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UNIQUE CONTINUATION FOR STOCHASTIC PARABOLIC EQUATIONS

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Abstract. This paper is devoted to a study of the unique continuation property for stochastic parabolic equations. Due to the adapted nature of solutions in the stochastic situation, classical approaches to treat the unique continuation problem for deterministic equations do not work. Our method is based on a suitable partial Holmgren coordinate transform and a stochastic version of Carleman estimate.

1. Introduction and main result

Let $T > 0$, $G \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a given bounded domain with a C^2 boundary ∂G, and $G_0 \neq G$ be a given subdomain of G. Put $Q \overset{\triangle}{=} (0,T) \times G$ and $Q_0 \overset{\triangle}{=} (0,T) \times G_0$. Throughout this paper, we assume that $a^{ij} \in C^{1,2}([0,T] \times G)$ satisfy $a^{ij} = a^{ji}$ $(i,j = 1,2,\cdots,n)$ and for any open subset G_1 of G, there is a constant $s_0 = s_0(G_1) > 0$ so that

$$\sum_{i,j} a^{ij}\xi^i\xi^j \geq s_0|\xi|^2, \tag{1.1}$$

for all $(t,x,\xi) \equiv (t,x,\xi^1,\cdots,\xi^n) \in (0,T) \times G_1 \times \mathbb{R}^n$. Here, we denote $\sum_{i,j=1}^n$ simply by $\sum_{i,j}$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space on which a 1 dimensional standard Brownian motion $\{w(t)\}_{t\geq 0}$ is defined. Let H be a Fréchet space. We denote by $L^2_{\mathcal{F}}(0,T;H)$ the Fréchet space consisting of all

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H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, with the canonical quasi-norm; by $L^\infty_{\mathcal{F}}(0,T;H)$ the Fréchet space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes, with the canonical quasi-norm; and by $L^2_{\mathcal{F}}(\Omega;C([0,T];H))$ the Fréchet space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes $X(\cdot)$ such that

$$\mathbb{E}(|X(\cdot)|^2_{C([0,T];H)}) < \infty,$$

with the canonical quasi-norm.

Let us consider the following stochastic parabolic equation:

$$\mathcal{F}z \equiv dz - \sum_{i,j}(a^{ij}z_i)_j dt = [\langle\, a, \nabla z\,\rangle + bz]dt + czdw(t) \qquad \text{in } Q. \tag{1.2}$$

Here, a, b and c are suitable coefficients. For simplicity, we use the notation $z_i \equiv z_i(x) = \partial z(x)/\partial x_i$, where x_i is the i-th coordinate of a generic point $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$. In a similar manner, in the sequel we use the notation u_i, v_i, etc., for the partial derivatives of u and v with respect to x_i. Also, we denote the scalar product in $\mathbb{R}^n$ by $\langle\,\cdot,\cdot\,\rangle$.

We begin with the following notion:

Definition 1.1. *We call* $z \in L^2_{\mathcal{F}}(\Omega;C([0,T];L^2_{loc}(G))) \cap L^2_{\mathcal{F}}(0,T;H^1_{loc}(G))$ *to be a solution of* (1.2) *if*

i) *For any nonempty open subset* G' *of* G,

$$z \in L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G'))) \cap L^2_{\mathcal{F}}(0,T;H^1(G')); \tag{1.3}$$

ii) *For any* $t \in [0,T]$ *and any* $\eta \in H^1_0(G')$, *it holds*

$$\begin{aligned} &\int_{G'} z(t,x)\eta(x)dx - \int_{G'} z(0,x)\eta(x)dx \\ &= \int_0^t \int_{G'} \Big\{ -\sum_{i,j} a^{ij}(s,x)z_i(s,x)\eta_j(x) \\ &\qquad + [\langle\, a(s,x), \nabla z(s,x)\,\rangle + b(s,x)z(s,x)]\eta(x)\Big\} dxds \\ &\quad + \int_0^t \int_{G'} c(s,x)z(s,x)\eta(x)dxdw(s), \qquad P-\text{a.s.} \end{aligned} \tag{1.4}$$

The main result of this paper is stated as follows:

Theorem 1.1. *Let* $a \in L^\infty_{\mathcal{F}}(0,T;L^\infty_{loc}(G;\mathbb{R}^n))$, $b \in L^\infty_{\mathcal{F}}(0,T;L^\infty_{loc}(G))$, *and* $c \in L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}_{loc}(G))$. *Then, any solution* $z \in L^2_{\mathcal{F}}(\Omega;C([0,T];L^2_{loc}(G))) \bigcap L^2_{\mathcal{F}}(0,T;H^1_{loc}(G))$ *of* (1.2) *vanishes identically in* $Q\times\Omega$, a.s. dP *provided that* $z = 0$ *in* $Q_0 \times \Omega$, a.s. dP.

