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PARAMETRIC ANALYTICITY OF FUNCTIONAL VARIATIONS OF DIRICHLET–NEUMANN OPERATORS

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Abstract. One of the important open questions in the theory of free–surface ideal fluid flows is the dynamic stability of traveling wave solutions. In a spectral stability analysis, the first variation of the governing Euler equations is required which raises both theoretical and numerical issues. With Zakharov and Craig and Sulem's formulation of the Euler equations in mind, this paper addresses the question of analyticity properties of first (and higher) variations of the Dirichlet–Neumann operator. This analysis will have consequences not only for theoretical investigations, but also for numerical simulations of spectral stability of traveling water waves.

1. Introduction

One of the central unresolved questions in the theory of ideal free–surface fluid flows is that of dynamic stability. It has been rigorously known for almost a century [20, 40] that traveling wave solutions exist to the governing Euler equations of these water wave flows. However, complete results concerning their dynamic stability are largely lacking (see [9]). A first step towards a comprehensive stability theory for the *full* Euler equations is to study the special case of spectral stability of periodic traveling wave solutions. This, in turn, involves the analysis of the first variation of the water wave equations about this traveling solution. A convenient formulation of the Euler equations due to Zakharov [42] involves only *surface* quantities and recognizes the Hamiltonian structure of the water wave equations. The explicit nature of this set of equations was clarified by Craig & Sulem [7] with the introduction of the Dirichlet–Neumann operator (DNO) and, with Zakharov's and Craig & Sulem's formulation in mind, in this paper we provide a mathematically rigorous analysis of analyticity properties of the first

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variation of the DNO. An important consequence of our method of proof is that it generates a sequence of recursions which is particularly amenable to stable, high–order numerical simulation.

Boundary integral operators, such as the DNO, arise in a wide array of free–boundary and boundary–value models in mathematical physics (see, e.g., [11, 41, 12, 18]). In these models, the DNO allows the volumetric statement of the problem to be simplified to one involving surface quantities alone. The crucial role played by the DNO is that, given values of the field variable at the interface (Dirichlet data), it produces surface normal derivatives of the field (Neumann data) giving closure to the set of surface equations.

In many free–boundary and boundary–value problems, the domain of definition is a small departure from a simple, separable geometry (e.g., a strip or half–space). In these cases, a perturbative approach is quite natural and, in fact, the DNO can be shown to depend *analytically* upon the size of the boundary deformation (provided that it is sufficiently smooth) [2, 3, 6, 4, 35, 37, 15, 38]. These results have proven crucial to not only theoretical developments, e.g. the traveling–wave existence proof of Craig & Nicholls [4], but also to numerical simulations, see e.g. [25, 26, 29, 30, 28, 27, 16, 41, 7, 36]. In the latter, the DNO is approximated by a truncation of its strongly convergent Taylor series producing highly accurate solutions, typically with a small number of terms.

In this work, we utilize the "Transformed Field Expansion" (TFE) approach of Nicholls & Reitich [35] to show that, for sufficiently smooth boundary perturbations, first and higher functional variations of the DNO are also parametrically analytic. Our theoretical developments are particularly important as they specify a stable, high–order algorithm for the numerical approximation of these variations. This is noteworthy as there is currently *no* stable, perturbative scheme for simulating these crucially important operators. Moreover, the recursions we derive are straightforward to implement (c.f. [37]), a task which we save for future work.

As mentioned above, we focus upon a spectral stability analysis of *periodic* traveling waves, i.e. the study of the time evolution of

$$u(x,t) = \bar{u}(x) + \delta e^{\lambda t} v(x), \qquad \delta \ll 1,$$

in a frame of reference moving with speed c, where $\bar{u}$ is a traveling wave (periodic with respect to a lattice Γ), and $\mathrm{Re}\{\lambda\} > 0$ indicates spectral instability. For this, Mielke's "Generalized Principle of Reduced Instability" [24] (essentially a generalization of Floquet's theory [8]) can be applied, provided that $v(x)$ is permitted to be "quasiperiodic" or "Bloch periodic" with respect to the lattice Γ:

$$v(x+\gamma) = e^{ip\cdot\gamma} v(x), \qquad \forall \gamma \in \Gamma,$$

for some real number p. Note that if p is rational, then v is periodic with respect to Γ, however, p irrational permits "perturbations" v with no periodicity properties relative to Γ. Mielke's theory enables a most general study of both super–harmonic and sub–harmonic instabilities simultaneously (see, e.g., [22, 23]).

We discuss the spectral stability analysis in such detail to contrast our new theorems with the only other known results on functional variations of the DNO (associated to water waves) with respect to domain shape, those due to Lannes [19]. Lannes required boundedness properties of these operators to carry out a Newton's iteration to solve the initial value problem, so Bloch periodicity never entered his theoretical developments. While these results are related, our new results improve on those of Lannes in a number of directions. First, our technique permits domain deformations with less smoothness (e.g., in the class of C^2, $C^{1+\alpha}$, or even Lipschitz functions depending on the Dirichlet data; see Remark 4.8). Also, with seamless presentation, we can accommodate the Bloch periodicity of the function v that our stability analysis requires. We point out that these differing periodicity requirements mandate that the "evaluation point," $\bar{u}$ (later, for the DNO, η), and the "direction," v (later, for the DNO, w), will *generically* lie in different function spaces. For this reason, we cannot appeal to previous results on analyticity of DNO which do not distinguish the very different roles these functions play (e.g., [2, 3, 6, 4, 35, 37, 15, 38]). Most importantly, our results differ from those of Lannes in that, as was demonstrated for the DNO in [37], our recursions lead to a stable, high–order numerical scheme for the approximation of variations of the DNO which Lannes' formulation does not provide (naturally, as this was not his intention).

