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ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS TO SOME CLASSES OF SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

Toka Diagana
Department of Mathematics, Howard University
2441 6th Street NW, Washington, DC 20059

Hernán Henriquez
Departamento de Matemática, Universidad de Santiago
USACH, Casilla 307, Correo-2, Santiago, Chile

Eduardo Hernández
Departamento de Matemática, I.C.M.C. Universidade de São Paulo
Caixa Postal 668, 13560-970, São Carlos SP, Brazil

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Abstract. In this paper, we study the existence of asymptotically almost periodic solutions to some classes of second-order partial functional-differential equations with unbounded delay. Our results are, subsequently, applied to studying the existence of asymptotically almost periodic solutions to some integrodifferential equations, which arise in the theory of heat conduction within fading memory materials.

1. Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. In this paper, we study the existence of asymptotically almost periodic solutions to the class of second-order abstract partial functional-differential equations of the form

$$\frac{d}{dt}\left[x'(t) - g(t, x_t)\right] = Ax(t) + f(t, x_t), \quad t \in I, \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

$$x'(0) = \xi \in \mathbb{X}, \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on $\mathbb{X}$, the history $x_t : (-\infty, 0] \to \mathbb{X}$,

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$x_t(\theta) := x(t+\theta)$, belongs to an abstract phase space $\mathcal{B}$ defined axiomatically, and f, g are some appropriate functions.

Neutral (functional) differential equations arise in many areas of applied mathematics. For this reason, these types of equations have been of a great interest during the past few decades. The literature relative to first and second-order ordinary neutral functional differential equations is very extensive. Thus, for the sake of briefness, we refer only to Hale-Lunel [7], Lakshmikantham-Wen-Zhang [19], and Kolmanovskii-Myshkis [18] as well as most of the references in these books. First-order abstract partial neutral functional differential equations have been studied by several authors including Adimy et al. [1, 2], Hale [8], Wu-Xia [27, 26] for finite delay equations, and Hernández et al. [10, 11, 12] for unbounded delays. In a similar way, second-order abstract neutral differential equations similar to (1.1)-(1.3) have recently been considered in the literature, see, e.g., Hernández et al. [9, 13, 14].

The abstract Cauchy system (1.1)-(1.3) arises, for instance, in the theory of heat conduction within materials with fading memories, see, e.g., Gurtin-Pipkin [6] and Nunziato [23]. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly upon the temperature $u(\cdot)$ as well as its gradient $\nabla u(\cdot)$. Under these conditions, the classic heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory for materials with fading memories. In the theory developed in [6, 23], the internal energy and the heat flux are described as functionals of u and u_x. Upon some physical conditions, they established that the temperature $u(t,\xi)$ satisfies the integro-differential equation

$$\begin{aligned} c\frac{\partial^2 u(t,\xi)}{\partial t^2} &= \beta(0)\frac{\partial u(t,\xi)}{\partial t} + \int_0^\infty \beta'(s)\frac{\partial u(t-s,\xi)}{\partial t}ds \\ &\quad +\alpha(0)\triangle u(t,\xi) + \int_0^\infty \alpha'(s)\triangle u(t-s,\xi)ds, \qquad (1.4) \end{aligned}$$

where $\beta(\cdot)$ is the energy relaxation function, $\alpha(\cdot)$ is the stress relaxation function and c is the density. Assuming that $\beta(\cdot)$ is smooth enough and that $\nabla u(t,\xi)$ is approximately constant at t, we can rewrite the previous equation in the form

$$\frac{\partial^2 u(t,\xi)}{\partial t^2} = \frac{\partial}{\partial t}\Big[\frac{\beta(0)}{c}u(t,\xi) + \frac{1}{c}\int_0^\infty \beta'(s)u(t-s,\xi)ds\Big] + d\triangle u(t,\xi).$$

By making the function $\beta(\cdot)$ explicitly dependent on the time t, we can consider the situation, in which, the material is submitted to an aging process so that the hereditary properties are lost as the time goes to infinity. In this case, the previous equation takes the form

$$\frac{\partial^2 u(t,\xi)}{\partial t^2} = \frac{\partial}{\partial t}\Big[\frac{\beta(t,0)}{c}u(t,\xi) + \frac{1}{c}\int_0^\infty \frac{\partial \beta(t,s)}{\partial s}u(t-s,\xi)ds\Big] + d\triangle u(t,\xi), \tag{1.5}$$

which can be transformed into the abstract system (1.1) -(1.3) by assuming that the solution $u(\cdot)$ is known on $[0,\infty)$. In Section 5, we will briefly discuss the existence of asymptotically almost periodic solutions to this system.

This paper is divided into five main sections. In Section 2, we review some basic concepts, notations and properties needed in the sequel. In Section 3, we discuss the existence of local and global mild solutions for the system (1.1)-(1.3) and, in Section 4, we establish the existence of asymptotically almost periodic mild solution for this system. Finally, in Section 5, some applications are considered.

2. Preliminaries

In what follows, the spaces $(\mathbb{Z}, \|\cdot\|_{\mathbb{Z}})$ and $(\mathbb{W}, \|\cdot\|_{\mathbb{W}})$ stand for Banach spaces. The notation $\mathcal{L}(\mathbb{Z},\mathbb{W})$ stands for the Banach space of bounded linear operators from $\mathbb{Z}$ into $\mathbb{W}$ endowed with the operator norm with $\mathcal{L}(\mathbb{Z},\mathbb{Z})$ denoted $\mathcal{L}(\mathbb{Z})$. On the other hand, $B_r(x,\mathbb{Z})$ denotes the closed ball in $\mathbb{Z}$, centered at x with radius r.

Throughout this paper, $(\mathbb{X}, \|\cdot\|)$ is a Banach space, the linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on $\mathbb{X}$ and $(S(t))_{t\in\mathbb{R}}$ denote the associated sine function, which is defined by

$$S(t)x = \int_0^t C(s)xds, \qquad x \in \mathbb{X},\ t \in \mathbb{R}.$$

Let us mention that in general, the second-order abstract Cauchy problem $u''(t) = Au(t)$ cannot be studied through the following system:

$$u_0'(t) = u_1(t), \quad u_1'(t) = Au_0(t).$$

In particular, when this system is considered on $\mathbb{X}\times\mathbb{X}$, this may lead to an inconsistent Cauchy problem, as u' may fail to depend continuously on $u(0)$, $u'(0)$. On other hand, if E is the space consisting of all vectors $x \in \mathbb{X}$ for which $C(\cdot)x$ is of class C^1 on $\mathbb{R}$, we know from [17] that E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0\le t\le a} \|AS(t)x\|, \;\; x \in E,$$

is a Banach space and that the operator-valued function

$$\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix},$$

is a strongly continuous group of bounded linear operators on $E \times \mathbb{X}$ whose generator is the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$.

For details on cosine function theory and their applications to the second order abstract Cauchy problem, we refer the reader to Fattorini [4], Kisyński [17] and Travis-Webb [24, 25].

The existence of solutions to the second-order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad t \in [0,b], \tag{2.1}$$

$$x(0) = w, \quad x'(0) = z, \tag{2.2}$$

where $h \in L^1([0,b],\mathbb{X})$, was investigated in [25]. The semilinear case was treated in [24]. Here, we merely recall that the function $x(\cdot)$ given by

$$x(t) = C(t)w + S(t)z + \int_0^t S(t-s)h(s)ds, \quad t \in [0,b], \tag{2.3}$$

is called a mild solution for the system (2.1)-(2.2).

In this work, we will make use of an axiomatic definition of the phase space $\mathcal{B}$, similar to the one utilized in [16]. More precisely, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty,0]$ into $\mathbb{X}$ endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. Moreover, we will assume that the following axioms hold:

(A) If $x : (-\infty,\sigma+b] \to \mathbb{X}$, $b > 0$, is such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma,\sigma+b]} \in C([\sigma,\sigma+b],\mathbb{X})$, then for every $t \in [\sigma,\sigma+b)$ the following conditions hold:
   (i) x_t is in $\mathcal{B}$,
   (ii) $\| x(t) \| \le H \| x_t \|_{\mathcal{B}}$,
   (iii) $\| x_t \|_{\mathcal{B}} \le K(t-\sigma)\sup\{\| x(s) \| : \sigma \le s \le t\} + M(t-\sigma) \| x_\sigma \|_{\mathcal{B}}$,
   where $H > 0$ is a constant, $K, M : [0,\infty) \to [1,\infty)$ with K being continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (**A**), the function $t \to x_t$ is continuous from $[\sigma,\sigma+a)$ into $\mathcal{B}$.

(B) The space $\mathcal{B}$ is complete.

