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ON HOMOGENIZATION OF A MIXED BOUNDARY OPTIMAL CONTROL PROBLEM

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Abstract. We study the asymptotic behaviour of an optimal control problem for the Ukawa equation in a thick multi-structure with different types and classes of admissible boundary controls. This thick multi-structure consists of a domain (the junction's body) and a large number of ε-periodically situated thin cylinders. We consider two types of boundary controls, namely, the Dirichlet $H^{1/2}$-controls on the bases Γ_ε of thin cylinders, and the Neumann L^2-controls on their 'vertical' sides. We present some ideas and results concerning of the asymptotic analysis for such problems as $\varepsilon \to 0$ and derive conditions under which the homogenized problem can be recovered in the explicit form. We show that the mathematical description of the homogenized optimal boundary control problem is different from the original one. These differences appear not only in the control constraints, limit cost functional, state equations, and boundary conditions, but also in the type of admissible controls for the limit problem - one of them is the Dirichlet L^2-control, whereas the second one is appeared as the distributed L^2-control.

1. Introduction

The goal of this work is to study one class of optimal control problems for the Ukawa equation in a domain $\Omega_\varepsilon \subset \mathbb{R}^n$ (thick multi-structure, or thick junctions), whose boundary $\partial\Omega_\varepsilon$ contains the very highly oscillating part with respect to ε, as $\varepsilon \to 0$. We consider this problem assuming that there are two types of the control influences via the Neumann and Dirichlet

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boundary conditions posed on the different parts of the oscillating boundary (for comparison see [8, 15, 39, 42]).

In this paper by a thick multi-structure Ω_ε we mean a domain in $\mathbb{R}^n$, which consists of some domain Ω^+ and a large number of cylinders with axes parallel to Ox_n and ε-periodically distributed along some manifold Σ on the boundary of Ω^+ (see Fig. 1). This manifold is called the joint zone and the domain Ω^+ is called the junction's body. Here ε is a small positive parameter, which characterizes the distance between the neighboring cylinders and their thickness. So, each attached cylinder has a small cross section of size ε and its limiting dimension (as $\varepsilon \to 0$) is equal to $n-1$. In view of this, such cylinders will be called thin domains. In a general case, each thick multi-structure $\Omega_\varepsilon \subset \mathbb{R}^n$ can be characterized by triplet of integer numbers $k : p : d$, which refer to the limiting dimensions of the junction's body, the joint zone, and each of the attached thin domains, respectively. This classification was given by T.A. Mel'nyk and S.A. Nazarov in [37]-[41], where the asymptotic methods were developed and basic results (convergence theorems, asymptotic approximations) were obtained for boundary-value problems in thick junctions of different types. For the further development of asymptotic methods in more complicated junctions (so-called thick multi-level junctions) we refer to [19, 20, 42].

Boundary value problems in thick junctions are prototypes of widely used engineering constructions as well as many other physical and biological systems with very distinct characteristic scales. The computational calculation of the solutions of these problems is very complicated due to singularities of the thick junctions. Indeed, increase in the size of computational domains for thick multi-structures naturally leads to longer computing time and makes it very difficult to keep an acceptable level of accuracy. Therefore, asymptotic analysis is one of the main approaches to study of boundary value problems in such domains because it gives the possibility to replace the original problem by the corresponding limit problem defined in a more "simpler" domain with preservation of the main variational property: both the optimal solution and the minimal value of the cost functional for the original problem converge to the corresponding characteristics of a limit optimal control problem as ε tends to zero.

It is well known that the asymptotic analysis of optimal control problems for the systems with distributed parameter is rather non trivial itself (for this we refer to [1, 4, 10, 11, 26, 27, 43]). The most typical algorithm of the homogenization consists of the following steps: first, we write down the necessary optimality conditions for the initial problem; next we find the corresponding limiting relations as $\varepsilon \to 0$ and interpret them as necessary optimality conditions for some control problem; then, using the limiting necessary optimality conditions, we recover an optimal control problem which is called the homogenized control problem to the initial one (see e.g. [12, 16, 28, 29, 44]).

While there are many great papers published on various aspects and methods in homogenization of different classes of optimization problems, only few papers deal with the problem of optimal control of partial differential equations in thick multi-structures (see [16, 22, 23, 34, 35]. In view of this we underline that there are still many open problems in this field, especially for the problems with Dirichlet and Neumann type boundary controls. Moreover, the thick multi-structures are non-convex domains with non-smooth boundaries. So, the solutions of boundary-value problems in such domains have only minimal smoothness. As the result the necessary optimality conditions for optimal control problems in thick multi-structures are baseless as a rule.

In this connection, we propose another approach to the homogenization of optimal boundary control problems in thick multi-structures, which is based on ideas of the theory of Γ-convergence and the concept of variational convergence of constrained minimization problems [4]–[7]. To investigate the asymptotic behavior of the considered optimal boundary control problem we apply the scheme of the direct homogenization, which was developed in [21], and we use homogenization results for solutions to boundary-value problems in thick multi-structures in [3], [14]-[19], [37]-[42]. This approach allows to reduce the procedure of the homogenization to the consecutive identification of the set of admissible solutions for the homogenized optimal control problem and then its cost functional. Some results concerning application of this approach to simpler optimal boundary control problems were obtained in [34, 35].

2. Statement of the problem and basic notation

Let $B = (0, a)^{n-1}$ and C be bounded open smooth domains in $\mathbb{R}^{n-1}$ $(n \geq 2)$ and $C \subset\subset (0,1)^{n-1}$, let $\gamma \in C^1(B)$ and $\beta \in C^1(B)$ be functions such that $0 < \gamma_0 = \inf_{x' \in B} \gamma(x')$; $0 < \beta_0 = \inf_{x' \in B} \beta(x')$, and let $\{\varepsilon\}$ be a sequence of positive numbers converging to zero. In the sequel we will always assume that $\varepsilon = a/N$, where N is a large positive integer. When we write $\varepsilon > 0$, we consider only the elements of this sequence. Let us denote a point of $\mathbb{R}^n$ by $x = (x_1, x_2, \ldots, x_n) = (x', x_n)$ and introduce the following sets:

$$
\begin{aligned}
\Omega &= \{(x', x_n) \in \mathbb{R}^n \ : \ -\beta(x') < x_n < \gamma(x'),\ x' \in B\}, \\
\Sigma &= B \times \{0\}, \\
\Omega^+ &= \{(x', x_n) \in \mathbb{R}^n \ : \ x' \in B,\ 0 < x_n < \gamma(x')\}, \\
\Omega^- &= \{(x', x_n) \in \mathbb{R}^n \ : \ x' \in B,\ -\beta(x') < x_n < 0\}, \\
\Gamma_0 &= \{(x', x_n) \in \mathbb{R}^n \ : \ x' \in B,\ x_n = -\beta(x')\}, \\
\theta_\varepsilon &= \{\mathbf{k} = (k_1, k_2, \dots, k_{n-1}) \in \mathbb{N}^{n-1} \ : \ \varepsilon C + \varepsilon \mathbf{k} \subset\subset B\}, \\
\Omega_\varepsilon &= \Omega^+ \cup \{(x', x_n) \ : \ x' \in \bigcup_{\mathbf{k}\in\theta_\varepsilon} (\varepsilon C + \varepsilon \mathbf{k}),\ -\beta(x') < x_n \le 0\}, \\
\Omega_\varepsilon^- &= \Omega_\varepsilon \cap \Omega^-.
\end{aligned} \tag{2.1}
$$

Then in view of our previous description the set Ω_ε is a thick multi-structure, which consists with the junction's body Ω^+ and a large number N^{n-1} of the thin cylinders $G_\varepsilon^{\mathbf{k}}$ with axis Ox_n and ε-periodically distributed on the basis Σ of Ω^+ (see Fig. 1 for 3-d example). Here, each cylinder $G_\varepsilon^{\mathbf{k}}$ is obtained with ε-homothety in the first $(n-1)$ variables, i.e.,

$$
G_\varepsilon^{\mathbf{k}} = \{(x', x_n) \ : \ x' \in \varepsilon C + \varepsilon \mathbf{k},\ -\beta(x') < x_n \le 0\}.
$$

It is easy to see that $\Omega_\varepsilon = \Omega^+ \cup \left(\bigcup_{\mathbf{k}\in\theta_\varepsilon} G_\varepsilon^{\mathbf{k}}\right)$ and the type of this multi-structure is $n : n-1 : n-2$.

Denote by Γ_ε the union of the lower bases $\Gamma_\varepsilon^{\mathbf{k}} = \{(x', x_n) \ : \ x' \in \varepsilon \cdot C + \varepsilon\mathbf{k},\ x_n = -\beta(x')\}$ of the thin cylinders $G_\varepsilon^{\mathbf{k}}$ when $\mathbf{k} \in \theta_\varepsilon$, and by S_ε the union of their boundaries along the axis Ox_n: $S_\varepsilon^{\mathbf{k}} = \{(x', x_n) \ : \ x' \in \varepsilon \cdot \partial C + \varepsilon \mathbf{k},\ -\beta(x') < x_n < 0\}$.

In Ω_ε, we consider the following optimal boundary control problem ($\mathrm{OBCP}_\varepsilon$):

$$
I_\varepsilon \ = \int_{\Omega^+} (y_\varepsilon - q_0)^2 dx \ + \int_{\Gamma_\varepsilon} u_\varepsilon^2 \, d\mathcal{H}^{n-1} \ + \ \varepsilon \int_{S_\varepsilon} p_\varepsilon^2 \, d\mathcal{H}^{n-1} \ \longrightarrow \ \inf, \tag{2.2}
$$

$$
\left.
\begin{aligned}
-\Delta_x \ y_\varepsilon(x) + y_\varepsilon &= f_\varepsilon(x), && x \in \Omega_\varepsilon, \\
\partial_\nu y_\varepsilon(x) &= \varepsilon\, p_\varepsilon(x), && x \in S_\varepsilon, \\
y_\varepsilon(x) &= u_\varepsilon(x), && x \in \Gamma_\varepsilon, \\
\partial_\nu y_\varepsilon(x) &= 0, && x \in \partial\Omega_\varepsilon \setminus (\Gamma_\varepsilon \cup S_\varepsilon)\,,
\end{aligned}
\right\} \tag{2.3}
$$

$$
u_\varepsilon \in U_\varepsilon = \left\{ u \in H^{1/2}(\Gamma_\varepsilon; \mathcal{H}^{n-1}), \quad \|u\|_{L^2(\Gamma_\varepsilon; \mathcal{H}^{n-1})} \le \mathbf{C_u} \right\}, \tag{2.4}
$$

$$
p_\varepsilon \in P_\varepsilon = \left\{ p \in L^2(S_\varepsilon; \mathcal{H}^{n-1}), \quad \varepsilon \|p\|^2_{L^2(S_\varepsilon; \mathcal{H}^{n-1})} \le \mathbf{C_p} \right\}, \tag{2.5}
$$

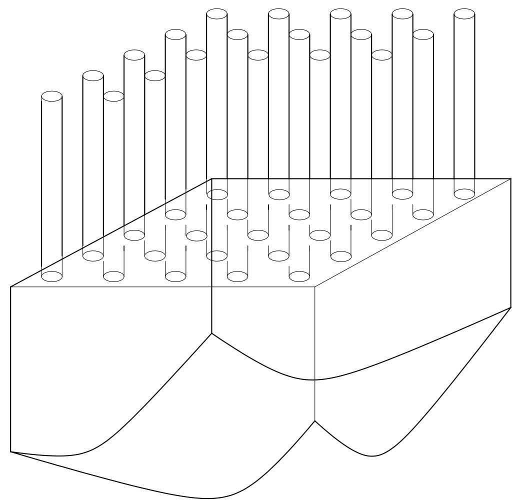

Figure 1. Thick multi-structure Ω_ε.

where σ and δ are positive constants; $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$, which coincides with ordinary "$(n-1)$-dimensional surface area"; $q_0 \in L^2(\Omega^+)$ and $f_\varepsilon \in L^2(\Omega)$ are given functions such that

$$f_\varepsilon \longrightarrow f_0 \quad \text{strongly in} \quad L_2(\Omega) \quad \text{as} \quad \varepsilon \to 0; \tag{2.6}$$

$\partial_\nu = \partial/\partial\nu$ is the outward normal derivative; $\mathbf{C_u} > 0$ and $\mathbf{C_p} > 0$ are some fixed constants that are independent of $\varepsilon, u_\varepsilon$ and p_ε; U_ε and P_ε are the sets of admissible controls, u_ε and p_ε are control functions, and $y_\varepsilon = y_\varepsilon(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)$ is the corresponding weak solution of the boundary value problem (2.3). In the sequel for simplicity we will always use the following notation: $H^{1/2}(\Gamma_\varepsilon) = H^{1/2}(\Gamma_\varepsilon; \mathcal{H}^{n-1})$, $L^2(\Gamma_\varepsilon) = L^2(\Gamma_\varepsilon; \mathcal{H}^{n-1})$, and $L^2(S_\varepsilon) = L^2(S_\varepsilon; \mathcal{H}^{n-1})$.