The organization of the paper is as follows, in § 2 we review the governing equations of ideal free–surface fluid mechanics and outline a spectral stability analysis in Zakharov and Craig & Sulem's formulation. In § 3, we recall the TFE method and review known analyticity results for DNO. In § 4 & § 5, we show how these results can be extended to first and higher variations of the DNO using the same TFE methodology.

2. Governing Equations

The Euler equations model the free–surface evolution of an ideal fluid under the influence of gravity (capillarity effects can easily be incorporated but we suppress them for this presentation). Consider the fluid domain

$$S_{h,\eta} := \{(x,y) \in \mathbf{R}^{d-1} \times \mathbf{R} : -h < y < \eta(x,t)\}, \tag{2.1}$$

where h is the mean depth, $d = 2, 3$ is the problem dimension, and η is the perturbation of the free air–fluid interface from its rest state at $y = 0$. Inside $S_{h,\eta}$ the well–known equations of motion are the Euler equations [18]

$$\Delta\varphi = 0 \qquad \text{in } S_{h,\eta} \tag{2.2a}$$

$$\partial_y \varphi = 0 \qquad \text{at } y = -h \tag{2.2b}$$

$$\partial_t \eta - \partial_y \varphi + \nabla_x \eta \cdot \nabla_x \varphi = 0 \qquad \text{at } y = \eta \tag{2.2c}$$

$$\partial_t \varphi + \tilde{g}\eta + \frac{1}{2}|\nabla\varphi|^2 = 0 \qquad \text{at } y = \eta, \tag{2.2d}$$

where $\varphi(x,y,t)$ is the velocity potential (the velocity is given by $\vec{u} = \nabla\varphi$) and $\tilde{g}$ is the gravitational constant. As we stated in the Introduction, we are interested in periodic traveling waves and their stability, so we enforce the condition that the x–dependence of φ and η is periodic with respect to a lattice $\Gamma \subset \mathbf{R}^{d-1}$.

V. Zakharov [42] pointed out that the water wave problem (2.2) can be restated in terms of *surface* quantities, $\eta(x,t)$ and $\xi(x,t) := \varphi(x, \eta(x,t), t)$, and, furthermore, that this system is Hamiltonian. As the formulation is somewhat implicit in nature, a clarifying contribution was made by Craig & Sulem [7] who introduced the Dirichlet–Neumann operator (DNO) to the problem. We define this DNO by considering the prototype elliptic problem motivated by (2.2)

$$\Delta v = 0 \qquad \text{in } S_{h,g} \tag{2.3a}$$

$$v(x, g(x)) = \xi(x) \tag{2.3b}$$

$$\partial_y v(x, -h) = 0 \tag{2.3c}$$

$$v(x+\gamma, y) = v(x,y) \qquad \forall\, \gamma \in \Gamma. \tag{2.3d}$$

If g is sufficiently smooth, then (2.3) admits a unique solution and we can compute the normal derivative of the solution at the surface $y = g$. The DNO carries out this procedure by mapping the Dirichlet data, ξ, to the Neumann data:

$$G(g)[\xi] := [\nabla v]_{y=g} \cdot N = \partial_y v(x, g(x)) - \nabla_x g \cdot \nabla_x v(x, g(x)).$$

In terms of this operator, the Euler equations can now be written [7]

$$\partial_t \eta = G(\eta)[\xi] \tag{2.4a}$$

$$\partial_t \xi = -\tilde{g}\eta - \frac{1}{2(1+|\nabla_x \eta|^2)} \Big[|\nabla_x \xi|^2 - (G(\eta)[\xi])^2 \\ -2\left(\nabla_x \xi \cdot \nabla_x \eta\right) G(\eta)[\xi] + |\nabla_x \xi|^2 \, |\nabla_x \eta|^2 - (\nabla_x \xi \cdot \nabla_x \eta)^2 \Big] . \tag{2.4b}$$

Any analysis of the Euler equations, in particular the dynamic stability of traveling waves, can be performed equivalently on the surface equations (2.4).

Before proceeding, we recall that while the dependence of the DNO upon the Dirichlet data, ξ, is linear, the g dependence is genuinely non-linear. In particular, this dependence is parametrically analytic [35] (see also [37, 15, 38]) which implies the strong convergence (see Theorem 3.2) of the following expansion:

$$G(g)[\xi] = G(\varepsilon f)[\xi] = \sum_{n=0}^{\infty} G_n(f)[\xi]\varepsilon^n, \tag{2.5}$$

for $g(x) = \varepsilon f(x)$ sufficiently small. Using this expansion, the action of the DNO can be approximated by the truncated Taylor series

$$G^N(g)[\xi] := \sum_{n=0}^{N} G_n(f)[\xi]\varepsilon^n,$$

a method which has been used with great success in a number of numerical simulations [7, 39, 31, 32, 5, 13, 14]. Our purpose is to justify a similar expansion for the first variation of the DNO for use in spectral stability simulations.