(**C2**) If $(\varphi^n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $C((-\infty,0];\mathbb{X})$ formed by functions with compact support and $\varphi^n \to \varphi$ in the compact-open topology, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Example 2.1. (**The phase space $\mathbf{C_r} \times \mathbf{L^p}(\rho,\mathbb{X})$**). Let $r \geq 0$, $1 \leq p < \infty$ and let $\rho : (-\infty,-r] \to \mathbb{R}$ be a non-negative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [16]. Briefly, this means that ρ is locally integrable and there exists a non-negative locally bounded function γ on $(-\infty,0]$ such that $\rho(\xi+\theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty,-r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty,-r)$ is a set whose Lebesgue measure zero. The space $C_r \times L^p(\rho,\mathbb{X})$ consists of all classes of functions $\varphi : (-\infty,0] \to \mathbb{X}$ such that φ is continuous on $[-r,0]$, Lebesgue-measurable, and $\rho\|\varphi\|^p$ is Lebesgue integrable on $(-\infty,-r)$. The seminorm on $C_r \times L^p(\rho,\mathbb{X})$ is defined by

$$\|\varphi\|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \Big(\int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|^p d\theta\Big)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(\rho;\mathbb{X})$ satisfies axioms **(A)**-**(A1)**-**(B)**. Moreover, when $r=0$ and $p=2$, we can take $H=1$, $M(t) = \gamma(-t)^{1/2}$ and

$$K(t) = 1 + \Big(\int_{-t}^{0} \rho(\theta)\, d\theta\Big)^{1/2},$$

for $t \geq 0$, see [16, Theorem 1.3.8] for details.

To obtain some of our results we need some additional properties in the space $\mathcal{B}$.

Definition 2.1. *Let $\mathcal{B}_0 = \{\psi \in \mathcal{B} : \psi(0) = 0\}$. The phase space $\mathcal{B}$ is called a fading memory space if $\| S(t)\psi\|_{\mathcal{B}} \to 0$ as $t \to \infty$ for every $\psi \in \mathcal{B}_0$. We say that $\mathcal{B}$ is a uniform fading memory space if $\|S(t)\|_{\mathcal{L}(\mathcal{B}_0)} \to 0$ as $t \to \infty$.*

For additional details regarding phase spaces, see, e.g., [16].

Let $I \subset \mathbb{R}$ be an interval. In this paper, $C_b(I,\mathbb{Z})$ and $C_0([0,\infty),\mathbb{Z})$ denote respectively the spaces

$$C_b(I,\mathbb{Z}) = \Big\{x : I \to \mathbb{Z},\, x \text{ is continuous and } \|x\| = \sup_{t\in I} \| x(t) \| < \infty\Big\},$$

$$C_0([0,\infty),\mathbb{Z}) = \Big\{x \in C_b([0,\infty),\mathbb{Z}) : \lim_{t\to\infty} \|x(t)\| = 0\Big\},$$

endowed with the uniform convergence norm.

For the sake of completeness, we recall here some concepts and results concerning almost periodic functions in the sense of Bohr.

Definition 2.2. *A function $F \in C_b(\mathbb{R}, \mathbb{Z})$ is almost periodic if for every $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, F, \mathbb{Z})$, such that*

$$\|F(t+\xi) - F(t)\|_{\mathbb{Z}} < \varepsilon,$$

for every $t \in \mathbb{R}$ and all $\xi \in \mathcal{H}(\varepsilon, F, \mathbb{Z})$.

Definition 2.3. *A function $F \in C_b([0,\infty), \mathbb{Z})$ is asymptotically almost periodic if there exists an almost periodic function $g(\cdot)$ and $w \in C_0([0,\infty), \mathbb{Z})$ such that $F = g + w$.*

The next properties are useful characterizations of almost periodic and asymptotically almost periodic functions.

Lemma 2.1. [28, p. 25] *A function $f \in C_b(\mathbb{R}, \mathbb{Z})$ is almost periodic if, and only if, the set of functions $\{H_t f : t \in \mathbb{R}\}$, where $(H_t f)(s) = f(s+t)$, is relatively compact in $C(\mathbb{R}, \mathbb{Z})$.*

Lemma 2.2. [28, Theorem 5.5] *A function $F \in C_b([0,\infty), \mathbb{Z})$ is asymptotically almost periodic if and only if, for every $\varepsilon > 0$ there exists $L(\varepsilon, F, \mathbb{Z}) > 0$ and a relatively dense subset of $[0,\infty)$, denoted by $\mathcal{H}(\varepsilon, F, \mathbb{Z})$, such that $\|F(t+\xi) - F(t)\|_{\mathbb{Z}} < \varepsilon$ for every $t \geq L(\varepsilon, F, \mathbb{Z})$ and every $\xi \in \mathcal{H}(\varepsilon, F, \mathbb{Z})$.*

In this paper, $AP(\mathbb{Z})$ and $AAP(\mathbb{Z})$ denote respectively the space of almost periodic functions and the space of asymptotically almost periodic functions on $\mathbb{Z}$. These spaces are endowed with the uniform convergence topology. We know from [28] that both $AP(\mathbb{Z})$ and $AAP(\mathbb{Z})$ are Banach spaces.

Our existence results are obtained with the help of the following well known fixed-point theorems.

Theorem 2.1. (Leray-Schauder Alternative) [5, Theorem 6.5.4] *Let D be a closed convex subset of a Banach space $\mathbb{Z}$ with $0 \in D$. Let $G : D \to D$ be a completely continuous map. Then, either G has a fixed point in D or the set $\{z \in D : z = \lambda G(z),\ 0 < \lambda < 1\}$ is unbounded.*

Theorem 2.2. [21, Theorem 4.3.2] *Let D be a convex, bounded and closed subset of a Banach space $\mathbb{Z}$ and $F : D \to D$ be a condensing map. Then F has a fixed point in D.*

3. EXISTENCE OF LOCAL AND GLOBAL MILD SOLUTIONS

In this section, we establish the existence of mild solutions to (1.1)-(1.3) in the cases when $I = [0, a]$ and $I = [0, \infty)$. In the sequel, $I = [0, a]$ or $I = [0, \infty)$ and$N, \widetilde{N}$ are positive constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \widetilde{N}$ for every $t \in I$.

To obtain our existence results, we will require the following general assumption.

($\mathbf{H_1}$) The functions $f, g : I \times \mathcal{B} \to \mathbb{X}$ satisfy the following conditions:

(i) The functions $f(t, \cdot)$, $g(t, \cdot) : I \times \mathcal{B} \to \mathbb{X}$ are continuous a.e. $t \in I$.

(ii) For each $\psi \in \mathcal{B}$, the functions $f(\cdot, \psi), g(\cdot, \psi) :\to \mathbb{X}$ are strongly measurable.

(ii) There exist integrable functions m_f, $m_g : I \to [0, \infty)$ and continuous non-decreasing functions $W_f, W_g : [0, \infty) \to (0, \infty)$ such that

$$\begin{aligned} \| f(t, \psi) \| &\leq m_f(t) W_f(\| \psi \|_{\mathcal{B}}), \quad (t, \psi) \in I \times \mathcal{B}, \\ \| g(t, \psi) \| &\leq m_g(t) W_g(\| \psi \|_{\mathcal{B}}), \quad (t, \psi) \in I \times \mathcal{B}. \end{aligned}$$

Let $x(\cdot)$ be a solution to (1.1)-(1.3). Let

$$h(t) = -\int_0^t g(s, x_s) ds,$$

$y(t) = x(t) + h(t)$, and assume that the function $t \to g(t, x_t)$ is differentiable. Then, from (2.3) we infer that

$$\begin{aligned} x(t) + h(t) &= C(t)\varphi(0) + S(t)[\xi - g(0, \varphi)] - \int_0^t AS(t - s)h(s, x_s)ds \\ &\quad + \int_0^t S(t - s)f(s, x_s)ds, \quad t \in I. \end{aligned} \tag{3.1}$$

On the other hand, from the proof of [24, Proposition 2.3] we infer that

$$\begin{aligned} A\int_0^t S(t - s)h(s, x_s)ds &= \int_0^t (C(t - s) - I)h'(s)ds \\ &= -\int_0^t C(t - s)g(s, u_s)ds - h(t), \end{aligned}$$

which enables us to rewrite (3.1) in the form

$$x(t) = C(t)\varphi(0) + S(t)[\xi - g(0, \varphi)] + \int_0^t C(t-s)g(s, x_s)ds + \int_0^t S(t-s)f(s, x_s)ds.$$

Motivated by the above, we introduce the following concept of a mild solution.

Definition 3.1. *A function* $x : (-\infty, 0] \cup I \to \mathbb{X}$ *is called a mild solution of the abstract Cauchy problem* (1.1)-(1.3) *on* I, *if* $x_0 = \varphi$

$$x(t) = C(t)\varphi(0) + S(t)[\xi - g(0,\varphi)] + \int_0^t C(t-s)g(s,x_s)ds + \int_0^t S(t-s)f(s,x_s)ds, \ \ t \in I.$$

For more rigorous discussions on the relation between the concept of mild solutions and strict solution of the system (1.1)-(1.3), we refer the reader to [13].

Throughout the rest of this paper, we set $W = \max\{W_f, W_g\}$.

3.1. **Solutions on Bounded Intervals.** The existence of mild solutions to (1.1)-(1.3) in the case when $I = [0, a]$ can be easily deduced from the results of [13]. However, for the sake of clarity we provide the reader with the proof of the next theorem, as some of the ideas in this proof are also useful in the sequel.

