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SMOOTHNESS OF INERTIAL MANIFOLDS FOR SEMILINEAR EVOLUTION EQUATIONS IN COMPLEX BANACH SPACES

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Abstract. We study inertial manifolds for a semilinear evolution equation $du/dt + Au = F(t,u)$ in a complex Banach space. It is known that various conditions ensure existence of inertial manifolds for the equation, however, Miklavčič gave a sharp but simple condition so as to show the existence of inertial manifolds. In this paper, we show smoothness of inertial manifolds using the sharp condition with additional assumptions on F, and also apply to a scalar reaction diffusion equation $u_t - u_{xx} = f(t,x,u,u_x)$ with the Dirichlet boundary conditions.

1. Introduction

We shall study inertial manifolds for a semilinear evolution equation

$$\frac{du}{dt} + Au = F(t,u),$$

in a complex Banach space with simple conditions on A and F. Inertial manifolds are invariant manifolds that attract all the solutions exponentially, and the notion of inertial manifolds is useful to analyze especially dynamics of nonlinear systems in infinite dimensional spaces since these manifolds have global attractors. This means that such dynamics on inertial manifolds can be written as some ordinary differential equations. As to the theory of inertial manifolds, refer to Chow and Lu [2], Foias et al. [5], Kobayasi [8], Kobayasi and Takagi [9], and Temam [12], for example. Chow et al. [3] considered smoothness of inertial manifolds for a nonlinear evolution equation $du/dt+Au = F(\lambda,u)$, where u belongs to a Hilbert space and the parameter λ belongs to a locally compact Hausdorff space. The proof of their results is based upon the contraction properties of a Lyapunov-Perron operator defined on appropriate spaces, and the fiber contraction principle is used to

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obtain the smoothness of the inertial manifold. As applications to the theory of inertial manifolds, Foias et al. [4], and Kobayasi [7] considered inertial manifolds for the Kuramoto-Sivashinsky equation $u_t + u_{xxxx} + u_{xx} + uu_x = 0$ with periodic boundary conditions. Mallet-Paret and Sell [10] applied to reaction diffusion equations in higher space dimensions.

Miklavčič [11] gave a sharp but simple condition to prove existence of inertial manifolds for a semilinear evolution equation, however, it has not been mentioned that the function whose graph constructs inertial manifolds converges in C^1 topology. In this paper, we shall assume new conditions on F which are more flexible to apply the perturbation theory, and show smoothness of inertial manifolds based upon the sharp condition by Miklavčič. The fixed point theorem and the iteration method in not complete normed spaces are used in the proof of our main result.

Let A be a sectorial operator in a complex Banach space X with the norm $\|\cdot\|$, that is, A is a closed densely defined operator such that for some $\theta \in (0, \pi/2)$, $C \geq 1$ and $\mu \in \mathbf{R}$, the sector

$$S_{\mu,\theta} := \{\lambda\,;\, \theta \leq |\arg(\lambda - \mu)| \leq \pi,\ \lambda \neq \mu\},$$

is in the resolvent set of A and

$$\|(A - \lambda I)^{-1}\| \ \leq \ \frac{C}{|\lambda - \mu|},$$

for any $\lambda \in S_{\mu,\theta}$, and for each $\alpha \geq 0$, let X^α be the domain of $(A - aI)^\alpha$ with the graph norm $\|x\|_\alpha := \|(A - aI)^\alpha x\|$ for $x \in X^\alpha$, where a is chosen as $\operatorname{Re}\sigma(A - aI) > 0$. This means that the real part of λ is positive whenever λ is a spectrum of $A - aI$. See Henry [6] for more precise definitions of these notations. We now consider a semilinear evolution equation

$$\text{(E)} \qquad \frac{du}{dt} + Au \ = \ F(t, u),$$

in X. Here, F is a continuous function from $\mathbf{R} \times X^\alpha$ to X and satisfies that for some $\alpha \in [0, 1)$, $m \in \mathbf{N}$ and $B_j \in \mathcal{B}(X^\alpha, X)$, $j = 1, 2, \cdots, m$,

$$\text{(F)} \qquad \|F(t, x) - F(t, y)\| \ \leq \ \sum_{j=1}^{m} \|B_j(x - y)\|, \qquad t \in \mathbf{R}, \quad x, y \in X^\alpha,$$

where $\mathcal{B}(X^\alpha, X)$ denotes a space of all bounded operators from X^α to X.

The simple condition given by Miklavčič [11] is as follows:

(SC)

Let $\lambda > 0$ be such that $\lambda + i\omega$ is in the resolvent set of A for any $\omega \in \mathbf{R}$.

Miklavčič studied existence of inertial manifolds for (E) using the condition (SC). It is known that various conditions ensure existence of inertial manifolds for the equation, however, (SC) is simple and sharp in a sense. Indeed, let X_1 be the range of the projection associated with the spectral set in the half plane $\operatorname{Re} z < \lambda$. It is obvious that A has an invariant subspace X_1, and therefore X_1 is an inertial manifold for $du/dt + Au = 0$. In a similar way, if $\lambda + i\omega$ is in the resolvent set of $A - B$ for any $\omega \in \mathbf{R}$, then X_1 is an inertial manifold for the equation $du/dt + Au = Bu$. Thus, it is the exact one way to ensure this to require that

$$\left\| B(A - (\lambda + i\omega)I)^{-1} \right\| \;<\; 1,$$

for any $\omega \in \mathbf{R}$.

Throughout this paper, we shall make the following assumptions along with Miklavčič [11]. Suppose first that there exists $M_0 > 0$ such that if $f \in C(\mathbf{R}\backslash\{0\}, X)$ and $\|f\| \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, then

$$\int_{-\infty}^{\infty} \left\| f(t) \right\|^2 dt \;\le\; M_0 \int_{-\infty}^{\infty} \left\| \widehat{f}(\omega) \right\|^2 d\omega \;\le\; M_0^2 \int_{-\infty}^{\infty} \left\| f(t) \right\|^2 dt,$$

where $\widehat{f}(\omega) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt$ for $\omega \in \mathbf{R}$. It is easy to see that if X is any Hilbert space, then this estimate holds with $M_0 = 1$. We also assume that

$$\int_{-\infty}^{0} \left\| e^{\mu t} F(t, 0) \right\|^2 dt \;<\; \infty,$$

for some $\mu < \lambda$ and

$$M_0 \sum_{j=1}^{m} \sup_{\omega \in \mathbf{R}} \left\| B_j (A - (\lambda + i\omega)I)^{-1} \right\| \;<\; 1.$$

We see from the definition of X^α that

$$\begin{aligned} &\left\| (A - (\lambda + i\omega)I)^{-1} \right\|_\alpha \\ &\qquad \le \quad C \left\| (A - aI)(A - (\lambda + i\omega)I)^{-1} \right\|^\alpha \left\| (A - (\lambda + i\omega)I)^{-1} \right\|^{1-\alpha}, \end{aligned}$$

with some constant $C > 0$ not depending on α, and then

$$\sup_{\omega \in \mathbf{R}} \left\| (A - (\lambda + i\omega)I)^{-1} \right\|_\alpha \;<\; \infty.$$

Choose $B_0 = b\,(A - aI)^\alpha$ with small $b > 0$ and put

$$\xi(\lambda) \;:=\; M_0 \sum_{j=0}^{m} \sup_{\omega \in \mathbf{R}} \left\| B_j (A - (\lambda + i\omega)I)^{-1} \right\|.$$

Then, we note that $\xi(\widetilde{\lambda}) < 1$ whenever $\left|\lambda - \widetilde{\lambda}\right|$ is sufficiently small, and also see that

$$\left\|F(t,x) - F(t,y)\right\| \ \leq \ L\left\|x - y\right\|_{\alpha},$$

for $t \in \mathbf{R}$ and $x, y \in X^{\alpha}$, where $L := \sum_{j=0}^{m} \left\|B_j\,(A - aI)^{-\alpha}\right\|$.