The above result is a unique continuation theorem for stochastic parabolic equations. There are numerous references on the unique continuation for deterministic parabolic equations (see, for example, [3, 6, 7, 10] and so on). However, to the author's best acknowledge, nothing is known for its stochastic counterpart.

There are two classical tools in the study of the unique continuation for deterministic partial differential equations. One is Holmgren-type uniqueness theorem, another is Carleman estimate. Note however, that the solution of a stochastic equation is generally non-analytic in time even if the coefficients of the equation are constants. Therefore, one cannot expect a Holmgren-type uniqueness theorem for the unique continuation for stochastic equations except some very special cases. On the other hand, the usual approach to employ Carleman estimate for the unique continuation needs to localize the problem. The difficulty of our present stochastic problem consists in the fact that one cannot simply localize the problem as usual because the classical localization technique may change the adaptedness of solutions, which is a key feature in the stochastic setting. In our equation (1.2), for the space variable x, we may proceed as in the classical argument. However, for the time variable t, due to the adaptedness requirement, we will have to treat it in a deliberate way. For this purpose, we shall introduce a suitable "partial Holmgren coordinate transform" and deduce a key stochastic version of Carleman estimate (see Theorem 2.1 in the next section).

It is well-known that, unique continuation is an important problem not only in partial differential equations itself, but also in some application problems such as controllability ([11]), inverse problems ([4]), optimal control ([5]) and so on. Numerous studies on unique continuation for deterministic partial differential equations can be found in [2, 12] and the rich references cited therein. It would be quite interesting to extend the deterministic unique continuation theorems to the stochastic ones, but there are many things which remain to be done, and some of which seem to be challenging. In this paper, in order to present the key idea in the simplest way, we do not pursue the full technical generality.

The rest of this paper is organized as follows. In Section 2, as a key preliminary, we show a Carleman estimate for stochastic parabolic operators. Section 3 is devoted to the proof of Theorem 1.1.

2. Carleman estimate for stochastic parabolic operators

For any nonnegative and nonzero function $\psi \in C^3(\overline{G})$, any $k \geq 2$, and any (large) parameters $\lambda > 1$ and $\mu > 1$, put

$$\ell = \lambda\alpha, \quad \alpha(t,x) = \frac{e^{\mu\psi(x)} - e^{2\mu|\psi|_{C(\overline{G})}}}{t^k(T-t)^k}, \quad \varphi(t,x) = \frac{e^{\mu\psi(x)}}{t^k(T-t)^k}. \tag{2.1}$$

In the sequel, we will use C to denote a generic positive constant depending only on T, G, G_0 and $(a^{ij})_{n\times n}$, which may change from line to line. Also, for $r \in \mathbb{N}$, we denote by $O(\mu^r)$ a function of order μ^r for large μ (which is independent of λ); by $O_\mu(\lambda^r)$ a function of order λ^r for fixed μ and for large λ. We recall the following known result.

Lemma 2.1. ([8, 9]) *Let $b^{ij} \in C^{1,2}(\overline{Q})$ satisfying $b^{ij} = b^{ji}$. Assume that either $(b^{ij})_{n\times n}$ or $-(b^{ij})_{n\times n}$ is a uniformly positive definite matrix, and $s_0(> 0)$ is its smallest eigenvalue. Let u be a $C^2(\overline{G})$-valued semimartingale. Set*

$$\theta = e^{\ell}, \quad v = \theta u, \quad \Psi = 2\sum_{i,j} b^{ij}\ell_{ij}. \tag{2.2}$$