Theorem 3.1. *Let assumption* $(\mathbf{H_1})$ *hold and assume that for every* $0 < t \le a$ *and* $r > 0$, *the sets* $U(t,r) = \{S(t)f(s,\psi) : s \in [0,t], \| \psi \|_{\mathcal{B}} \le r\}$ *and* $g(I \times B_r(0,\mathcal{B}))$ *are relatively compact in* $\mathbb{X}$. *If*

$$K_a \int_0^a (Nm_g(s) + \widetilde{N}m_f(s))\, ds < \int_c^\infty \frac{ds}{W(s)}, \tag{3.2}$$

where $c = (K_a N H + M_a) \| \varphi \|_{\mathcal{B}} + K_a \widetilde{N}(\|\xi\| + \|g(0,\varphi)\|)$, $K_a = \sup_{s\in[0,a]} K(s)$ *and* $M_a = \sup_{s\in[0,a]} M(s)$, *then* (1.1)-(1.3) *has a mild solution.*

Proof. Let $\mathcal{B}C = \{x : (-\infty, a] \to \mathbb{X} : x|_{(-\infty,0]} \in \mathcal{B},\ x|_{[0,a]} \in C([0,a], \mathbb{X})\}$ endowed with the norm $\|x\|_{\mathcal{B}C} = \|x|_{(-\infty,0]}\|_{\mathcal{B}} + \|x|_{[0,a]}\|_a$. On this space, we define the map $\Gamma : \mathcal{B}C \to \mathcal{B}C$ by $(\Gamma x)_0 = \varphi$ and

$$\Gamma x(t) = C(t)\varphi(0) + S(t)[\xi - g(0,\varphi)] + \int_0^t C(t-s)g(s,x_s)ds + \int_0^t S(t-s)f(s,x_s)ds, \quad t \in I. \tag{3.3}$$

It is then easy to see that Γx is well-defined and that $\Gamma x \in \mathcal{B}C$. Moreover, by using the phase space axioms and the Lebesgue Dominated Convergence Theorem, one can prove that Γ is a continuous function from $\mathcal{B}C$ into $\mathcal{B}C$.

In order to apply Theorem 2.1, we establish an *a priori* estimate for the solution of the integral equation $x = \lambda\Gamma x$, $\lambda \in (0,1)$. Let $x^\lambda \in \mathcal{B}C$ be a solution of $x = \lambda\Gamma x$, $\lambda \in (0,1)$. For $t \in I$, we get

$$\begin{aligned} \| x^\lambda(t) \| \leq NH \| \varphi \|_{\mathcal{B}} + \widetilde{N}(\| \xi \| + \| g(0,\varphi) \|) \\ + \int_0^t (Nm_g(s) + \widetilde{N}m_f(s))W(\| x_s^\lambda \|_{\mathcal{B}})ds, \end{aligned} \tag{3.4}$$

which implies that

$$\begin{aligned} \| x_t^\lambda \|_{\mathcal{B}} \leq & (K_aNH + M_a) \| \varphi \|_{\mathcal{B}} + K_a\widetilde{N}(\| \xi \| + \| g(0,\varphi) \|) \\ & + K_a \int_0^t (Nm_g(s) + \widetilde{N}m_f(s))W(\| x_s^\lambda \|_{\mathcal{B}})ds. \end{aligned}$$

Denoting by $\beta_\lambda(t)$ the right hand side of the last inequality, we find that

$$\beta_\lambda'(t) \leq K_a(Nm_g(t) + \widetilde{N}m_f(t))W(\beta_\lambda(t)),$$

and hence,

$$\int_{\beta_\lambda(0)=c}^{\beta_\lambda(t)} \frac{ds}{W(s)} \leq K_a \int_0^t (Nm_g(s) + \widetilde{N}m_f(s))\, ds < \int_c^\infty \frac{ds}{W(s)},$$

which permits to conclude that the set of functions $\{\beta_\lambda : \lambda \in (0,1)\}$ is bounded. As a consequence of the previous fact, $\{x^\lambda : \lambda \in (0,1)\}$ is bounded in $C(I,X)$, as $\| x^\lambda(t) \| \leq H \| x_t^\lambda \| \leq \beta_\lambda(t)$ for every $t \in I$.

On the other hand, from [9, Lemma 3.1] we infer that Γ is completely continuous on $\mathcal{B}C$.

The existence of a mild solution for system (1.1)-(1.3) is now a consequence of Theorem 2.1. The proof is complete. □

In most of the situations of practical interests, the sine function is compact. This is the motivation for the next result.

Corollary 3.1. *Assume that assumption* $(\mathbf{H_1})$ *holds,* $S(t)$ *is compact for all* $t \geq 0$ *and the set* $g(I \times B_r(0,\mathcal{B}))$ *is relatively compact in* $\mathbb{X}$ *for every* $r > 0$. *If the inequality* (3.2) *holds, then the system* (1.1)-(1.3) *has a mild solution.*

Remark 3.1. In relation with the above result, let us point out that except when the space $\mathbb{X}$ is a finite dimensional space, the cosine function is not compact, and that for this reason, we cannot remove the compactness assumption on the function g. For additional details regarding this issue, see, e.g., Travis and Webb [25, pp. 557].

Following along the same lines as in the proof of Theorem 3.1, we can prove the following local existence result.

Theorem 3.2. *Assume that assumption* ($\mathbf{H_1}$) *holds and that for every* $0 < t \leq a$ *and* $r > 0$*, the sets* $U(t,r) = \{S(t)f(s,\psi) : s \in [0,t], \|\psi\|_{\mathcal{B}} \leq r\}$ *and* $g(I \times B_r(0,\mathcal{B}))$ *are relatively compact in* $\mathbb{X}$*. Then there exists a mild solution to* (1.1)-(1.3) *on* $[0,b]$ *for some* $0 < b \leq a$.

Note that assuming that both f and g satisfy Lipschitz condition, one can also establish the existence of a mild solution.

Theorem 3.3. *Suppose that* f, g *are continuous and that there exist positive constants* L_f, L_g *such that*

$$\begin{aligned} \| f(t,\psi_1) - f(t,\psi_2) \| &\leq L_f \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \\ \| g(t,\psi_1) - g(t,\psi_2) \| &\leq L_g \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \end{aligned}$$

for every $t \in I, \psi_1, \psi_2 \in \mathcal{B}$*. If* $aK_a[NL_g + \tilde{N}L_f] < 1$*, then there exists a unique mild solution to* (1.1)-(1.3).

Proof. It is easy to see that the inequality (3.2) holds for some small $0 < b \leq a$ in the place of a.

3.2. **Global Solutions.** In the sequel of this section, we discuss the existence of solutions defined on $I = [0,\infty)$. To this end, and for the rest of this paper, we assume that M, K are positive constants such that $M(t) \leq M$ and $K(t) \leq K$ for every $t \geq 0$ and that the functions m_f, m_g are locally integrable. In what follows, we make use of the notations $W = \max\{W_f, W_g\}$, $m = \max\{m_f, m_g\}$ and $\gamma(s) = Nm_g(s) + \widetilde{N}m_f(s)$.

Remark 3.2. If $\mathcal{B}$ is a fading memory space, then the functions $M(\cdot)$, $K(\cdot)$ are bounded on $[0,\infty)$, see [16, Proposition 7.1.5] for details.

We need to introduce some additional concepts and results. Let $h : [0,\infty) \to (0,\infty)$ be a continuous non-decreasing function with $h(0) = 1$ and such that $h(t) \to \infty$ as $t \to \infty$. In what follows, $C_{0,h}(\mathbb{X})$ denotes the space

$$C_{0,h}(\mathbb{X}) = \{x \in C([0,\infty),\mathbb{X}) : \lim_{t\to\infty} \frac{\| x(t) \|}{h(t)} = 0\},$$

endowed the norm $\|x\|_h = \sup_{t\geq 0} \frac{\|x(t)\|}{h(t)}$.

We now recall, without proof, the following compactness criterion.

Lemma 3.1. *A set* $B \subset C_0([0,\infty),\mathbb{X})$ *is relatively compact in* $C_0([0,\infty),\mathbb{X})$ *if, and only if,*

(a) *B is equicontinuous;*
(b) $\lim_{t\to\infty} \| x(t) \| = 0$, *uniformly for $x \in B$;*
(c) *The set $B(t) = \{x(t) : x \in B\}$ is relatively compact in $\mathbb{X}$ for every $t \geq 0$.*

We can now establish the principal result of this section.

Theorem 3.4. *Under assumption $(\mathbf{H_1})$, if the following conditions are satisfied:*

(a) *for every $t \in I$ and each $r \geq 0$ the sets $\{S(t)f(s,\psi) : (s,\psi) \in [0,t] \times B_r(0,\mathcal{B})\}$ and $g([0,t] \times B_r(0,\mathcal{B}))$ are relatively compact in $\mathbb{X}$;*
(b) *for every $L \geq 0$,*

$$\frac{1}{h(t)} \int_0^t m(s) W(Lh(s))\, ds \to 0, \quad \text{as } t \to \infty$$

and

$$\limsup_{r\to\infty} \frac{1}{r} \int_0^\infty \gamma(s) \frac{W((K+M)rh(s))}{h(s)} ds < 1.$$

Then the system (1.1)-(1.3) has a mild solution on $[0,\infty)$.