We define two spectral sets σ_1 and σ_2 as $\sigma_1 := \left\{z \in \sigma(A)\,;\, \mathrm{Re}\, z < \lambda\right\}$ and $\sigma_2 := \left\{z \in \sigma(A)\,;\, \mathrm{Re}\, z > \lambda\right\}$, respectively. Note that $\sigma(A) = \sigma_1 \cup \sigma_2$ and σ_1 is bounded. Let $P_1 \in \mathcal{B}(X)$ be a projection associated with σ_1, $P_2 = I - P_1$ and $X_i = P_i X$ for $i = 1, 2$. Choose $\lambda_1,\ \lambda_2 \in \mathbf{R}$ such that

$$\sup\, \mathrm{Re}\, \sigma_1 \ < \ \lambda_1 \ < \ \lambda \ < \ \lambda_2 \ < \ \inf\, \mathrm{Re}\, \sigma_2.$$

Then, we see from Henry [6] that $X_1 \subset \mathcal{D}(A)$ and $AX_1 \subset X_1$. Furthermore, denoting A restricted to X_1 as A_1, we also see that $A_1 \in \mathcal{B}(X_1)$, $P_i e^{-At} = e^{-At} P_i$ for $t \geq 0$, $i = 1, 2$, $e^{-A_1 z} = \sum_{n=0}^{\infty} (-A_1 z)^n (n!)^{-1}$ for $z \in \mathbf{C}$, $e^{-A_1 t} x = e^{-At} x$ for $x \in X_1$, $t \geq 0$, and there exists $M > 0$ such that for any $x \in X$,

$$\begin{aligned}
\left\|e^{-At}\right\| &\leq Me^{-at}, & t &\geq 0,\\
\left\|e^{-At}x\right\|_{\alpha} &\leq Mt^{-\alpha}e^{-at}\left\|x\right\|, & t &> 0,\\
\left\|e^{-A_1 t}P_1 x\right\| &\leq Me^{-\lambda_1 t}\left\|x\right\|, & t &\leq 0,\\
\left\|e^{-A_1 t}P_1 x\right\|_{\alpha} &\leq Me^{-\lambda_1 t}\left\|x\right\|, & t &\leq 0,\\
\left\|e^{-At}P_2 x\right\| &\leq Me^{-\lambda_2 t}\left\|x\right\|, & t &\geq 0,\\
\left\|e^{-At}P_2 x\right\|_{\alpha} &\leq Mt^{-\alpha}e^{-\lambda_2 t}\left\|x\right\|, & t &> 0.
\end{aligned}$$

This paper is organized as follows. We introduce new assumptions on F and mention our main result in Section 2. Section 3 is devoted to prove our main result using some important lemmas. We apply our theory to a scalar reaction diffusion equation $u_t - u_{xx} = f(t, x, u, u_x)$ with the Dirichlet boundary conditions in Section 4.

2. The main result

Let $\mathbf{R}^{-} := (-\infty, 0]$ and $\mathbf{R}^{+} := [0, \infty)$. We begin with the definition of a manifold $\mathcal{M}(\tau) \subset X$.

Definition 2.1. *For $\tau \in \mathbf{R}$, we say $x \in \mathcal{M}(\tau)$ if there exists $v \in C(\mathbf{R}^{-}, X^{\alpha})$ such that*

$$\begin{aligned}
v(0) &= x,\\
v(t) &= e^{-A(t-T)}v(T) + \int_{T}^{t} e^{-A(t-s)}F(s+\tau, v(s))\, ds,
\end{aligned}$$

for $-\infty < T \le t \le 0$*, and* $\int_{-\infty}^{0} \|e^{\lambda t} v(t)\|_\alpha^p \, dt$ *is finite for* $p = 1, 2$*.*

It is easy to see that for each $\tau \in \mathbf{R}$ and $x \in X^\alpha$, there exists a unique $u \in C([\tau, \infty), X^\alpha)$ such that

$$u(t) \;=\; e^{-A(t-\tau)}x + \int_\tau^t e^{-A(t-s)} F(s, u(s))\, ds,$$

for $t \ge \tau$, and therefore if $x \in \mathcal{M}(\tau)$, then there exists $u \in C(\mathbf{R}, X^\alpha)$ such that $u(\tau) = x,\ u(t) \in \mathcal{M}(t)$ for all $t \in \mathbf{R}$ and

$$u(t) \;=\; e^{-A(t-T)}u(T) + \int_T^t e^{-A(t-s)} F(s, u(s))\, ds,$$

for $-\infty < T \le t < \infty$. Hence, $\mathcal{M}$ is invariant.

We here assume new conditions on F instead of (F) so as to show smoothness of inertial manifolds for (E).

(N1) For each $t \in \mathbf{R}$, $F(t, x)$ is Fréchet differentiable with respect to x.
(N2) $DF \in C_s(\mathbf{R} \times X^\alpha, \mathcal{B}(X^\alpha, X))$ with $D := \partial/\partial x$. The space $C_s(\mathbf{R} \times X^\alpha, \mathcal{B}(X^\alpha, X))$ is defined as a space of all functions ϕ such that $\phi(t, x) \in \mathcal{B}(X^\alpha, X)$ for each $t \in \mathbf{R}$, $x \in X^\alpha$, and $\phi(\cdot, \cdot)z \in C(\mathbf{R} \times X^\alpha, X)$ for each $z \in X^\alpha$.
(N3) For each $T < 0$ and $r > 0$, there exists a continuous and nondecreasing function β on $\mathbf{R}^+$ such that $\beta(0) = 0$ and

$$\big\|DF(t, x) - DF(t, y)\big\|_{\mathcal{B}(X^\alpha, X)} \;\le\; \beta(\|x - y\|_\alpha),$$

for all $t \in [T, 0]$ and $x, y \in X^\alpha$ with $\|x\|_\alpha,\ \|y\|_\alpha \le r$.
(N4) There exist $B_j \in \mathcal{B}(X^\alpha, X),\ \ j = 1, \cdots, m,$ such that

$$\big\|DF(t, x)y\big\| \;\le\; \sum_{j=1}^m \big\|B_j\, y\big\|,$$

for $t \in \mathbf{R}$ and $x, y \in X^\alpha$.