Then for any $x \in G$ and $\omega \in \Omega$ (a.s. dP),

$$\begin{aligned}
&2\int_0^T \theta\Big[-\sum_{i,j}(b^{ij}v_i)_j + Av\Big]\Big[du - \sum_{i,j}(b^{ij}u_i)_j dt\Big] + 2\int_0^T \sum_{i,j}(b^{ij}v_i dv)_j \\
&+ \int_0^T \Big(\theta^2 \sum_{i,j} b^{ij}\ell_i (du)^2\Big)_j + 2\int_0^T \sum_{i,j}\Big[\sum_{i',j'}\Big(2b^{ij}b^{i'j'}\ell_{i'}v_i v_{j'} - b^{ij}b^{i'j'}\ell_i v_{i'}v_{j'}\Big) \\
&\qquad + \Psi b^{ij}v_i v - b^{ij}\Big(A\ell_i + \frac{\Psi_i}{2}\Big)v^2\Big]_j dt \\
&\geq 2\sum_{i,j}\int_0^T c^{ij}v_i v_j dt + \int_0^T Bv^2 dt + \int_0^T \Big| - \sum_{i,j}(b^{ij}v_i)_j + Av\Big|^2 dt \\
&\quad - \int_0^T \theta^2 \sum_{i,j} b^{ij} du_i du_j - \int_0^T \theta^2\Big[A - \sum_{i,j}\Big(b^{ij}\ell_i\ell_j + (b^{ij}\ell_i)_j\Big)\Big](du)^2,
\end{aligned} \tag{2.3}$$

where

$$A \stackrel{\triangle}{=} -\sum_{i,j}\Big[b^{ij}\ell_i\ell_j - (b^{ij}\ell_i)_j\Big] - \Psi,$$

$$B \stackrel{\triangle}{=} 2\Big[A\Psi - \sum_{i,j}(Ab^{ij}\ell_i)_j\Big] - A_t - \sum_{i,j}(b^{ij}\Psi_j)_i - \ell_t^2,$$

$$c^{ij} \stackrel{\triangle}{=} \sum_{i',j'} \Big[2b^{ij'}(b^{i'j}\ell_{i'})_{j'} - (b^{ij}b^{i'j'}\ell_{i'})_{j'}\Big] - \frac{b^{ij}_t}{2} + \Psi b^{ij}.$$

Moreover, for λ and μ large enough, it holds

$$A = -\lambda^2\mu^2\varphi^2 \sum_{i,j} b^{ij}\psi_i\psi_j + \lambda\varphi O(\mu^2), \tag{2.4}$$

$$\begin{aligned} B \geq\ & 2s_0^2\lambda^3\mu^4\varphi^3|\nabla\psi|^4 + \lambda^3\varphi^3 O(\mu^3) + \lambda^2\varphi^2 O(\mu^4) + \lambda\varphi O(\mu^4) \\ & + \lambda^2\varphi^{2+2k^{-1}} O(e^{4\mu|\psi|_{C(\overline{G})}}) + \lambda^2\varphi^{2+k^{-1}} O(\mu^2) + \lambda\varphi^{1+k^{-1}} O(\mu^2), \end{aligned}$$

$$\sum_{i,j} c^{ij} v_i v_j \geq [s_0^2\lambda\mu^2\varphi|\nabla\psi|^2 + \lambda\varphi O(\mu)]|\nabla v|^2.$$

We now show a Carleman estimate for stochastic parabolic operators as follows:

Theorem 2.1. *Let $b^{ij} \in C^{1,2}(\overline{Q})$ satisfying $b^{ij} = b^{ji}$. Assume that either $(b^{ij})_{n\times n}$ or $-(b^{ij})_{n\times n}$ is a uniformly positive definite matrix. Let $\psi \in C^3(\overline{G})$ satisfy*

$$\min_{x\in G} |\nabla\psi(x)| > 0. \tag{2.5}$$

Then, there is some $\mu_0 > 0$ such that for all $\mu \geq \mu_0$, one can find two constants $C = C(\mu) > 0$ and $\lambda_1 = \lambda_1(\mu)$ so that for all

$$u \in L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G))) \cap L^2_{\mathcal{F}}(0,T; H_0^2(G)),$$

$f \in L^2_{\mathcal{F}}(0,T; L^2(G))$ and $g \in L^2_{\mathcal{F}}(0,T; H^1(G))$ with

$$du - \sum_{i,j}(b^{ij}u_i)_j dt = f dt + g dw(t), \qquad \text{in } Q, \tag{2.6}$$

and all $\lambda \geq \lambda_1$, it holds

$$\begin{aligned} & \lambda^3\mu^4\mathbb{E}\int_Q \varphi^3\theta^2u^2dxdt + \lambda\mu^2\mathbb{E}\int_Q \varphi\theta^2|\nabla u|^2dxdt \\ & \leq C\Big\{\mathbb{E}\int_Q \theta^2 f^2dxdt + \mathbb{E}\int_Q \theta^2\sum_{i,j} b^{ij}g_ig_jdxdt \\ & \quad + \mathbb{E}\int_Q \theta^2\Big[A - \sum_{i,j}\Big(b^{ij}\ell_i\ell_j + (b^{ij}\ell_i)_j\Big)\Big]g^2dxdt\Big\}, \end{aligned} \tag{2.7}$$

where

$$A \stackrel{\triangle}{=} -\sum_{i,j}(b^{ij}\ell_i\ell_j - b^{ij}_j\ell_i + b^{ij}\ell_{ij}). \tag{2.8}$$