Proof. On the space $\mathcal{B}C_{0,h}(\mathbb{X}) = \{x : \mathbb{R} \to \mathbb{X} : x_0 \in \mathcal{B},\, x|_I \in C_{0,h}(\mathbb{X})\}$ endowed with the norm $\|x\|_{\mathcal{B}C_{0,h}} = \|x_0\|_{\mathcal{B}} + \|x|_I\|_h$, we define the map $\Gamma : \mathcal{B}C_{0,h}(\mathbb{X}) \to \mathcal{B}C_{0,h}(\mathbb{X})$ by $(\Gamma x)_0 = \varphi$ and

$$\Gamma x(t) = C(t)\varphi(0) + S(t)[\xi - g(0,\varphi)] + \int_0^t C(t-s) g(s,x_s) ds + \int_0^t S(t-s) f(s,x_s) ds, \quad t \geq 0.$$

It is easy to prove that the expression $\Gamma x(\cdot)$ is well defined for each $x \in \mathcal{B}C_{0,h}(\mathbb{X})$. On the other hand, using the fact that

$$\|x_s\|_{\mathcal{B}} \leq (K+M)\|x\|_{\mathcal{B}C_{0,h}} h(s)$$

for $s \in I$, we find that

$$\frac{\|\Gamma x(t)\|}{h(t)} \leq \frac{NH\|\varphi\|_{\mathcal{B}} + (\|\xi\| + \|g(0,\varphi)\|)}{h(t)} + \frac{1}{h(t)} \int_0^t [N m_g(s) + \widetilde{N} m_f(s)] W((K+M) \| x \|_{\mathcal{B}C_{0,h}} h(s)) ds, \tag{3.5}$$

which implies, from condition (**c**), that $\frac{\|\Gamma x(t)\|}{h(t)}$ converges to zero as $t \to \infty$. This shows that Γ is a well-defined map from $\mathcal{B}C_{0,h}(\mathbb{X})$ into $\mathcal{B}C_{0,h}(\mathbb{X})$. Note that the inequality (3.5) shows also that $\frac{\|\Gamma x(t)\|}{h(t)} \to 0$, as $t \to \infty$, uniformly for x in bounded sets of $\mathcal{B}C_{0,h}(\mathbb{X})$.

In the sequel, we prove that Γ verifies the hypotheses of Theorem 2.1. We begin by proving that Γ is continuous. Let $(u^n)_n$ be a sequence in $\mathcal{B}C_{0,h}(\mathbb{X})$ and $u \in \mathcal{B}C_{0,h}(\mathbb{X})$ such that $u^n \to u$ as $n \to \infty$. Clearly, $g(s, u^n_s) \to g(s, u_s)$, $f(s, u^n_s) \to f(s, u_s)$ a.e. $s \in I$ as $n \to \infty$, and

$$\|f(s, u^n_s)\| \leq m_f(s) W_f(\beta h(s)), \quad s \geq 0,$$
$$\|g(s, u^n_s)\| \leq m_g(s) W_g(\beta h(s)), \quad s \geq 0,$$

where $\beta = (K+M)L$ and $L > 0$ is such that $\sup\{\| u \|_{\mathcal{B}C_{0,h}(\mathbb{X})}, \|u^n\|_{\mathcal{B}C_{0,h}(\mathbb{X})} : n \in \mathbb{N}\} \leq L$. Since the functions on the right hand side are integrable on $[0,t]$, we conclude that $\| \Gamma u^n(t) - \Gamma u(t) \| \to 0$, when $n \to \infty$, uniformly for t in bounded intervals. Moreover, using the argument that the set of functions $\{u^n : n \in \mathbb{N}\}$ is bounded in $\mathcal{B}C_{0,h}(\mathbb{X})$, for each $\epsilon > 0$ there exists $T_\epsilon > 0$ such that $\frac{\|\Gamma u^n(t) - \Gamma u(t)\|}{h(t)} \leq \epsilon$, for all $n \in \mathbb{N}$ and every $t \geq T_\epsilon$. Combining these properties we obtain that $\Gamma u^n \to \Gamma u$ in $\mathcal{B}C_{0,h}(\mathbb{X})$. Thus, Γ is continuous.

On the other hand, if $x^\lambda \in \mathcal{B}C_{0,h}(\mathbb{X})$ is a solution of the integral equation $\lambda \Gamma z = z$, $0 < \lambda < 1$, for $t \geq 0$, we obtain that

$$\begin{aligned}\frac{\|x^\lambda(t)\|}{h(t)} &\leq \frac{NH \| \varphi \|_{\mathcal{B}} + \widetilde{N}(\| \xi \| + \|g(0,\varphi)\|)}{h(t)} \\ &\quad + \frac{1}{h(t)} \int_0^t \gamma(s) W((K+M) \| x^\lambda \|_{\mathcal{B}C_{0,h}(\mathbb{X})} h(s)) ds,\end{aligned}$$

and hence

$$\begin{aligned}\| x^\lambda \|_{\mathcal{B}C_{0,h}(\mathbb{X})} &\leq (1 + NH) \| \varphi \|_{\mathcal{B}} + \widetilde{N}(\| \xi \| + \|g(0,\varphi)\|) \\ &\quad + \int_0^\infty \gamma(s) \frac{W((K+M) \| x^\lambda \|_{\mathcal{B}C_{0,h}(\mathbb{X})} h(s))}{h(s)} ds.\end{aligned}$$

From the previous estimates, if the set $\{\| x^\lambda \|_{\mathcal{B}C_{0,h}(\mathbb{X})} : 0 < \lambda < 1\}$ is unbounded, we deduce the existence of a sequence $(r^n)_{n\in\mathbb{N}}$ with $r^n \to \infty$ such that

$$1 \leq \liminf_{n\to\infty} \frac{1}{r_n} \int_0^\infty \gamma(s) \frac{W((K+M) r_n h(s))}{h(s)} ds,$$

which is absurd, therefore the set $\{\| x^\lambda \|_{\mathcal{B}C_{0,h}(\mathbb{X})} : 0 < \lambda < 1, \}$ is bounded.

Arguing as in the proof of Theorem 3.1, we can prove that $\{\Gamma x(t) : x \in B_r(0, \mathcal{B}C_{0,h}(\mathbb{X})))\}$ is relatively compact in $\mathbb{X}$ for every $t \geq 0$ and that $\{\frac{\Gamma x}{h} : x \in B_r(0, \mathcal{B}C_{0,h}(\mathbb{X})))\}$ is equicontinuous on $[0, \infty)$. Moreover, from our previous remarks we know that $\frac{\Gamma x(t)}{h(t)} \to 0$ as $t \to \infty$, uniformly for $x \in B_r(0, \mathcal{B}C_{0,h}(\mathbb{X}))$. Consequently, we have shown that the set $\{\frac{\Gamma x}{h} : x \in B_r(0, \mathcal{B}C_{0,h}(\mathbb{X}))\}$ fulfills the conditions of Lemma 3.1, which implies that it is relatively compact in $C_0(\mathbb{X})$. This proves that $\Gamma B_r(0, \mathcal{B}C_{0,h}(\mathbb{X}))$ is relatively compact in $\mathcal{B}C_{0,h}(\mathbb{X})$.

Finally, the existence of a mild solution for the system (1.1)-(1.3) on $[0, \infty)$ follows from Theorem 2.1. The proof is complete. □

4. Existence of asymptotically almost periodic solutions

This section is devoted to the existence of asymptotically almost periodic solutions for the abstract system (1.1)-(1.3). Throughout the rest of this paper, we assume that $N, \widetilde{N}$ are positive constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \widetilde{N}$ for every $t \geq 0$.

For the sake of completeness, in the next definition, we recall some well-known concepts.

Definition 4.1. *Let $F : [0, \infty) \to \mathcal{L}(\mathbb{Z}, \mathbb{W})$ be an operator function.*

- **(a)** *The operator function F is said strongly continuous if for every each $x \in \mathbb{Z}$, the function $F(\cdot)x : [0, \infty) \to \mathbb{W}$ is continuous.*
- **(b)** *The operator function F is said pointwise almost periodic (respectively, pointwise asymptotically almost periodic) if $F(\cdot)x \in AP(\mathbb{W})$ for every $x \in \mathbb{Z}$ (respectively, $F(\cdot)x \in AAP(\mathbb{W})$ for every $x \in \mathbb{Z}$).*
- **(c)** *The operator function F is said almost periodic (respectively, asymptotically almost periodic) if $F(\cdot) \in AP(\mathcal{L}(\mathbb{Z}, \mathbb{W}))$ (respectively, $F(\cdot) \in AAP(\mathcal{L}(\mathbb{Z}, \mathbb{W}))$).*

We refer the reader to [3] for the characterization of almost periodic cosine functions and to [15] for similar results for almost periodic sine functions.

Remark 4.1. Note that if the sine function $S(\cdot)$ is uniformly bounded and pointwise almost periodic, then the cosine function $C(\cdot)$ is also pointwise almost periodic, see [15, Lemma 3.1] and [15, Theorem 3.2] for details.

Our next results are based upon some well-known compactness criterion in $AP(\mathbb{X})$ and $AAP(\mathbb{X})$

Lemma 4.1. [28, Chapter 6] *Let $V \subseteq AP(\mathbb{X})$ be a set with the following properties:*

- **(a)** *V is uniformly equicontinuous on $\mathbb{R}$;*
- **(b)** *for each $t \in \mathbb{R}$, the set $V(t) = \{x(t) : x \in V\}$ is relatively compact in $\mathbb{X}$;*

(c) *V is equi-almost periodic, that is, for every $\epsilon > 0$ there is a relatively dense set $\mathcal{H}(\varepsilon, V, \mathbb{X}) \subset \mathbb{R}$ such that*

$$\|x(t+\tau) - x(t)\| \leq \epsilon, \quad x \in V, \quad \tau \in \mathcal{H}(\varepsilon, V, \mathbb{X}),\ t \in \mathbb{R}.$$

Then V is relatively compact in $AP(\mathbb{X})$.