We are now in the position to mention our main result.

Theorem 2.2. *Suppose that* (SC) *and* (N1)-(N4) *hold. Then* $\mathcal{M}$ *defined as above is a* C^1 *inertial manifold.*

3. Proof of Theorem 2.2

We first claim that the assumptions (N1)-(N4) imply (F). Indeed, we see from Taylor's formula that

$$F(t,y) \;=\; F(t,x) + \int_0^1 DF(t, x+s(y-x))(y-x)\,ds,$$

for $t \in \mathbf{R}$ and $x, y \in X^\alpha$, and this implies that

$$\big\|F(t,x) - F(t,y)\big\| \;\le\; \sum_{j=1}^{m} \big\|B_j(x-y)\big\|.$$

We also note that $\big\|DF(t,x)\big\|_{\mathcal{B}(X^\alpha,X)} \le L$ since

$$\lim_{y\to x} \frac{\big\|F(t,x)-F(t,y)\big\|}{\big\|x-y\big\|_\alpha} \;=\; \big\|DF(t,x)\big\|_{\mathcal{B}(X^\alpha,X)} \qquad \text{in } X^\alpha.$$

Thanks to Miklavčič [11], the existence of the inertial manifold for (E) are assured.

Now we define two useful functional spaces Y and Z by

$$Y \;:=\; \left\{ v \in C(\mathbf{R}^-, X^\alpha)\,;\, \int_{-\infty}^0 \big\|e^{\lambda t} v(t)\big\|_\alpha^p\,dt < \infty \qquad \text{for} \;\; p = 1,2 \right\},$$

with the norm

$$\big|v\big|_Y \;:=\; \sum_{j=0}^{m} \left(\int_{-\infty}^0 \big\|e^{\lambda t} B_j v(t)\big\|^2\,dt \right)^{1/2},$$

for $v \in Y$, and

$$Z \;:=\; \Bigg\{ \Phi \in C_s(\mathbf{R}^-, \mathcal{B}(X_1, X^\alpha))\;; \sup_{z \in X_1,\; \|z\|=1} \int_{-\infty}^0 \big\|e^{\lambda t}\Phi(t) z\big\|_\alpha^p\,dt < \infty \qquad \text{for} \;\; p = 1,2 \Bigg\},$$

respectively. Recall that $C_s(\mathbf{R}^-, \mathcal{B}(X_1, X^\alpha))$ denotes a space of all functions Φ such that $\Phi(t) \in \mathcal{B}(X_1, X^\alpha)$ for each $t \in \mathbf{R}^-$ and $\Phi(\cdot)z \in C(\mathbf{R}^-, X^\alpha)$ for each $z \in X_1$. For each $\tau \in \mathbf{R}$ and $v \in Y$, we define a mapping $S(\cdot) = S(\cdot\,;\tau, v)$ from Z into itself by

$$S(\Phi)(t) \;:=\; e^{-A_1 t} + \int_0^t e^{-A_1(t-s)} P_1 DF(s+\tau, v(s))\Phi(s)\,ds$$

$$+\int_{-\infty}^{t} e^{-A(t-s)} P_2 DF(s+\tau, v(s))\Phi(s)\, ds,$$

for $t \in \mathbf{R}^-$ and $\Phi \in Z$.

Lemma 3.1. *Let $\tau \in \mathbf{R}$ and $v \in Y$. Then*

$$\left|S(\Phi_1)(\cdot)z - S(\Phi_2)(\cdot)z\right|_Y \;\le\; \xi(\lambda)\left|\Phi_1(\cdot)z - \Phi_2(\cdot)z\right|_Y \tag{3.1}$$

for $\Phi_1, \Phi_2 \in Z$ and $z \in X_1$.

Proof. Let $z \in X_1$ and put

$$\begin{aligned} c_z &:= S(\Phi_1)(0)z - S(\Phi_2)(0)z, \\ g_z(t) &:= \begin{cases} e^{\lambda t} e^{-At} c_z & \text{if } t > 0 \\ e^{\lambda t}(S(\Phi_1)(t)z - S(\Phi_2)(t)z) & \text{if } t \le 0, \end{cases} \\ f_z(t) &:= \begin{cases} 0 & \text{if } t > 0 \\ e^{\lambda t} DF(t+\tau, v(t))(\Phi_1(t)z - \Phi_2(t)z) & \text{if } t \le 0. \end{cases} \end{aligned}$$

We note from $\int_0^\infty \left\|g_z(t)\right\|_\alpha^p dt$ is finite for $p = 1, 2$ that $g_z \in C(\mathbf{R}, X^\alpha)$, $\left\|g_z\right\|_\alpha \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ and $\left\|f_z\right\| \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Indeed, since $c_z \in X^\alpha \cap X_2$, we see that for $t > 0$, $\left\|g_z(t)\right\|_\alpha \le M e^{(\lambda-\lambda_2)t}\left\|c_z\right\|_\alpha$, and thus $\left\|g_z\right\|_\alpha \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. We also note that

$$g_z(t) = \int_0^t e^{\lambda(t-s)} e^{-A_1(t-s)} P_1 f_z(s)\, ds + \int_{-\infty}^t e^{\lambda(t-s)} e^{-A(t-s)} P_2 f_z(s)\, ds,$$

for $t \le 0$ and $c_z = \int_{-\infty}^0 e^{-\lambda s} e^{As} P_2 f_z(s)\, ds$. Let

$$\widehat{g_z}(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} g_z(t)\, dt.$$

Then, we see that $\widehat{g_z}(\omega) = (A - (\lambda + i\omega)I)^{-1} \widehat{f_z}(\omega)$ and

$$\begin{aligned} &\int_{-\infty}^0 \left\|e^{\lambda t} B_j(S(\Phi_1)(t)z - S(\Phi_2)(t)z)\right\|^2 dt \\ &\qquad \le\; M_0^2 \rho_j(\lambda)^2 \int_{-\infty}^0 \left\|e^{\lambda t} DF(t+\tau, v(t))(\Phi_1(t)z - \Phi_2(t)z)\right\|^2 dt \\ &\qquad \le\; M_0^2 \rho_j(\lambda)^2 \left(\sum_{k=1}^m \left(\int_{-\infty}^0 \left\|e^{\lambda t} B_k(\Phi_1(t)z - \Phi_2(t)z)\right\|^2 dt\right)^{1/2}\right)^2, \end{aligned}$$

where $\rho_j(\lambda) := \sup_{\omega\in\mathbf{R}} \left\|B_j(A-(\lambda+i\omega)I)^{-1}\right\|$. Consequently, we obtain the inequality (3.1). □