2.1. **Spectral Stability.** To motivate our subsequent theoretical developments, we briefly review the spectral stability analysis of evolution equations (see, e.g., Deconinck & Kutz [8] for a full description and a complete list of references). Consider the generic dynamical system

$$\partial_t u = F(u), \tag{2.6}$$

which possesses the solution $u(x,t) = \bar{u}(x)$; for instance F could be the right–hand side of (2.4) modified to follow solutions in a reference frame moving with speed c, and $\bar{u}$ could be a periodic Stokes wave [33]. To decide upon the dynamic stability of $\bar{u}$, one adds a (small) perturbation

$$u(x,t) = \bar{u}(x) + \delta\tilde{u}(x,t), \quad \delta \ll 1,$$

and then studies the evolution of $\tilde{u}$. It is not difficult to show that $\tilde{u}$ satisfies the evolution equation

$$\partial_t \tilde{u} = \delta_u F(\bar{u})[\tilde{u}] + \mathcal{O}(\delta), \tag{2.7}$$

where $\delta_u F$ is the first variation of the right–hand side, F, evaluated at the steady solution $\bar{u}$. At this point, we can now see how the first variation of the DNO will play a role in stability calculations for traveling water waves as it appears explicitly in the variation of the right–hand side of (2.4). If we ignore the order δ correction our analysis becomes a *linear* stability analysis, and if we further assume the separable form

$$\tilde{u}(x,t) = e^{\lambda t} w(x),$$

then, ignoring $\mathcal{O}(\delta)$ terms, (2.7) becomes

$$\lambda w = \delta_u F(\bar{u})[w] =: \mathcal{A}(x)[w]. \tag{2.8}$$

The study of solutions (λ, w) of this equation constitutes a *spectral* stability analysis. The solutions of interest to us are the periodic traveling waves of the Euler equations, so we point out that if the solutions $\bar{u}(x)$ are periodic (with respect to a lattice Γ), then the operator $\mathcal{A}(x)$ will also inherit this periodicity.

The final specification we make for our spectral stability problem, (2.8), is the boundary conditions which w must satisfy. For guidance, we follow the "Generalized Principle of Reduced Instability" developed by Mielke [24], which is essentially the Floquet theory of differential equations with periodic coefficients [8]. This method distills the general setting of L^2 perturbations to the study of the "Bloch waves," e.g.

$$w(x) = e^{ip\cdot x} W(x),$$

where $W(x)$ is periodic with respect to Γ, the period lattice of the linear operator $\mathcal{A}(x)$. The theory shows that the full L^2 spectral stability problem can be decided by simply considering Bloch waves with $p \in P(\Gamma')$, the fundamental cell of wavenumbers (e.g., if $\Gamma = (2\pi)\mathbf{Z}$, then $\Gamma' = \mathbf{Z}$, and $P(\Gamma') = [0,1]$). Thus we are left with the spectral problem [24]

$$\mathcal{A}_p[W] = \lambda W,$$

c.f. (2.8), where $\mathcal{A}_p$ is the "Bloch operator"

$$\mathcal{A}_p[W] := e^{-ip\cdot x} \mathcal{A}[e^{ip\cdot x} W].$$

The crucial spectral identity (see [24], Theorems 2.1 and A.4) is:

$$L^2\text{–spec}(\mathcal{A}) = L^2_{lu}\text{–spec}(\mathcal{A}) = \text{closure}\Big(\bigcup_{p\in P(\Gamma')} \text{spec}(\mathcal{A}_p) \Big), \tag{2.9}$$

where L^2_{lu} is the space of uniformly local L^2 functions. Thus, we can obtain information about stability with respect to *all* of these perturbations by simply considering periodic perturbations $W(x)$ and $\mathcal{A}_p$ with $p \in P(\Gamma')$ appearing as a parameter [24].

For the current theoretical developments, this Bloch analysis is equivalent to considering the linear operator $\mathcal{A}$ acting on "Bloch periodic" (quasiperiodic) functions $w(x)$ which satisfy the "Bloch boundary conditions":

$$w(x+\gamma) = e^{ip\cdot\gamma} w(x), \quad \forall\, \gamma \in \Gamma.$$

Notice that, if p is a rational number then, these functions will be periodic with respect to the lattice Γ.

3. Transformed Field Expansions

As we mentioned earlier, for many systems of partial differential equations of physical interest (including water waves), the problem domain is a small departure from a separable geometry and quantities of interest (such as DNO) depend analytically upon these deformations. A useful technique for establishing such analyticity results is the method of "Transformed Field Expansions" (TFE) [35]. The TFE strategy begins by selecting a change of the independent variables which maps the problem domain to the simpler geometry. This has the effect of not only simplifying the problem coordinates, but it also moves the perturbation quantity (the domain shape) from the geometry to the right–hand side of the governing equations. At this point, one expands the solution in a Taylor series in the perturbation variable which can typically be shown to converge strongly.