Remark 4.2. As an immediate consequence of this characterization, one can assert that if $F : \mathbb{R} \to \mathcal{L}(\mathbb{X}, \mathbb{Y})$ is almost periodic and U is a relatively compact subset of $\mathbb{X}$, then $V = \{F(\cdot)x : x \in U\}$ is relatively compact in $AP(\mathbb{Y})$. For the sine function, we can strengthen this property.

Proposition 4.1. *Assume that the sine function $S(\cdot)$ is almost periodic and that $U \subseteq \mathbb{X}$. If the set $\{S(t)x : x \in U,\ t \geq 0\}$ is relatively compact in $\mathbb{X}$, then $V = \{S(\cdot)x : x \in U\}$ is relatively compact in $AP(\mathbb{X})$.*

Proof. Let us fix $\delta > 0$. Since $S(\delta)U$ is relatively compact in $\mathbb{X}$, by the previous remark, we can affirm that $V_\delta = \{S(\cdot)S(\delta)x : x \in U\}$ is relatively compact in $AP(\mathbb{X})$. On the other hand, for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|(I - C(s))S(t)x\| \leq \varepsilon,$ for all $0 \leq s \leq \delta$, every $x \in U$ and all $t \geq 0$. It follows from this that

$$\| S(t)x - \frac{1}{\delta} S(t)S(\delta)x \| = \| \frac{1}{\delta} \int_0^\delta (I - C(s))S(t)x ds \| \leq \varepsilon,$$

for every $t \geq 0$. This property and the decomposition

$$S(\cdot)x = \frac{1}{\delta} S(\cdot)S(\delta)x + S(\cdot)x - \frac{1}{\delta} S(\cdot)S(\delta)x,$$

imply that $V \subseteq \frac{1}{\delta} V_\delta + B_\epsilon(0, C_b(\mathbb{X}))$, which in turn proves that V is relatively compact in $AP(\mathbb{X})$. The proof is complete. □

Remark 4.3. Note that the assumption on the compactness of the set $\{S(t)x : x \in U,\ t \geq 0\}$ considered in the previous proposition is verified, for instance, in the case when the sine function is almost periodic.

Lemma 4.2. *Assume that $S(\cdot)$ is pointwise almost periodic and that U is a bounded subset of $\mathbb{X}$. If one of the following conditions holds:*

(i) *U is relatively compact.*

(ii) *$S(\cdot)$ is almost periodic and $S(t)$ is compact for every $t \in \mathbb{R}$.*

Then $\{S(t)x : x \in U,\ t \geq 0\}$ is relatively compact in $\mathbb{X}$.

We note by the results of Lutz in [20] that the case (ii) includes the periodic sine functions.

For asymptotically almost periodic functions, we have a similar characterization of compactness given in the next lemma.

Lemma 4.3. *Let $V \subseteq AAP(\mathbb{X})$ be a set with the following properties:*

(a) *V is uniformly equicontinuous on $[0,\infty)$;*
(b) *for each $t \geq 0$, the set $V(t) = \{x(t) : x \in V\}$ is relatively compact in $\mathbb{X}$;*
(c) *V is equi-asymptotically almost periodic, that is, for every $\varepsilon > 0$ there are $L(\varepsilon, V, \mathbb{X}) \geq 0$ and a relatively dense set $\mathcal{H}(\varepsilon, V, \mathbb{X}) \subseteq [0,\infty)$ such that*

$$\|x(t+\tau) - x(t)\| \leq \varepsilon, \quad x \in V,\ t \geq L(\varepsilon, V, \mathbb{X}),\ \tau \in \mathcal{H}(\varepsilon, V, \mathbb{X}).$$

Then V is relatively compact in $AAP(\mathbb{X})$.

Remark 4.4. We have already mentioned in Definition 2.3 that each function $f \in AAP(\mathbb{X})$ can be decomposed in unique form as $f = f_1 + f_2$, where $f_1 \in AP(\mathbb{X})$ and $f_2 \in C_0(\mathbb{X})$. Let $V \subseteq AAP(\mathbb{X})$ and $V_i = \{f_i : f \in V\}$, $i = 1, 2$. It follows from the above-mentioned results that V is relatively compact in $AAP(\mathbb{X})$ if, and only if, V_1 is relatively compact in $AP(\mathbb{X})$ and V_2 is relatively compact in $C_0(\mathbb{X})$.

We will repeatedly use the following property.

Proposition 4.2. *Let $(\mathbb{Z}_i, \|\cdot\|_{\mathbb{Z}_i})$, $i = 1, 2$, be Banach spaces and $V \subseteq L^1([0,\infty), \mathbb{Z}_1)$. If $F_1 : [0,\infty) \to \mathcal{L}(\mathbb{Z}_1, Z_2)$ and $F_2 : [0,\infty) \to \mathcal{L}(\mathbb{Z}_2)$ are strongly continuous functions of bounded linear operators which satisfy*

(a) *$\int_L^\infty F_1(s)x(s)ds \to 0$ in $\mathbb{Z}_2$ when $L \to \infty$, uniformly for $x \in V$;*
(b) *For each $t \geq 0$, the set $\{x(s) : x \in V,\ 0 \leq s \leq t\}$ is relatively compact in $\mathbb{Z}_1$,*

then the sets $W(t) = \{\int_0^t F_1(s)x(s)ds : x \in V\}$, $t \geq 0$, and $W = \bigcup_{0 \leq t \leq \infty} W(t)$ are relatively compact in $\mathbb{Z}_2$. Moreover, if F_2 is uniformly bounded on $[0,\infty)$ and $\int_t^{t+h} F_1(s)x(s)ds \to 0$, as $h \to 0$, uniformly for $x \in V$, then the set $U = \{z_x : x \in V\}$, where $z_x(t) = F_2(t)\int_t^\infty F_1(s)x(s)ds$, is relatively compact in $C_0(\mathbb{Z}_2)$.

Proof. Let $(K_t)_{t \geq 0}$ be a family of compacts sets such that $\{x(s) : x \in V, s \in [0,t]\} \subseteq K_t$ for every $t \geq 0$. Since F_1 is strongly continuous, then the set $F_1 K_t = \{F_1(s)y : y \in K_t, 0 \leq s \leq t\}$ is relatively compact in $\mathbb{Z}_2$. Let $(\widetilde{K}_t)_{t \geq 0}$ be a non-decreasing family of compact and absolutely convex sets such that $F_1 K_t \subset \widetilde{K}_t$ for every $t \geq 0$.

From the mean value theorem for the Bochner integral (see [21, Lemma 2.1.3]), we infer that $W(t) \subseteq t\widetilde{K}_t$ for all $t > 0$. On the other hand, for each $\varepsilon > 0$ there is a constant $L \geq 0$ such that $\|\int_L^\infty F_1(s)x(s)ds\|_{\mathbb{Z}_2} \leq \varepsilon$, for all $x \in V$. We obtain from the properties of the sets $\widetilde{K}_t$ that $W \subseteq L\widetilde{K}_L + B_\varepsilon(0, \mathbb{Z}_2)$, which shows that W is relatively compact in $\mathbb{Z}_2$. Thus, the sets $W(t)$, $t \geq 0$, and W are relatively compact in $\mathbb{Z}_2$.

To establish the last assertion, we make use of Lemma 3.1. The hypothesis (**b**) of Lemma 3.1 can be easily obtained as an immediate consequence of (**a**) and the fact that F_2 is uniformly bounded. Moreover, for every $t \geq 0$ and $x \in V$, we have that

$$\int_t^\infty F_1(s)x(s)ds \in \overline{W - W(t)} \subset \overline{W - W} = W_1,$$

which proves that the set $U(t) = \{F_2(t) \int_t^\infty F_1(s)x(s)ds : x \in V\}$ is relatively compact in $\mathbb{Z}_2$ for every $t \geq 0$.

Finally, we prove that U is equicontinuous. To this end, we fix $t \geq 0$. Since the elements $\int_t^\infty F_1(\xi)x(\xi)\, d\xi$, $x \in V$, are in the compact set W_1 (which is independent of t), and the family $(F_2(t))_{t\geq 0}$ is strongly continuous in $\mathbb{Z}_2$, for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} \| F_2(t+s)x - F_2(t)x \| &\leq \varepsilon, \quad x \in W_1, \\ \left\| \int_t^{t+s} F_1(\xi)x(\xi)d\xi \right\| &\leq \varepsilon, \quad x \in V, \end{aligned}$$

for every $0 < |s| < \delta$ with $t + s \geq 0$. Consequently, for $x \in V$ and $0 <| s |< \delta$ such that $t + s \geq 0$, we get,

$$\begin{aligned} &\| F_2(t+s) \int_{t+s}^\infty F_1(\xi)x(\xi)\, d\xi - F_2(t) \int_t^\infty F_1(\xi)x(\xi)\, d\xi \| \\ &\leq \left\| (F_2(t+s) - F_2(t)) \int_{t+s}^\infty F_1(\xi)x(\xi)d\xi \right\| + \| F_2(t) \| \left\| \int_{t\wedge(t+s)}^{t\vee(t+s)} F_1(\xi)x(\xi)\, d\xi \right\| \\ &\leq \sup\{\| (F_2(t+s)y - F_2(t)y \|: y \in W_1\} + \| F_2(t) \| \varepsilon, \\ &\leq (1 + \sup_{\theta\geq 0} \| F_2(\theta) \|)\varepsilon, \end{aligned}$$

which implies that U is equicontinuous at t. This complete the proof that U is relatively compact in $C_0(\mathbb{Z}_2)$. The proof is complete. $\square$