We next define two spaces W and W_D by

$$\begin{aligned} W &:= \left\{u \in C(\mathbf{R}^- \times X_1, X^\alpha)\,;\, u(\cdot,x) \in Y \quad \text{for each} \quad x \in X_1\right\}, \\ W_D &:= \Big\{u \in W\,;\, u(t,\cdot) \in C^1(X_1, X^\alpha) \qquad \text{for each} \quad t \in \mathbf{R}^- \qquad \text{and} \\ &\qquad Du(\cdot,x) \in Z \quad \text{for each} \quad x \in X_1\Big\}, \end{aligned}$$

respectively. For each $\tau \in \mathbf{R}$, we also define a mapping $V(\cdot) = V(\cdot\,;\tau)$ from W_D into itself by

$$\begin{aligned} V(u)(t,x) \;:=\; & e^{-A_1 t}x + \int_0^t e^{-A_1(t-s)}P_1F(s+\tau, u(s,x))\,ds \\ & + \int_{-\infty}^t e^{-A(t-s)}P_2F(s+\tau, u(s,x))\,ds, \end{aligned}$$

for $(t,x) \in \mathbf{R}^- \times X_1$. Let $\tau \in \mathbf{R}$. Then, we see that

$$DV(u)(t,x) \;=\; S(Du(\cdot,x)\,;\tau, u(\cdot,x))(t),$$

for $(t,x) \in \mathbf{R}^- \times X_1$. To this end, considering first the case of $s \in (-\infty,t)$, we have

$$\begin{aligned} &\int_{-\infty}^t \left\|e^{-A(t-s)}P_2DF(s+\tau,u(s,x))Du(s,x)\,z\right\|ds \\ &\quad\le\; e^{-\lambda t}M\sum_{j=1}^m \left\|B_j\right\|_{\mathcal{B}(X^\alpha,X)}\left\|z\right\| \\ &\qquad\qquad \times \int_{-\infty}^t (t-s)^{-\alpha}e^{(\lambda-\lambda_2)(t-s)}\left\|e^{\lambda s}Du(s,x)\right\|_{\mathcal{B}(X_1,X^\alpha)}\,ds \;<\; \infty, \end{aligned}$$

for $z \in X_1$. On the other hand, as to the case of $s \in (t,0]$, we also see that

$$\begin{aligned} &\int_t^0 \left\|e^{-A_1(t-s)}P_1DF(s+\tau,u(s,x))Du(s,x)\,z\right\|, ds \\ &\le e^{-\lambda t}M\sum_{j=1}^m \left\|B_j\right\|_{\mathcal{B}(X^\alpha,X)}\left\|z\right\| \int_t^0 e^{(\lambda-\lambda_1)(t-s)}\left\|Du(s,x)\right\|_{\mathcal{B}(X_1,X^\alpha)}\,ds < \infty, \end{aligned}$$

for $z \in X_1$. Therefore, by the Lebesgue convergence theorem, these observations indicate the conclusion.

Let $u_0 = 0$ and $V(u_n\,;\,\tau) = u_{n+1}$ for $n \geq 0$. We see from the definitions of mappings V and S that

$$\begin{aligned} Du_{n+1}(t,x) &= e^{-A_1 t} + \int_0^t e^{-A_1(t-s)} P_1 DF(s+\tau, u_n(s,x)) Du_n(s,x)\,ds \\ &\quad + \int_{-\infty}^t e^{-A(t-s)} P_2 DF(s+\tau, u_n(s,x)) Du_n(s,x)\,ds. \end{aligned}$$

Lemma 3.2. *Let $\tau \in \mathbf{R}$ and $u \in W$. Then, for each $x \in X_1$ there exists a unique $\Phi(\cdot,x) \in Z$ such that $S(\Phi(\cdot,x)\,;\,\tau, u(\cdot,x)) = \Phi(\cdot,x)$.*

Proof. Let $\tau \in \mathbf{R}$ and $u \in W$, and fix $x \in X_1$. Let $\Phi_0 = 0$, $\Phi_{n+1} = S(\Phi_n\,;\,\tau, u(\cdot,x))$ for $n \geq 0$. For each $z \in X_1$, we obtain that

$$\begin{aligned} &\left\| e^{\lambda t}(\Phi_{n+2}(t)z - \Phi_{n+1}(t)z) \right\|_\alpha \\ &\quad \leq \int_t^0 \left\| e^{-A_1(t-s)} P_1 DF(s+\tau, u(s,x))(\Phi_{n+1}(s)z - \Phi_n(s)z) \right\| ds \\ &\qquad + \int_{-\infty}^t \left\| e^{-A(t-s)} P_2 DF(s+\tau, u(s,x))(\Phi_{n+1}(s)z - \Phi_n(s)z) \right\| ds \\ &\quad \leq M \int_t^0 e^{(\lambda-\lambda_1)(t-s)} \sum_{j=1}^m \left\| e^{\lambda s} B_j (\Phi_{n+1}(s)z - \Phi_n(s)z) \right\| ds \\ &\qquad + M \int_{-\infty}^t e^{(\lambda-\lambda_2)(t-s)} \sum_{j=1}^m \left\| e^{\lambda s} B_j (\Phi_{n+1}(s)z - \Phi_n(s)z) \right\| ds \\ &\quad \leq C_0\, \xi(\lambda)^n \left\| z \right\|, \end{aligned}$$

where

$$C_0 := M^2 \left(\frac{1}{\sqrt{2(\lambda-\lambda_1)}} + \frac{1}{\sqrt{2(\lambda_2-\lambda)}} \right) \frac{1}{\sqrt{2(\lambda-\lambda_1)}} \sum_{j=0}^m \left\| B_j \right\|_{\mathcal{B}(X^\alpha, X)}.$$

Furthermore, we see that

$$\Phi_{n+1}(t) = e^{-A(t-T)}\Phi_{n+1}(T) + \int_T^t e^{-A(t-s)} DF(s+\tau, u(s,x)) \Phi_n(s)\,ds,$$

implies

$$\begin{aligned} &\left\| \Phi_{n+2}(t)z - \Phi_{n+1}(t)z \right\|_\alpha \\ &\leq M(t-T)^{-\alpha} e^{-a(t-T)} \left\| \Phi_{n+2}(T)z - \Phi_{n+1}(T)z \right\|_\alpha \end{aligned}$$

$$+ ML \int_T^t (t-s)^{-\alpha} e^{-a(t-s)} \big\| \Phi_{n+1}(s)z - \Phi_n(s)z \big\|_\alpha \, ds$$
$$\leq C\,(t-T)^{-\alpha} \xi(\lambda)^n \big\| z \big\| + C'\,\xi(\lambda) \int_T^t (t-s)^{-\alpha} \big\| \Phi_{n+1}(s)z - \Phi_n(s)z \big\|_\alpha \, ds,$$

with positive constants C and C'. Thus, in a similar way as Miklavčič [11, Lemma 3.3], there exists another constant $\widetilde{C} > 0$ such that

$$\big\| \Phi_{n+1}(t)z - \Phi_n(t)z \big\|_\alpha \;\leq\; \xi(\lambda)^{n-1} (t-T)^{-\alpha}\, \widetilde{C}\, \big\| z \big\|,$$