Remark 2.1. Similar to the above Theorem 2.1, there is a Carleman estimate in [1, Theorem 3.1] which is however only for the special case that $\left(a^{ij}\right)_{1\le i,j\le n} = I$ (the identity matrix), $a \equiv 0$ and $b \in L^\infty_{\mathcal{F}}(0,T;\, W^{1,\infty}_{loc}(G))$. Note further that the weight function ψ used in this paper (which plays a key role in the sequel) is quite different from that in [1]. It seems that the Carleman estimate in [1] can not be applied to prove our main result, Theorem 2.1. Indeed, the weight function ψ in [1, Theorem 3.1] is supposed to vanish on the boundary of G, and therefore it does not satisfy the indispensable monotonicity condition used later in our proof.

Proof of Theorem 2.1. Recalling that $k \ge 2$ and (2.1), we get

$$\begin{aligned}&|\lambda^2\varphi^2 O(\mu^4) + \lambda\varphi O(\mu^4) + \lambda^2\varphi^{2+2k^{-1}} O(e^{4\mu|\psi|_{C(\overline{G})}})\\&\quad + \lambda^2\varphi^{2+k^{-1}}O(\mu^2) + \lambda\varphi^{1+k^{-1}}O(\mu^2)| \le \varphi^3 O_\mu(\lambda^2).\end{aligned} \tag{2.9}$$

Integrating (2.3) (in Lemma 2.1) on G, taking mean value in both sides, and noting (2.4) (in Lemma 2.1) and (2.9), recalling that u, and hence v, belongs to $L^2_{\mathcal{F}}(0,T;H^2_0(G))$, we conclude that there is a constant $c_0 > 0$ such that

$$\begin{aligned}&2\mathbb{E}\int_Q \theta\Big[-\sum_{i,j}(b^{ij}v_i)_j + Av\Big]\Big[du - \sum_{i,j}(b^{ij}u_i)_j dt\Big]dx\\&\ge 2c_0\mathbb{E}\int_Q [\lambda\mu^2\varphi|\nabla\psi|^2 + \lambda\varphi O(\mu)]|\nabla v|^2 dtdx\\&\quad + 2c_0\mathbb{E}\int_Q \Big[\lambda^3\mu^4\varphi^3|\nabla\psi|^4 + \lambda^3\varphi^3 O(\mu^3) + \varphi^3 O_\mu(\lambda^2)\Big]v^2 dtdx\\&\quad + \mathbb{E}\int_Q \Big|-\sum_{i,j}(b^{ij}v_i)_j + Av\Big|^2 dtdx - \mathbb{E}\int_Q \theta^2\sum_{i,j} b^{ij}du_i du_j dx\\&- \mathbb{E}\int_Q \theta^2\Big[A - \sum_{i,j}\Big(b^{ij}\ell_i\ell_j + (b^{ij}\ell_i)_j\Big)\Big](du)^2 dx.\end{aligned} \tag{2.10}$$

By (2.6), we have

$$\begin{aligned}&2\mathbb{E}\int_Q \theta\Big[-\sum_{i,j}(b^{ij}v_i)_j + Av\Big]\Big[du - \sum_{i,j}(b^{ij}u_i)_j dt\Big]dx\\&= 2\mathbb{E}\int_Q \theta\Big[-\sum_{i,j}(b^{ij}v_i)_j + Av\Big][fdt + gdw(t)]dx\end{aligned}$$

$$
\begin{aligned}
&= 2\mathbb{E}\int_Q \theta\Big[-\sum_{i,j}(b^{ij}v_i)_j + Av\Big] f dtdx \\
&\le \mathbb{E}\int_Q \Big|-\sum_{i,j}(b^{ij}v_i)_j + Av\Big|^2 dtdx + \mathbb{E}\int_Q \theta^2 f^2 dtdx,
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
&\mathbb{E}\int_Q \theta^2 \sum_{i,j} b^{ij} du_i du_j dx + \mathbb{E}\int_Q \theta^2 \Big[A - \sum_{i,j}\Big(b^{ij}\ell_i\ell_j + (b^{ij}\ell_i)_j\Big)\Big](du)^2 dx \\
&= \mathbb{E}\int_Q \theta^2 \sum_{i,j} b^{ij} g_i g_j dxdt \quad + \mathbb{E}\int_Q \theta^2 \Big[A - \sum_{i,j}\Big(b^{ij}\ell_i\ell_j + (b^{ij}\ell_i)_j\Big)\Big] g^2 dxdt.
\end{aligned}
\tag{2.12}
$$