In the next results, for a locally integrable function $x : [0, \infty) \to X$, we denote by z_x, $y_x : [0, \infty) \to \mathbb{X}$ the functions given by

$$z_x(t) = \int_0^t C(t-s)x(s)ds \quad \text{and} \quad y_x(t) = \int_0^t S(t-s)x(s)ds.$$

Proposition 4.3. *Assume that $S(\cdot)$ is pointwise almost periodic and that $V \subseteq L^1([0, \infty), \mathbb{X})$ is a set with the following properties:*

(a) $\int_L^\infty \|x(s)\| ds \to 0$ *when* $L \to \infty$*, uniformly for* $x \in V$*;*
(b) $\int_t^{t+s} \|x(\xi)\| \, d\xi \to 0$*, when* $s \to 0$*, uniformly for* $x \in V$ *and* $t \geq 0$*;*
(c) *for each* $t \geq 0$ *the set* $\{x(s) : 0 \leq s \leq t, \ x \in V\}$ *is relatively compact.*

Then the sets $\{y_x : x \in V\}$ *and* $\{z_x : x \in V\}$ *are relatively compact in* $AAP(\mathbb{X})$*.*

Proof. We establish initially that each function y_x is asymptotically almost periodic. For $x \in V$, we can write

$$\begin{aligned} y_x(t) &= S(t)\int_0^t C(s)x(s)\,ds \;-\; C(t)\int_0^t S(s)x(s)\,ds \\ &= S(t)\int_0^\infty C(s)x(s)\,ds \;-\; S(t)\int_t^\infty C(s)x(s)\,ds \\ &\quad - C(t)\int_0^\infty S(s)x(s)\,ds \;+\; C(t)\int_t^\infty S(s)x(s)\,ds. \end{aligned}$$

Since the sine function $S(\cdot)$ is pointwise almost periodic, it follows from [15, Lemma 3.1] and [15, Theorem 3.2] that $C(\cdot)$ is also pointwise almost periodic. Therefore, the first and third terms on the right hand side define almost periodic functions while the second and fourth terms are functions that vanish at ∞. Thus, $y_x \in AAP(\mathbb{X})$.

From Proposition 4.2, we know that the integrals $\int_0^\infty C(s)x(s)ds$ and $\int_0^\infty S(s)x(s)ds$, $x \in V$, are included in a compact subset of $\mathbb{X}$, which implies that the set formed by the functions

$$S(\cdot)\int_0^\infty C(s)x(s)ds - C(\cdot)\int_0^\infty S(s)x(s)ds, \quad x \in V,$$

is relatively compact in $AP(\mathbb{X})$. The same Proposition enables us to infer that the set

$$\Big\{t \to C(t)\int_t^\infty S(s)x(s)ds - S(t)\int_t^\infty C(s)x(s)ds : x \in V\Big\},$$

is relatively compact in $C_0(\mathbb{X})$. This shows that $\{y_x : x \in V\}$ is relatively compact in $AAP(\mathbb{X})$.

We now prove that the set $\{z_x : x \in V\}$ is relatively compact in $AAP(\mathbb{X})$. For that we first show that the functions z_x, $x \in V$, are uniformly continuous. First of all, fix $L > 0$. Since $C(\cdot)$ is pointwise almost periodic, from (**c**) we have that $\|(C(t+s) - C(t))x(\xi)\| \to 0$, as $s \to 0$, uniformly for $t \geq 0$,

$0 \leq \xi \leq L$ and $x \in V$. Therefore,

$$\begin{aligned}\|z_x(t+s) - z_x(t)\| \leq & \int_0^{t\wedge(t+s)} \|C(t+s-\xi)x(\xi) - C(t-\xi)x(\xi)\| d\xi \\ & + \| \int_{t\wedge(t+s)}^{t\vee(t+s)} C(t+s-\xi)x(\xi)d\xi \| \\ \leq & \int_0^L \sup_{t\geq 0, x\in V} \|(C(t+s-\xi) - C(t-\xi))x(\xi)\| d\xi \\ & + 2N \int_L^\infty \|x(\xi)\| d\xi + N \int_{t\wedge(t+s)}^{t\vee(t+s)} \|x(\xi)\| d\xi.\end{aligned}$$

Using conditions (**a**) and (**b**) we can appropriately choose L to show that the right hand side of the above inequality converges to 0 as $s \to 0$, uniformly in $t \geq 0$ and $x \in V$, which proves that each function z_x is uniformly continuous on $[0, \infty)$. Moreover, from the above, is clear that the set $\{z_x : x \in V\}$ is uniformly equicontinuous on $[0, \infty)$

Since z_x is the derivative of y_x, it follows from [28, Theorem 5.2] that $\{z_x : x \in V\}$ is a uniformly equicontinuous subset of $AAP(\mathbb{X})$. Moreover, from Proposition 4.2 we obtain that $\{z_x(t) : x \in V\}$ is relatively compact, for all $t \geq 0$.

Finally, we establish that $\{z_x : x \in V\}$ is equi-asymptotically almost periodic. For a given $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that $\int_{L_\varepsilon}^\infty \|x(s)\| ds \leq \varepsilon/6N$, for all $x \in V$. In addition, since the set $\{C(\cdot)x(s) : 0 \leq s \leq L_\varepsilon\}$ is equi-almost periodic, there is a relatively dense set $P_\varepsilon \subseteq [0, \infty)$ such that

$$\|C(\xi + \tau)x(s) - C(\xi)x(s)\| \leq \frac{\varepsilon}{3L_\varepsilon},$$

for all $\xi \geq 0$, $0 \leq s \leq L_\varepsilon$ and every $\tau \in P_\varepsilon$. Hence, for $t \geq L_\varepsilon$ and $\tau \in P_\varepsilon$, we obtain

$$\begin{aligned}\|z_x(t+\tau) - z_x(t)\| \leq & \int_0^t \|C(t+\tau-s)x(s) - C(t-s)x(s)\| ds \\ & + \int_t^{t+\tau} \|C(t+\tau-s)x(s)\| \, ds \\ \leq & \int_0^{L_\varepsilon} \|C(t+\tau-s)x(s) - C(t-s)x(s)\| ds + 3N \int_{L_\varepsilon}^\infty \|x(s)\| \, ds \leq \varepsilon,\end{aligned}$$

which shows the assertion. One completes the proof by applying Lemma 4.3 to the set $\{z_x : x \in V\}$. □

Using this result and proceeding as in the proof of Proposition 4.1 we obtain the compactness of $\{y_x : x \in V\}$ with some weaker conditions.

Proposition 4.4. *Assume that* $S(\cdot)$ *is almost periodic and that* $V \subseteq L^1([0,\infty),\mathbb{X})$ *is uniformly bounded and satisfies the following properties:*

- **(a)** $\int_L^\infty \| x(s) \| \, ds \to 0$, *when* $L \to \infty$, *uniformly for* $x \in V$;
- **(b)** $\int_t^{t+s} \|x(\xi)\| \, d\xi \to 0$, *as* $s \to 0$, *uniformly for* $t \geq 0$ *and* $x \in V$;
- **(c)** *for each* $t, \delta \geq 0$, *the set* $\{S(\delta)x(s) : 0 \leq s \leq t,\ x \in V\}$ *is relatively compact in* $\mathbb{X}$.

Then $\{y_x : x \in V\}$ *is relatively compact in* $AAP(\mathbb{X})$.

Proof. For $x \in V$ we define the function $\widetilde{y}_x(t) = \int_0^t S(s)x(s)ds$. Let $0 < \varepsilon < t \leq a$. Since the function $s \to S(s)$ is Lipschitz continuous, we can choose points $0 = t_1 < t_2 ... < t_n = t$ such that $\| S(s) - S(s') \| \leq \varepsilon$ for $s, s' \in [t_i, t_{i+1}]$ and $i = 1, 2, ...n-1$. For $x \in V$, then from the Mean Value Theorem for the Bochner integral (see [21, Lemma 2.1.3]), we find that

$$\begin{aligned} \widetilde{y}_x(t) &= \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} (S(s) - S(t_i))x(s)ds + \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} S(t_i)x(s)ds \\ &\in \mathcal{C}_\varepsilon + \sum_{i=1}^{n-1} (t_{i+1} - t_i) \overline{co(\{S(t_i)z(s) : s \in [0, t_i], z \in V\})} \\ &\subset \mathcal{C}_\varepsilon + \mathcal{K}_\varepsilon, \end{aligned}$$

where $\mathcal{K}_\varepsilon$ is compact and $diam(\mathcal{C}_\varepsilon) \to 0$ as $\varepsilon \to 0$. This prove that $W_1(t) = \{\widetilde{y}_x(t);\ x \in V\}$ is relatively compact in $\mathbb{X}$. Moreover, proceeding as in the proof of Proposition 4.2, we infer that $W = \bigcup_{0 \leq t \leq \infty} W(t)$ and $U = \{\int_t^\infty S(s)x(s)ds : x \in V,\ t \geq 0\}$ are also relatively compact in $\mathbb{X}$.