for $n \geq 1$ and $t \in (T, 0]$. Thus, we see from the uniformly bounded theorem that there exists $\Phi(\cdot, x) \in C_s(\mathbf{R}^-, \mathcal{B}(X_1, X^\alpha))$ such that

$$\lim_{n\to\infty} \sup_{t\in[T,0]} \big\| \Phi_n(t) - \Phi(t,x) \big\|_{\mathcal{B}(X_1, X^\alpha)} \;=\; 0,$$

for $n \geq 1$ and $T < 0$. Now we note that

$$\big|\Phi_n z\big|_Y \leq \sum_{i=0}^{n-1} \big|\Phi_{i+1} z - \Phi_i z\big|_Y \leq \frac{M}{1-\xi(\lambda)} \cdot \frac{1}{\sqrt{2(\lambda-\lambda_1)}} \sum_{j=0}^m \big\|B_j\big\|_{\mathcal{B}(X^\alpha, X)} \big\|z\big\|,$$
$$\big|\Phi_n z\big|_Y = \sum_{j=0}^m \left(\int_{-\infty}^0 \big\| e^{\lambda t} B_j \Phi_n(t) z \big\|^2 \, dt \right)^{1/2} \geq b \left(\int_T^0 \big\| e^{\lambda t} \Phi_n(t) z \big\|_\alpha^2 \, dt \right)^{1/2},$$

and therefore we see that

$$\int_T^0 \big\| e^{\lambda t} \Phi_n(t) z \big\|_\alpha^2 dt \;\leq \Big(\frac{M}{b\,(1-\xi(\lambda))} \sum_{j=0}^m \big\|B_j\big\|_{\mathcal{B}(X^\alpha, X)} \Big)^2 \frac{1}{2(\lambda-\lambda_1)} \big\|z\big\|^2,$$

which implies

$$\int_{-\infty}^0 \big\| e^{\lambda t} \Phi(t,x) z \big\|_\alpha^2 dt \;\leq \Big(\frac{M}{b\,(1-\xi(\lambda))} \sum_{j=0}^m \big\|B_j\big\|_{\mathcal{B}(X^\alpha, X)} \Big)^2 \frac{1}{2(\lambda-\lambda_1)} \big\|z\big\|^2,$$

by letting $n \to \infty$ first and then $T \to -\infty$. If $|\widetilde{\lambda} - \lambda|$ is small enough, then $\xi(\widetilde{\lambda}) < 1$ and Φ_n is not concerned with $\widetilde{\lambda}$, so we may replace λ with $\widetilde{\lambda}$. This implies that

$$\int_{-\infty}^0 \big\| e^{\widetilde{\lambda} t} \Phi(t,x) z \big\|_\alpha^2 \, dt \;\leq\; C\, \big\|z\big\|^2, \tag{3.2}$$

for any $\widetilde{\lambda} \in \mathbf{R}$ with $|\widetilde{\lambda} - \lambda|$ sufficiently small, $z \in X_1$ and $C > 0$.

On the other hand, we also note that

$$\begin{aligned}\big|\Phi_{n+k}z-\Phi_n z\big|_Y &\le \frac{M\xi(\lambda)^n}{(1-\xi(\lambda))\sqrt{2(\lambda-\lambda_1)}}\sum_{j=0}^m \big\|B_j\big\|_{\mathcal{B}(X^\alpha,X)}\big\|z\big\|,\\ \big|\Phi_{n+k}z-\Phi_n z\big|_Y &\ge b\left(\int_T^0 \big\|e^{\lambda t}(\Phi_{n+k}(t)z-\Phi_n(t)z)\big\|_\alpha^2\,dt\right)^{1/2}.\end{aligned}$$

Passing to the limit as $k\to\infty$ first and then $T\to-\infty$, these estimates implicate that

$$\int_{-\infty}^0 \|e^{\lambda t}(\Phi(t,x)z-\Phi_n(t)z)\|_\alpha^2 dt \le \Big(\frac{\xi(\lambda)^n}{1-\xi(\lambda)}M\sum_{j=0}^m \|B_j\|_{\mathcal{B}(X^\alpha,X)}\Big)^2\frac{\|z\|^2}{2(\lambda-\lambda_1)},$$

and this follows that

$$\begin{aligned}\big|\Phi(\cdot,x)z-\Phi_n z\big|_Y &\le \sum_{j=0}^m \big\|B_j\big\|_{\mathcal{B}(X^\alpha,X)}\Big(\int_{-\infty}^0 \big\|e^{\lambda t}(\Phi(t,x)z-\Phi_n(t)z)\big\|_\alpha^2\,dt\Big)^{\frac12}\\ &\to 0\end{aligned}$$

as $n\to\infty$ and

$$\begin{aligned}&\big|\Phi(\cdot,x)z-S(\Phi(\cdot,x)\,;\,\tau,u(\cdot,x))z\big|_Y\\ &\quad\le \big|\Phi(\cdot,x)z-\Phi_n z\big|_Y+\xi(\lambda)\big|\Phi_{n-1}z-\Phi(\cdot,x)z\big|_Y \to 0,\end{aligned}$$

as $n\to\infty$. Hence, we conclude that $\Phi(\cdot,x)=S(\Phi(\cdot,x)\,;\,\tau,u(\cdot,x))$. Moreover, we see from (3.2) that

$$\begin{aligned}\int_{-\infty}^0 \big\|e^{\lambda t}\Phi(t,x)z\big\|_\alpha\,dt &\le \Big(\int_{-\infty}^0 e^{2(\lambda-\widetilde{\lambda})t}\,dt\Big)^{1/2}\Big(\int_{-\infty}^0 \big\|e^{\widetilde{\lambda}t}\Phi(t,x)z\big\|_\alpha^2\,dt\Big)^{1/2}\\ &\le C\big\|z\big\|\end{aligned}$$

whenever $\widetilde{\lambda}<\lambda$ and $|\widetilde{\lambda}-\lambda|$ small enough with constant $C>0$, and it implies $\Phi(\cdot,x)\in Z$. □

Thanks to Lemma 3.2, we see that there exists a unique $u\in C(\mathbf{R}^-\times X_1,X^\alpha)$ such that

$$\lim_{n\to\infty}\sup_{T\le t\le 0}\big\|u_n(t,x)-u(t,x)\big\|_\alpha = 0, \tag{3.3}$$

for $x\in X_1$, and for this $u\in W$ and $x\in X_1$, there exists a unique $\Phi(\cdot,x)\in Z$ such that $S(\Phi(\cdot,x)\,;\,\tau,u(\cdot,x))=\Phi(\cdot,x)$.

