in[0,T]} \|v_n(t)\|_{\infty,\delta_2} < K$$

for all n.

Proof. (1) We show the estimates involved in three cases: (i) $0 \leq \delta \leq N$, (ii) $\delta < 0$, and (iii) $\delta > N$.

For the case $0 \leq \delta \leq N$, using Lemma 2.1 in [9], we can see that

$$\|e^{t\Delta}|x|^\sigma \langle x\rangle^a\|_{\infty,\delta} \leq \|e^{t\Delta}\langle x\rangle^{\sigma+a}\|_{\infty,\delta} \leq \begin{cases} C, & (0 \leq \delta < N), \\ C\log(2+t), & (\delta = N). \end{cases}$$

In the case $\delta < 0$, we have

$$\begin{aligned} e^{t\Delta}|x|^\sigma\langle x\rangle^a &= (4\pi t)^{-N/2}\int_{\boldsymbol{R}^N} \exp\Big(-\frac{|y|^2}{4t}\Big)|x-y|^\sigma\langle x-y\rangle^a dy \\ &\leq (4\pi t)^{-N/2}\int_{\boldsymbol{R}^N} \exp\Big(-\frac{|y|^2}{4t}\Big)\langle x-y\rangle^{\sigma+a} dy \\ &\leq Ct^{-N/2}\int_{|y|\leq|x|/2} \exp\Big(-\frac{|y|^2}{4t}\Big)\langle x\rangle^{\sigma+a} dy \\ &\quad + Ct^{-N/2}\int_{|y|>|x|/2} \exp\Big(-\frac{|y|^2}{4t}\Big)\langle y\rangle^{\sigma+a} dy \\ &\leq C\langle x\rangle^{\sigma+a} + C(1+t)^{(\sigma+a)/2}. \end{aligned}$$

Finally, for $\delta > N$, we have

$$\begin{aligned} e^{t\Delta}|x|^\sigma\langle x\rangle^a &= (4\pi t)^{-N/2}\int_{\boldsymbol{R}^N} \exp\Big(-\frac{|x-y|^2}{4t}\Big)|y|^\sigma\langle y\rangle^a dy \\ &\leq Ct^{-N/2}\int_{|y|\leq|x|/2} \exp\Big(-\frac{|x|^2}{16t}\Big)\langle y\rangle^{\sigma+a} dy \end{aligned}$$

$$+ Ct^{-N/2}\int_{|y|>|x|/2}\exp\Big(-\frac{|x-y|^2}{4t}\Big)\langle x\rangle^{\sigma+a}dy$$
$$\leq Ct^{-N/2}|x|^N\langle x\rangle^{\sigma+a}\langle x\rangle^{-(N+\sigma+a)}\exp\Big(-\frac{|x|^2}{16t}\Big)+C\langle x\rangle^{\sigma+a}$$
$$\leq C\langle x\rangle^{\sigma+a}(1+t)^{-(N+\sigma+a)/2}.$$

(2) We first estimate $\|e^{t\Delta}u_0\|_{\infty,\delta_1}$. By (3.3), we have

$$\|e^{t\Delta}u_0\|_{\infty,\delta_1}\leq\|u_0\|_{\infty,\delta_1}\|e^{t\Delta}\langle\,\cdot\,\rangle^{-\delta_1}\|_{\infty,\delta_1}$$
$$\leq\begin{cases}C\|u_0\|_{\infty,\delta_1} & (0\leq\delta_1<N),\\ C\|u_0\|_{\infty,\delta_1}(1+T)^{\kappa_1} & (\text{otherwise}),\end{cases}$$

where $\kappa_1=\kappa_1(\delta_1,N)>0$.
From the above, there exists a constant $\tilde{C}=\tilde{C}(T)>0$ satisfying

$$\sup_{t\in[0,T]}\|e^{t\Delta}u_0\|_{\infty,\delta_1}\leq\tilde{C}\|u_0\|_{\infty,\delta_1},$$
$$\sup_{t\in[0,T]}\|e^{t\Delta}v_0\|_{\infty,\delta_2}\leq\tilde{C}\|v_0\|_{\infty,\delta_2}$$

for fixed $T>0$.