Combining (2.10)–(2.12), we arrive at

$$
\begin{aligned}
&2c_0\mathbb{E}\int_Q \varphi[\lambda\mu^2|\nabla\psi|^2 + \lambda O(\mu)]|\nabla v|^2 dtdx \\
&\quad + 2c_0\mathbb{E}\int_Q \varphi^3\Big[\lambda^3\mu^4|\nabla\psi|^4 + \lambda^3 O(\mu^3) + O_\mu(\lambda^2)\Big] v^2 dtdx \\
&\le \mathbb{E}\int_Q \theta^2 f^2 dxdt + \mathbb{E}\int_Q \theta^2 \sum_{i,j} b^{ij} g_i g_j dxdt \\
&\quad + \mathbb{E}\int_Q \theta^2\Big[A - \sum_{i,j}\Big(b^{ij}\ell_i\ell_j + (b^{ij}\ell_i)_j\Big)\Big] g^2 dxdt.
\end{aligned}
\tag{2.13}
$$

Finally, combining (2.13) and (2.5), and returning v to u, we obtain the desired estimate (2.7). □

3. Proof of Theorem 1.1

The proof is divided into several steps.

Step 1. First of all, for any given subdomain G_0 in G, any neighborhood $\mathcal{O}$ of G_0 can be covered by a finite number of the images of the following open subset of $\mathbb{R}^n$

$$
G' = \Big\{(w_1, \cdots, w_n) \in \mathbb{R}^n : 0 < w_n < 1 - \sum_{i=1}^{n-1} w_i^2\Big\}
$$

under diffeomorphisms $x_j = x_j(w_1, \cdots, w_n)$ $(1 \le j \le n)$ of class $C^2(\overline{G'})$ so that the image of $\partial G' \cap \{w_n = 0\}$ is contained in ∂G_0. Such diffeomorphisms change the coefficients of the parabolic operator $\mathcal{F}$ in (1.2), but do not change its parabolicity and the adaptedness of solutions. Therefore, it suffices to consider $G = G'$. Note also that those diffeomorphisms do not change the time variable. Hence, to simplify the notations and noting that the original z vanishes in $(0,T)\times G_0 \times \Omega$, we may assume the resulting parabolic equation in $(0,T)\times G'\times\Omega$ reads

$$\begin{cases} \mathcal{F}z \equiv dz - \sum_{i,j}(a^{ij}z_i)_j dt = [\langle\, a, \nabla z\,\rangle + bz]dt + czdw(t), \\ \hfill \text{in } (0,T)\times G'\times\Omega, \\ \operatorname{supp} z \subset (0,T)\times\Big\{(x',x_n) : x_n \ge 0\Big\}\times\Omega, \end{cases}$$

where $x' = (x_1, \cdots, x_{n-1})$ and $x = (x', x_n)$.

Next, we introduce a "partial Holmgren coordinate transform" $F:\ G' \to \mathbb{R}^n$ as follows:

$$\tilde{x}' = x', \qquad \tilde{x}_n = |x'|^2 + x_n. \tag{3.1}$$

It is easy to see that

$$F(G') = \Big\{(\tilde{x}', \tilde{x}_n) : |\tilde{x}'|^2 < \tilde{x}_n < 1\Big\}.$$