Lemma 3.3. *Let C and $\widetilde{C}$ be positive constants not depending on n and $\widetilde{T}$, and let*

$$\nu_{n,\widetilde{T}} \;:=\; C\,(\xi(\lambda)^n + \omega_n(\widetilde{T}) + \eta(\widetilde{T})),$$

with ω_n, η satisfying $\omega_n(\widetilde{T}) \to 0$ as $n \to \infty$ for each $\widetilde{T} < 0$, and $\eta(\widetilde{T}) \to 0$ as $\widetilde{T} \to -\infty$. Then for any bounded set E in X_1,

$$\sup_{x\in E} \left\|Du_{n+1}(t,x) - \Phi(t,x)\right\|_{\mathcal{B}(X_1,X^\alpha)} \leq \sum_{j=1}^{n+1} \nu_{n-j+1,\widetilde{T}}\, \widetilde{C}^{j-1}\, \frac{\Gamma(1-\alpha)^j}{\Gamma(j-j\alpha)}\,(t-T)^{j-1-j\alpha},$$

for $\widetilde{T} \le T < t \le 0$.

Proof. Let $x \in Z_1$ and put

$$r_n(s) \;:=\; e^{\lambda s}(DF(s+\tau, u_n(s,x))Du_n(s,x) - DF(s+\tau, u(s,x))\Phi(s,x)).$$

We see from the explicit forms of $\Phi(t,x)$ and $Du_{n+1}(t,x)$ that

$$e^{\lambda t}(Du_{n+1}(t,x) - \Phi(t,x)) = \int_0^t e^{-\lambda(t-s)}e^{-A_1(t-s)}P_1 r_n(s)\,ds + \int_{-\infty}^t e^{-\lambda(t-s)}e^{-A(t-s)}P_2 r_n(s)\,ds,$$

and, also see that

$$\|r_n(s)z\| \le e^{\lambda s}\sum_{j=1}^{m}\left\|B_j(Du_n(s,x)z - \Phi(s,x)z)\right\|_{\mathcal{B}(X^\alpha,X)} + \left\|DF(s+\tau,u_n(s,x)) - DF(s+\tau,u(s,x))\right\|_{\mathcal{B}(X^\alpha,X)}\|e^{\lambda s}\Phi(s,x)z\|_\alpha.$$

Now we let E be a bounded set in X_1 and $\widetilde{T} < 0$, and set

$$K(t) := \begin{cases} e^{(\lambda-\lambda_1)t} & \text{if } \ t \le 0 \\ t^{-\alpha}e^{(\lambda-\lambda_2)t} & \text{if } \ t > 0, \end{cases}$$

$$\omega_n(\widetilde{T}) := \sup_{s\in[2\widetilde{T},0],\ x\in E} \left\|DF(s+\tau,u_n(s,x)) - DF(s+\tau,u(s,x))\right\|_{\mathcal{B}(X^\alpha,X)}.$$

Note that $K \in L^1(\mathbf{R})$, and also note from (N3) and (3.3) that for each $\widetilde{T} < 0$, $\omega_n(\widetilde{T}) \to 0$ as $n \to \infty$. If $x \in E$, then we have

$$\left\|e^{\lambda t}(Du_{n+1}(t,x)z - \Phi(t,x)z)\right\|_\alpha$$

$$
\begin{aligned}
&\le\ M\int_{-\infty}^{0} K(t-s)e^{\lambda s}\sum_{j=1}^{m}\big\|B_j(Du_n(s,x)z-\Phi(s,x)z)\big\|\,ds\\
&\quad + M\int_{-\infty}^{0} K(t-s)\big\|DF(s+\tau,u_n(s,x))-DF(s+\tau,u(s,x))\big\|_{\mathcal{B}(X^\alpha,X)}\\
&\qquad\qquad \times\big\|e^{\lambda s}\Phi(s,x)z\big\|_\alpha\,ds\\
&\le\ M\Big(\frac{1}{\sqrt{2(\lambda-\lambda_1)}}+\frac{1}{\sqrt{2(\lambda_2-\lambda)}}\Big)\ \big|Du_n(\cdot,x)z-\Phi(\cdot,x)z\big|_Y\\
&\quad + M\omega_n(\widetilde{T})\Big(\int_{2\widetilde{T}}^{0}\big|K(t-s)\big|^2\,ds\Big)^{1/2}\Big(\int_{-\infty}^{0}\big\|e^{\lambda s}\Phi(s,x)z\big\|_\alpha^2\,ds\Big)^{1/2}\\
&\quad + 2ML\Big(\int_{-\infty}^{2\widetilde{T}}\big|K(t-s)\big|^2\,ds\Big)^{1/2}\Big(\int_{-\infty}^{0}\big\|e^{\lambda s}\Phi(s,x)z\big\|_\alpha^2\,ds\Big)^{1/2}.
\end{aligned}
$$

Here we note that for $\widetilde{T}<t\le 0$,

$$
\begin{aligned}
\Big(\int_{2\widetilde{T}}^{0}\big|K(t-s)\big|^2\,ds\Big)^{1/2} &\le \Big(\frac{1}{\sqrt{2(\lambda-\lambda_1)}}+\frac{1}{\sqrt{2(\lambda_2-\lambda)}}\Big)^{1/2},\\
\Big(\int_{-\infty}^{2\widetilde{T}}\big|K(t-s)\big|^2\,ds\Big)^{1/2} &\le \frac{e^{(\lambda_2-\lambda)\widetilde{T}}}{\sqrt{2(\lambda-\lambda_1)}}.
\end{aligned}
$$

Thus, we obtain from these estimates that

$$
\begin{aligned}
&\big\|e^{\lambda t}(Du_{n+1}(t,x)z-\Phi(t,x)z)\big\|_\alpha\\
&\qquad\le\ C_1\big|Du_{n+1}(\cdot,x)z-\Phi(\cdot,x)z\big|_Y+C_2(\omega_n(\widetilde{T})+e^{(\lambda_2-\lambda)\widetilde{T}})\big\|z\big\|,
\end{aligned}
$$

for $\widetilde{T}<t\le 0,\ x\in E,\ z\in X_1$ and positive constants C_1, C_2.

On the other hand, we have

$$
\begin{aligned}
&\big|Du_{n+1}(\cdot,x)z-\Phi(\cdot,x)z\big|_Y\le\xi(\lambda)\big|Du_n(\cdot,x)z-\Phi(\cdot,x)z\big|_Y\\
&\quad+\sum_{j=0}^{m}\Big(\int_{-\infty}^{0}\big\|e^{\lambda t}B_j(S(\Phi(\cdot,x)\,;\,\tau,u_n(\cdot x))(t)z\\
&\qquad\quad -S(\Phi(\cdot,x)\,;\,\tau,u(\cdot,x))(t)z)\big\|^2dt\Big)^{1/2},
\end{aligned}
$$

and

$$
\big\|e^{\lambda t}B_j(S(\Phi(\cdot,x)\,;\,\tau,u_n(\cdot,x))(t)z-S(\Phi(\cdot,x)\,;\,\tau,u(\cdot,x))(t)z)\big\|
$$