To complete the proof, we consider one more time the decomposition

$$\begin{aligned} y_x(t) &= S(t) \int_0^\infty C(s)x(s)ds - S(t) \int_t^\infty C(s)x(s)ds \\ &\quad - C(t) \int_0^\infty S(s)x(s)ds + C(t) \int_t^\infty S(s)x(s)ds. \end{aligned}$$

Since the cosine function is pointwise almost periodic (see Remark 4.1), we infer from Remark 4.2 and Lemma 3.1 that the set of functions

$$\left\{ t \to -C(t) \int_0^\infty S(s)x(s) \, ds + C(t) \int_t^\infty S(s)x(s) \right\}$$

is relatively compact in $AAP(\mathbb{X})$. Moreover, using the fact that $S(\cdot)$ is almost periodic and that $S(t)$ is a compact operator for every $t \geq 0$, we can prove from Remark 4.2 and Lemma 3.1 that the set of functions

$$\left\{t \to S(t)\int_0^\infty C(s)x(s)ds - S(t)\int_t^\infty C(s)x(s)ds : x \in V\right\}$$

is also completely continuous in $AAP(\mathbb{X})$. □

We are now in conditions to establish the main results of this work.

Theorem 4.1. *Assume that $S(\cdot)$ is almost periodic and that condition $(\mathbf{H_1})$ holds with $m_f(\cdot)$ and $m_g(\cdot)$ in $L^1([0,\infty))$. Suppose, in addition, that for every $t \geq 0$ and each $r \geq 0$ the sets $\{S(t)f(s,\psi) : (s,\psi) \in [0,t] \times B_r(0,\mathcal{B})\}$ and $g([0,t] \times B_r(0,\mathcal{B}))$ are relatively compact in $\mathbb{X}$. If*

$$K\int_0^\infty (Nm_g(s) + \widetilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W(s)}, \tag{4.1}$$

where $c = (KNH + M) \parallel \varphi \parallel_{\mathcal{B}} + K\widetilde{N}(\parallel \xi \parallel + \parallel g(0,\varphi) \parallel)$, then there exists a mild solution $u(\cdot) \in AAP(\mathcal{B},\mathbb{X})$ of (1.1)-(1.3).

Proof. Let $\mathcal{B}AAP = \{x : \mathbb{R} \to \mathbb{X} : x_0 \in \mathcal{B},\ x\mid_{[0,\infty)} \in AAP(\mathbb{X})\}$ endowed with the semi-norm $\|x\|_{\mathcal{B}AAP} := \|x_0\|_{\mathcal{B}} + \sup_{t\geq 0}\|x(t)\|$ and $\Gamma : \mathcal{B}AAP \to \mathcal{B}AAP$ be the operator defined by

$$\begin{aligned}\Gamma x(t) &= C(t)\varphi(0) + S(t)[\xi - g(0,\varphi)] \\ &\quad + \int_0^t C(t-s)g(s,x_s)ds + \int_0^t S(t-s)f(s,x_s)ds,\end{aligned}$$

for $t \geq 0$, and $(\Gamma x)_0 = \varphi$.

By the integrability of the functions $m_f(\cdot)$ and $m_g(\cdot)$ and proceeding as in the proof of Proposition 4.3 for the functions $f(s,x_s)$ and $g(s,x_s)$, we infer that $\Gamma(x) \in AAP(\mathcal{B},\mathbb{X})$. Furthermore, if we take a sequence $(x^n)_n$ that converges to x in the space $AAP(\mathcal{B},\mathbb{X})$, then $S(t-s)f(s,x^n_s) \to S(t-s)f(s,x_s)$ and $C(t-s)g(s,x^n_s) \to C(t-s)g(s,x_s)$, as $n \to \infty$, a.e. for $t,s \in [0,\infty]$. Let $L = \sup\{\|x\|_{\mathcal{B}C}, \|x^n\|_{\mathcal{B}C} : n \in \mathbb{N}\}$ and $\beta = (K+M)L$. From the inequalities

$$\begin{aligned}\|C(t-s)g(s,x^n_s) - C(t-s)g(s,x_s)\| &\leq 2Nm_g(s)W_g(\beta), \\ \|S(t-s)f(s,x^n_s) - S(t-s)f(s,x_s)\| &\leq 2\widetilde{N}m_f(s)W_f(\beta),\end{aligned}$$

and using the integrability of $m_f(\cdot)$ and $m_g(\cdot)$, we conclude that $\parallel \Gamma x^n - \Gamma x \parallel_{\mathcal{B}AAP} \to 0$ when $n \to \infty$. Thus, Γ is a continuous map from $AAP(\mathcal{B},\mathbb{X})$ into $AAP(\mathcal{B},\mathbb{X})$.

On the other hand, proceeding as in the proof of Theorem 3.1, we conclude that the set of functions $\{x^\lambda \in AAP(\mathcal{B},\mathbb{X}) : \lambda\Gamma(x^\lambda) = x^\lambda,\, 0 < \lambda < 1\}$ is uniformly bounded on $[0,\infty)$.

Finally, we show that Γ is completely continuous. In order to establish this assertion, we take a bounded set $V \subseteq AAP(\mathcal{B},\mathbb{X})$. Since the sets of functions $\Lambda_1 = \{s \to g(s,x_s) : x \in V\}$ and $\Lambda_2 = \{s \to f(s,x_s) : x \in V\}$ satisfy the hypotheses of Proposition 4.3 and Proposition 4.4, respectively, we infer that $\Gamma(V)$ is relatively compact in $AAP(\mathbb{X})$. The assertion is now a consequence of Theorem 2.1. □

In Theorem 4.2, below, we prove the existence of an asymptotically almost periodic mild solution to (1.1)-(1.3) by assuming that $g(\cdot)$ satisfies an appropriate Lipschitz condition. For that, we need the following lemma.

Lemma 4.4. *If $\mathcal{B}$ is a fading memory space and $z \in C_b(\mathbb{R};\mathbb{X})$ is a function such that $z_0 \in \mathcal{B}$ and $z \in AAP(\mathbb{X})$, then $t \to z_t \in AAP(\mathcal{B})$.*

Theorem 4.2. *Assume that the sine function $S(\cdot)$ is almost periodic and that $\mathcal{B}$ is a fading memory space. Suppose, in addition, that the following conditions hold:*

- **(a)** *For every $t \geq 0$ and each $r \geq 0$, the set $\{S(t)f(s,\psi) : (s,\psi) \in [0,t] \times B_r(0,\mathcal{B})\}$ is relatively compact in $\mathbb{X}$.*
- **(b)** *There exists a function $L_g \in L^1([0,\infty))$ such that*

$$\| g(t,\psi_1) - g(t,\psi_2) \| \quad \leq \quad L_g(t) \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \quad (t,\psi_j) \in [0,\infty) \times \mathcal{B}.$$

- **(c)** *The condition $(\mathbf{H_1})$ is valid with m_g, m_f in $L^1([0,\infty))$ and*

$$(K+M)\liminf_{\xi\to\infty} \frac{W(\xi)}{\xi}\int_0^\infty (Nm_g(s) + \widetilde{N}m_f(s))ds < 1. \tag{4.2}$$

Then there exists a mild solution $u(\cdot) \in AAP(\mathbb{X})$ of (1.1)-(1.3).

Proof. Let $\mathcal{B}AAP = \{x : \mathbb{R} \to \mathbb{X} : x_0 \in \mathcal{B},\, x\,|_{[0,\infty)} \in AAP(\mathbb{X})\}$ endowed with the semi-norm defined by $\|x\|_{\mathcal{B}AAP} = \|x_0\|_{\mathcal{B}} + \sup_{t\geq 0}\|x(t)\|$. On this space, we define the operators $\Gamma_i : \mathcal{B}AAP \to \mathcal{B}AAP$, $i = 1, 2$, by

$$\begin{aligned}\Gamma_1 x(t) &= C(t)\varphi(0) + S(t)[\xi - g(0,\varphi)] + \int_0^t C(t-s)g(s,x_s)ds,\\ \Gamma_2 x(t) &= \int_0^t S(t-s)f(s,x_s)ds,\end{aligned}$$

for $t \geq 0$, and $(\Gamma_1 x)_0 = \varphi$ and $(\Gamma_2 x)_0 = 0$.

From the proof of Proposition 4.3, we infer that the functions $\zeta(t) = \int_0^t S(t-s)g(s,x_s)ds$ and $\Gamma_2 x$ are asymptotically almost periodic. It is easy to see that $\zeta'(t) = \int_0^t C(t-s)g(s,x_s)ds$. Moreover, since the function $s \to g(s,x_s)$ is integrable on $[0,\infty)$ and

$$\|\zeta'(t+h) - \zeta'(t)\| \leq \int_0^h N \parallel g(s,x_s) \parallel ds \\ + N\int_0^\infty \parallel g(s+h,x_{s+h}) - g(s,x_s) \parallel ds,$$

converge to zero as $h \to 0$, uniformly for $t \in [0,\infty)$, we can conclude from [28, Theorem 5.2] that $\Gamma_1 x$ is also asymptotically almost periodic. This proof that $\Gamma_1 x, \Gamma_2 x$ are well defined and that Γ_1, Γ_2 are functions defined from $\mathcal{B}AAP$ into $\mathcal{B}AAP$.