It is well known that for every fixed ε and for any control functions $u_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ and $p_\varepsilon \in L^2(S_\varepsilon)$ the boundary value problem (2.3) admits a unique solution $y_\varepsilon \in H^1(\Omega_\varepsilon)$ such that the following integral identity

$$\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla \varphi \, dx + \int_{\Omega_\varepsilon} y_\varepsilon \, \varphi \, dx = \int_{\Omega_\varepsilon} f_\varepsilon(x) \, \varphi(x) \, dx + \varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi \, d\mathcal{H}^{n-1} \quad \forall \, \varphi \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon), \tag{2.7}$$

holds and the trace of y_ε equals to u_ε on Γ_ε. The function y_ε is called the weak solution to problem (2.3). Here $H^1(\Omega_\varepsilon;\Gamma_\varepsilon) = \{\varphi \in H^1(\Omega_\varepsilon) : \ \varphi = 0 \ \text{ on } \ \Gamma_\varepsilon\}$. In addition, from [41] it follows that the weak solution satisfies a priori norm–estimate

$$\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le c_1 \left(\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{H^{1/2}(\Gamma_\varepsilon)} + \sqrt{\varepsilon}\, \|p_\varepsilon\|_{L^2(S_\varepsilon)} \right) \tag{2.8}$$

if $u \in H^{1/2}(\Gamma_\varepsilon)$ and $p_\varepsilon \in L^2(S_\varepsilon)$.

The aim of our research is to study the asymptotic behavior of this problem as $\varepsilon \to 0$, that is when the number of attached thin cylinders infinitely increases and their thickness vanishes.

It should be stressed here that if the small parameter ε is changed, then all components of this control problems (the domain Ω_ε, the constraint sets U_ε and P_ε, the cost functional I_ε, and the set, where we seek its infimum) are changed as well. Let us observe also that the volume of the material included in the set Ω_ε^- does not converge to zero as $\varepsilon \to 0$. Moreover, $\lim_{\varepsilon\to 0} \Omega_\varepsilon^- = \Omega^-$ in the Hausdorff metric and $|\Omega_\varepsilon^-| \to |\Omega^-|$, i.e., the set Ω^- is filled up by the thin cylinders in the limit passage as $\varepsilon \to 0$. It produces the fact that the Neumann boundary controls p_ε will transform (as $\varepsilon \to 0$) to some distributed control function $p \in L^2(\Omega^-)$ in the right-hand side of the homogenized equation (see for comparison [17, 35, 41, 42]).

The scheme of the investigation is as follows. The sense of the original problem at a fixed value of the parameter ε consists in the control of the state of the original system through the set of the thin cylinders $\Omega_\varepsilon^- = \cup_{\mathbf{k}\in\theta_\varepsilon} G_\varepsilon^{\mathbf{k}}$. We consider two control zones: the bases Γ_ε of the thin cylinders, where the Dirichlet conditions for admissible controls u_ε are given, and the second one is the other part S_ε of $\partial\Omega_\varepsilon$, where the Neumann type of controls p_ε is considered. It should be noted that boundary value problems in thick multi-structures with both types of controls on the boundaries of the thin domains are not investigated in full and such boundary conditions make the process of homogenization more complicated. We begin in Section 3 with the description of the class of Neumann boundary controls in the terms of singular measures. After that we reformulate the original optimal control problem (see (3.5)–(3.6)), give a priori norm–estimate for its solutions. We study also the solvability of the original optimal control problem at a fixed value of ε and the uniqueness of its solution.

In Section 4 we choose the topology for the homogenization of the optimal control problem and introduce the convergence concept of admissible triplets (see Definitions 4.1, 4.2, and 4.3). We prove that any sequence of admissible triplets for the initial problem is relatively compact with respect to so-called w-convergence. In Section 5 we give the definition of the w-homogenized problem (see Definitions 5.2 and 5.3) and establish its main variational properties.

Section 6 deals with the analytical representation of the set Ξ_0, which is the limit in Kuratowski's sense of sets of admissible triplets for the original optimal control problem (see Definition 5.1). We show that this set can be represented in an explicit form (see Theorem 6.4) as set of admissible quaternaries for some limit boundary value problem.

In Section 7 we identify the limit cost functional I_0 for the homogenized constrained minimization problem. We show that this functional has different analytical representation from the original one and prove the main result of homogenization for problem (2.2)–(2.5) as $\varepsilon \to 0$.

The Section 8 is devoted to variational properties of the homogenized optimal control problem.

3. On solvability of the original optimal boundary control problem

Following Zhikov's approach ([47, 48], see also [2, 9]) we will describe the geometry of the set S_ε, and hence the class of admissible boundary controls, in the terms of so-called singular measure in $\mathbb{R}^n$. We note that this measure does not satisfy the regularity property with respect to the corresponding Lebesgue measure.

Let μ_0 be a periodic finite positive Borel measure in $\mathbb{R}^{n-1}$. Let $\square = [0,1)^{n-1}$ be the cell or torus of periodicity for μ_0. We assume that Borel measure μ_0 is the probability measure in $\mathbb{R}^{n-1}$, concentrated and uniformly distributed on the set ∂C, so $\int_\square d\mu_0 = 1$.

Remark 3.1. Note that by definition $\mu_0(\square \setminus \partial C) = 0$. Therefore any functions, taking the same values on the manifold ∂C, coincide as elements of $L^2(\square, d\mu_0)$. Here the Lebesgue space $L^2(\square, d\mu_0)$ with respect to the measure μ_0 is defined in a usual way with the corresponding norm

$$\|f\|^2_{L^2(\square,\, d\mu_0)} = \int_\square |f(x)|^2 \, d\mu_0$$

(we adopt the standard notation $L^2(\square)$ when μ_0 is the Lebesgue measure).

Now we set $\square_n = \square \times [0,1) = [0,1)^n$, and consider the following measure $d\mu = d\mu_0 \times dx_n$ in $\square_n$. It is easy to see that this measure concentrated on the set $\partial C \times [0,1)$, and for any smooth function g we have

$$\int_{\square_n} g\, d\mu = \int_0^1 \int_\square g\, dx_n\, d\mu_0 \;=\; \left[\mathcal{H}^{n-1}(\partial C \times [0,1))\right]^{-1} \int_{\partial C \times [0,1)} g\, d\mathcal{H}^{n-1}.$$

However, as follows from the properties of the Hausdorff measure (see [24]), we have $\mathcal{H}^k = \mathcal{L}^k$ for every integer k. Since the Lebesgue measure $\mathcal{L}^k$ can be defined as the k-fold product of one-dimensional Lebesgue measure $\mathcal{L}^1$, it follows that $\mathcal{H}^{n-1}(\partial C \times [0,1)) = \mathcal{H}^{n-2}(\partial C)$. Using in the sequel the notation $|\partial C|_H = \mathcal{H}^{n-2}(\partial C)$, the previous relation can be rewritten in the following form

$$\int_{\square_n} g\, d\mu = \int_0^1 \int_{\square} g\, dx_n\, d\mu_0 \;=\; |\partial C|_H^{-1} \int_{\partial C \times (0,1)} g\, d\mathcal{H}^{n-1}. \tag{3.1}$$

Thus, $|\partial C|_H$ is the $(n-2)$-dimensional Hausdorff measure of the manyfold ∂C. For instance, let us consider the plane thick multi-structure $\Omega_\varepsilon \subset \mathbb{R}^2$. Then $n = 2$, and the set C is some part of the segment $(0,1)$, for example, $C = \{x_1 \in (0,1) \;:\; |x_1 - 1/2| < h/2\}$, where $h \in (0,1)$ is a fixed number. So, in this case $|\partial C|_H = 2$ and the 1-periodic measure μ_0 in $\mathbb{R}^1$ can be defined by the rule

$$\mu_0 = \frac{1}{|\partial C|_H} \left(\delta_{M_1} + \delta_{M_2}\right) = \frac{1}{2} \left(\delta_{M_1} + \delta_{M_2}\right),$$

where $M_i = \frac{1}{2} + \left(i - \frac{3}{2}\right) h$, $i = 1, 2$. Here, by δ_{M_i} we denote Dirac measures located at the points M_i. Thus, the multiplier $|\partial C|_H^{-1}$ in (3.1) is equal to $1/2$.

Let Λ be any Borel set of $\mathbb{R}^n$. We introduce so-called "scaling" measure μ_ε by the rule $\mu_\varepsilon(\Lambda) = \varepsilon^n \mu(\varepsilon^{-1}\Lambda)$. This measure has a period ε, and moreover, since $\mu(\varepsilon\square_n) = \varepsilon \cdot \mu_0(\varepsilon\square)$ by definition of μ, it follows that

$$\mu_\varepsilon(\varepsilon\square_n) = \varepsilon^n \int_0^\varepsilon \int_{\varepsilon\square} d\mu_0(x'/\varepsilon)\, d(x_n/\varepsilon) = \varepsilon^n \int_0^1 \int_{\square} d\mu_0\, dx_n = \varepsilon^n.$$

It means that the measure μ_ε weakly converges to Lebesgue measure in $\mathbb{R}^n$ as $\varepsilon \to 0$ (in symbols $d\mu_\varepsilon \to dx$), that is,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \varphi\, d\mu_\varepsilon = \int_{\mathbb{R}^n} \varphi\, dx$$

for all functions $\varphi \in C_0^\infty(\mathbb{R}^n)$ (see [49]).

Now we turn back to the definition of the weak solution $y_\varepsilon \in H^1(\Omega_\varepsilon)$ of boundary value problem (2.3) (see (2.7)). Since the Sobolev space $H^1(\Omega_\varepsilon)$ can be viewed as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\left(\int_{\Omega_\varepsilon} (y^2 + |\nabla y|^2)\, dx \right)^2,$$

it follows that a function $y_\varepsilon \in H^1(\Omega_\varepsilon)$ is a weak solution of the above mentioned problem whenever

$$\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla \varphi \, dx + \int_{\Omega_\varepsilon} y_\varepsilon \, \varphi \, dx = \int_{\Omega_\varepsilon} f_\varepsilon(x) \, \varphi(x) \, dx + \delta \, \varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi \, d\mathcal{H}^{n-1} \tag{3.2}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$, where we denoted by $C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$ the set of all functions from $C_0^\infty(\mathbb{R}^n)$ such that $\varphi|_{\Gamma_\varepsilon} = 0$.

Let us consider the last term in the left part of identity (3.2). Using the notation from the section 2, we may write down

$$\begin{aligned}
\varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi \, d\mathcal{H}^{n-1} &= \varepsilon \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\partial C + \mathbf{k}_j)} \int_{-\beta(x')}^{0} p_\varepsilon \varphi \, d\mathcal{H}^{n-2} \, dx_n \\
&= \varepsilon |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square + \mathbf{k}_j)} \int_{-\beta(x')}^{0} \widehat{p}_\varepsilon \varphi \, \varepsilon^{n-2} \, d\mu_0(x'/\varepsilon) \, dx_n \\
&= |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square + \mathbf{k}_j)} \int_{-\beta(x')}^{0} \widehat{p}_\varepsilon \varphi \; \varepsilon^{n} \, d\mu_0(x'/\varepsilon) \, d(x_n/\varepsilon) \\
&= |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square + \mathbf{k}_j)} \int_{-\beta(x')}^{0} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon = |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon.
\end{aligned}$$

Here $\widehat{p}_\varepsilon$ is a function of $L^2(\Omega^-, d\mu_\varepsilon)$ taking the same values with $p_\varepsilon \in L^2(S_\varepsilon)$ on S_ε. We note also that the integral $\int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon$ is well defined for every function $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$. Indeed, since the set Ω^- is bounded, and $\widehat{p}_\varepsilon \, d\mu_\varepsilon$ is a Radon measure, it follows that $\int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon$ is a linear continuous functional on $C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$.

Definition 3.1. *We say that a function $y_\varepsilon \in H^1(\Omega_\varepsilon)$ is a weak solution of the boundary value problem* (2.3)*, if for given function $u_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ and $\widehat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)$ the following integral identity*

$$\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla \varphi \, dx + \int_{\Omega_\varepsilon} y_\varepsilon \varphi \, dx = |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon + \int_{\Omega_\varepsilon} f_\varepsilon(x) \varphi(x) \, dx \tag{3.3}$$

holds for every $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$, and the trace of y_ε equals to u_ε on Γ_ε.