We next estimate the nonlinear terms. Define $\Phi_1(u,v)$ and $\Phi_2(u,v)$ by

$$\Phi_1(u,v)(t)=\int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_1}u(s)^{p_1}v(s)^{q_1}ds,$$
$$\Phi_2(u,v)(t)=\int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_2}u(s)^{p_2}v(s)^{q_2}ds.$$

Applying (3.3) again, we obtain

$$\sup_{t\in[0,T]}\|\Phi_1(u_n,v_n)(t)\|_{\infty,\delta_1}\leq C(T)\sup_{t\in[0,T]}\|u_n(t)\|_{\infty,\delta_1}^{p_1}\sup_{t\in[0,T]}\|v_n(t)\|_{\infty,\delta_2}^{q_1},$$
$$\sup_{t\in[0,T]}\|\Phi_2(u_n,v_n)(t)\|_{\infty,\delta_2}\leq C(T)\sup_{t\in[0,T]}\|u_n(t)\|_{\infty,\delta_1}^{p_2}\sup_{t\in[0,T]}\|v_n(t)\|_{\infty,\delta_2}^{q_2},$$

where $C(T)\downarrow 0$ $(T\downarrow 0)$. Indeed, for example, we can see

$$\|\Phi_1(u,v)(t)\|_{\infty,\delta_1}$$
$$\leq\sup_{t\in[0,T]}\|u(t)\|_{\infty,\delta_1}^{p_1}\sup_{t\in[0,T]}\|v(t)\|_{\infty,\delta_2}^{q_1}\int_0^t\|e^{(t-s)\Delta}|\cdot|^{\sigma_1}\langle\,\cdot\,\rangle^{-p_1\delta_1-q_1\delta_2}\|_{\infty,\delta_1}ds$$
$$\leq\begin{cases}C\sup_{t\in[0,T]}\|u(t)\|_{\infty,\delta_1}^{p_1}\sup_{t\in[0,T]}\|v(t)\|_{\infty,\delta_2}^{q_1}T & (0\leq\delta_1<N),\\ C\sup_{t\in[0,T]}\|u(t)\|_{\infty,\delta_1}^{p_1}\sup_{t\in[0,T]}\|v(t)\|_{\infty,\delta_2}^{q_1}T(1+T)^{\kappa_2} & (\text{otherwise}),\end{cases}$$

where $\kappa_2 = \kappa_2(\sigma_1, p_1, q_1, \delta_1, \delta_2, N) > 0$.

Here, we put $R = \max(\|u_0\|_{\infty,\delta_1}, \|v_0\|_{\infty,\delta_2})$. Taking large $K > 0$ and small $T > 0$ such that

$$K > 2\tilde{C}R,\ C(T) < \frac{K - \tilde{C}R}{K^{p_1+q_1} + K^{p_2+q_2}},$$

we have

$$\begin{aligned}
&\|e^{t\Delta}u_0 + \Phi_1(u,v)(t)\|_{\infty,\delta_1} \\
&\le \tilde{C}\|u_0\|_{\infty,\delta_1} + C(T) \sup_{t\in[0,T]} \|u_n(t)\|_{\infty,\delta_1}^{p_1} \sup_{t\in[0,T]} \|v_n(t)\|_{\infty,\delta_2}^{q_1} \\
&\le \tilde{C}R + \frac{K - \tilde{C}R}{K^{p_1+q_1} + K^{p_2+q_2}} K^{p_1+q_1} \le \tilde{C}R + K - \tilde{C}R = K.
\end{aligned}$$

This completes the proof. □

Returning to the proof of Proposition 3.2, one can see from Lemma 3.3(2) that

$$\sup_{t\in[0,T]} \|\langle\cdot\rangle^{\delta_1} u_n(t)\|_\infty < K, \qquad \sup_{t\in[0,T]} \|\langle\cdot\rangle^{\delta_2} v_n(t)\|_\infty < K$$

for all n. The monotonicity of the heat kernel gives

$$u_n \le u_{n+1}, \quad v_n \le v_{n+1}$$

for all n. Therefore, there exist $\widetilde{u}(x,t) = \lim_{n\to\infty} u_n(x,t)$ and $\widetilde{v}(x,t) = \lim_{n\to\infty} v_n(x,t)$ on $\boldsymbol{R}^N \times [0,T]$, and we have

$$\sup_{t\in[0,T]} \|\widetilde{u}(t)\|_{\infty,\delta_1} \le K, \qquad \sup_{t\in[0,T]} \|\widetilde{v}(t)\|_{\infty,\delta_2} \le K.$$

Moreover, from Lebesgue's monotone convergence theorem, we can easily see that $(\widetilde{u}, \widetilde{v})$ are local solutions of (3.1) and (3.2). This completes the proof of Proposition 3.2. □

Next, we improve the regularity of the local solutions given in Proposition 3.2.