$$
\begin{aligned}
&\le M\|B_j\|_{\mathcal{B}(X^\alpha,X)}\int_t^0 e^{(\lambda-\lambda_1)(t-s)}\\
&\times\|DF(s+\tau,u_n(s,x))-DF(s+\tau,u(s,x))\|_{\mathcal{B}(X^\alpha,X)}\|e^{\lambda s}\Phi(s,x)z\|_\alpha\,ds\\
&+M\|B_j\|_{\mathcal{B}(X^\alpha,X)}\int_{-\infty}^t e^{(\lambda-\lambda_2)(t-s)}(t-s)^{-\alpha}\\
&\times\|DF(s+\tau,u_n(s,x))-DF(s+\tau,u(s,x))\|_{\mathcal{B}(X^\alpha,X)}\|e^{\lambda s}\Phi(s,x)z\|_\alpha\,ds,
\end{aligned}
$$

and therefore we see that for $t\in[\widetilde{T},0]$,

$$
\begin{aligned}
&\|e^{\lambda t}B_j(S(\Phi(\cdot,x)\,;\,\tau,u_n(\cdot,x))(t)z-S(\Phi(\cdot,x)\,;\,\tau,u(\cdot,x))(t)z)\|\\
&\le M\|B_j\|_{\mathcal{B}(X^\alpha,X)}\omega_n(\widetilde{T})\int_{-\infty}^0 K(t-s)\|e^{\lambda s}\Phi(s,x)z\|_\alpha\,ds\\
&+2ML\|B_j\|_{\mathcal{B}(X^\alpha,X)}\int_{-\infty}^{2\widetilde{T}} e^{(\lambda-\lambda_2)\widetilde{T}/2}e^{(\lambda-\lambda_2)(t-s)/2}(t-s)^{-\alpha}\|e^{\lambda s}\Phi(s,x)z\|_\alpha\,ds.
\end{aligned}
$$

Thus, we obtain that

$$
\begin{aligned}
&\Big(\int_{\widetilde{T}}^0\|e^{\lambda t}B_j(S(\Phi(\cdot,x)\,;\,\tau,u_n(\cdot,x))(t)z-S(\Phi(\cdot,x)\,;\,\tau,u(\cdot,x))(t)z)\|^2\,dt\Big)^{1/2}\\
&\le M\|B_j\|_{\mathcal{B}(X^\alpha,X)}\omega_n(\widetilde{T})\|K\|_{L^1(\mathbf{R})}\Big(\int_{-\infty}^0\|e^{\lambda s}\Phi(s,x)z\|_\alpha^2\,ds\Big)^{1/2}\\
&+2ML\|B_j\|_{\mathcal{B}(X^\alpha,X)}e^{(\lambda_2-\lambda)\widetilde{T}/2}\Big(\int_0^\infty e^{(\lambda-\lambda_2)t/2}t^{-\alpha}\,dt\Big)\\
&\times\Big(\int_{-\infty}^0\|e^{\lambda s}\Phi(s,x)z\|_\alpha^2\,ds\Big)^{1/2}\le C_3(\omega_n(\widetilde{T})+\eta_1(\widetilde{T}))\|z\|,
\end{aligned}
$$

with $\eta_1(\widetilde{T})\to 0$ as $\widetilde{T}\to-\infty$ and constant $C_3>0$, and

$$
\begin{aligned}
&\Big(\int_{-\infty}^0\|e^{\lambda t}B_j(S(\Phi(\cdot,x)\,;\,\tau,u_n(\cdot,x))(t)z-S(\Phi(\cdot,x)\,;\,\tau,u(\cdot,x))(t)z)\|^2\,dt\Big)^{1/2}\\
&\le C_3(\omega_n(\widetilde{T})+\eta_1(\widetilde{T}))\|z\|\\
&+e^{(\lambda-\widetilde{\lambda})\widetilde{T}}\|B_j\|_{\mathcal{B}(X^\alpha,X)}\bigg(\Big(\int_{-\infty}^0\|e^{\widetilde{\lambda}t}S(\Phi(\cdot,x)\,;\,\tau,u_n(\cdot,x))(t)z\|_\alpha^2\,dt\Big)^{1/2}\\
&\qquad+\Big(\int_{-\infty}^0\|e^{\widetilde{\lambda}t}S(\Phi(\cdot,x)\,;\,\tau,u(\cdot,x))(t)z\|_\alpha^2\,dt\Big)^{1/2}\bigg)
\end{aligned}
$$

$$\leq C_4(\omega_n(\widetilde{T}) + \eta_2(\widetilde{T}))\|z\|,$$

for $\lambda > \widetilde{\lambda}$, with $\eta_2(\widetilde{T}) \to 0$ as $\widetilde{T} \to -\infty$ and constant $C_4 > 0$. These estimates imply that

$$\begin{aligned} &\left|Du_{n+1}(\cdot,x)z - \Phi(\cdot,x)z\right|_Y \\ &\quad \leq \xi(\lambda)\left|Du_n(\cdot,x)z - \Phi(\cdot,x)z\right|_Y + C_4(\omega_n(\widetilde{T}) + \eta_2(\widetilde{T}))\|z\| \\ &\quad \leq C_5\xi(\lambda)^n\|z\| + \frac{C_4}{1-\xi(\lambda)}(\omega_n(\widetilde{T}) + \eta_2(\widetilde{T}))\|z\|, \end{aligned}$$

with positive constant C_5, and therefore we obtain

$$\left\|e^{\lambda t}(Du_{n+1}(t,x)z - \Phi(t,x)z)\right\|_\alpha \leq C_6(\xi(\lambda)^n + \omega_n(\widetilde{T}) + \eta(\widetilde{T}))\|z\|,$$

with $\eta(\widetilde{T}) \to 0$ as $\widetilde{T} \to -\infty$ and constant $C_6 > 0$.

Next, we let $z \in X_1$ and $\widetilde{T} \leq T \leq t \leq 0$, and set

$$\begin{aligned} Du_{n+1}(t,x) &:= e^{-A(t-T)}Du_{n+1}(T,x) \\ &\quad + \int_T^t e^{-A(t-s)}DF(s+\tau, u_n(s,x))Du_n(s,x)\,ds, \\ \Phi(t,x) &:= e^{-A(t-T)}\Phi(T,x) + \int_T^t e^{-A(t-s)}DF(s+\tau,u(s,x))\Phi(s,x)\,ds. \end{aligned}$$

Then we have

$$\begin{aligned} &\left\|Du_{n+1}(t,x)z - \Phi(t,x)z\right\|_\alpha \\ &\quad \leq C\,(\xi(\lambda)^n + \omega_n(\widetilde{T}) + \eta(\widetilde{T}))(t-T)^{-\alpha}\|z\| \\ &\qquad + \widetilde{C}\int_T^t (t-s)^{-\alpha}\left\|Du_n(s,x)z - \Phi(s,x)z\right\|_\alpha ds, \end{aligned}$$

and note that C and $\widetilde{C}$ are positive constants not depending on n and $\widetilde{T}$. Set

$$\nu_{n,\widetilde{T}} := C\,(\xi(\lambda)^n + \omega_n(\widetilde{T}) + \eta(\widetilde{T})),$$

and then we have

$$\begin{aligned} &\left\|Du_{n+1}(t,x)z - \Phi(t,x)z\right\|_\alpha \\ &\quad \leq (t-T)^{-\alpha}\nu_{n,\widetilde{T}}\|z\| + \widetilde{C}\int_T^t (t-s)^{-\alpha}\left\|Du_n(s,x)z - \Phi(s,x)z\right\|_\alpha ds \\ &\quad \leq \sum_{j=1}^n \nu_{n-j+1,\widetilde{T}}\widetilde{C}^{j-1}\frac{\Gamma(1-\alpha)^j}{\Gamma(j-j\alpha)}(t-T)^{j-1-j\alpha}\|z\| \end{aligned}$$

$$+ \widetilde{C}^n \frac{\Gamma(1-\alpha)^n}{\Gamma(n-n\alpha)} \int_T^t (t-s)^{n-1-n\alpha} \big\| Du_1(s,x)z - \Phi(s,x)z \big\|_\alpha ds.$$