Let us introduce the following set

$$\widehat{P}_\varepsilon = \{ p \in L^2(\Omega^-, d\mu_\varepsilon) \; : \; \|p\|^2_{L^2(\Omega^-, d\mu_\varepsilon)} \le |\partial C|_H^{-1} \, \mathbf{C_p} \} \tag{3.4}$$

Then taking into account the above obtained result, we may conclude: there is a one to one correspondence between the sets P_ε and $\widehat{P}_\varepsilon$, that is for every admissible boundary control $p_\varepsilon \in P_\varepsilon$ one can find a function $\widehat{p}_e \in$

$L^2(\Omega^-, d\mu_\varepsilon)$ such that $p_\varepsilon = \widehat{p}_\varepsilon$ on S_ε, $p_\varepsilon \in P_\varepsilon$ if and only if $\widehat{p}_e \in \widehat{P}_\varepsilon$. In particular, we have

$$\varepsilon \int_{S_\varepsilon} p_\varepsilon^2 \, d\mathcal{H}^{n-1} = |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon^{\,2} \, d\mu_\varepsilon.$$

In view of this result we may reformulate the original optimal boundary control problem (2.2)–(2.5) as follows: find a triplet $(u_\varepsilon^0, \widehat{p}_\varepsilon^{\,0}, y_\varepsilon^0)$ such that $(u_\varepsilon^0, \widehat{p}_\varepsilon^{\,0}, y_\varepsilon^0) \in \mathbb{X}_\varepsilon \equiv H^{1/2}(\Gamma_\varepsilon) \times L^2(\Omega^-, d\mu_\varepsilon) \times H^1(\Omega_\varepsilon)$ and

$$\begin{aligned} I_\varepsilon(u_\varepsilon^0, \widehat{p}_\varepsilon^{\,0}, y_\varepsilon^0) &= \inf_{(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \qquad (3.5) \\ &= \inf_{(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} \left[\int_{\Omega^+} (y_\varepsilon - q_0)^2 dx \, + \int_{\Gamma_\varepsilon} u_\varepsilon^2 \, d\mathcal{H}^{n-1} \, + |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon^{\,2} \, d\mu_\varepsilon \right], \end{aligned}$$

where the set $\Xi_\varepsilon \subset \mathbb{X}_\varepsilon$ has the form

$$\Xi_\varepsilon = \left\{ (u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \;\middle|\; \begin{array}{r} u_\varepsilon \in U_\varepsilon, \quad \widehat{p}_\varepsilon \in \widehat{P}_\varepsilon, \quad y_\varepsilon \in H^1(\Omega_\varepsilon), \\ y_\varepsilon|_{\Gamma_\varepsilon} = u_\varepsilon \quad \mathcal{H}^{n-1} \text{ a. e. on } \Gamma_\varepsilon, \\ \displaystyle\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla \varphi \, dx + \int_{\Omega_\varepsilon} y_\varepsilon \varphi \, dx = |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon \\ \displaystyle + \int_{\Omega_\varepsilon} f_\varepsilon(x) \varphi(x) \, dx, \quad \textit{for every } \varphi \in C_0^\infty(\mathbb{R}^n, \Gamma_\varepsilon), \end{array} \right\}. \qquad (3.6)$$

In the sequel optimal control problem (3.5)–(3.6) will be called the $\widehat{\mathbb{P}}_\varepsilon$-problem.

Due to the norm estimate (2.8) we have the following obvious result:

Proposition 3.1. *Let $u_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ and $\widehat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)$ be any admissible control functions for $\widehat{\mathbb{P}}_\varepsilon$-problem. Then there exists a constant $c > 0$, independent of ε, such that for every $\varepsilon > 0$,*

$$\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le c \left(\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{H^{1/2}(\Gamma_\varepsilon)} + \|\widehat{p}_\varepsilon\|_{L^2(\Omega^-, d\mu_\varepsilon)} \right). \qquad (3.7)$$

Let τ_ε be the product of the weak topologies of $H^{1/2}(\Gamma_\varepsilon)$, $L^2(\Omega^-, d\mu_\varepsilon)$, and $H^1(\Omega_\varepsilon)$, and let τ_ε^h be the product of the weak topologies of $L^2(\Gamma_\varepsilon)$, $L^2(\Omega^-, d\mu_\varepsilon)$, and $H^1(\Omega_\varepsilon)$.

Definition 3.2. *The triplet $(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \mathbb{X}_\varepsilon$ is called admissible for the problem $\widehat{\mathbb{P}}_\varepsilon$ if $u_\varepsilon \in U_\varepsilon$, $\widehat{p}_\varepsilon \in \widehat{P}_\varepsilon$, and $y_\varepsilon = y_\varepsilon(u_\varepsilon, \widehat{p}_\varepsilon)$ is the corresponding weak solution of problem (2.3) whose trace on Γ_ε is equal to u_ε.*

As follows from this definition and representation (3.6), Ξ_ε is the set of all admissible triplets for $\widehat{\mathbb{P}}_\varepsilon$-problem. It is easy to see that this set is nonempty, convex, and τ_ε-closed for every $\varepsilon > 0$. As consequence we have the following result.

Theorem 3.2 (Existence and uniqueness)**.** *For every value of ε the optimal boundary control problem $\widehat{\mathbb{P}}_\varepsilon$ has the unique solution, i.e., there exists a unique triplet $(u_\varepsilon^0, \widehat{p}_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ such that*

$$I_\varepsilon(u_\varepsilon^0, \widehat{p}_\varepsilon^{\,0}, y_\varepsilon^0) = \inf_{(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon).$$

Let us denote by $\mathrm{cl}_{\tau_\varepsilon^h}\Xi_\varepsilon$ the closure of Ξ_ε with respect to τ_ε^h-topology and consider the following constrained minimization problem

$$\Big\langle \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathrm{cl}_{\tau_\varepsilon^h}\Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) \Big\rangle. \tag{3.8}$$

It is clear that this problem is solvable for every ε. Indeed, $\mathrm{cl}_{\tau_\varepsilon^h}\Xi_\varepsilon$ is a convex, closed and bounded subset of $L^2(\Gamma_\varepsilon) \times L^2(\Omega^-, d\mu_\varepsilon) \times H^1(\Omega_\varepsilon)$, $I_\varepsilon : L^2(\Gamma_\varepsilon) \times L^2(\Omega^-, d\mu_\varepsilon) \times H^1(\Omega_\varepsilon) \to \mathbb{R}$ is the strictly convex τ_ε^h-lower semicontinuous functional. Then by applying the direct method of Calculus of Variations, the minimization problem (3.8) admits a unique solution $(u_\varepsilon^*, p_\varepsilon^*, y_\varepsilon^*) \in \mathrm{cl}_{\tau_\varepsilon^h}\Xi_\varepsilon$. Moreover, it can be easily proved that for every fixed $\varepsilon > 0$ the unique solution $(u_\varepsilon^*, p_\varepsilon^*, y_\varepsilon^*)$ of the constrained minimization problem (3.8) coincides with the solution of the $\widehat{\mathbb{P}}_\varepsilon$-problem, i.e.,

$$\begin{aligned} I_\varepsilon(u_\varepsilon^*, p_\varepsilon^*, y_\varepsilon^*) &= \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathrm{cl}_{\tau_\varepsilon^h}\Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) \\ &= \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) = I_\varepsilon(u_\varepsilon^0, \widehat{p}_\varepsilon^{\,0}, y_\varepsilon^0). \end{aligned} \tag{3.9}$$

4. The choice of topology for homogenization

At it was noted in Introduction the main questions of our paper are devoted to the study of the asymptotic behaviour of the optimal boundary control problem (2.2)–(2.5) or, what is equivalent, $\widehat{\mathbb{P}}_\varepsilon$-problem (3.5)–(3.6) as $\varepsilon \to 0$. With this aim we represent the $\widehat{\mathbb{P}}_\varepsilon$-problem for various values of ε in the form of the sequence of corresponding constrained minimization problems

$$\Big\{ \Big\langle \inf_{(u,p,y) \in \Xi_\varepsilon} I_\varepsilon(u, p, y) \Big\rangle ;\, \varepsilon = a/N \to 0 \Big\}, \tag{4.1}$$

where the cost functional $I_\varepsilon : \Xi_\varepsilon \to \mathbb{R}$ and the sets of admissible triplets are defined in (3.5) and (3.6), respectively. Then the definition of an appropriate homogenized optimal control problem to the family (3.5)–(3.6) as $\varepsilon \to 0$ can be reduced to the analysis of the limit properties of the sequences (4.1). This will be done through the concept of variational convergence of constrained minimization problems [1]–[33].

Before the short representation of the necessary formalism of such convergence we note that whatever the procedure for the limit analysis is, it has to preserve the main variational property: both optimal triplet and minimal value of the cost functional for $\widehat{\mathbb{P}}_\varepsilon$-problem converge to the corresponding characteristics of a limit optimal control problem as ε tends to zero. Thus, one has to define the most natural topology for the homogenization that takes into account the above mentioned variational property ([4], [31]), i.e with respect to which the sequence of optimal triplets $\left\{(u^0_\varepsilon, \widehat{p}_\varepsilon{}^0, y^0_\varepsilon);\, \varepsilon \to 0\right\}$ is a sequentially compact. However, it should be stressed that in our case for every fixed ε the admissible triplets $(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon)$ belong to the corresponding functional spaces $\mathbb{X}_\varepsilon = H^{1/2}(\Gamma_\varepsilon) \times L^2(\Omega^-, d\mu_\varepsilon) \times H^1(\Omega_\varepsilon)$ depending on the small parameter ε. Moreover, in view of the specific construction of the domains Ω_ε, the scale of Sobolev spaces $\{\mathbb{X}_\varepsilon\}_{\varepsilon>0}$ does not satisfy the classical "condition of strong or weak connectedness" (see [37]-[39]), i.e., there are no any uniformly bounded in ε sequences of extension operators. So, in this section we focus our attention on the convergence formalism in this scale.

Let $\left\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{Y}_\varepsilon \equiv L^2(\Gamma_\varepsilon) \times L^2(\Omega^-, d\mu_\varepsilon) \times H^1(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ be any sequence of triplets. We denote by $\widetilde{u}_\varepsilon$ the trivial extension by zero of a function $u_\varepsilon \in L^2(\Gamma_\varepsilon)$ to the set Γ_0 (see (2.1)); then $\widetilde{u}_\varepsilon \in L^2(\Gamma_0)$.

Definition 4.1. *We will say that a sequence of controls* $\left\{u_\varepsilon \in L^2(\Gamma_\varepsilon)\right\}_{\varepsilon>0}$ *is weakly convergent to a function* u^* *with respect to the space* $L^2(\Gamma_0)$ *if* $\widetilde{u}_\varepsilon \to u^*$ *weakly in* $L^2(\Gamma_0)$.

In fact, in view of (2.4), we have the following obvious result.

Lemma 4.1. *Any sequence of admissible controls* $\{u_\varepsilon \in U_\varepsilon\}$ *is relatively compact with respect to the above introduced weak convergence. Moreover, its weak limit* u^* *satisfies the following condition* $\|u^*\|_{L^2(\Gamma_0)} \le \mathbf{C_u}$.

Now, following to V.Zhikov [47], we give the convergence formalism of the sequence of controls $\left\{p_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\right\}_{\varepsilon>0}$ in the scale of Lebesgue spaces with varying measures. First of all we recall that the sequence of Borel measures $\{d\mu_\varepsilon\}$ weakly converges to Lebesgue measure dx. Let $\left\{p_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\right\}_{\varepsilon>0}$ be any bounded sequence, i.e.,

$$\limsup_{\varepsilon\to 0} \int_{\Omega^-} p_\varepsilon^2 \, d\mu_\varepsilon < +\infty.$$

Definition 4.2. *We say that a bounded sequence* $\left\{p_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\right\}_{\varepsilon>0}$ *is weakly convergent if there exists an element* $p^* \in L^2(\Omega^-)$ *such that*

$$\lim_{\varepsilon\to 0}\int_{\Omega^-} \varphi p_\varepsilon\, d\mu_\varepsilon = \int_{\Omega^-} \varphi p^*\, dx \quad \textit{for any function} \;\; \varphi \in C_0^\infty(\mathbb{R}^n).$$

The simplest example of weakly convergent sequence is the following one: $p_\varepsilon = \psi$, where $\psi \in C_0^\infty(\Omega^-)$. Note also that the class of test functions in Definition 4.2 can be essentially extended. Indeed, as follows from [49] (see Lemmas 4.5–4.6) the following results take place:

Proposition 4.2. *If a bounded sequence* $\{p_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\}$ *converges weakly to* $p^* \in L^2(\Omega^-)$ *and a function* $\psi \in L^2(\Omega^-)$ *is such that* $|\psi(x)| \le$ *const almost everywhere on* Ω^-*, then*

$$\lim_{\varepsilon\to 0}\int_{\Omega^-} \psi p_\varepsilon\, d\mu_\varepsilon = \int_{\Omega^-} \psi p^*\, dx; \quad \lim_{\varepsilon\to 0}\int_{\Omega^-} \psi\, d\mu_\varepsilon = \int_{\Omega^-} \psi\, dx.$$

Proposition 4.3. *Let* $\{p_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\}_{\varepsilon>0}$ *be any bounded sequence. Then*

(i) *this sequence is relatively compact with respect to the weak convergence;*
(ii) *if* $p_\varepsilon \longrightarrow p^*$ *weakly as* $\varepsilon \to 0$*, then*

$$\liminf_{\varepsilon\to 0}\int_{\Omega^-} p_\varepsilon^2\, d\mu_\varepsilon \ge \int_{\Omega^-} (p^*)^2\, dx.$$

Taking this proposition into account, it is easy to establish the following fact.

Lemma 4.4. *Any sequence of admissible controls* $\{\widehat{p}_\varepsilon \in \widehat{P}_\varepsilon\}$ *is relatively compact with respect to the weak convergence, and its weak limit* $p^* \in L^2(\Omega^-)$ *satisfies the following condition:* $\|p^*\|^2_{L^2(\Omega^-)} \le |\partial C|_H^{-1}\, \mathbf{C_p}$.

Now we introduce the convergence formalism for the following type of sequences $\left\{y_\varepsilon \in H^1(\Omega_\varepsilon)\right\}_{\varepsilon\to 0}$. Using the fact that the boundaries of the thin rods S_ε are rectilinear, similar as in [3] (see also [14, 16, 42]), we extend the y_ε by zero into to the hole domain Ω, namely,

$$\widetilde{y}_\varepsilon(x) := \begin{cases} y_\varepsilon(x), & x \in \Omega_\varepsilon, \\ 0, & x \in \Omega \setminus \Omega_\varepsilon. \end{cases} \tag{4.2}$$

Let us introduce the following notations $y_\varepsilon^+(x) := y_\varepsilon(x)$ if $x \in \Omega^+$, and $\widetilde{y}_\varepsilon^{\,-}(x) := \widetilde{y}_\varepsilon(x)$ if $x \in \Omega^-$. Thanks to the rectilinear boundaries of S_ε with respect to x_n,

$$\partial_{x_n}(\widetilde{y}_\varepsilon^-) = \widetilde{\partial_{x_n}(y_\varepsilon^-)} \quad \text{in} \quad \Omega^-. \tag{4.3}$$

This means that $\widetilde{y}_\varepsilon^- \in W_2^{0,1}(\Omega^-)$, where $W_2^{0,1}(\Omega^-)$ is the anizotropic Sobolev space $\{v \in L^2(\Omega^-) : \partial_{x_n} v \in L^2(\Omega^-)\}$.