Proposition 3.4. *Let* $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$*, and let* $(u,v) \in E_T$ *be solutions of* (3.1) *and* (3.2) *in* $\boldsymbol{R}^N \times (0,T]$*. Assume that there exists a constant* $C > 0$ *such that*

$$\|u(t)\|_{\infty,\delta_1} < C,\ \|v(t)\|_{\infty,\delta_2} < C \qquad (0 \le t \le T). \tag{3.4}$$

Then (u,v) *are classical solutions of* (1.1) *in* $\boldsymbol{R}^N \times (0,T)$*.*

Proof. From the assumptions, we can easily see that $|x|^{\sigma_j} u^{p_j} v^{q_j}$ $(j = 1, 2)$ are locally θ-Hölder continuous in x $(0 < \theta \le 1)$; that is, for any $\varepsilon > 0$ and for any compact set $K \subset \boldsymbol{R}^N$, there exists a constant $C > 0$ such that

$$||x_1|^{\sigma_j} u^{p_j} v^{q_j}(x_1, t) - |x_2|^{\sigma_j} u^{p_j} v^{q_j}(x_2, t)| < C|x_1 - x_2|^{\theta},$$

for any $t \in (\varepsilon, T)$, and $x_1, x_2 \in K$. It follows from the Hölder continuity and the standard regularity argument in [3] Section I.3 that (u, v) are classical solutions. □

This completes the proof of Theorem 3.1. □

4. Proof of Theorems 2.1–2.4

In proving Theorems 2.1–2.4, we use a comparison theorem and the existence of supersolutions. First, we show the following comparison theorem.

Proposition 4.1. (*cf.* [5] *Lemma* A.1) *Let $f(u, v)$ and $g(u, v)$ be strictly monotone increasing in u and v for u, $v \ge 0$. Assume that $\bar{u}$, $\bar{v}$, $\underline{u}$, and $\underline{v}$ are non-negative and satisfy*

$$\begin{cases} \bar{u}_t - \Delta \bar{u} \ge |x|^{\sigma_1} f(\bar{u}, \bar{v}), \\ \bar{v}_t - \Delta \bar{v} \ge |x|^{\sigma_2} g(\bar{u}, \bar{v}), \\ \underline{u}_t - \Delta \underline{u} \le |x|^{\sigma_1} f(\underline{u}, \underline{v}), \\ \underline{v}_t - \Delta \underline{v} \le |x|^{\sigma_2} g(\underline{u}, \underline{v}), \end{cases} \qquad \text{in } \boldsymbol{R}^N \times (0, T),$$

$$\begin{cases} \bar{u}(x, 0) - \underline{u}(x, 0) \ge 0, \not\equiv 0, \\ \bar{v}(x, 0) - \underline{v}(x, 0) \ge 0, \not\equiv 0. \end{cases} \qquad x \in \boldsymbol{R}^N.$$

Then we have $\bar{u}(x, t) \ge \underline{u}(x, t)$ and $\bar{v}(x, t) \ge \underline{v}(x, t)$ on $\boldsymbol{R}^N \times (0, T)$.

Proof. Put

$$t_1 = \inf\{\tau \in (0, T) : \exists x' \in \boldsymbol{R}^N \text{ s.t. } \bar{u}(x', \tau) < \underline{u}(x', \tau)\},$$
$$t_2 = \inf\{\tau \in (0, T) : \exists x' \in \boldsymbol{R}^N \text{ s.t. } \bar{v}(x', \tau) < \underline{v}(x', \tau)\}.$$

If $\bar{u}(x, t) \ge \underline{u}(x, t)$ for any $(x, t) \in \boldsymbol{R}^N \times (0, T)$, then let $t_1 = \infty$. And if $\bar{v}(x, t) \ge \underline{v}(x, t)$ for any $(x, t) \in \boldsymbol{R}^N \times (0, T)$, then let $t_2 = \infty$.

We assume that $t_1 \le t_2$ and $t_1 < T$. By the definition of t_1 and the continuity argument, we have

$$\bar{u}(x_0, t_1) = \underline{u}(x_0, t_1) \qquad \text{for some } x_0 \in \boldsymbol{R}^N, \tag{4.1}$$

$$\bar{u}(x, t) \ge \underline{u}(x, t) \qquad \text{for any } (x, t) \in \boldsymbol{R}^N \times (0, t_1), \tag{4.2}$$

$$\bar{v}(x,t) \geq \underline{v}(x,t) \qquad \text{for any } (x,t) \in \boldsymbol{R}^N \times (0,t_1). \tag{4.3}$$

From the associated integral inequalities of $\bar{u}$ and $\underline{u}$, we have

$$\begin{aligned}
&\bar{u}(x_0,t_1) - \underline{u}(x_0,t_1)\\
&\geq e^{t_1\Delta}(\bar{u}(x,0) - \underline{u}(x,0))|_{x=x_0} + \int_0^{t_1} e^{(t_1-s)\Delta}|x|^{\sigma_1}(f(\bar{u},\bar{v}) - f(\underline{u},\underline{v}))ds\Big|_{x=x_0}\\
&> \int_0^{t_1} e^{(t_1-s)\Delta}|x|^{\sigma_1}(f(\bar{u},\bar{v}) - f(\underline{u},\underline{v}))ds\Big|_{x=x_0}.
\end{aligned}$$

Positivity of the heat kernel and (4.1) imply that there exist $(y,s) \in \boldsymbol{R}^N \times (0,t_1)$ such that

$$(f(\bar{u},\bar{v}) - f(\underline{u},\underline{v}))(y,s) < 0.$$

This contradicts (4.2), (4.3) and the monotone increasing property of the function f.