On the other hand, we see that

$$\begin{aligned} &\int_T^t (t-s)^{n-1-n\alpha} \big\| e^{-A_1 s} z - \Phi(s,x)z \big\|_\alpha \, ds \\ &\quad \leq \int_T^t (t-s)^{n-1-n\alpha} \left(M e^{-\lambda_1 s} + \sup_{s \in [T,0]} \big\| \Phi(s,x) \big\|_{\mathcal{B}(X_1, X^\alpha)} \right) \big\| z \big\| \, ds \\ &\quad \leq \frac{C_7}{(1-\alpha)n} (t-T)^{n-1-n\alpha} \big\| z \big\|, \end{aligned}$$

with positive constant C_7. We thus obtain that

$$\begin{aligned} &\widetilde{C}^n \frac{\Gamma(1-\alpha)^n}{\Gamma(n-n\alpha)} \int_T^t (t-s)^{n-1-n\alpha} \big\| Du_1(s,x)z - \Phi(s,x)z \big\|_\alpha \, ds \\ &\leq \widetilde{C}^n \frac{\Gamma(1-\alpha)^{n+1}}{\Gamma((1-\alpha)(n+1))} (t-T)^{n-(n+1)\alpha} \big\| z \big\| \; \frac{C_7 \, \Gamma((1-\alpha)(n+1))(t-T)^\alpha}{\Gamma(n-n\alpha)\Gamma(1-\alpha)(1-\alpha)n}, \end{aligned}$$

and see from Stirling's formula that

$$\lim_{n \to \infty} \frac{C_7 \, \Gamma((1-\alpha)(n+1))(t-T)^\alpha}{\Gamma(n-n\alpha)\Gamma(1-\alpha)(1-\alpha)n} \; = \; 0.$$

Therefore, for sufficiently large n, we have

$$\begin{aligned} &\widetilde{C}^n \frac{\Gamma(1-\alpha)^n}{\Gamma(n-n\alpha)} \int_T^t (t-s)^{n-1-n\alpha} \big\| Du_1(s,x)z - \Phi(s,x)z \big\|_\alpha \, ds \\ &\quad \leq \nu_{0,\widetilde{T}} \widetilde{C}^n \frac{\Gamma(1-\alpha)^{n+1}}{\Gamma((1-\alpha)(n+1))} (t-T)^{n-(n+1)\alpha} \big\| z \big\|, \end{aligned}$$

which implies that

$$\big\| Du_{n+1}(s,x)z - \Phi(s,x)z \big\|_\alpha \leq \sum_{j=1}^{n+1} \nu_{n-j+1,\widetilde{T}} \widetilde{C}^{j-1} \frac{\Gamma(1-\alpha)^j}{\Gamma(j-j\alpha)} (t-T)^{j-1-j\alpha} \big\| z \big\|,$$

for $\widetilde{T} \leq T \leq t \leq 0, \; x \in E, \; z \in X_1$. Hence, we conclude that

$$\begin{aligned} &\sup_{x \in E} \big\| Du_{n+1}(t,x) - \Phi(t,x) \big\|_{\mathcal{B}(X_1, X^\alpha)} \\ &\leq \sum_{j=1}^{n+1} \nu_{n-j+1,\widetilde{T}} \widetilde{C}^{j-1} \frac{\Gamma(1-\alpha)^j}{\Gamma(j-j\alpha)} (t-T)^{j-1-j\alpha} \end{aligned}$$

for $\widetilde{T} \leq T \leq t \leq 0$. □

Observing that $\nu_{n,\widetilde{T}} \geq 0$, $\nu_{n,\widetilde{T}} \to 0$ as $n \to \infty$ first and then $\widetilde{T} \to -\infty$,

$$\widetilde{C}^{j-1} \frac{\Gamma(1-\alpha)^j}{\Gamma(j-j\alpha)} (t-T)^{j-1-j\alpha} \geq 0$$

$$\sum_{j=1}^{\infty} \widetilde{C}^{j-1} \frac{\Gamma(1-\alpha)^j}{\Gamma(j-j\alpha)} (t-T)^{j-1-j\alpha} \ < \ \infty,$$

we obtain that

$$\lim_{n\to\infty} \sup_{x\in E} \left\| Du_{n+1}(t,x) - \Phi(t,x) \right\|_{\mathcal{B}(X_1, X^\alpha)} \ = \ 0. \tag{3.4}$$

We see from (3.3) and (3.4) that $u \in C^1$ with respect to x and $Du(t,x) = \Phi(t,x)$. Hence, we complete the proof of our main result.

4. Application

We now apply our theory to the problem of smoothness of inertial manifolds for the following scalar reaction diffusion equation

$$\text{(RD)} \qquad u_t - u_{xx} \ = \ f(t,x,u,u_x), \qquad t > 0, \ x \in \Omega := (0,\pi),$$

with the Dirichlet boundary conditions $u(t,0) = u(t,\pi) = 0$. Here, f is a continuous function from $\mathbf{R} \times \mathbf{R} \times \mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$. Let $X = L^2_{per}(\Omega)$, $A = -\partial^2/\partial x^2$ with the domain $\mathcal{D}(A) = H^1_{0,per}(\Omega) \cap H^2_{per}(\Omega)$. Define $F(t,u)(x) := f(t,x,u,u_x)$, and assume that F satisfies the conditions (N1)-(N4). In this case, it is known that the spectrum of A is $\{\lambda_n\,;\, n > 0\}$ with $\lambda_n = n^2$. Putting $\lambda := (\lambda_n + \lambda_{n+1})/2 \in (\lambda_n, \lambda_{n+1})$, we may denote P_1 and P_2 the spectral projections corresponding to $\sigma_1 := \{\lambda_1, \cdots, \lambda_n\}$ and $\sigma_2 := \{\lambda_{n+1}, \cdots\}$, respectively. If we also assume that

$$\int_{-\infty}^{0} \int_{0}^{\pi} \left| e^{\mu t} f(t,x,0,0) \right|^2 dxdt \ < \ \infty,$$

for some $\mu < \lambda$, then we see that all assumptions of Theorem 2.2 are satisfied for sufficiently large n. By Brunovský and Tereščák [1], the existence of an invariant manifold are assured, and the smoothness result are also obtained.

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