Let χ_C be the $\square$–periodic characteristic function of the set C. It is well known that $\chi_C(\cdot/\varepsilon) \to |C|$ weakly-$*$ in $L^\infty(B)$ as $\varepsilon \to 0$. Here, $B = (0,a)^{n-1}$, and $|C|$ is $(n-1)$–dimensional Lebesgue measure of C. From this fact it follows that

$$\chi_{\Omega_\varepsilon^-} \longrightarrow |C| \quad \text{weakly-}* \quad \text{in} \quad L^\infty(\Omega^-) \quad \text{as} \quad \varepsilon \to 0, \tag{4.4}$$

$$\chi_{\Omega_\varepsilon^- \cap \Sigma} \longrightarrow |C| \quad \text{weakly-}* \quad \text{in} \quad L^\infty(\Sigma; \mathcal{H}^{n-1}) \quad \text{as} \quad \varepsilon \to 0, \tag{4.5}$$

where $\chi_{\Omega_\varepsilon^-}$, $\chi_{\Omega_\varepsilon \cap \Sigma}$ are the characteristic functions of sets Ω_ε^- and $\overline{\Omega}_\varepsilon \cap \Sigma$ respectively.

Thus, if for a sequence $\left\{y_\varepsilon \in H^1(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ there exists a constant $\mathbf{C} > 0$ independent of ε such that $\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq \mathbf{C}$, then

$$\|y_\varepsilon^+\|_{H^1(\Omega^+)} + \|\widetilde{y}_\varepsilon^-\|_{W_2^{(0,1)}(\Omega^-)} \leq \mathbf{C}$$

and there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ (still denoted by ε) and elements $y_0^+ \in H^1(\Omega^+)$, $y_0^- \in L^2(\Omega^-)$, $\gamma_n \in L^2(\Omega^-)$ such that

$$\left.\begin{array}{rcll} y_\varepsilon^+ & \to & y_0^+ & \text{weakly in } H^1(\Omega^+); \\ \widetilde{y}_\varepsilon^- & \to & v = |C|(|C|^{-1}v) =: |C| y_0^- & \text{weakly in } L^2(\Omega^-); \\ \partial_{x_n}\widetilde{y}_\varepsilon^- & \to & |C|\,\partial_{x_n} y_0^- & \text{weakly in } L^2(\Omega^-). \end{array}\right\} \tag{4.6}$$

The last limit is consequence of (4.3).

Similar as in [3] it is easy to prove the following relation

$$y_*^+ = y_*^- \quad \text{a.e. on} \quad \Sigma. \tag{4.7}$$

i.e., the traces of the limit functions coincide on Σ. Note that the assertions (4.6) and (4.7) were proved without any suppositions that y_ε has to be a solution of the boundary value problem (2.3). Due to these facts we give the following definitions.

Definition 4.3. *We say that a sequence $\left\{y_\varepsilon \in H^1(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ is weakly convergent to a function $y_* = (y_*^+, y_*^-)$ with respect to the space $H^1(\Omega^+) \times W_2^{(0,1)}(\Omega^-)$ as ε tends to zero (in symbols, $y_\varepsilon \rightsquigarrow y_* = (y_*^+, y_*^-)$), if*

(a) $y_\varepsilon^+ \longrightarrow y_*^+$ weakly in $H^1(\Omega^+)$;

(b) $\widetilde{y}_\varepsilon^- \longrightarrow |C|\, y_*^-$ weakly in $W_2^{(0,1)}(\Omega^-)$.

Definition 4.4. *We say that a sequence of triplets* $\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ *is* w*-convergent to a quadruple* (u, p, y^+, y^-) *, in symbols,* $(u_\varepsilon, p_\varepsilon y_\varepsilon) \xrightarrow{w} (u, p, y^+, y^-)$ *with respect to the space* $\mathbb{Y}_0 \equiv L^2(\Gamma_0) \times L^2(\Omega^-) \times H^1(\Omega^+) \times W_2^{(0,1)}(\Omega^-)$ *as* ε *tends to zero, if*

(1) $u_\varepsilon \to u$ *in the sense of Definition* 4.1;
(2) $p_\varepsilon \to p$ *in the sense of Definition* 4.2;
(3) $y_\varepsilon \rightsquigarrow (y^+, y^-)$ *in the sense of Definition* 4.3.

From results obtained in this section and estimate (3.7)) it follows that every sequence of admissible triplets for $\widehat{\mathbb{P}}_\varepsilon$-problem is relatively compact with respect to w-convergence. Thus, the following statement holds.

Proposition 4.5. *Let* $\{(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *be any sequence of admissible triplets for the* $\widehat{\mathbb{P}}_\varepsilon$*-problem. Then there exist a subsequence* $\{(u_{\varepsilon'}, \widehat{p}_{\varepsilon'}, y_{\varepsilon'})\}_{\varepsilon'>0}$ *and a quaternary* $(u, p, y^+, y^-) \in \mathbb{Y}_0$ *such that* $u \in U_0$*,* $p \in P_0$*, and* $(u_{\varepsilon'}, \widehat{p}_{\varepsilon'}, y_{\varepsilon'}) \xrightarrow{w} (u, p, y^+, y^-)$*, where*

$$U_0 = \left\{u \in L^2(\Gamma_0) \ : \ \|u\|_{L^2(\Gamma_0)} \le \mathbf{C_u}\right\}, \tag{4.8}$$

$$P_0 = \left\{p \in L^2(\Omega^-) \ : \ \|p\|^2_{L^2(\Omega^-)} \le |\partial C|^{-1}_H \, \mathbf{C_p}\right\}. \tag{4.9}$$

Remark 4.1. Taking into account Proposition 4.5, we may give the following conclusion: the sequences of optimal triplets $\left\{(u^0_\varepsilon, \widehat{p}^{\,0}_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ for $\widehat{\mathbb{P}}_\varepsilon$-problem is relatively compact with respect to the topology associated with w-convergence on $\mathbb{Y}_0$. So, this topology can be chosen as the most natural one for the homogenization of the original optimal boundary control problem.

5. The definition of homogenized problem and its properties

We begin this section with some notions that will be used for the definition of homogenized boundary control problem. First of all we introduce the convergence concept of the sets of admissible triplets Ξ_ε as $\varepsilon \to 0$. As follows from the previous results, each of these sets Ξ_ε belongs to the corresponding Sobolev space $H^{1/2}(\Gamma_\varepsilon) \times L^2(\Omega^-, d\mu_\varepsilon) \times H^1(\Omega_\varepsilon)$. So, we put as the basis of such convergence the w-sequential version of the set convergence in Kuratowski's sense.

Definition 5.1. *We say that a set* $\Xi \subset \mathbb{Y}_0$ *is the sequential* w*-limit in the Kuratowski's sense* (*or* $K(w)$*-limit*) *of the sequence*

$$\left\{\Xi_\varepsilon \subset H^{1/2}(\Gamma_\varepsilon) \times L^2(\Omega^-, d\mu_\varepsilon) \times H^1(\Omega_\varepsilon)\right\}_{\varepsilon>0} \tag{5.1}$$

with respect to the space $\mathbb{Y}_0$ *if the following conditions are satisfied:*

(1) *for every quadruple* $r = (u, p, y^+, y^-) \in \Xi$ *there exist a sequence of triplets* $\{z_\varepsilon = (u_\varepsilon, p_\varepsilon, y_\varepsilon)\}$, *w-converging to* r, *and a positive value* $\varepsilon_0 > 0$ *such that* $z_\varepsilon \in \Xi_\varepsilon$ *for every* $\varepsilon < \varepsilon_0$;

(2) *if a sequence* $\{z_k = (u_k, p_k, y_k)\}$ *w-converges to* $r = (u, p, y^+, y^-)$, *and there exists a subsequence* $\{\varepsilon_k\}$ *of* $\{\varepsilon\}$ *such that: (i)* $\varepsilon_k \to 0$ *as* $k \to \infty$; *(ii)* $z_k \in \Xi_{\varepsilon_k}$ *for all* k. *Then* $r \in \Xi$.

In view of the main question of this paper our intention next is to characterize a "limit" minimization problem of the sequence (4.1) as ε tends to zero.

Definition 5.2. *We say that a minimization problem on* $\mathbb{Y}_0$

$$\left\langle \inf_{(u,p,y^+,y^-)\in \Xi_0} I_0(u, p, y^+, y^-) \right\rangle \tag{5.2}$$

is the variational limit of the sequence (4.1) *with respect to the w-convergence* (*or variational w-limit*) *if the following conditions are satisfied:*

(i) Ξ_0 *is a* $K(w)$*-limit of the sets* $\{\Xi_\varepsilon\}$;

(ii) *for every quadruple* $(u, p, y^+, y^-) \in \Xi_0$ *and for every sequence of admissible triplets* $\{(u_k, \widehat{p}_k, y_k)\}_{k\in\mathbb{N}}$, *which is w-converging to* (u, p, y^+, y^-) *and such that* $(u_k, \widehat{p}_k, y_k) \in \Xi_{\varepsilon_k}$ *for some* $\varepsilon_k \to 0$ *as* $k \to \infty$, *it is*

$$I_0(u, p, y^+, y^-) \le \liminf_{k\to\infty} I_{\varepsilon_k}(u_k, \widehat{p}_k, y_k); \tag{5.3}$$

(iii) *for every quadruple* $(u, p, y^+, y^-) \in \Xi_0$ *there exist a positive constant* ε_0 *and a sequence* $\{(u_\varepsilon, p_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ *such that*

$$(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon \quad \text{for every } \varepsilon \le \varepsilon_0;\ (u_\varepsilon, p_\varepsilon, y_\varepsilon) \xrightarrow{w} (u, p, y^+, y^-);$$

$$I_0(u, p, y^+, y^-) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon). \tag{5.4}$$

Remark 5.1. Note that Definition 5.2 can be interpreted as the natural extension of the well-known notion of Γ-convergence. We do not want here to enter into details of Γ-convergence theory, but only to emphasize that the variational properties of w-limit cost functional $I_0 : \Xi_0 \to R$ in the sense of Definition 5.2 are similar to these of Γ-limit. In fact, in the following theorem we will prove that the variational w-convergence of the sequence (4.1) to the problem (5.2) implies the convergence of the minimum values of I_ε on Ξ_ε to the minimum of I_0 on Ξ_0, and moreover, we'll prove that in this case every w-cluster "point" of the sequence of the minimizers of I_ε is a minimizer of I_0.

Theorem 5.1. *Assume that the constrained minimization problem* (5.2) *is the variational w-limit of the sequence* (4.1) *and this problem has a unique solution* $(u^0, p^0, (y^0)^+, (y^0)^-)$ *in* Ξ_0. *Let* $\left\{(u^0_\varepsilon, \widehat{p}_\varepsilon^{\,0}, y^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ *be a sequence of optimal triplets for* $\mathbb{P}_\varepsilon$*-problems. Then*

$$(u^0_\varepsilon, \widehat{p}_\varepsilon^{\,0}, y^0_\varepsilon) \xrightarrow{\ w\ } (u^0, p^0, (y^0)^+, (y^0)^-), \tag{5.5}$$

$$\inf_{(u,\,p,\,y^+,\,y^-)\in\Xi_0} I_0(u,p,y^+,y^-) = I_0\left(u^0, p^0, (y^0)^+, (y^0)^-\right)$$
$$= \lim_{\varepsilon\to 0} I_\varepsilon(u^0_\varepsilon, \widehat{p}_\varepsilon^{\,0}, y^0_\varepsilon) = \lim_{\varepsilon\to 0} \inf_{(u,p,y)\in\Xi_\varepsilon} I_\varepsilon(u,p,y). \tag{5.6}$$

Proof. Let $\left\{(u^0_{\varepsilon_k}, \widehat{p}^{\,0}_{\varepsilon_k}, y^0_{\varepsilon_k})\right\}_{k\in\mathbb{N}}$ be any w-convergent subsequence of the sequence of minimizers. Note that in view of Proposition 4.5 and Remark 5.2 such choice is always possible. Let $(u^*, p^*, (y^*)^+, (y^*)^-)$ be its w-limit. Then, by Definition 5.1, we have $(u^*, p^*, (y^*)^+, (y^*)^-) \in \Xi_0$. Moreover, due to the property (ii) of Definition 5.2, one gets

$$\liminf_{k\to\infty} \min_{(u,p,y)\in\Xi_{\varepsilon_k}} I_{\varepsilon_k}(u,p,y) = \liminf_{k\to\infty} I_{\varepsilon_k}(u^0_{\varepsilon_k}, \widehat{p}^{\,0}_{\varepsilon_k}, y^0_{\varepsilon_k})$$
$$\geq I_0(u^*, p^*, (y^*)^+, (y^*)^-) \geq \min_{(u,\,p,\,y^+,\,y^-)\in\Xi_0} I_0(u,p,y^+,y^-)$$
$$= I_0(u^0, p^0, (y^0)^+, (y^0)^-), \tag{5.7}$$

where $(u^0, p^0, (y^0)^+, (y^0)^-) \in \Xi_0$ is the unique solution of the limit problem (5.2). It follows that there exist a constant $\varepsilon^0 > 0$ and a sequence $\{(u_\varepsilon, p_\varepsilon, y_\varepsilon)\}$, w-converging to $(u^0, p^0, (y^0)^+, (y^0)^-)$, such that $(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon$ for all $\varepsilon \leq \varepsilon^0$, and

$$I_0(u^0, p^0, (y^0)^+, (y^0)^-) \geq \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon).$$