In the case $t_1 \geq t_2$ and $t_2 < T$, we can derive a contradiction in the same way as above by replacing u with v and f with g. This completes the proof. □

We next show the existence of supersolutions of (1.1) in several cases.

Proposition 4.2. (i) *Let $p_1 > 1$, $q_2 > 1$ or $p_2q_1 - (1-p_1)(1-q_2) > 0$, and let $p_1 + q_1 > 1$, $p_2 + q_2 > 1$. Assume that one of the following conditions is satisfied:*

(A) $p_1, q_2 > 1$, $p_1 + q_1 > 1 + (2+\sigma_1)/N$, $p_2 + q_2 > 1 + (2+\sigma_2)/N$.

(B) $p_1 > 1 > q_2$, $p_1 + q_1 > 1 + (2+\sigma_1)/N$, $\alpha < N/2$.

(C) $p_1, q_2 < 1$, $p_2q_1 - (1-p_1)(1-q_2) > 0$, $\alpha, \beta < N/2$.

Then there exist C_1, C_2, α_1, $\beta_1 > 0$, and $t_0 > 1$ such that

$$\bar{u}(x,t) = C_1(t+t_0)^{\alpha_1 - \frac{N}{2}} \exp\Big(-\frac{|x|^2}{4(t+t_0)}\Big), \tag{4.4}$$

$$\bar{v}(x,t) = C_2(t+t_0)^{\beta_1 - \frac{N}{2}} \exp\Big(-\frac{|x|^2}{4(t+t_0)}\Big) \tag{4.5}$$

are supersolutions of (1.1).

(ii) *Let $p_1 > 1$, $q_2 > 1$ or $p_2q_1 - (1-p_1)(1-q_2) > 0$. And let $p_1 + q_1 > 1$, $p_2 + q_2 \leq 1$. Assume that one of the following conditions is satisfied:*

(D) $p_1 > 1 > q_2$, $p_1 + q_1 > 1 + (2+\sigma_1)/N$, $\alpha < N/2$,

(E) $p_1, q_2 \leq 1$, $p_2q_1 - (1-p_1)(1-q_2) > 0$, $\alpha, \beta < N/2$.

Then there exist C_1 ,C_2, α_1, $\beta_1 > 0$, $t_0 > 1$, and $a > 0$ such that

$$\bar{u}(x,t) = C_1(t+t_0)^{\alpha_1 - \frac{N}{2}} \exp\Big(- \frac{|x|^2}{4(t+t_0)}\Big), \tag{4.6}$$

$$\bar{v}(x,t) = C_2(t+t_0)^{\beta_1 - \frac{Na}{2}} \exp\Big(- \frac{a|x|^2}{4(t+t_0)}\Big) \tag{4.7}$$

are supersolutions of (1.1).

(iii) *Let $p_1 < 1$, $q_2 < 1$ and $p_2 q_1 - (1-p_1)(1-q_2) < 0$. Then there exist C_1, C_2, k, $a > 0$ such that*

$$\bar{u}(x,t) = C_1 \langle x \rangle^{-2\delta_1} \exp(kt), \tag{4.8}$$

$$\bar{v}(x,t) = C_2 \langle x \rangle^{-2\delta_2} \exp(akt) \tag{4.9}$$

are supersolutions of (1.1).

Proof. (i) Put

$$\bar{u}(x,t) = C_1(t+t_0)^{\alpha_1 - \frac{N}{2}} \exp\Big(- \frac{|x|^2}{4(t+t_0)}\Big), \tag{4.10}$$

$$\bar{v}(x,t) = C_2(t+t_0)^{\beta_1 - \frac{N}{2}} \exp\Big(- \frac{|x|^2}{4(t+t_0)}\Big), \tag{4.11}$$

where C_1, C_2, α_1, $\beta_1 > 0$, and $t_0 > 1$. We can see that $(\bar{u}, \bar{v})$ are global supersolutions for small $C_1, C_2 > 0$ and large $t_0 > 1$, provided that