3.1. **Change of Variables.** To begin, we consider the change of variables

$$x' = x, \qquad y' = h\Big(\frac{y-g(x)}{h+g(x)}\Big), \tag{3.1}$$

which transforms the domain $S_{h,g}$ to $S_{h,0}$. The differential operators transform by:

$$\begin{aligned}(h+g(x))\nabla_x &= (h+g(x'))\nabla_{x'} - (h+y')(\nabla_{x'}g(x'))\partial_{y'}\\ (h+g(x))\mathrm{div}_x &= (h+g(x'))\mathrm{div}_{x'} - (h+y')(\nabla_{x'}g(x'))\cdot\partial_{y'}\\ (h+g(x))\partial_y &= h\partial_{y'},\end{aligned}$$

and the system (2.3) becomes a PDE and boundary conditions for the unknown transformed field

$$u(x',y') := v\Big(x', \frac{y'(h+g(x'))}{h} + g(x')\Big).$$

These equations are, upon dropping primes,

$$\Delta u = F(x,y;g,u) \quad \text{in } S_{h,0} \tag{3.2a}$$

$$u(x,0) = \xi(x) \tag{3.2b}$$

$$\partial_y u(x,-h) = 0 \tag{3.2c}$$

$$u(x+\gamma,y) = u(x,y) \quad \forall\ \gamma \in \Gamma, \tag{3.2d}$$

where

$$F = \operatorname{div}_x [F_x] + \partial_y F_y + F_h, \tag{3.2e}$$

and the x–derivative, y–derivative, and homogeneous parts of F are given by:

$$F_x = -\frac{2}{h} g \nabla_x u - \frac{1}{h^2} g^2 \nabla_x u + \frac{h+y}{h} \nabla_x g \partial_y u + \frac{h+y}{h^2} g \nabla_x g \partial_y u, \tag{3.2f}$$

$$F_y = \frac{h+y}{h} \nabla_x g \cdot \nabla_x u + \frac{h+y}{h^2} g \nabla_x g \cdot \nabla_x u - \frac{(h+y)^2}{h^2} \left|\nabla_x g\right|^2 \partial_y u, \tag{3.2g}$$

and

$$F_h = \frac{1}{h} \nabla_x g \cdot \nabla_x u + \frac{1}{h^2} g \nabla_x g \cdot \nabla_x u - \frac{h+y}{h^2} \left|\nabla_x g\right|^2 \partial_y u. \tag{3.2h}$$

Additionally, the DNO transforms to

$$G(g)[\xi] = \partial_y u(x,0) + H(x;g,u), \tag{3.3a}$$

where

$$H = -\frac{1}{h} g G(g)[\xi] - \nabla_x g \cdot \nabla_x u(x,0) - \frac{1}{h} g \nabla_x g \cdot \nabla_x u(x,0) + \left|\nabla_x g\right|^2 \partial_y u(x,0). \tag{3.3b}$$

The reason for the particular gathering of terms in these equations is that both F and H are $\mathcal{O}(g)$.

3.2. **Analyticity.** Now that we have implemented the "transformation" in the TFE method, all that remains is to expand the field, u, and the DNO, G, in a power series in a parameter which measures the boundary deformation, e.g. ε in the relationship $g(x) = \varepsilon f(x)$. Using this approach, several authors (see, e.g., [35, 15]) have shown that if ε is small and f is smooth, then the expansion

$$u(x,y,\varepsilon) = \sum_{n=0}^{\infty} u_n(x,y) \varepsilon^n, \tag{3.4}$$

converges strongly in an appropriate function space, and each u_n satisfies

$$\Delta u_n = F_n(x,y) \quad \text{in } S_{h,0} \tag{3.5a}$$

$$u_n(x,0) = \delta_{n,0}\ \xi(x) \tag{3.5b}$$

$$\partial_y u_n(x,-h) = 0 \tag{3.5c}$$

$$u_n(x+\gamma, y) = u_n(x,y) \qquad \forall\, \gamma \in \Gamma, \tag{3.5d}$$

where $\delta_{n,m}$ is the Kronecker delta,

$$F_n = \mathrm{div}_x\left[F_{x,n}\right] + \partial_y F_{y,n} + F_{h,n}, \tag{3.5e}$$

$$F_{x,n} = -\frac{2}{h} f \nabla_x u_{n-1} - \frac{1}{h^2} f^2 \nabla_x u_{n-2} + \frac{h+y}{h} \nabla_x f \partial_y u_{n-1} + \frac{h+y}{h^2} f \nabla_x f \partial_y u_{n-2}, \tag{3.5f}$$

$$F_{y,n} = \frac{h+y}{h} \nabla_x f \cdot \nabla_x u_{n-1} + \frac{h+y}{h^2} f \nabla_x f \cdot \nabla_x u_{n-2} - \frac{(h+y)^2}{h^2} \left|\nabla_x f\right|^2 \partial_y u_{n-2}, \tag{3.5g}$$

and

$$F_{h,n} = \frac{1}{h} \nabla_x f \cdot \nabla_x u_{n-1} + \frac{1}{h^2} f \nabla_x f \cdot \nabla_x u_{n-2} - \frac{h+y}{h^2} \left|\nabla_x f\right|^2 \partial_y u_{n-2}. \tag{3.5h}$$

In these formulas, any function with a negative index should be replaced by zero. Under the same hypotheses [35, 15], the expansion (2.5) can be shown to converge strongly, and the G_n can be computed via

$$G_n(f)[\xi] = \partial_y u_n(x,0) + H_n(x), \tag{3.6a}$$

where

$$H_n = -\frac{1}{h} f G_{n-1}(f)[\xi] - \nabla_x f \cdot \nabla_x u_{n-1}(x,0) - \frac{1}{h} f \nabla_x f \cdot \nabla_x u_{n-2}(x,0) + \left|\nabla_x f\right|^2 \partial_y u_{n-2}(x,0). \tag{3.6b}$$

The recursions above can be used *directly* to establish the strong convergence of (3.4) and (2.5). The details are given in [35, 37, 15], but the results are summarized here for use in future sections.