Again, the coordinate transform F does not change the parabolicity of $\mathcal{F}$ and the adaptedness of solutions. Hence, to simplify the notations, we may assume the resulting parabolic equation to be the following:

$$\begin{cases} \mathcal{F}z \equiv dz - \sum_{i,j}(a^{ij}z_i)_j dt = [\langle\, a, \nabla z\,\rangle + bz]dt + czdw(t), \\ \hfill \text{in } (0,T)\times U\times\Omega, \\ \operatorname{supp} z \subset (0,T)\times\Big\{(x',x_n) : x_n \ge |x'|^2\Big\}\times\Omega, \end{cases} \tag{3.2}$$

where $U = \Big\{(x', x_n) : |x'|^2 < x_n < 1\Big\}$. It suffices to show that

$$z \equiv 0, \qquad \text{in } (0,T)\times U\times\Omega. \tag{3.3}$$

Finally, fix any r_0 and r_1 such that $0 < r_0 < r_1 < 1$, we choose a function $\rho \in C^\infty[0,1]$ so that $0 \le \rho(x_n) \le 1$ for $x_n \in [0,1]$, $\rho(x_n) \equiv 1$ for $0 \le x_n \le r_0$ and $\rho(x_n) \equiv 0$ for $r_1 \le x_n \le 1$. Put

$$u = u(t, x', x_n) \stackrel{\triangle}{=} \rho(x_n) z(t, x', x_n), \qquad (t,x) \in (0,T)\times U\times\Omega. \tag{3.4}$$

Then, by the first equation in (3.2), we have

$$du - \sum_{i,j}(a^{ij}u_i)_j dt = d(\rho z) - \sum_{i,j}\big(a^{ij}(\rho z)_i\big)_j dt \tag{3.5}$$

$$
\begin{aligned}
&= \rho\mathcal{F}z - \sum_{i,j}\Big[(a^{ij}\rho_i z)_j + a^{ij}\rho_j z_i\Big]dt\\
&= \Big\{\rho(\langle\, a, \nabla z\,\rangle + bz) - \sum_{i,j}\Big[(a^{ij}\rho_i z)_j + a^{ij}\rho_j z_i\Big]\Big\}dt + \rho czdw(t),
\end{aligned}
$$

in $(0,T)\times U\times\Omega$; while, by the second equation in (3.2), one has

$$
u = 0, \qquad \text{on } (0,T)\times\partial U\times\Omega. \tag{3.6}
$$

Step 2. The above transforms do not change the adaptedness of z, and hence that of u. We now apply Theorem 2.1 to u given by (3.4), Q replaced by $(0,T)\times U$, and

$$
\psi = \psi(x) = 1 - x_n, \qquad x\in\overline{U}. \tag{3.7}
$$

Recalling $a^{ij}\in C^{1,2}([0,T]\times G)$ satisfying $a^{ij}=a^{ji}$ $(i,j=1,2,\cdots,n)$ and the uniformly elliptic condition in (1.1), by (2.7) in Theorem 2.1, and noting (3.5), we conclude that there is a constant $C>0$ such that for any sufficiently large λ and μ, it holds

$$
\begin{aligned}
&\lambda^3\mu^4\mathbb{E}\int_0^T\int_U \varphi^3\theta^2u^2dxdt + \lambda\mu^2\mathbb{E}\int_0^T\int_U\varphi\theta^2|\nabla u|^2dxdt\\
&\le C\Big[\mathbb{E}\int_0^T\int_U\theta^2\Big\{\rho(\langle\, a,\nabla z\,\rangle + bz) - \sum_{i,j}\Big[(a^{ij}\rho_i z)_j + a^{ij}\rho_j z_i\Big]\Big\}^2dxdt\\
&\quad + \mathbb{E}\int_0^T\int_U\theta^2\sum_{i,j}a^{ij}(\rho cz)_i(\rho cz)_j dxdt\\
&\quad + \mathbb{E}\int_0^T\int_U\theta^2\Big[A - \sum_{i,j}\Big(b^{ij}\ell_i\ell_j + (b^{ij}\ell_i)_j\Big)\Big](\rho cz)^2dxdt\Big],
\end{aligned} \tag{3.8}
$$

where $A = -\sum_{i,j}(a^{ij}\ell_i\ell_j - a^{ij}_j\ell_i + a^{ij}\ell_{ij})$.

By the first estimate in (2.4) and noting our assumptions on a, b and c, we get

$$
\begin{aligned}
&\mathbb{E}\int_0^T\int_U\theta^2\Big\{\rho(\langle\, a,\nabla z\,\rangle + bz) - \sum_{i,j}\Big[(a^{ij}\rho_i z)_j + a^{ij}\rho_j z_i\Big]\Big\}^2dxdt \qquad (3.9)\\
&+\mathbb{E}\int_0^T\int_U\theta^2\sum_{i,j}a^{ij}(\rho cz)_i(\rho cz)_j dxdt
\end{aligned}
$$

$$
+\mathbb{E}\int_0^T\int_U \theta^2\Big[A-\sum_{i,j}\Big(b^{ij}\ell_i\ell_j+(b^{ij}\ell_i)_j\Big)\Big](\rho cz)^2dxdt
$$
$$
\leq C\mathbb{E}\int_0^T\int_U \theta^2(\lambda^2\mu^2\varphi^2z^2+|\nabla z|^2)dxdt.
$$