$$\begin{cases} \alpha_1 - N/2 - 1 > p_1(\alpha_1 - N/2) + q_1(\beta_1 - N/2) + \sigma_1/2, \ \text{and} \\ \beta_1 - N/2 - 1 > p_2(\alpha_1 - N/2) + q_2(\beta_1 - N/2) + \sigma_2/2, \end{cases}$$

which is equivalent to

$$(p_1 - 1)\alpha_1 + q_1\beta_1 < (p_1 + q_1 - 1)N/2 - (\sigma_1 + 2)/2, \ \text{and} \tag{4.12}$$

$$p_2\alpha_1 + (q_2 - 1)\beta_1 < (p_2 + q_2 - 1)N/2 - (\sigma_2 + 2)/2. \tag{4.13}$$

Now, we shall show the existence of α_1, $\beta_1 > 0$ on the (α_1, β_1)-plane in each case of Proposition 4.2.

Case (A): $p_1, q_2 > 1$, $p_1 + q_1 > 1 + (2+\sigma_1)/N$, and $p_2 + q_2 > 1 + (2+\sigma_2)/N$.

Since the right-hand sides of (4.12) and (4.13) are positive, we can take small $\alpha_1, \beta_1 > 0$ satisfying (4.12) and (4.13).

Case (B): $p_1 > 1 > q_2$, $p_1 + q_1 > 1 + (2+\sigma_1)/N$, and $\alpha < N/2$.

We remark that the intersection of

$$(p_1 - 1)\alpha_1 + q_1\beta_1 = (p_1 + q_1 - 1)N/2 - (\sigma_1 + 2)/2, \ \text{and}$$
$$p_2\alpha_1 + (q_2 - 1)\beta_1 = (p_2 + q_2 - 1)N/2 - (\sigma_2 + 2)/2$$

is $(\alpha_1, \beta_1) = (N/2 - \alpha, N/2 - \beta)$. From the assumption, we can see that the intersection lies above the α_1-axis and that the boundary of (4.12) lies above the origin. For $\varepsilon_1, \varepsilon_2 > 0$, put $(\alpha_1, \beta_1) = (\varepsilon_1, \{(p_1 + q_1 - 1)N/2 - (\sigma_1 + 2)/2\}/q_1 + \varepsilon_2)$. Then there exist small constants $\varepsilon_1, \varepsilon_2 > 0$ such that (α_1, β_1) satisfy (4.12) and (4.13).

Case (C): $p_1, q_2 < 1$, $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$, $\alpha, \beta < N/2$.

From the assumption, we can see that the intersection lies in the first quadrant. Since $p_1, q_2 < 1$ and $p_2 q_1 - (1 - p_1)(1 - q_1) > 0$, we have $(1 - p_1)/q_1 < p_2/(1 - q_2)$, that is, the angular coefficient of (4.13) is larger than that of (4.12). Hence, there exist small constants $\varepsilon_1, \varepsilon_2 > 0$ such that $(\alpha_1, \beta_1) = (N/2 - \alpha - \varepsilon_1, N/2 - \beta - \varepsilon_2)$ satisfy (4.12) and (4.13). ☐

(ii) Case (D): $p_1 > 1 > q_2$, $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, $\alpha < N/2$.
Put $a > 0$ such that

$$\max\Big\{0, \frac{(1 - p_1)N + (\sigma_1 + 2)}{q_1 N}\Big\} < a < \frac{p_2}{1 - q_2}. \tag{4.14}$$

In fact, since $q_2 < 1$, $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$ and $\alpha < N/2$, we have

$$\begin{aligned}
&\frac{p_2}{1 - q_2} - \frac{(1 - p_1)N + (\sigma_1 + 2)}{q_1 N} \\
&= \frac{1}{N q_1 (1 - q_2)} \{N q_1 p_2 - N(1 - q_2)(1 - p_1) - (1 - q_2)(\sigma_1 + 2)\} \\
&= \frac{2\{p_2 q_1 - (1 - p_1)(1 - q_2)\}}{N q_1 (1 - q_2)} \Big\{\frac{N}{2} - \frac{(1 - q_2)(\sigma_1 + 2)}{2(p_2 q_1 - (1 - p_1)(1 - q_2))}\Big\} \\
&\geq \frac{2\{p_2 q_1 - (1 - p_1)(1 - q_2)\}}{N q_1 (1 - q_2)} \Big(\frac{N}{2} - \alpha\Big) > 0.
\end{aligned}$$