Theorem 3.1. *Given an integer $s \geq 0$, if $f \in C^{s+2}$ and $\xi \in H^{s+3/2}$, then the series* (3.4) *converges strongly. In other words, there exist constants $\tilde{C}_0$ and $\tilde{K}_0$ such that*

$$\|u_n\|_{H^{s+2}} \leq \tilde{K}_0 B_0^n,$$

for any $B_0 > \tilde{C}_0 \left|f\right|_{C^{s+2}}$.

Theorem 3.2. *Given an integer $s \geq 0$, if $f \in C^{s+2}$ and $\xi \in H^{s+3/2}$, then the series* (2.5) *converges strongly as an operator from $H^{s+3/2}$ to $H^{s+1/2}$. In other words, there exist constants C_0 and K_0 such that*

$$\|G_n(f)[\xi]\|_{H^{s+1/2}} \leq K_0 B_0^n,$$

for any $B_0 > C_0 \left|f\right|_{C^{s+2}}$.

Remark 3.3. As was shown in [35, 15], the constants $\tilde{K}_0$ and K_0 depend linearly on $\|\xi\|_{H^{s+3/2}}$. The same realization can be made for subsequent constants $\tilde{K}_j$ and K_j appearing in § 4 & § 5.

Remark 3.4. For convenience of presentation, we suppress the domain dependence of the function spaces where no confusion exists. The surface spaces (for the functions f, ξ, etc.) are defined on the fundamental period cell $P(\Gamma)$, while the volumetric spaces (for y–dependent functions like u) are defined on $P(\Gamma) \times [-h, 0]$.

Remark 3.5. While Theorem 3.1 establishes the parametric analyticity of the transformed field u under rather mild assumptions on the boundary shape, this does not generically extend to the original field v due to the change of variables, (3.1). However, if the regularity assumptions on f are sufficiently strengthened (e.g., f analytic) then the same general conclusions can be realized for v.

4. First Variation

We now proceed to the parametric analyticity of the first variation of the field, u, and the DNO, G, with respect to the boundary deformation g. Our analysis is greatly simplified by the change of variables (3.1) as this geometric perturbation is now included solely as an inhomogeneity in our elliptic PDE. We recall [21] Gateaux's definition of the variation of a functional F with respect to a function φ at φ_0 in the direction ψ as

$$\delta_\varphi F(\varphi_0)\{\psi\} := \lim_{\tau \to 0} \frac{1}{\tau} \left[F(\varphi_0 + \tau\psi) - F(\varphi_0) \right].$$

It is easy to derive equations for the first variations of the field and DNO in the direction w:

$$u^{(1)}(x, y; g)\{w\} := \delta_g u(x, y; g)\{w\}, \qquad G^{(1)}(g)[\xi]\{w\} := \delta_g G(g)[\xi]\{w\}.$$

First, the first variation of the field, $u^{(1)}$, satisfies the following elliptic problem:

$$\Delta u^{(1)} = F^{(1)}(x, y) \qquad \text{in } S_{h,0} \tag{4.1a}$$

$$u^{(1)}(x, 0) = 0 \tag{4.1b}$$

$$\partial_y u^{(1)}(x, -h) = 0 \tag{4.1c}$$

$$u^{(1)}(x + \gamma, y) = u^{(1)}(x, y) \qquad \forall\, \gamma \in \Gamma, \tag{4.1d}$$

where

$$F^{(1)} = \operatorname{div}_x \left[F_x^{(1)} \right] + \partial_y F_y^{(1)} + F_h^{(1)}, \tag{4.1e}$$

$$F_x^{(1)} = -\frac{2}{h} w \nabla_x u - \frac{2}{h} g \nabla_x u^{(1)} - \frac{2}{h^2} g w \nabla_x u - \frac{1}{h^2} g^2 \nabla_x u^{(1)} + \frac{h+y}{h} \nabla_x w \partial_y u + \frac{h+y}{h} \nabla_x g \partial_y u^{(1)} + \frac{h+y}{h^2} w \nabla_x g \partial_y u + \frac{h+y}{h^2} g \nabla_x w \partial_y u + \frac{h+y}{h^2} g \nabla_x g \partial_y u^{(1)}, \quad (4.1f)$$

$$F_y^{(1)} = \frac{h+y}{h} \nabla_x w \cdot \nabla_x u + \frac{h+y}{h} \nabla_x g \cdot \nabla_x u^{(1)} + \frac{h+y}{h^2} w \nabla_x g \cdot \nabla_x u + \frac{h+y}{h^2} g \nabla_x w \cdot \nabla_x u + \frac{h+y}{h^2} g \nabla_x g \cdot \nabla_x u^{(1)} - \frac{2(h+y)^2}{h^2} \nabla_x w \cdot \nabla_x g \partial_y u - \frac{(h+y)^2}{h^2} \left| \nabla_x g \right|^2 \partial_y u^{(1)}, \quad (4.1g)$$