On the other hand, by (3.4), one finds

$$
\begin{aligned}
&\lambda\mu^2\mathbb{E}\int_0^T\int_U \varphi\theta^2|\nabla u|^2dxdt+\lambda^3\mu^4\mathbb{E}\int_0^T\int_U \varphi^3\theta^2u^2dxdt\\
&\geq \mathbb{E}\int_0^T\int_{U\cap\{0<x_n<r_0\}}\theta^2\Big[\lambda\mu^2\varphi|\nabla z|^2+\lambda^3\mu^4\varphi^3z^2\Big]dxdt.
\end{aligned}
\tag{3.10}
$$

Hence, combining (3.8)–(3.10), and choosing λ and μ large enough, we arrive at

$$
\begin{aligned}
&\mathbb{E}\int_0^T\int_{U\cap\{0<x_n<r_0\}}\theta^2\Big[\lambda\mu^2\varphi|\nabla z|^2+\lambda^3\mu^4\varphi^3z^2\Big]dxdt\\
&\leq C\mathbb{E}\int_0^T\int_{U\cap\{r_0<x_n<1\}}\theta^2(\lambda^2\mu^2\varphi^2z^2+|\nabla z|^2)dxdt.
\end{aligned}
\tag{3.11}
$$

Step 3. From now on, we fix μ. Also, we fix any $\kappa_1\in(0,r_0)$. Noting the definition of ψ in (3.7) implies that $\theta=\theta(t,x_n)$ is decreasing with respect to x_n, from (3.11), we deduce that

$$
\begin{aligned}
&\lambda^3\mu^4\mathbb{E}\int_0^T\int_{U\cap\{0<x_n<\kappa_1\}}|\theta(t,\kappa_1)|^2\varphi^3z^2dxdt\\
&\leq \mathbb{E}\int_0^T\int_{U\cap\{0<x_n<\kappa_1\}}|\theta(t,\kappa_1)|^2\Big[\lambda\mu^2\varphi|\nabla z|^2+\lambda^3\mu^4\varphi^3z^2\Big]dxdt\\
&\leq \mathbb{E}\int_0^T\int_{U\cap\{0<x_n<\kappa_1\}}\theta^2\Big[\lambda\mu^2\varphi|\nabla z|^2+\lambda^3\mu^4\varphi^3z^2\Big]dxdt\\
&\leq \mathbb{E}\int_0^T\int_{U\cap\{0<x_n<r_0\}}\theta^2\Big[\lambda\mu^2\varphi|\nabla z|^2+\lambda^3\mu^4\varphi^3z^2\Big]dxdt\\
&\leq C\mathbb{E}\int_0^T\int_{U\cap\{r_0<x_n<1\}}\theta^2(\lambda^2\mu^2\varphi^2z^2+|\nabla z|^2)dxdt\\
&\leq C\mathbb{E}\int_0^T\int_{U\cap\{r_0<x_n<1\}}|\theta(t,r_0)|^2(\lambda^2\mu^2\varphi^2z^2+|\nabla z|^2)dxdt,
\end{aligned}
\tag{3.12}
$$

for a constant $C>0$, independent of λ.

Further, fix any $\kappa_2 \in (0,1)$. Noting that $\theta = \theta(t, x_n)$ is increasing (*resp.* decreasing) with respect to t in $[0, T/2]$ (*resp.* $(T/2, 1]$), we deduce that

$$\begin{aligned} &\mathbb{E}\int_0^T \int_{U\cap\{0<x_n<\kappa_1\}} |\theta(t,\kappa_1)|^2\varphi^3 z^2 dxdt \\ &\geq \mathbb{E}\int_{(1-\kappa_2)T/2}^{(1+\kappa_2)T/2} \int_{U\cap\{0<x_n<\kappa_1\}} |\theta(t,\kappa_1)|^2\varphi^3 z^2 dxdt \\ &\geq |\theta((1-\kappa_2)T/2,\kappa_1)|^2\mathbb{E}\int_{(1-\kappa_2)T/2}^{(1+\kappa_2)T/2} \int_{U\cap\{0<x_n<\kappa_1\}} \varphi^3 z^2 dxdt, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\mathbb{E}\int_0^T \int_{U\cap\{r_0<x_n<1\}} |\theta(t,r_0)|^2(\lambda^2\mu^2\varphi^2 z^2 + |\nabla z|^2)dxdt \\ &\leq |\theta(T/2,r_0)|^2\mathbb{E}\int_0^T \int_{U\cap\{r_0<x_n<1\}} (\lambda^2\mu^2\varphi^2 z^2 + |\nabla z|^2)dxdt. \end{aligned} \tag{3.14}$$