Therefore, we can take $a > 0$ satisfying (4.14). Let

$$\bar{u}(x, t) = C_1 (t + t_0)^{\alpha_1 - \frac{N}{2}} \exp\Big(-\frac{|x|^2}{4(t + t_0)}\Big), \tag{4.15}$$

$$\bar{v}(x, t) = C_2 (t + t_0)^{\beta_1 - \frac{Na}{2}} \exp\Big(-\frac{a|x|^2}{4(t + t_0)}\Big), \tag{4.16}$$

where C_1, C_2, α_1, $\beta_1 > 0$, $t_0 > 1$. We can see that $(\bar{u}, \bar{v})$ are global supersolutions provided that

$$\begin{cases} \alpha_1 - N/2 - 1 > p_1(\alpha_1 - N/2) + q_1(\beta_1 - Na/2) + \sigma_1/2, \text{ and} \\ \beta_1 - Na/2 - 1 > p_2(\alpha_1 - N/2) + q_2(\beta_1 - Na/2) + \sigma_2/2 \end{cases} \tag{4.17}$$

for small $C_1, C_2 > 0$ and large $t_0 > 1$. The condition (4.17) is equivalent to

$$(p_1 - 1)\alpha_1 + q_1\beta_1 < (p_1 + aq_1 - 1)N/2 - (\sigma_1 + 2)/2, \quad \text{and} \tag{4.18}$$
$$p_2\alpha_1 + (q_2 - 1)\beta_1 < (p_2 + aq_2 - a)N/2 - (\sigma_2 + 2)/2. \tag{4.19}$$

We remark that the intersection of

$$(p_1 - 1)\alpha_1 + q_1\beta_1 = (p_1 + aq_1 - 1)N/2 - (\sigma_1 + 2)/2, \quad \text{and}$$
$$p_2\alpha_1 + (q_2 - 1)\beta_1 = (p_2 + aq_2 - a)N/2 - (\sigma_2 + 2)/2$$

is $(\alpha_1, \beta_1) = (N/2-\alpha, Na/2-\beta)$. From the assumption $\alpha < N/2$, we see that the intersection lies above the α_1-axis. From $a > \{(1-p_1)N+(\sigma_1+2)\}/q_1N$, we can easily see that the boundary of (4.18) lies above the origin. Hence, we can prove the existence of (α_1, β_1) satisfying (4.18) and (4.19) in the same way as in Case (B).

Case (E): $p_1, q_2 \leq 1$, $p_2q_1 - (1 - p_1)(1 - q_2) > 0$, $\alpha, \beta < N/2$.
Putting $a > 0$ satisfying

$$\max\left\{\frac{1 - p_1}{q_1}, \frac{2\beta}{N}\right\} < a < \frac{p_2}{1 - q_2}, \tag{4.20}$$

we can prove this case in the same way as in Case (C). In fact, since $q_2 < 1$, $p_2q_1 - (1 - p_1)(1 - q_2) > 0$ and $\alpha < N/2$, we have

$$\begin{aligned}
&\frac{p_2}{1 - q_2} - \frac{2\beta}{N} \\
&= \frac{p_2N\{p_2q_1 - (1 - p_1)(1 - q_2)\} - (1 - p_1)p_2(\sigma_1 + 2) - p_2q_1(\sigma_2 + 2)}{(1 - q_2)\{p_2q_1 - (1 - p_1)(1 - q_2)\}} \\
&+ \frac{p_2q_1(\sigma_2 + 2) - (1 - p_1)(1 - q_2)(\sigma_2 + 2)}{(1 - q_2)\{p_2q_1 - (1 - p_1)(1 - q_2)\}} \\
&= \frac{2p_2N}{1 - q_2}\left(\frac{N}{2} - \alpha\right) + \frac{\sigma_2 + 2}{1 - q_2} > 0,
\end{aligned}$$

and since $p_1, q_2 \leq 1$, $p_2q_1 - (1 - p_1)(1 - q_2) > 0$, we have $(1 - p_1)/q_1 < p_2/(1 - q_2)$. Therefore, we can take $a > 0$ satisfying (4.20). □

(iii) Let $a = \frac{p_2}{1-q_2}$. Put

$$\bar{u}(x, t) = C_1\langle x\rangle^{-2\delta_1} \exp{(kt)}, \tag{4.21}$$
$$\bar{v}(x, t) = C_2\langle x\rangle^{-2\delta_2} \exp{(akt)}, \tag{4.22}$$

where $C_1, C_2, k > 0$. We can see that $(\bar{u}, \bar{v})$ are global supersolutions for large $k > 0$. □

We are now in a position to prove Theorems 2.1–2.4.