Using this fact we have

$$\min_{(u,\,p\,,\,y^+,\,y^-)\in\Xi_0} I_0(u,\,p\,,\,y^+,\,y^-) = I_0(u^0, p^0, (y^0)^+, (y^0)^-)$$
$$\geq \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) \geq \limsup_{\varepsilon\to 0} \min_{(u,p,y)\in\Xi_\varepsilon} I_\varepsilon(u,p\,,y)$$
$$\geq \limsup_{k\to\infty} \min_{(u,p,y)\in\Xi_{\varepsilon_k}} I_{\varepsilon_k}(u,p,y) = \limsup_{k\to\infty} I_{\varepsilon_k}(u^0_{\varepsilon_k}, \widehat{p}^{\,0}_{\varepsilon_k}, y^0_{\varepsilon_k}). \tag{5.8}$$

Therefore, in view of (5.7) we have obtained the following inequality

$$\liminf_{k\to\infty} I_{\varepsilon_k}(u^0_{\varepsilon_k}, \widehat{p}^{\,0}_{\varepsilon_k}, y^0_{\varepsilon_k}) \geq \limsup_{k\to\infty} I_{\varepsilon_k}(u^0_{\varepsilon_k}, \widehat{p}^{\,0}_{\varepsilon_k}, y^0_{\varepsilon_k}).$$

In the result, combining (5.7) and (5.8), we conclude

$$I_0(u^*, p^*, (y^*)^+, (y^*)^-) = I_0(u^0, p^0, (y^0)^+, (y^0)^-)$$
$$= \min_{(u,p,y^+,y^-)\in \Xi_0} I_0(u, p, y^+, y^-),$$
$$I_0(u^0, p^0, (y^0)^+, (y^0)^-) = \lim_{k\to\infty} \min_{(u,p,y)\in \Xi_{\varepsilon_k}} I_{\varepsilon_k}(u, p, y).$$

Using these relations and the fact that an optimal quaternary for the problem (5.2) is unique, we obtain $(u^*, p^*, (y^*)^+, (y^*)^-) = (u^0, p^0, (y^0)^+, (y^0)^-)$. Since this equality holds for the limits of all subsequences of $\{(u^0_\varepsilon, \widehat{p}^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon>0}$, it follows that these limits are coincident and therefore $(u^0, p^0, (y^0)^+, (y^0)^-)$ is the w-limit of the whole sequence $\left\{(u^0_\varepsilon, \widehat{p}_\varepsilon^{\,0}, y^0_\varepsilon)\right\}_{\varepsilon>0}$. Hence, making for the sequence of minimizers what we did before with a subsequence $\left\{(u^0_{\varepsilon_k}, \widehat{p}^{\,0}_{\varepsilon_k}, y^0_{\varepsilon_k})\right\}_{k\in\mathbb{N}}$, we have

$$\liminf_{\varepsilon\to 0} \min_{(u,p,y)\in \Xi_\varepsilon} I_\varepsilon(u, p, y) = \liminf_{\varepsilon\to 0} I_\varepsilon(u^0_\varepsilon, \widehat{p}_\varepsilon^{\,0}, y^0_\varepsilon)$$
$$\geq I_0(u^0, p^0, (y^0)^+, (y^0)^-) = \min_{(u,p,y^+,y^-)\in \Xi_0} I_0(u, p, y^+, y^-)$$
$$\geq \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) \geq \limsup_{\varepsilon\to 0} \min_{(u,p,y)\in\Xi_\varepsilon} I_\varepsilon(u, p, y) = \limsup_{\varepsilon\to 0} I_\varepsilon(u^0_\varepsilon, \widehat{p}_\varepsilon^{\,0}, y^0_\varepsilon).$$

Thus, we have obtained the required. This proof is complete. □

Definition 5.3. *We say that the family of optimal control problems* (2.2)-(2.5) *admits the homogenization as* ε *tends to zero with respect to the* w*-convergence, if for the corresponding sequence of constrained minimization problems* (4.1) *there exists a variational limit which can be recovered in the form of some optimal control problem.*

6. Analytical representation of the limit set of admissible solutions

The main objects of our consideration in this section are the sequence of the sets of admissible triplets

$$\left\{\Xi_\varepsilon \in L^2(\Gamma_\varepsilon) \times L^2(\Omega^-, d\mu_\varepsilon) \times H^1(\Omega_\varepsilon)\right\}_{\varepsilon>0} \tag{6.1}$$

for the original optimal boundary control problem $\widehat{\mathbb{P}}_\varepsilon$ and its w-limiting properties in Kuratowski sense. In spite of the fact that we have no result concerning the $K(w)$-compactness properties of uniformly bounded sequences of sets in the scale $\{L^2(\Gamma_\varepsilon)\times L^2(\Omega^-, d\mu_\varepsilon)\times H^1(\Omega_\varepsilon)\}$, we will show that in actual fact $K(w)$-limit set exists for the sequence (6.1) and it can be recovered in the analytical form. As it follows from the definition of $K(w)$-limit, the main difficulty is connected with the passage to the limit in integral identity (3.3) for problem (2.3) as $\varepsilon \to 0$.

But at first we refer to the result (see Kesavan and Saint Jean Paulin [29]) which will be not only useful in the sequel but it seems to be interesting per se.

Proposition 6.1. *For every bounded sequence* $\{u_\varepsilon \in L^2(\Gamma_\varepsilon)\}_{\varepsilon>o}$ *such that* $\widetilde{u}_\varepsilon \longrightarrow u_*$ *weakly in* $L^2(\Gamma_0)$, *the following inequality holds*

$$\liminf_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} u_\varepsilon^2 \, d\mathcal{H}^{n-1} \geq |C|^{-1} \int_{\Gamma_0} u_*^2 \, d\mathcal{H}^{n-1} \tag{6.2}$$

(*recall that by* $\widetilde{u}_\varepsilon$ *we denote the trivial extension of a function* $u_\varepsilon \in L^2(\Gamma_\varepsilon)$ *into* Γ_0).

Proposition 6.2. *Let* $w_* \in L^\infty(\Omega^-)$ *and* $p_* \in L^2(\Omega^-)$ *be elements such that*

$$L^\infty(\Omega^-) \ni w_\varepsilon \longrightarrow w_* \quad \textit{strongly in} \ \ L^\infty(\Omega^-), \tag{6.3}$$

$$L^2(\Omega^-, d\mu_\varepsilon) \ni p_\varepsilon \longrightarrow p_* \quad \textit{weakly in the sense of Definition } 4.2. \tag{6.4}$$

Then

$$\lim_{\varepsilon\to 0} \int_{\Omega^-} w_\varepsilon \, p_\varepsilon \, \varphi \, d\mu_\varepsilon = \int_{\Omega^-} w_* \, p_* \, \varphi \, dx$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Proof. First of all we note that for every $\varepsilon > 0$ each of the integrals $\int_{\Omega^-} w_\varepsilon \, p_\varepsilon \, \varphi \, d\mu_\varepsilon$ is well defined. Indeed, using the Cauchy-Schwartz inequality, we have

$$\Big| \int_{\Omega^-} w_\varepsilon \, p_\varepsilon \, \varphi \, d\mu_\varepsilon \Big| \leq \|w_\varepsilon\|_{L^\infty(\Omega^-)} \, |\mu_\varepsilon(\overline{\Omega^-})|^{1/2} \, \|p_\varepsilon\|_{L^2(\Omega^-, d\mu_\varepsilon)} \cdot \sup_{x\in\Omega^-} |\varphi(x)|.$$

Since the sequence of Borel measures $\{d\mu_\varepsilon\}$ is weakly convergent to the Lebesgue measure dx, the sequences

$$\{w_\varepsilon \in L^\infty(\Omega^-)\}_{\varepsilon>0}, \qquad \{p_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\}_{\varepsilon>0}$$

are bounded, and for every compact set $K \subset \mathbb{R}^n$ the following inequality $\limsup_{\varepsilon\to 0} \mu_\varepsilon(K) \leq |K|$ holds (see V.V.Zhikov [49]), it follows that

$$\left| \int_{\Omega^-} w_\varepsilon \, p_\varepsilon \, \varphi \, d\mu_\varepsilon \right| \leq C \sup_{x\in\Omega^-} |\varphi(x)|.$$

Thus, $(w_\varepsilon \, p_\varepsilon) \, d\mu_\varepsilon$ is a Radon measure. Moreover, from this we immediately have that $p_\varepsilon w \longrightarrow p_* w$ weakly in $\{L^2(\Omega^-, d\mu_\varepsilon)\}$ for any function

$w \in L^{\infty}(\Omega^-)$. Then taking (6.3)–(6.4) into account, and passing to the limits in the following inequality

$$\left|\int_{\Omega^-} w_\varepsilon\, p_\varepsilon\, d\mu_\varepsilon - \int_{\Omega^-} w_*\, p_*\, dx\right| \le \left|\int_{\Omega^-} p_\varepsilon\,(w_\varepsilon\, - w_*)\, d\mu_\varepsilon\right| + \left|\int_{\Omega^-} w_*\, p_\varepsilon\, d\mu_\varepsilon - \int_{\Omega^-} w_*\, p_*\, dx\right|$$
$$\le \|w_\varepsilon - w_*\|_{L^\infty(\Omega^-)} \int_{\Omega^-} |p_\varepsilon\, \varphi|\, d\mu_\varepsilon + \left|\int_{\Omega^-} w_*\, p_\varepsilon\, d\mu_\varepsilon - \int_{\Omega^-} w_*\, p_*\, dx\right|$$

as ε tends to zero, we immediately obtain the required. □

The crucial point in the analytical investigation of $K(w)$-limit properties for the sequence of admissible triplets (6.1) is next theorem on the convergence of the solution to problem (2.3). As was mentioned in the Sec. 4 the thick multi-structure are not strong or weak connected domains. This fact creates one of the main difficulties in investigations of boundary-value problems in multi-structures.

There are different methods to prove such convergence theorems. The first convergence theorems for solutions to boundary-value problems in thick multi-structures of different types were proved in [37, 38, 39] using special extension operators, which preserves the space class of solution and whose norms in H^1 are uniformly bounded in ε, and using special test functions. Such approach allows to prove the convergence theorems in case that the boundaries of the thin domains of thick junctions are not rectilinear with respect some variables and smooth (see [18, 19]) and in case of different boundary conditions on the boundaries of thin domains (see [41, 17]); in the last case special integral identities were used.

It should be emphasized here that there is a big difference between the asymptotic investigation of boundary-value problems in thick multi-structures and in domains with rapidly oscillating boundaries [3], where the uniform Neumann problem was considered. The extension in [3] was constructed without conservation of the class of the space (only in $H^1_{loc}(\Omega_1^+)$, where $\Omega_1^+ \subset \mathbb{R}^2$ is a domain that in the limit is filled up by the oscillating boundary) and under the assumption that the right-hand side $f \in H^1$. In addition, the function h, which defines the oscillating boundary in [3], must be a continuously differentiable periodic function and the reciprocal functions of h on some intervals have to be existed to construct such an extension operator. Also in the paper [3] it was shown that in case of the rectilinear boundaries of the thin domains of thick junctions it is possible to prove the convergence theorems using extension by zero (see (4.2)). This approach was used to prove the convergence theorem for a boundary-value problem in a thick multi-level junction with the non-uniform Neumann and Robin conditions on the boundaries of the thin cylinders (see [42]). For non-linear problems this approach was used in [14].

Since the lateral boundaries S_ε of the thin cylinders are plane with respect to x_n it is possible to use the approach of the papers [3].

Let us introduce the following anizotropic Sobolev space

$$\mathcal{V}(\Omega;\Gamma_0)=\{v\in L^2(\Omega):\partial_{x_n}v\in L^2(\Omega),\ v\in H^1(\Omega^+),\ v=0\ \mathcal{H}^{n-1}\text{ a.e. on }\Gamma_0\}$$

with the scalar product

$$(y,v)_{\mathcal{V}(\Omega;\Gamma_0)} = \int_{\Omega^+}(y\,v+\nabla y\cdot\nabla v)\,dx + |C|\int_{\Omega^-}(y\,v+\partial_{x_n}y\,\partial_{x_n}v)\,dx. \tag{6.5}$$

It should be stressed here that functions from this space have trace on any horizontal interval in $\overline{\Omega^-}$ (see [45, 46]) and the space $H^1(\Omega;\Gamma_0)$ is dense in $\mathcal{V}(\Omega;\Gamma_0)$. Moreover, for any function $v\in\mathcal{V}(\Omega)$ the relation (4.7) holds.