and

$$F_h^{(1)} = \frac{1}{h} \nabla_x w \cdot \nabla_x u + \frac{1}{h} \nabla_x g \cdot \nabla_x u^{(1)} + \frac{1}{h^2} w \nabla_x g \cdot \nabla_x u + \frac{1}{h^2} g \nabla_x w \cdot \nabla_x u + \frac{1}{h^2} g \nabla_x g \cdot \nabla_x u^{(1)} - \frac{2(h+y)}{h^2} \nabla_x w \cdot \nabla_x g \partial_y u - \frac{h+y}{h^2} \left| \nabla_x g \right|^2 \partial_y u^{(1)}. \quad (4.1h)$$

Next, the variation of the DNO satisfies the formula

$$G^{(1)}(g)[\xi]\{w\} = \partial_y u^{(1)}(x,0) + H^{(1)}(x), \quad (4.2a)$$

where

$$H^{(1)} = -\frac{1}{h} w G(g)[\xi] - \frac{1}{h} g G^{(1)}(g)[\xi]\{w\} - \nabla_x w \cdot \nabla_x u(x,0) - \nabla_x g \cdot \nabla_x u^{(1)}(x,0) - \frac{1}{h} w \nabla_x g \cdot \nabla_x u(x,0) - \frac{1}{h} g \nabla_x w \cdot \nabla_x u(x,0) - \frac{1}{h} g \nabla_x g \cdot \nabla_x u^{(1)}(x,0) + 2 \nabla_x w \cdot \nabla_x g \partial_y u(x,0) + \left| \nabla_x g \right|^2 \partial_y u^{(1)}(x,0). \quad (4.2b)$$

4.1. **Analyticity of the First Variation.** As with the case of the field and DNO, the TFE methodology can be utilized to show that the expansions

$$u^{(1)}(x,y,\varepsilon)\{w\} = \sum_{n=0}^{\infty} u_n^{(1)}(x,y)\{w\}\, \varepsilon^n, \quad G^{(1)}(\eta)[\xi]\{w\} = \sum_{n=0}^{\infty} G_n^{(1)}[\xi]\{w\}\, \varepsilon^n, \quad (4.3)$$

converge strongly; see Theorems 4.2 and 4.3. Given these expansions it is not difficult to see that the $u_n^{(1)}$ must satisfy

$$\Delta u_n^{(1)} = F_n^{(1)}(x,y) \quad \text{in } S_{h,0} \quad (4.4a)$$

$$u_n^{(1)}(x,0) = 0 \tag{4.4b}$$

$$\partial_y u_n^{(1)}(x,-h) = 0 \tag{4.4c}$$

$$u_n^{(1)}(x+\gamma,y) = u_n^{(1)}(x,y) \qquad \forall\ \gamma \in \Gamma, \tag{4.4d}$$

where

$$F_n^{(1)} = \operatorname{div}_x \left[F_{x,n}^{(1)}\right] + \partial_y F_{y,n}^{(1)} + F_{h,n}^{(1)}, \tag{4.4e}$$

and

$$\begin{aligned} F_{x,n}^{(1)} = &-\frac{2}{h} w \nabla_x u_n - \frac{2}{h} f \nabla_x u_{n-1}^{(1)} - \frac{2}{h^2} w f \nabla_x u_{n-1} - \frac{1}{h^2} f^2 \nabla_x u_{n-2}^{(1)} \\ &+ \frac{h+y}{h} \nabla_x w \partial_y u_n + \frac{h+y}{h} \nabla_x f \partial_y u_{n-1}^{(1)} \\ &+ \frac{h+y}{h^2} w \nabla_x f \partial_y u_{n-1} + \frac{h+y}{h^2} f \nabla_x w \partial_y u_{n-1} + \frac{h+y}{h^2} f \nabla_x f \partial_y u_{n-2}^{(1)}, \end{aligned} \tag{4.4f}$$

$$\begin{aligned} F_{y,n}^{(1)} = &\frac{h+y}{h} \nabla_x w \cdot \nabla_x u_n + \frac{h+y}{h} \nabla_x f \cdot \nabla_x u_{n-1}^{(1)} \\ &+ \frac{h+y}{h^2} w \nabla_x f \cdot \nabla_x u_{n-1} + \frac{h+y}{h^2} f \nabla_x w \cdot \nabla_x u_{n-1} + \frac{h+y}{h^2} f \nabla_x f \cdot \nabla_x u_{n-2}^{(1)} \\ &- \frac{2(h+y)^2}{h^2} \nabla_x w \cdot \nabla_x f \partial_y u_{n-1} - \frac{(h+y)^2}{h^2} \left|\nabla_x f\right|^2 \partial_y u_{n-2}^{(1)}, \end{aligned} \tag{4.4g}$$

and

$$\begin{aligned} F_{h,n}^{(1)} = &\frac{1}{h} \nabla_x w \cdot \nabla_x u_n + \frac{1}{h} \nabla_x f \cdot \nabla_x u_{n-1}^{(1)} \\ &+ \frac{1}{h^2} w \nabla_x f \cdot \nabla_x u_{n-1} + \frac{1}{h^2} f \nabla_x w \cdot \nabla_x u_{n-1} + \frac{1}{h^2} f \nabla_x f \cdot \nabla_x u_{n-2}^{(1)} \\ &- \frac{2(h+y)}{h^2} \nabla_x w \cdot \nabla_x f \partial_y u_{n-1} - \frac{h+y}{h^2} \left|\nabla_x f\right|^2 \partial_y u_{n-2}^{(1)}. \end{aligned} \tag{4.4h}$$