Combining (3.12)–(3.14), we end up with

$$\begin{aligned} &\lambda^3\mu^4|\theta((1-\kappa_2)T/2,\kappa_1)|^2\mathbb{E}\int_{(1-\kappa_2)T/2}^{(1+\kappa_2)T/2} \int_{U\cap\{0<x_n<\kappa_1\}} \varphi^3 z^2 dxdt \\ &\leq C|\theta(T/2,r_0)|^2\mathbb{E}\int_0^T \int_{U\cap\{r_0<x_n<1\}} (\lambda^2\mu^2\varphi^2 z^2 + |\nabla z|^2)dxdt. \end{aligned} \tag{3.15}$$

By (2.2), (2.1) and (3.7), we find

$$\begin{aligned} |\theta((1-\kappa_2)T/2,\kappa_1)|^2 &= \exp\Big\{\frac{2\lambda[e^{(1-\kappa_1)\mu} - e^{2\mu}]4^k}{(1-\kappa_2^2)^k T^{2k}}\Big\}, \\ |\theta(T/2,r_0)|^2 &= \exp\Big\{\frac{2\lambda[e^{(1-r_0)\mu} - e^{2\mu}]4^k}{T^{2k}}\Big\}. \end{aligned} \tag{3.16}$$

We now choose κ_2 to be

$$\kappa_2 = \sqrt{1 - \sqrt[k]{\frac{e^{2\mu} - e^{(1-\kappa_1)\mu}}{e^{2\mu} - e^{(1-r_0)\mu}}}}. \tag{3.17}$$

Since $\kappa_1 \in (0, r_0)$, one sees that $\kappa_2 \in (0,1)$. Moreover, by (3.17), we have

$$\frac{e^{(1-\kappa_1)\mu} - e^{2\mu}}{(1-\kappa_2^2)^k} = e^{(1-r_0)\mu} - e^{2\mu}. \tag{3.18}$$

Combining (3.16) and (3.18), it follows that

$$|\theta((1-\kappa_2)T/2,\kappa_1)|^2 = |\theta(T/2,r_0)|^2. \tag{3.19}$$

Now, by (3.15) and noting (3.19), we conclude that

$$\begin{aligned} &\lambda^3\mu^4\mathbb{E}\int_{(1-\kappa_2)T/2}^{(1+\kappa_2)T/2}\int_{U\cap\{0<x_n<\kappa_1\}}\varphi^3 z^2 dxdt \\ &\le C\mathbb{E}\int_0^T\int_{U\cap\{r_0<x_n<1\}}(\lambda^2\mu^2\varphi^2 z^2+|\nabla z|^2)dxdt. \end{aligned} \tag{3.20}$$

Letting $\lambda\to+\infty$ in (3.20), we conclude that

$$z\equiv 0, \qquad \text{in } \big((1-\kappa_2)T/2,(1+\kappa_2)T/2\big)\times(U\cap\{0<x_n<\kappa_1\})\times\Omega.$$

Hence,

$$z(T/2,\cdot)\equiv 0, \qquad \text{in } (U\cap\{0<x_n<\kappa_1\})\times\Omega.$$

Since r_0 (*resp.* κ_1) can be chosen as close to 1 (*resp.* r_0) as one likes, one concludes that

$$z(T/2,\cdot)\equiv 0, \qquad \text{in } U\times\Omega.$$

Replace T by any given $t_0\in(0,T)$. Then, the above argument yields $z(t_0/2,\cdot)\equiv 0$ in $U\times\Omega$. Hence,

$$z\equiv 0, \qquad \text{in } (0,T/2]\times U\times\Omega.$$

Applying this argument to $z(\cdot+T/2,\cdot)$, it follows that

$$z\equiv 0, \qquad \text{in } (T/2,3T/4]\times U\times\Omega.$$

Repeating the above procedure, we arrive at the desired inequality (3.3). This completes the proof of Theorem 1.1. □

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