Theorem 6.3. *Let $\{u_\varepsilon\in U_\varepsilon\}_{\varepsilon>0}$ and $\{\widehat{p}_\varepsilon\in\widehat{P}_\varepsilon\}_{\varepsilon>0}$ be any sequences of admissible boundary controls for $\widehat{\mathbb{P}}_\varepsilon$-problems. Let $u_0\in L^2(\Gamma_0)$ and $p_0\in L^2(\Omega^-)$ are their weak limits in the sense of Definitions* 4.1–4.2*, respectively. Let $\{y_\varepsilon\in H^1(\Omega_\varepsilon)\}$ be the corresponding weak solutions of the problem* (2.3)*. Then $(u_\varepsilon,\widehat{p}_\varepsilon,y_\varepsilon)\xrightarrow{w}(u_0,p_0,v_0^+,v_0^-)$ as $\varepsilon\to 0$ with respect to the space $L^2(\Gamma_0)\times L^2(\Omega^-)\times H^1(\Omega^+)\times W_2^{(0,1)}(\Omega^-)$, where*

$$\widehat{v}_0(x)=\begin{cases}v_0^+(x), & x\in\Omega^+,\\ v_0^-(x), & x\in\Omega^-,\end{cases} \tag{6.6}$$

is the unique weak solution in $\mathcal{V}(\Omega)$ of the following limit problem

$$\left.\begin{array}{rcll}-\Delta_x v_0^+(x)+v_0^+(x) &=& f_0(x), & x\in\Omega^+,\\ -\,\partial^2_{x_nx_n}v_0^-(x)+v_0^- &=& f_0(x)+\delta\,|\partial C|_H\,|C|^{-1}\cdot p_0(x), & x\in\Omega^-,\\ \partial_\nu v_0^+(x) &=& 0, & x\in\partial\Omega^+\setminus\Sigma,\\ v_0^- &=& |C|^{-1}u_0, & \text{on }\ \Gamma_0,\\ v_0^+ &=& v_0^-, & \text{on }\ \Sigma,\\ \partial_{x_n}v_0^+ &=& |C|\,\partial_{x_n}v_0^-, & \text{on }\ \Sigma.\end{array}\right\} \tag{6.7}$$

Proof. As it follows from Proposition 4.5 there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and a quaternary $(u_0, p_0, v_0^+, v_0^-) \in L^2(\Gamma_0) \times L^2(\Omega^-) \times H^1(\Omega^+) \times W_2^{(0,1)}(\Omega^-)$ such that $(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \xrightarrow{w} (u_0, p_0, v_0^+, v_0^-)$ as $\varepsilon \to 0$. Similar as in section 4 (see relations (4.6) and (4.7)) we prove that

$$\begin{aligned} y_\varepsilon^+ &\to v_0^+ && \text{weakly in } H^1(\Omega^+);\\ \widetilde{y_\varepsilon^-} &\to |C|\, v_0^- && \text{weakly in } L^2(\Omega^-);\\ v_0^+ &= v_0^- && \mathcal{H}^{n-1} \text{ almost everywhere on } \Sigma;\\ \partial_{x_n} v_0^+ &= |C| \partial_{x_n} v_0^-, && \mathcal{H}^{n-1} \text{ almost everywhere on } \Sigma. \end{aligned} \tag{6.8}$$

In addition, there exist functions $\gamma_i \in L^2(\Omega^-)$ $(i = 1, \dots, n-1)$ such that

$$\widetilde{\nabla_x y_\varepsilon^-} \to \big(\gamma_1, \dots, \gamma_{n-1}, |C| \partial v_0^- / \partial x_n\big) \quad \text{weakly in } \big[L^2(\Omega^-)\big]^n. \tag{6.9}$$

By the same arguments as in [35] we can prove that

$$v_0^- = |C|^{-1} u_0 \quad \mathcal{H}^{n-1} \text{ almost everywhere on } \Gamma_0.$$

Now let us show that the function v_0 is the unique weak solution of the boundary value problem (6.7). With this aim we rewrite integral identity (3.3) in the following way

$$\begin{aligned} \int_{\Omega^+} \nabla y_\varepsilon^+ \cdot \nabla \varphi \, dx + \int_{\Omega^-} \widetilde{\nabla y_\varepsilon^-} \cdot \nabla \varphi \, dx + \int_{\Omega^+} y_\varepsilon^+ \varphi \, dx + \int_{\Omega^-} \widetilde{y_\varepsilon^-} \varphi \, dx \\ = \int_{\Omega^+} f_\varepsilon \varphi \, dx + \delta \, |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon + \int_{\Omega^-} \chi_{\Omega_\varepsilon^-} f_\varepsilon \varphi \, dx \end{aligned} \tag{6.10}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_0)$, where $C_0^\infty(\mathbb{R}^n; \Gamma_0) = \{\varphi \in C_0^\infty(\mathbb{R}^n) : \varphi = 0 \text{ on } \Gamma_0\}$. Passing in the limit in (6.10) as $\varepsilon \to 0$ and taking (6.8) (see also (4.4)) into account, we obtain

$$\begin{aligned} &\int_{\Omega^+} \nabla v_0^+ \cdot \nabla \varphi \, dx + \int_{\Omega^-} \sum_{i=1}^{n-1} \gamma_i \, (\partial \varphi / \partial x_i) \, dx \\ &\quad + |C| \int_{\Omega^-} (\partial v_0^- / \partial x_n)(\partial \varphi / \partial x_n) \, dx + \int_{\Omega^+} v_0^+ \varphi \, dx + |C| \int_{\Omega^-} v_0^- \varphi \, dx \\ &\qquad = \delta \, |\partial C|_H \int_{\Omega^-} p_0 \varphi \, dx + \int_{\Omega^+} f_0 \varphi \, dx + |C| \int_{\Omega^-} f_0 \varphi \, dx \end{aligned} \tag{6.11}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_0)$.

Let us prove that $\gamma_i = 0$ a.e. in Ω^- for all $i \in \{1, \dots, n-1\}$. For this let us fix $i \in \{1, \dots, n-1\}$ and let w_ε^i be a sequence in $W^{1,\infty}(\Omega^-)$ satisfying the following conditions:

$$w_\varepsilon^i \longrightarrow x_i \quad \text{strongly in} \quad L^\infty(\Omega^-), \tag{6.12}$$

$$D\, w_\varepsilon^i = 0 \quad \text{a.e. in} \quad \Omega_\varepsilon^- \tag{6.13}$$

for every $\varepsilon > 0$. The existence of such sequence is proved in [3], [14].

Take the following test-functions $\varphi = w_\varepsilon^i\, \phi$ and $\varphi = x_i\, \phi$ with $\phi \in C_0^\infty(\Omega^-)$ in (6.10). Hence, by virtue of (6.13), we get

$$\int_{\Omega^-} \widetilde{\nabla y_\varepsilon^-} \cdot \nabla\phi\, w_\varepsilon^i\, dx + \int_{\Omega^-} \widetilde{y_\varepsilon^-}\, \phi\, w_\varepsilon^i\, dx = \delta\, |\partial C|_H \int_{\Omega^-} \widehat{p_\varepsilon}\, \phi\, w_\varepsilon^i\, d\mu_\varepsilon$$
$$+ \int_{\Omega^-} \chi_{\Omega_\varepsilon^-}\, f_\varepsilon\, \phi\, w_\varepsilon^i\, dx \quad \text{for every } \phi \in C_0^\infty(\Omega^-), \tag{6.14}$$

$$\int_{\Omega^-} \widetilde{\nabla y_\varepsilon^-} \cdot \nabla(\phi\, x_i)\, dx + \int_{\Omega^-} \widetilde{y_\varepsilon^-}\, \phi\, x_i\, dx = \delta\, |\partial C|_H \int_{\Omega^-} \widehat{p_\varepsilon}\, \phi\, x_i\, d\mu_\varepsilon$$
$$+ \int_{\Omega^-} \chi_{\Omega_\varepsilon^-}\, f_\varepsilon\, \phi\, x_i\, dx \quad \text{for every } \phi \in C_0^\infty(\Omega^-), \tag{6.15}$$

for every $\varepsilon > 0$. Then passing to the limit in (6.14) and (6.15) as $\varepsilon \to 0$, and using the properties (4.4), (6.8), (6.12), and Proposition 6.2, we have, for every $\phi \in C_0^\infty(\Omega^-)$,

$$\int_{\Omega^-} \sum_{k=1}^{n-1} \gamma_k (\partial\phi/\partial x_k) x_i dx + |C| \int_{\Omega^-} (\partial v_0^-/\partial x_n)\, (\partial\phi/\partial x_n) x_i dx$$
$$+|C| \int_{\Omega^-} v_0^- \phi x_i dx = \delta |\partial C|_H \int_{\Omega^-} p_0 \phi x_i\, dx + |C| \int_{\Omega^-} f_0 \phi x_i\, dx, \tag{6.16}$$

$$\int_{\Omega^-} \sum_{k=1}^{n-1} \gamma_k (\partial(\phi x_i)/\partial x_k)\, dx + |C| \int_{\Omega^-} (\partial v_0^-/\partial x_n)(\partial\phi/\partial x_n) x_i\, dx$$
$$+|C| \int_{\Omega^-} v_0^- \phi x_i\, dx = \delta |\partial C|_H \int_{\Omega^-} p_0 \phi x_i\, dx + |C| \int_{\Omega^-} f_0 \phi x_i\, dx. \tag{6.17}$$

Making comparison (6.16) with (6.17), we conclude that $\int_{\Omega^-} \gamma_k \phi\, dx = 0$ $\forall k \in \{1, \dots, n-1\}$, for every $\phi \in C_0^\infty(\Omega^-)$, that is $\gamma_i = 0$ a.e. in Ω^-. Thus, the function v_0 satisfies the following identity, for every $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_0)$

$$\int_{\Omega^+} \nabla v_0^+ \cdot \nabla\varphi\, dx + |C| \int_{\Omega^-} (\partial v_0^-/\partial x_n)(\partial\varphi/\partial x_n)\, dx + \int_{\Omega^+} v_0^+ \varphi\, dx$$
$$+ |C| \int_{\Omega^-} v_0^- \varphi\, dx = \delta |\partial C|_H \int_{\Omega^-} p_0 \varphi\, dx + \int_{\Omega^+} f_0 \varphi\, dx + |C| \int_{\Omega^-} f_0 \varphi\, dx. \tag{6.18}$$

As a result, if we recall that $\widehat{v}_0 = \left(v_0^+, v_0^-\right)$, where $v_0^+ \in H^1(\Omega_0)$ and $v_0^- \in W_2^{0,1}(D)$, and the definition of space $\mathcal{V}(\Omega;\Gamma_0)$ (see (6.5)), the identity (6.18) can be rewritten in the form

$$\left(\widehat{v}_0, \varphi\right)_{\mathcal{V}} = \int_{\Omega^+} f_0\,\varphi\,dx + |C| \int_{\Omega^-} f_0\,\varphi\,dx + \delta\,|\partial C|_H \int_{\Omega^-} p_0\,\varphi\,dx \quad (6.19)$$

for every $\varphi \in \mathcal{V}(\Omega;\Gamma_0)$.

We say that a function $\widehat{v}_0 \in \mathcal{V}(\Omega)$ is a weak solution to problem (6.7) if it satisfies the identity (6.19) and the trace of v_0^- on Γ_0 equals $|C|^{-1}u_0$. Then using standard Hilbert space methods, we can state that the function $\widehat{v}_0$, which was defined in (6.6), is a unique weak solution of problem (6.7) in the space $\mathcal{V}(\Omega)$.

Due to the uniqueness of the solution to problem (6.7), the above reasoning holds for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. Therefore, the theorem is proved. □

We now in a position to state the main result of this section which deals with the $K(w)$-limit analysis of set sequence (6.1).

Theorem 6.4. *For the sequence of the sets of admissible triplets for $\widehat{\mathbb{P}}_\varepsilon$-problems $\{\Xi_\varepsilon\}_{\varepsilon>0}$ there exists non-empty $K(w)$-limit set $\Xi_0 \subset L^2(\Gamma_0) \times L^2(\Omega^-) \times H^1(\Omega^+) \times W_2^{(0,1)}(\Omega^-)$ which has the following representation*

$$\Xi_0 = \left\{ (u,p,v^+,v^-) \;\middle|\; \begin{array}{r} u \in \widehat{U}_0 = \left\{ u \in L^2(\Gamma_0) \;:\; \|u\|_{L^2(\Gamma_0)} \le \sqrt{|C|}\cdot \mathbf{C_u} \right\}, \\ p \in \widehat{P}_0 = \left\{ p \in L^2(\Omega^-) \;:\; \|p\|^2_{L^2(\Omega^-)} \le |\partial C|_H^{-1} \cdot \mathbf{C_p} \right\}, \\ -\Delta_x v^+(x) + v^+ = f_0(x),\ x \in \Omega^+, \\ -\dfrac{\partial^2 v^-(x)}{\partial x_n^2} + v^-(x) = \\ = f_0(x) + \delta\,|\partial C|_H\,|C|^{-1}\cdot p(x),\ x \in \Omega^-, \\ \partial_\nu v^+(x) = 0,\ x \in \partial\Omega^+ \setminus \Sigma, \\ v^- = |C|^{-1}u, \quad on\ \ \Gamma_0, \\ v^+ = v^-,\ on\ \ \Sigma, \\ \partial_{x_n} v^+ = |C|\,\partial_{x_n} v^-, \quad on\ \ \Sigma. \end{array} \right\} \quad (6.20)$$

Proof. We show that in actual fact $K(w)$-limit set exists for whole sequence (6.1) and it can be represented in the form (6.20). To obtain the representation (6.20), in accordance with the definition of $K(w)$-limit, we have to verify the conditions (1) and (2) of Definition 5.1. For this we break down the proof into several steps.