The $G_n^{(1)}$ can be computed via

$$G_n^{(1)}(f)[\xi]\{w\} = \partial_y u_n^{(1)}(x,0) + H_n^{(1)}(x), \tag{4.5a}$$

where

$$\begin{aligned} H_n^{(1)} = &-\frac{1}{h} w G_n(f)[\xi] - \frac{1}{h} f G_{n-1}^{(1)}(f)[\xi]\{w\} - \nabla_x w \cdot \nabla_x u_n(x,0) \\ &- \nabla_x f \cdot \nabla_x u_{n-1}^{(1)}(x,0) - \frac{1}{h} w \nabla_x f \cdot \nabla_x u_{n-1}(x,0) \\ &- \frac{1}{h} f \nabla_x w \cdot \nabla_x u_{n-1}(x,0) - \frac{1}{h} f \nabla_x f \cdot \nabla_x u_{n-2}^{(1)}(x,0) \end{aligned}$$

$$+ 2\nabla_x w \cdot \nabla_x f \partial_y u_{n-1}(x,0) + |\nabla_x f|^2 \partial_y u^{(1)}_{n-2}(x,0). \quad (4.5b)$$

The primary result of this section is the parametric analyticity of the first variation of the DNO, $G^{(1)}$, with respect to the boundary variation $g = \varepsilon f$. This can be shown directly from the next result on parametric analyticity of the first variation of the field, $u^{(1)}$. To make this precise, we define the quantities D_1 and $\tilde{D}_1$ which help characterize the disk of convergence of the Taylor series of $G^{(1)}$ and $u^{(1)}$.

Definition 4.1. *For any positive real number B_0 (see Theorems 3.1 and 3.2), and functions $f, w \in C^{s+2}$, let*

$$D_1 := |f|_{C^{s+2}} + B_0 |w|_{C^{s+2}}$$
$$\tilde{D}_1 := |f|^2_{C^{s+2}} + B_0 |f|_{C^{s+2}} |w|_{C^{s+2}}.$$

Theorem 4.2. *Given an integer $s \geq 0$, if $f \in C^{s+2}$, $\xi \in H^{s+3/2}$, and $w \in C^{s+2}$ then the series for $u^{(1)}$ in (4.3) converges strongly. In other words, there exist constants $\tilde{C}_1$ and $\tilde{K}_1$ such that*

$$\left\| u^{(1)}_n \right\|_{H^{s+2}} \leq \tilde{K}_1 B_1^n, \quad (4.6)$$

for any $B_1 > \max\{B_0, 2C_e\tilde{C}_1 D_1, \sqrt{2C_e\tilde{C}_1\tilde{D}_1}\}$. C_e is given in Lemma 4.5 and B_0 is given by Theorem 3.1 which holds with the hypotheses given above.

The parametric analyticity of $G^{(1)}$ now follows.

Theorem 4.3. *Given an integer $s \geq 0$, if $f \in C^{s+2}$, $\xi \in H^{s+3/2}$, and $w \in C^{s+2}$, then the series for $G^{(1)}$ in (4.3) converges strongly as an operator from $H^{s+3/2}$ to $H^{s+1/2}$. In other words, there exist constants C_1 and K_1 such that*

$$\left\| G^{(1)}_n(f)[\xi]\{w\} \right\|_{H^{s+1/2}} \leq K_1 B_1^n, \quad (4.7)$$

for any $B_1 > \max\{B_0, C_1 D_1, C_1\sqrt{\tilde{D}_1}\}$.

A key element in the proof of these results is an "Algebra Property" of the function spaces H^s and C^s [1, 35].

Lemma 4.4. *For any integer $s \geq 0$ and any $\varepsilon > 0$, if $f \in C^s(P(\Gamma))$, $u \in H^s(P(\Gamma) \times [-h, 0])$, $g \in C^{s+1/2+\varepsilon}(P(\Gamma))$, and $\mu \in H^{s+1/2}(P(\Gamma))$, then*

$$\|fu\|_{H^s(P(\Gamma)\times[-h,0])} \leq M |f|_{C^s(P(\Gamma))} \|u\|_{H^s(P(\Gamma)\times[-h,0])}$$
$$\|g\mu\|_{H^{s+1/2}(P(\Gamma))} \leq M |g|_{C^{s+1/2+\varepsilon}(P(\Gamma))} \|\mu\|_{H^{s+1/2}(P(\Gamma))},$$

where M is a constant depending only on s and the dimension d.

We note that in [34] a significant variation on this Lemma was utilized to allow for Dirichlet data in a weak Sobolev space, $H^{-1/2}$, at the expense of additional smoothness on the boundary deformation f; this could also be pursued in the present context. Another invaluable tool in our analysis is the following well–known "Elliptic Estimate" [17, 10].