Proof of Theorems 2.1(i), 2.2 and 2.4. Let T^* be the maximal existence time of the classical solutions for (1.1). From the local existence theorem in Section 3, it is clear that $T^* \neq 0$. Assume $T^* < \infty$. If the initial data (u_0, v_0) are sufficiently small, then the solutions (u, v) are dominated from above by the supersolutions in Proposition 4.2. Using Proposition 3.2 and Proposition 3.4, we can extend the solutions (u, v) with new initial data $(u(T^*), v(T^*))$ to time $T^{**} > T^*$. This contradicts the maximality of T^*. Hence $T^* = \infty$. □

Proof of Theorem 2.3. By symmetry, it is clear from Theorem 2.2. □

Proof of Theorem 2.1 (ii). The constants C_1 and $C_2 > 0$ in Proposition 4.2 (iii) have no restriction. Hence, the argument as above works for arbitrary initial data in $I^{\delta_1} \times I^{\delta_2}$. □

5. Proof of Theorem 2.7

First, we prepare some notation. For $\gamma > 0$, we put

$$\eta_\gamma(t, x) = e^{t\Delta}\langle x\rangle^{-\gamma}.$$

We define the Banach space E_η by

$$E_\eta = \{(u(t), v(t)) : u, v \geq 0,\ \|(u, v)\|_{E_\eta} < \infty\}$$

with the norm

$$\|(u, v)\|_{E_\eta} = \left\|\frac{u}{\eta_a}\right\|_\infty + \left\|\frac{v}{\eta_b}\right\|_\infty,$$

where $\|w\|_\infty = \sup_{t\in(0,\infty)} \|w(t)\|_\infty$. Let $m = \|u_0\|_{\infty,a} + \|v_0\|_{\infty,b}$. We define

$$\Psi(u, v) = (\Psi_1(u, v), \Psi_2(u, v)),$$

where

$$\Psi_1(u, v) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_1} u(s)^{p_1} v(s)^{q_1}\, ds,$$

$$\Psi_2(u, v) = e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_2} u(s)^{p_2} v(s)^{q_2}\, ds.$$

In this section, we use the following lemma to show that $\Psi(u, v)$ is a contraction mapping from $B(E_\eta, 2m) = \{(u, v) \in E_\eta : \|(u, v)\|_{E_\eta} \leq 2m\}$ into itself.

Lemma 5.1. *Let $0 < \gamma < N$ and $0 \leq \delta \leq \gamma$. Then we have*

$$\|\eta_\gamma(t)\|_{\infty,\delta} \leq C(1 + t)^{(\delta-\gamma)/2} \qquad \textit{for } t > 0.$$

Proof. See Lemma 2.1 in [9]. □

Proof of Theorem 2.7. First, we show that Ψ is a mapping from $B(E_\eta, 2m)$ into itself. Assume that $(u_0, v_0) \in I^a \times I^b$. Then we can easily see that

$$|e^{t\Delta}u_0(x)| \le \|u_0\|_{\infty,a}|e^{t\Delta}\langle x\rangle^{-a}| = \|u_0\|_{\infty,a}\eta_a.$$

Hence, we obtain

$$\|(e^{t\Delta}u_0, e^{t\Delta}v_0)\|_{E_\eta} \le m.$$

We next estimate the nonlinear parts. Since $\sigma_1/(p_1+q_1-1) < a, b < N$ and $\sigma_1 - a(p_1-1) - bq_1 < -2$, we have from Lemma 5.1

$$\begin{aligned}
&\int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_1}u(s)^{p_1}v(s)^{q_1}ds \\
&= \int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_1}\eta_a(s)^{p_1-1}\eta_b(s)^{q_1}\eta_a(s)\Big|\frac{u(s)}{\eta_a(s)}\Big|^{p_1}\Big|\frac{v(s)}{\eta_b(s)}\Big|^{q_1}ds \\
&\le \int_0^t e^{(t-s)\Delta}\|\eta_a(s)\|^{p_1-1}_{\infty,\frac{\sigma_1}{p_1+q_1-1}}\|\eta_b(s)\|^{q_1}_{\infty,\frac{\sigma_1}{p_1+q_1-1}}\eta_a(s)\Big|\frac{u(s)}{\eta_a(s)}\Big|^{p_1}\Big|\frac{v(s)}{\eta_b(s)}\Big|^{q_1}ds \\
&\le C\int_0^t e^{(t-s)\Delta}(1+s)^{\{\sigma_1-a(p_1-1)-bq_1\}/2}\eta_a(s)ds\Big\|\frac{u}{\eta_a}\Big\|^{p_1}_\infty\Big\|\frac{v}{\eta_b}\Big\|^{q_1}_\infty \\
&= C\eta_a(t)\int_0^t (1+s)^{\{\sigma_1-a(p_1-1)-bq_1\}/2}ds\Big\|\frac{u}{\eta_a}\Big\|^{p_1}_\infty\Big\|\frac{v}{\eta_b}\Big\|^{q_1}_\infty \\
&\le C\eta_a(t)(2m)^{p_1+q_1}.
\end{aligned}$$

Hence, we obtain

$$\|(\Psi_1(u,v), \Psi_2(u,v))\|_{E_\eta} \le 2m$$

for sufficiently small $m > 0$.