Step 1. Let (u, p, y^+, y^-) be any quaternary of the set Ξ_0 (from previous lemma we see that the set Ξ_0 is non-empty). We have to construct a w-convergent sequence $\{(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon)\}_{\varepsilon>0}$ to (u, p, y^+, y^-) such that $(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi_\varepsilon$ for every $\varepsilon < \varepsilon_0$, where $\varepsilon_0 > 0$ is some positive value. We consider two cases: the given quaternary belongs to a relative interior of the set Ξ_0, namely $\|u\|_{L^2(\Gamma_0)} < \sqrt{|C|}\cdot\mathbf{C_u}$, and, in the second case, $\|u\|_{L^2(\Gamma_0)} = \sqrt{|C|}\cdot\mathbf{C_u}$. In the first case we construct a weak convergent sequence $\{\widehat{u}_\varepsilon\}_{\varepsilon>0}$ to u as follows: let $u_\varepsilon \in L^2(\Gamma_\varepsilon)$ be the restriction of $|C|^{-1}u$ on Γ_ε. Then

$$\widetilde{u}_\varepsilon = |C|^{-1}\chi_{\Gamma_\varepsilon} u \longrightarrow u \quad \text{weakly in } \ L^2(\Gamma_0). \tag{6.21}$$

Besides, since

$$\|u_\varepsilon\|_{L^2(\Gamma_\varepsilon)} = |C|^{-1}\Big(\int_{\Gamma_0} \chi_{\Gamma_\varepsilon} u^2 d\mathcal{H}^{n-1}\Big)^{\frac{1}{2}} \longrightarrow \sqrt{|C|^{-1}}\Big(\int_{\Gamma_0} u^2 d\mathcal{H}^{n-1}\Big)^{\frac{1}{2}} < \mathbf{C_u}$$

as $\varepsilon \to 0$, it follows that $u_\varepsilon = |C|^{-1}u|_{\Gamma_\varepsilon} \in U_\varepsilon$ for ε sufficiently small (recall that we have $\chi^2_{\Gamma_\varepsilon} = \chi_{\Gamma_\varepsilon}$, and $\chi_{\Gamma_\varepsilon} \to |C|$ weakly-$*$ in $L^\infty(\Gamma_0)$ and weakly in $L^2(\Gamma_0)$).

Further, using the fact that Sobolev space $H^{1/2}(\Gamma_\varepsilon)$ is dense in $L^2(\Gamma_\varepsilon)$ we get: for every $\varepsilon > 0$ can be found an element $\widehat{u}_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ such that $\|u_\varepsilon - \widehat{u}_\varepsilon\|_{L^2(\Gamma_\varepsilon)} < \varepsilon^2$ and $\widehat{u}_\varepsilon \in U_\varepsilon$ for ε sufficiently small. Then taking into account (6.21), we obtain

$$\left|\int_{\Gamma_0} g\left(\widetilde{(\widehat{u}_\varepsilon)} - u\right) d\mathcal{H}^{n-1}\right| \equiv \left|< g, \widetilde{(\widehat{u}_\varepsilon)} - u >\right| \leq \|g\|_{L^2(\Gamma_0)}\|\widehat{u}_\varepsilon - u_\varepsilon\|_{L^2(\Gamma_\varepsilon)} + |< g, \widetilde{u}_\varepsilon - u >| \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

for every $g \in L^2(\Gamma_0)$, i.e., $\widehat{u}_\varepsilon \to u$ weakly in the sense of Definition 4.1 and $\widehat{u}_\varepsilon \in U_\varepsilon$ for ε sufficiently small.

Now we may consider the second case, namely, when the given quaternary (u, p, y^+, y^-) is such that $\|u\|_{L^2(\Gamma_0)} = \sqrt{|C|}\cdot\mathbf{C_u}$. Then there exists controls sequence $\{\overline{u}_\varepsilon \in L^2(\Gamma_0)\}$ satisfying the condition: $\overline{u}_\varepsilon \longrightarrow u$ weakly in $L^2(\Gamma_0)$, and $\|\overline{u}_\varepsilon\|_{L^2(\Gamma_0)} < \sqrt{|C|}\cdot\mathbf{C_u}$ for every $\varepsilon > 0$. Since the weak topology of $L^2(\Gamma_0)$ is metrizable on the set

$$\widehat{U}_0 = \Big\{u \in L^2(\Gamma_0) \,:\, \|u\|_{L^2(\Gamma_0)} \leq \sqrt{|C|}\cdot\mathbf{C_u}\Big\}$$

one can construct a sequence $\{w_\varepsilon \in L^2(\Gamma_0)\}_{\varepsilon>0}$ such that each of w_ε is a convex envelope of finite amount of elements $\{\overline{u}_\varepsilon\}_{\varepsilon>0}$, and $w_\varepsilon \longrightarrow u$ strongly in $L^2(\Gamma_0)$. Note that in this case we also have $\|w_\varepsilon\|_{L^2(\Gamma_0)} < \sqrt{|C|}\cdot \mathbf{C_u}$ for every $\varepsilon > 0$. So, we construct a weak convergent sequence $\{\widehat{u}_\varepsilon\}_{\varepsilon>0}$ to u as follows: $\widehat{u}_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ are elements such that $\|\widehat{u}_\varepsilon - |C|^{-1}w_\varepsilon\|_{L^2(\Gamma_\varepsilon)} < \varepsilon^2$. Then, since

$$\||C|^{-1}w_\varepsilon\|_{L^2(\Gamma_\varepsilon)} = \sqrt{|C|^{-2}\int_{\Gamma_0}\chi_{\Gamma_\varepsilon}w_\varepsilon^2 d\mathcal{H}^{n-1}} \longrightarrow \sqrt{|C|^{-1}\int_{\Gamma_0}u^2 d\mathcal{H}^{n-1}} < \mathbf{C_u},$$

$$|C|^{-1}\chi_{\Gamma_\varepsilon}w_\varepsilon \to u \quad \text{weakly in} \quad L^2(\Gamma_0),$$

(as the limit of the product of weakly and strongly convergent sequences), it follows that $\widetilde{(\widehat{u}_\varepsilon)} \to u$ weakly in $L^2(\Gamma_0)$ and $\widehat{u}_\varepsilon \in U_\varepsilon$ for ε sufficiently small.

Step 2. For above given element $p \in \widehat{P}_0$ we have $p \in L^2(\Omega^-)$ and $\|p\|^2_{L^2(\Omega^-)} \le |\partial C|^{-1}_H \cdot \mathbf{C_p}$. Let $\{\zeta_\varepsilon\}$ be any sequence of smooth functions such that $\zeta_\varepsilon \to p$ strongly in $L^2(\Omega^-)$ as $\varepsilon \to 0$. Further we use the following result.

Lemma 6.5.

$$\lim_{\varepsilon\to 0}\int_{\Omega^-}\zeta_\varepsilon\,\varphi\,d\mu_\varepsilon = \int_{\Omega^-}p\,\varphi\,dx \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^n). \tag{6.22}$$

Proof. Indeed, let us partition Ω^- into cubes with edges ε and denote these cubes by the symbols $\varepsilon\Box_n^j$. Then

$$\begin{aligned}\int_{\Omega^-}\zeta_\varepsilon\,\varphi\,d\mu_\varepsilon &= \sum\int_{\varepsilon\Box_n^j}\zeta_\varepsilon(x)\,\varphi(x)\,d\mu_\varepsilon + \sum\int_{\Omega^-\cap\,\varepsilon\Box_n^j}\zeta_\varepsilon(x)\,\varphi(x)\,d\mu_\varepsilon\\ &= \sum\zeta_\varepsilon(x_j)\,\varphi(x_j)\int_{\varepsilon\Box_n^j}d\mu_\varepsilon + \sum\int_{\Omega^-\cap\,\varepsilon\Box_n^j}\zeta_\varepsilon(x)\,\varphi(x)\,d\mu_\varepsilon,\end{aligned} \tag{6.23}$$

where x_j is a point of the cube $\varepsilon\Box_n^j$ and the second sum is calculated over the set of 'boundary' cubes. By definition of the measure μ_ε we have

$$\int_{\varepsilon\Box_n^j}d\mu_\varepsilon = \varepsilon^n\int_{\Box_n}d\mu = \varepsilon^n.$$

Hence, the first term in (6.23) is a Riemann sum for the integral $\int_{\Omega^-}\zeta_\varepsilon\varphi\,dx$. Moreover,

$$\left|\sum\int_{\Omega^-\cap\,\varepsilon\Box_n^j}\zeta_\varepsilon(x)\,\varphi(x)\,d\mu_\varepsilon\right| \le \sup_{x\in\Omega^-}|\zeta_\varepsilon(x)\varphi(x)|\varepsilon^n\cdot D(\varepsilon),$$

where $D(\varepsilon)$ is a quantity of 'boundary' cubes, and $\varepsilon^n D(\varepsilon) \to 0$ by the measurability of the set $\partial\Omega^-$. Thus, summarizing the above cited facts, we have

$$\lim_{\varepsilon\to 0}\int_{\Omega^-}\zeta_\varepsilon\,\varphi\,d\mu_\varepsilon = \lim_{\varepsilon\to 0}\sum\zeta_\varepsilon(x_j)\,\varphi(x_j)\varepsilon^n = \int_{\Omega^-}p\,\varphi\,dx,$$

that completes the proof of this lemma. □

Taking the property (6.22) into account we note that for the selected sequence $\{\zeta_\varepsilon\}$ there exist a numerical sequence $\{d_\varepsilon\in\mathbb{R}\}_{\varepsilon>0}$ and a value $\varepsilon_0>0$ such that

$$\lim_{\varepsilon\to 0}\sqrt{d}_\varepsilon = 1, \quad \text{and} \quad d_\varepsilon\|\zeta_\varepsilon\|^2_{L^2(\Omega^-)}\le|\partial C|^{-1}_H\cdot\mathbf{C_p} \quad \text{for all } \ \varepsilon<\varepsilon_0.$$

Thus, the required sequence of admissible boundary controls

$$\{\widehat{p}_\varepsilon\in L^2(\Omega^-,d\mu_\varepsilon)\}_{\varepsilon>0},$$

which is weakly convergent to p, can be chosen as $\widehat{p}_\varepsilon=\sqrt{d}_\varepsilon\cdot\zeta_\varepsilon$ for every $\varepsilon>0$.

Step 3. Let $\widehat{y}_\varepsilon$ be the corresponding to $\widehat{u}_\varepsilon$ and $\widehat{p}_\varepsilon$ weak solution of the boundary value problem (2.3) in the sense of Definition 3.1. Then due to Theorem 6.3, we have $(\widehat{u}_\varepsilon,\widehat{p}_\varepsilon,\widehat{y}_\varepsilon)\xrightarrow{w}(u,p,v_0^-,v_0^+)$, where $\widehat{v}=(v_0^+,v_0^-)$ is a weak solution in $\mathcal{V}(\Omega)$ of the limit problem (6.7). Since this problem has a unique solution we immediately deduce that $(u,p,v_0^-,v_0^+)=(u,p,y^-,y^+)$. Thus, the property (1) of Definition 5.1 holds for any quaternary $(u,p,y^+,y^-)\in\Xi_0$.

Step 4. We now verify the second property of Definition 5.1. Let

$$\{(u_k,p_k,y_k)\}_{k\in\mathbb{N}}$$

be a w-convergent sequence for which there exists a sequence $\{\varepsilon_k\to 0\}$ such that $(u_k,p_k,y_k)\in\Xi_{\varepsilon_k}$ for all $k\in\mathbb{N}$. Let (u,p,y^+,y^-) be its w-limit. Then by Proposition 6.1 we immediately have

$$\mathbf{C_0}\ge\liminf_{k\to\infty}\|u_k\|_{L^2(\Gamma_{\varepsilon_k})}\ge\sqrt{|C|^{-1}}\cdot\|u\|_{L^2(\Gamma_0)},$$

that is, $u\in\widehat{U}_0$.

Due to the Proposition 4.3 and the definition of constrained set $\widehat{P}_\varepsilon$ (see (3.4)) we also have

$$|\partial C|^{-1}_H\,\mathbf{C_p}\ge\liminf_{k\to 0}\int_{\Omega^-}p_k^2\,d\mu_{\varepsilon_k}\ge\int_{\Omega^-}p^2\,dx.$$

It implies that $p\in\widehat{P}_0$. In conclusion, it remained only to apply Theorem 6.3. Thus, $(u,y^+,y^-)\in\Xi_0$, and we have obtained the required. This concludes the proof. □

7. Identification of the limit cost functional

In this section we show that the cost functional in the limit constrained minimization problem (5.2) can be recovered in the explicit form, and moreover, its analytical representation is different from the original one (2.2).

Theorem 7.1. *For the sequence of constrained minimization problems* (4.1) *there exists a variational w-limit* (5.2) (*in the sense of Definition* 5.2) *as ε tends to zero, where the set Ξ_0 is defined in* (6.20), *and*

$$I_0(u,p,y^+,y^-)=\int_{\Omega^+}(y^+-q_0)^2dx+|C|^{-1}\int_{\Gamma_0}u^2d\mathcal{H}^{n-1}+\rho|\partial C|_H\int_{\Omega^-}p^2dx. \tag{7.1}$$

Proof. In order to obtain the result of identification for the limit cost functional $I_0 : L^2(\Gamma_0)\times L^2(\Omega^-)\times H^1(\Omega^+)\times W_2^{(0,1)}(\Omega^-)\to\mathbb{R}$ in the form (7.1) we have to verify the conditions (ii)–(iii) of Definition 5.2.