Let $y : \mathbb{R} \to \mathbb{X}$ be the extension of φ to $\mathbb{R}$ such that $y(t) = C(t)\varphi(0) + S(t)[\xi - g(0,\varphi)]$ for $t \geq 0$ and $\Gamma : \mathcal{B}AAP \to \mathcal{B}AAP$ be the map $\Gamma = \Gamma_1 + \Gamma_2$. We next prove that there exists $r > 0$ such that $\Gamma(B_r(y,\mathcal{B}AAP)) \subset B_r(y,\mathcal{B}AAP)$. Proceeding by contradiction, we suppose that for each $r > 0$ there exist $u^r \in B_r(y,\mathcal{B}AAP)$ and $t^r \geq 0$ such that $\parallel \Gamma u^r(t^r) - y(t^r) \parallel > r$. Consequently,

$$\begin{aligned} r &\leq \parallel \Gamma u^r(t^r) - y(t^r) \parallel \\ &\leq \int_0^{t^r} (Nm_g(s) + \widetilde{N}m_f(s))W(\parallel u_s^r - y_s \parallel_{\mathcal{B}} + \parallel y_s \parallel_{\mathcal{B}})ds \\ &\leq \int_0^\infty (Nm_g(s) + \widetilde{N}m_f(s))W((K+M)r + \rho)ds, \end{aligned}$$

where $\rho = (M + KNH) \parallel \varphi \parallel_{\mathcal{B}} + K\widetilde{N} \parallel \xi - g(0,\varphi) \parallel$, which yields

$$1 \leq (K+M)\liminf_{\xi\to\infty} \frac{W(\xi)}{\xi} \int_0^\infty (Nm_g(s) + \widetilde{N}m_f(s))ds.$$

Since this inequality contradicts (4.2), we obtain the assertion.

Let $r > 0$ such that $\Gamma(B_r(0,\mathcal{B}AAP)) \subset B_r(0,\mathcal{B}AAP)$. Proceeding as in the proof of Theorem 4.1, we can show that the map Γ_2 is completely continuous. Moreover, from the estimate

$$\parallel \Gamma_1 u(t) - \Gamma_1 v(t) \parallel \leq NK \int_0^t L_g(s)ds \parallel u - v \parallel_{\mathcal{B}AAP},$$

we infer that Γ_1 is a contraction on $\mathcal{B}AAP$, which enables us to conclude that Γ is a condensing on $B_r(0,\mathcal{B}AAP)$. Now, the assertion is a consequence of Theorem 2.2. □

5. Applications

Motivated by the integro-differential system (1.5), in this section we discuss the existence of asymptotically almost periodic mild solutions for the system

$$\frac{\partial}{\partial t}\left[\frac{\partial u(t,\xi)}{\partial t}+\eta(t)u(t,\xi)+\int_{-\infty}^{t}\alpha_1(t,s)u(s,\xi)ds\right] \tag{5.1}$$
$$=\frac{\partial^2 u(t,\xi)}{\partial \xi^2}+\int_{-\infty}^{t}\alpha_2(t,s)u(s,\xi)ds,$$

for $t\geq 0$ and $\xi\in J=[0,\pi]$, subject to the initial conditions

$$u(t,0) = u(t,\pi)=0, \quad t\geq 0, \tag{5.2}$$
$$u(\theta,\xi) = \varphi(\theta,\xi), \quad \theta\leq 0,\ \xi\in J, \tag{5.3}$$
$$\frac{\partial u(0,\xi)}{\partial t} = z(\xi), \quad \xi\in J, \tag{5.4}$$

where $\eta(\cdot):\mathbb{R}\to\mathbb{R}$, $\alpha_i:\mathbb{R}^2\to\mathbb{R}$ are continuous functions and φ,ξ are some appropriate functions.

To transform this system into the abstract framework utilized in the previous sections, we consider the space $\mathbb{X}=L^2([0,\pi];\|\cdot\|_2)$ and the operator $A:D(A)\subset\mathbb{X}\to\mathbb{X}$ defined by

$$D(A)=\{x\in W^{2,2}(0,\pi):x(0)=x(\pi)=0\},\quad Ax=x'',\ \ x\in D(A).$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function, $(C(t))_{t\in\mathbb{R}}$ on $\mathbb{X}$. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n\in\mathbb{N}$, with corresponding normalized eigenvectors given by $z_n(\xi)=\left(\frac{2}{\pi}\right)^{1/2}\sin(n\xi)$. Moreover, the following properties are fulfilled:

- **(a)** The set $\{z_n:n\in\mathbb{N}\}$ is an orthonormal basis of $\mathbb{X}$;
- **(b)** For $x\in\mathbb{X}$, $C(t)x=\sum_{n=1}^{\infty}\cos(nt)\langle x,z_n\rangle z_n$. It follows from this expression that

  $$S(t)x=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}\langle x,z_n\rangle z_n.$$

  Moreover, the sine function $S(\cdot)$ is periodic and $S(t)$ is a compact operator for all $t\in\mathbb{R}$ and $\max\{\|C(t)\|,\|S(t)\|\}\leq 1$, for every $t\in\mathbb{R}$.
- **(c)** If G denotes the group of translations on $\mathbb{X}$ defined by $G(t)x(\xi)=\tilde{x}(\xi+t)$, where $\tilde{x}$ is the odd extension of x with period 2π, then

$C(t) = \frac{1}{2}[G(t) + G(-t)]$. Hence, it follows ([4]) that $A = B^2$, where B is the infinitesimal generator of the group G and

$$E = \left\{x \in H^1(0, \pi) : x(0) = x(\pi) = 0\right\}.$$

As a phase space we choose the space $\mathcal{B} = C_r \times L^p(\rho, \mathbb{X})$, $r \geq 0$, $1 \leq p < \infty$ (see Example 2.1) and assume that the conditions (g-5)-(g-7) in the terminology of [16] are valid. Note that, under these conditions, the space $\mathcal{B}$ is a fading memory space and that there exists $\mathfrak{K} > 0$ such that $\max\{K(t), M(t)\} \leq \mathfrak{K}$ for all $t \geq 0$, see [16, Example 7.1.8] and [16, Proposition 7.1.5] for details. By assuming that

$$\begin{aligned} L_g(t) &= |\eta(t)| + \Big(\int_{-\infty}^0 \Big[\frac{\alpha_1(t, t+\theta)}{\rho(\theta)}\Big]^2 d\theta\Big)^{1/2}, \\ m_f(t) &= \Big(\int_{-\infty}^0 \Big[\frac{\alpha_2(t, t+\theta)}{\rho(\theta)}\Big]^2 d\theta\Big)^{1/2}, \end{aligned}$$

are finite, for every $t \geq 0$, we can define the operators $g, f : \mathbb{R}^+ \times \mathcal{B} \to \mathbb{X}$ by the mean of the expressions

$$\begin{aligned} g(t, \psi)(\xi) &= \eta(t)\psi(0, \xi) + \int_{-\infty}^0 \alpha_1(t, t+s)\psi(s, \xi)ds, \\ f(t, \psi)(\xi) &= \int_{-\infty}^0 \alpha_2(t, t+s)\psi(s, \xi)ds. \end{aligned}$$

It is easy to see that $g(t, \cdot)$ and $f(t, \cdot)$ are bounded linear operators, as $\| g(t, \cdot) \|_{\mathcal{L}(\mathcal{B}, \mathbb{X})} \leq L_g(t)$ and $\| f(t, \cdot) \|_{\mathcal{L}(\mathcal{B}, X)} \leq m_f(t)$ for every $t \geq 0$. The next results is a direct consequence of Theorem 4.2. Thus, the details of the proof will be omitted.

Proposition 5.1. *Assume $\varphi \in \mathcal{B}$, $\eta \in \mathbb{X}$ and that $L_g(\cdot), m_f(\cdot)$ are functions in $L^1([0, \infty))$. If*

$$2\mathfrak{K} \int_0^\infty (L_g(s) + m_f(s))ds < 1, \tag{5.5}$$

then there exists an asymptotically almost periodic mild solution to (5.1)-(5.4).

To complete this section, we study briefly the existence of asymptotically almost periodic solutions for the system (1.5). To simplify the description and for sake of brevity, we consider the case when $d = 1$. Assume that the functions $\beta(\cdot)$ and $\frac{\partial\beta(\cdot)}{\partial s}$ are continuous and that the expression

$$L_g(t) = \Big|\frac{\beta(t,0)}{c}\Big| + \frac{1}{|\,c\,|}\Big(\int_{-\infty}^{0}\Big[\frac{\partial\beta(t,s)}{\partial s}\rho^{-1}(s)\Big]^2 ds\Big)^{1/2},$$

define a function in $L^1([0,\infty))$. By assuming that the solution $u(\cdot)$ of (1.5) is known on $[0,\infty)$, and defining the function $g:\mathbb{R}\times\mathcal{B}\to\mathbb{X}$ by

$$g(t,\psi)(\xi) \;=\; \frac{\beta(t,0)}{c}\psi(0,\xi) + \frac{1}{c}\int_0^{\infty}\frac{\partial\beta(t,s)}{\partial s}\psi(-s,\xi)ds,$$

we can transform system (1.5) into the abstract system (1.1)-(1.3).

Corollary 5.1. *For every $\varphi\in\mathcal{B}$ and $\xi\in\mathbb{X}$, there exists an asymptotically almost periodic mild solution of* (1.5) *with $u_0=\varphi$.*

Proof. This result is a particular case of Proposition 5.1. We only observe that the inequality (4.2) is automatically satisfied, as $m_f\equiv 0$.

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