At last, we show that Ψ is contraction. By $p_1, q_1 > 1$, we have

$$\begin{aligned}
&|\Psi_1(u,v) - \Psi_1(\tilde{u},\tilde{v})| \le \int_0^t e^{(t-s)\Delta}|\cdot|^{\sigma_1}|u(s)^{p_1}v(s)^{q_1} - \tilde{u}(s)^{p_1}\tilde{v}(s)^{q_1}|ds \\
&\le \int_0^t e^{(t-s)\Delta}\|\eta_a(s)\|^{p_1-1}_{\infty,\frac{\sigma_1}{p_1+q_1-1}}\|\eta_b(s)\|^{q_1}_{\infty,\frac{\sigma_1}{p_1+q_1-1}}\eta_a(s) \\
&\qquad\cdot\Big\{\Big|\frac{u(s)}{\eta_a(s)}\Big|^{p_1}\Big|\Big(\frac{v(s)}{\eta_b(s)}\Big)^{q_1} - \Big(\frac{\tilde{v}(s)}{\eta_b(s)}\Big)^{q_1}\Big| \\
&\qquad\qquad + \Big|\frac{\tilde{v}(s)}{\eta_b(s)}\Big|^{q_1}\Big|\Big(\frac{u(s)}{\eta_a(s)}\Big)^{p_1} - \Big(\frac{\tilde{u}(s)}{\eta_a(s)}\Big)^{p_1}\Big|\Big\}ds
\end{aligned}$$

$$\leq C\eta_a(t)\int_0^t (1+s)^{\{\sigma_1-a(p_1-1)-bq_1\}/2}ds$$
$$\cdot\Big\{(2m)^{p_1}q_1(2m)^{q_1-1}\Big\|\frac{v}{\eta_b}-\frac{\tilde{v}}{\eta_b}\Big\|_\infty+(2m)^{q_1}p_1(2m)^{p_1-1}\Big\|\frac{u}{\eta_a}-\frac{\tilde{u}}{\eta_a}\Big\|_\infty\Big\}$$
$$\leq C\eta_a(t)(2m)^{p_1+q_1-1}\|(u-\tilde{u},v-\tilde{v})\|_{E_\eta}.$$
$$\leq A\eta_a(t)\|(u-\tilde{u},v-\tilde{v})\|_{E_\eta}$$

for sufficiently small $m>0$, where $A<1$. Hence we obtain

$$\|\Psi(u-\tilde{u},v-\tilde{v})\|_{E_\eta}<A\|(u-\tilde{u},v-\tilde{v})\|_{E_\eta}.$$

This completes the proof. □

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	$\alpha < 0$	$0 < \alpha < \frac{N}{2}$	$\alpha \geq \frac{N}{2}$
$\beta < 0$	G.E.	G.E. and B.L.	B.L.
$0 < \beta < \frac{N}{2}$	G.E. and B.L.	G.E. and B.L.	B.L.
$\beta \geq \frac{N}{2}$	B.L.	B.L.	B.L.

TABLE 1. In case $p_1 < 1$, $q_2 < 1$

	$p_1 + q_1 > 1 + \frac{\sigma_1+2}{N}$	$p_1 + q_1 \leq 1 + \frac{\sigma_1+2}{N}$
$\alpha < \frac{N}{2}$	G.E. and B.L.	B.L.
$\alpha \geq \frac{N}{2}$	B.L.	B.L.

TABLE 2. In case $p_1 > 1$, $q_2 < 1$

	$p_2 + q_2 > 1 + \frac{\sigma_2+2}{N}$	$p_2 + q_2 \leq 1 + \frac{\sigma_2+2}{N}$
$\beta < \frac{N}{2}$	G.E. and B.L.	B.L.
$\beta \geq \frac{N}{2}$	B.L.	B.L.

TABLE 3. In case $p_1 < 1$, $q_2 > 1$

	$p_1 + q_1 > 1 + \frac{\sigma_1+2}{N}$	$p_1 + q_1 \leq 1 + \frac{\sigma_1+2}{N}$
$p_2 + q_2 > 1 + \frac{\sigma_2+2}{N}$	G.E. and B.L.	B.L.
$p_2 + q_2 \leq 1 + \frac{\sigma_2+2}{N}$	B.L.	B.L.

TABLE 4. In case $p_1 > 1$, $q_2 > 1$