Let (u,p,y^+,y^-) be any quadruple of Ξ_0, and let $\{(u_k,p_k,y_k)\}_{k\in\mathbb{N}}$ be a w-convergent sequence such that $(u_k,p_k,y_k)\xrightarrow{w}(u,p,y^+,y^-)$, $(u_k,p_k,y_k)\in\Xi_{\varepsilon_k}$ for every $k\in\mathbb{N}$, where $\{\varepsilon_k\}$ is a subsequence of $\{\varepsilon\}$ converging to zero. Then using Propositions 4.3, 6.1, and the properties of w-convergence, we get

$$\liminf_{k\to\infty}\int_{\Gamma_{\varepsilon_k}}u_k^2\,d\mathcal{H}^{n-1}=\liminf_{k\to\infty}\int_{\Gamma_0}\widetilde{u}_k^2\,d\mathcal{H}^{n-1}\ge|C|^{-1}\int_{\Gamma_0}u^2\,d\mathcal{H}^{n-1},$$

$$\liminf_{k\to\infty}\int_{\Omega^-}p_k^2\,d\mu_{\varepsilon_k}\ge\int_{\Omega^-}p^2\,dx.$$

Therefore,

$$\begin{aligned}\liminf_{k\to\infty}I_{\varepsilon_k}(u_k,p_k,y_k)&=\lim_{k\to\infty}\int_{\Omega^+}(y_k^+-q_0)^2\,dx+\liminf_{k\to\infty}\int_{\Gamma_{\varepsilon_k}}u_k^2\,d\mathcal{H}^{n-1}\\&\quad+\rho\,|\partial C|_H\liminf_{k\to\infty}\int_{\Omega^-}p_k^2\,d\mu_{\varepsilon_k}\ge\int_{\Omega^+}(y^+-q_0)^2\,dx\\&\quad+|C|^{-1}\int_{\Gamma_0}u^2\,d\mathcal{H}^{n-1}+\rho\,|\partial C|_H\int_{\Omega^-}p^2\,dx,\end{aligned}$$

i.e., the property (ii) of Definition 5.2 is valid.

To verify the correctitude of the "contrary" inequality (5.4), for arbitrary quaternary $(u,p,y^+,y^-)\in\Xi_0$ we have to construct the special 'realizing sequence' $\{(u_\varepsilon,p_\varepsilon,y_\varepsilon)\}$ that would be satisfied the condition (iii) of Definition 5.2. With this aim we construct the w-convergent sequence

$\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}$ to (u, p, y^+, y^-) in the same way as we did it under the proof of Theorem 6.4. Namely, $u_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$, $\|u_\varepsilon - |C|^{-1} w_\varepsilon\|_{L^2(\Gamma_\varepsilon)} < \varepsilon^2$, where the elements $w_\varepsilon \in L^2(\Gamma_0)$ are such that

$$w_\varepsilon \longrightarrow u \quad \text{strongly in } L^2(\Gamma_0), \text{ and } \|w_\varepsilon\|_{L^2(\Gamma_0)} < \sqrt{|C|} \cdot \mathbf{C_u}.$$

So, we may always suppose that the elements u_ε are able to have the following representation $u_\varepsilon = |C|^{-1} \widehat{w}_\varepsilon|_{\Gamma_\varepsilon}$, where the sequence $\{\widehat{w}_\varepsilon\}$ is such that $L^2(\Gamma_0) \ni \widehat{w}_\varepsilon \to u$ strongly in $L^2(\Gamma_0)$ and $\|\widehat{w}_\varepsilon\|_{L^2(\Gamma_0)} < \sqrt{|C|} \cdot \mathbf{C_u}$ for every $\varepsilon > 0$. Then

$$\begin{aligned} \lim_{\varepsilon\to 0} \|u_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} &= \lim_{\varepsilon\to 0} \||C|^{-1} \chi_{\Gamma_\varepsilon} \widehat{w}_\varepsilon\|^2_{L^2(\Gamma_0)} \\ &= |C|^{-2} \lim_{\varepsilon\to 0} \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} \widehat{w}^2_\varepsilon \, d\mathcal{H}^{n-1} = |C|^{-1} \|u\|^2_{L^2(\Gamma_0)}, \end{aligned}$$

that is for ‘realizing sequence’ of Dirichlet boundary controls $\{u_\varepsilon\}$ we have

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} u^2_\varepsilon \, d\mathcal{H}^{n-1} = |C|^{-1} \int_{\Gamma_0} u^2 \, d\mathcal{H}^{n-1},$$

and $u_\varepsilon \in U_\varepsilon$ for all $\varepsilon > 0$.

As for the ’realizing sequence’ for the function $p \in L^2(\Omega^-)$ we take any collection of smooth functions $\{p_\varepsilon\}_{\varepsilon>0}$ such that $p_\varepsilon \to p$ strongly in $L^2(\Omega^-)$. Then $\lim_{\varepsilon\to 0} \int_{\Omega^-} p^2_\varepsilon \, dx = \int_{\Omega^-} p^2 \, dx$, and we may always suppose that there exists a value $\varepsilon_0 > 0$ such that $\int_{\Omega^-} p^2_\varepsilon \, dx \le \int_{\Omega^-} p^2 \, dx$ for every $\varepsilon < \varepsilon_0$. So, using the same arguments as under the proof of Lemma 6.5, we obtain

$$\lim_{\varepsilon\to 0} \int_{\Omega^-} p^2_\varepsilon \, d\mu_\varepsilon = \int_{\Omega^-} p^2 \, dx \quad \textit{and } p_\varepsilon \in \widehat{P}_\varepsilon \quad \forall\, \varepsilon < \varepsilon_0.$$

For the conclusion we should to take y_ε as the corresponding to u_ε and p_ε solutions of the boundary value problem (2.3). Then by Theorem 6.3, we have $(u_\varepsilon, p_\varepsilon, y_\varepsilon) \xrightarrow{w} (u, p, y^+, y^-)$ and, therefore,

$$\begin{aligned} \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) &= \lim_{\varepsilon\to 0} \int_{\Omega^+} (y^+_\varepsilon - q_0)^2 \, dx + \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} u^2_\varepsilon \, d\mathcal{H}^{n-1} \\ &+ \rho\, |\partial C|_H \liminf_{\varepsilon\to 0} \int_{\Omega^-} p^2_\varepsilon \, d\mu_\varepsilon \ge \int_{\Omega^+} (y^+ - q_0)^2 \, dx \\ &+ |C|^{-1} \int_{\Gamma_0} u^2 \, d\mathcal{H}^{n-1} + \rho\, |\partial C|_H \int_{\Omega^-} p^2 \, dx, \end{aligned}$$

We have obtained the required. This completes the proof. $\square$

It is easy to see now that constrained minimization problems (5.2) can be recovered in the form of some optimal control problem. So, taking into account the Theorems 6.4 and 7.1, and Definition 5.3, we may give the following conclusion:

For the optimal control problem (2.2)–(2.5) *there exists the unique homogenized one with respect to w-convergence and it can be represented in the form (here $|C|$ is the $(n-1)$-dimensional Lebesgue measure of set C, and $|\partial C|_H = \mathcal{H}^{n-2}(\partial C)$) :*

$$I_0(u,p,y^+,y^-) = \int_{\Omega^+} (y^+ - q_0)^2 dx + |C|^{-1} \int_{\Gamma_0} u^2 d\mathcal{H}^{n-1} + \rho|\partial C|_H \int_{\Omega^-} p^2 dx \longrightarrow \inf \tag{7.2}$$

$$u \in \widehat{U}_0 = \left\{ u \in L^2(\Gamma_0) \;:\; \|u\|_{L^2(\Gamma_0)} \leq \sqrt{|C|}\cdot \mathbf{C_u} \right\}, \tag{7.3}$$

$$p \in \widehat{P}_0 = \left\{ p \in L^2(\Omega^-) \;:\; \|p\|^2_{L^2(\Omega^-)} \leq |\partial C|_H^{-1} \cdot \mathbf{C_p} \right\}. \tag{7.4}$$

$$\left.\begin{array}{rcll} -\Delta_x y^+(x) + y^+(x) &=& f_0(x), & x \in \Omega^+, \\ -\partial^2 y^-(x)/\partial x_n^2 + y^-(x) &=& f_0(x) + \delta|\partial C|_H|C|^{-1}p(x), & x \in \Omega^-, \\ \partial_\nu y^+(x) &=& 0, & x \in \partial\Omega^+ \setminus \Sigma, \\ y^- &=& |C|^{-1}u, & \text{on } \Gamma_0, \\ y^+ &=& y^-, & \text{on } \Sigma, \\ \partial_{x_n} y^+ &=& |C|\partial_{x_n} y^-, & \text{on } \Sigma. \end{array}\right\} \tag{7.5}$$

8. VARIATIONAL PROPERTIES OF THE HOMOGENIZED PROBLEM

Taking into account the weak lower semicontinuity of the cost functionals $I_0 : \Xi_0 \to \mathbb{R}$, topological properties of its domain Ξ_0, we just conclude that homogenized optimal control problems (7.2)–(7.5) has a unique solution. Let us denote by $(u^0, p^{\,0}, (y^0)^+, (y^0)^-) \in L^2(\Gamma_0)\times L^2(\Omega^-)\times H^1(\Omega^+)\times W_2^{(0,1)}(\Omega^-)$ the optimal quaternary for the limit problem (7.2)–(7.5). Now we are able to state the result concerning variational properties of w-homogenized problem (7.2)–(7.5).

Lemma 8.1. *Assume that $f_\varepsilon \to f_0$ strongly in $L^2(\Omega)$. Then*

$$\lim_{\varepsilon\to 0} I_\varepsilon(u^0_\varepsilon, \widehat{p}_\varepsilon^{\,0} y^0_\varepsilon) = \lim_{\varepsilon\to 0} \inf_{(u_\varepsilon,\widehat{p}_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(u_\varepsilon,\widehat{p}_\varepsilon,y_\varepsilon)$$

$$= \inf_{(u,p,y^+,y^-)\in\Xi_0} I_0(u,p,y^+,y^-) = I_0(u^0, p^{\,0}, (y^0)^+, (y^0)^-), \quad (8.1)$$

$$(u^0_\varepsilon, \widehat{p}_\varepsilon{}^0 y^0_\varepsilon) \xrightarrow{w} (u^0, p^0, (y^0)^+, (y^0)^-), \quad (8.2)$$

$$\widetilde{u}_\varepsilon - |C|^{-1}\chi_{\Gamma_\varepsilon} u^0 \longrightarrow 0 \quad \textit{strongly in} \ \ L^2(\Gamma_0). \quad (8.3)$$

Proof. As follows from Theorem 3.2, for every value of ε the optimal control problem (2.2)-(2.5) has the unique solution $(u^0_\varepsilon, \widehat{p}_\varepsilon{}^0, y^0_\varepsilon) \in \Xi_\varepsilon$. Since the sequences $\{u^0_\varepsilon \subset U_\varepsilon\}_{\varepsilon>0}$ and $\{\widehat{p}_\varepsilon{}^0 \subset \widehat{P}_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded, there exists a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$, which we again denote by $\{\varepsilon\}$, such that $u^0_\varepsilon \to u^0 \in \widehat{U}_0$ weakly with respect to the space $L^2(\Gamma_0)$ and $\widehat{p}_\varepsilon{}^0 \to p^{\,0} \in \widehat{P}_0$ weakly in $L^2(\Omega^-, d\mu_\varepsilon)$ as $\varepsilon \to 0$. Then in view of Theorem 6.3, we have $(u^0_\varepsilon, \widehat{p}_\varepsilon{}^0, y^0_\varepsilon) \xrightarrow{w} (u^*, p^*, (y^*)^-, (y^*)^+)$ as $\varepsilon \to 0$, where the quaternary $(u^*, p^*, (y^*)^-, (y^*)^+)$ is the unique solution of problem (7.5) with the Dirichlet condition $y^- = u^0 \quad \mathcal{H}^{n-1}$ a. e. on Γ_0. By Theorem 5.1 we immediately conclude that $(u^*, p^*, (y^*)^-, (y^*)^+)$ is an optimal solution of homogenized problem (7.2)–(7.5). Hence $(u^*, p^*, (y^*)^-, (y^*)^+) = (u^0, p^0, (y^0)^+, (y^0)^-)$, and in addition the property (8.1) are valid. To conclusion the proof, it remains to verify assertion (8.3). Using (6.21), and the fact that $\|u^0_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} \longrightarrow |C|^{-1}\|u^0\|^2_{L^2(\Gamma_0)}$ (see (8.1)), we get

$$\int_{\Gamma_0} \left(\widetilde{u}^0_\varepsilon - |C|^{-1}\chi_{\Gamma_\varepsilon} u^0\right)^2 d\mathcal{H}^{n-1} = \int_{\Gamma_e} (u^0_\varepsilon)^2 d\mathcal{H}^{n-1} - 2|C|^{-1}\int_{\Gamma_0} u^0 \widetilde{u}^0_\varepsilon d\mathcal{H}^{n-1}$$
$$+|C|^{-2}\int_{\Gamma_0} \chi_{\Gamma_\varepsilon}(u^0)^2 d\mathcal{H}^{n-1} \longrightarrow |C|^{-1}\int_{\Gamma_0}(u^0)^2 d\mathcal{H}^{n-1} - 2|C|^{-1}\int_{\Gamma_0}(u^0)^2 d\mathcal{H}^{n-1}$$
$$+ |C|^{-1}\int_{\Gamma_0}(u^0)^2 d\mathcal{H}^{n-1} = 0 \quad \text{ as } \ \varepsilon \to 0,$$

which yields (8.3). Thus, we have obtained the required. The Theorem is proved. □

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