Lemma 4.5. *For any integer $s \geq 0$ there exists a constant C_e such that for any $F \in H^s$, $\xi \in H^{s+3/2}$, the solution $W \in H^{s+2}$ of*

$$\begin{aligned}
&\Delta W(x,y) = F(x,y) && \text{in } S_{h,0}\\
&W(x,0) = \xi(x)\\
&\partial_y W(x,-h) = 0\\
&W(x+\gamma,y) = W(x,y) && \forall\ \gamma \in \Gamma,
\end{aligned}$$

satisfies

$$\|W\|_{H^{s+2}} \leq C_e \left\{ \|F\|_{H^s} + \|\xi\|_{H^{s+3/2}} \right\}.$$

Our proof is inductive in nature relying upon the relation (4.4) for $u_n^{(1)}$; therefore a recursive estimate on the right–hand side $F_n^{(1)}$ is essential.

Lemma 4.6. *Let $s \geq 0$ be an integer and let $f, w \in C^{s+2}$. Assume*

$$\|u_n\|_{H^{s+2}} \leq \tilde{K}_0 B_0^n \qquad \forall n \tag{4.8a}$$

$$\left\| u_n^{(1)} \right\|_{H^{s+2}} \leq \tilde{K}_1 B_1^n \qquad n < N, \tag{4.8b}$$

and constants $\tilde{K}_0, \tilde{K}_1, B_0, B_1 > 0$. Then if $B_1 > B_0$, $\tilde{K}_1 > \tilde{K}_0$, there exists a constant $\tilde{C}_1$ such that

$$\left\| F_N^{(1)} \right\|_{H^s} \leq \tilde{C}_1 \tilde{K}_1 \left\{ D_1 B_1^{N-1} + \tilde{D}_1 B_1^{N-2} \right\}.$$

Proof. We recall that $F_N^{(1)} = \operatorname{div}_x \left[F_{x,N}^{(1)} \right] + \partial_y F_{y,N}^{(1)} + F_{h,N}^{(1)}$ and focus our attention upon $F_{x,N}^{(1)}$ as the other terms can be handled in a similar fashion. Using Lemma 4.4,

$$\begin{aligned}
&\left\| \operatorname{div}_x \left[F_{x,N}^{(1)} \right] \right\|_{H^s} \leq \left\| F_{x,N}^{(1)} \right\|_{H^{s+1}}\\
&\leq \frac{2M}{h} |w|_{C^{s+1}} \|u_N\|_{H^{s+2}} + \frac{2M}{h} |f|_{C^{s+1}} \left\| u_{N-1}^{(1)} \right\|_{H^{s+2}}\\
&\quad + \frac{2M^2}{h^2} |w|_{C^{s+1}} |f|_{C^{s+1}} \|u_{N-1}\|_{H^{s+2}} + \frac{M^2}{h^2} |f|_{C^{s+1}}^2 \left\| u_{N-2}^{(1)} \right\|_{H^{s+2}}\\
&\quad + \frac{YM}{h} |w|_{C^{s+2}} \|u_N\|_{H^{s+2}} + \frac{YM}{h} |f|_{C^{s+2}} \left\| u_{N-1}^{(1)} \right\|_{H^{s+2}}
\end{aligned}$$

$$+ \frac{YM^2}{h^2} |w|_{C^{s+1}} |f|_{C^{s+2}} \|u_{N-1}\|_{H^{s+2}} + \frac{YM^2}{h^2} |f|_{C^{s+1}} |w|_{C^{s+2}} \|u_{N-1}\|_{H^{s+2}}$$
$$+ \frac{YM^2}{h^2} |f|_{C^{s+1}} |f|_{C^{s+2}} \left\| u^{(1)}_{N-2} \right\|_{H^{s+2}},$$

where we have used

$$\|(h+y)v\|_{H^s} \leq Y \|v\|_{H^s},$$

for some constant $Y = Y(s,d)$. By using $|f|_{C^{s+1}} \leq |f|_{C^{s+2}}$, $|w|_{C^{s+1}} \leq |w|_{C^{s+2}}$, and the inductive hypotheses (4.8), it is easy to show that

$$\left\| \mathrm{div}_x \left[F^{(1)}_{x,N} \right] \right\|_{H^s} \leq \frac{(2+Y)M}{h} |w|_{C^{s+2}} \tilde{K}_0 B_0^N$$
$$+ \frac{2M^2(1+Y)}{h^2} |w|_{C^{s+2}} |f|_{C^{s+2}} \tilde{K}_0 B_0^{N-1}$$
$$+ \frac{(2+Y)M}{h} |f|_{C^{s+2}} \tilde{K}_1 B_1^{N-1} + \frac{M^2(1+Y)}{h^2} |f|^2_{C^{s+2}} \tilde{K}_1 B_1^{N-2}$$
$$\leq \tilde{K}_1 \Big(\frac{(2+Y)M}{h} |w|_{C^{s+2}} B_0 B_1^{N-1} + \frac{2M^2(1+Y)}{h^2} |w|_{C^{s+2}} |f|_{C^{s+2}} B_0 B_1^{N-2}$$
$$+ \frac{(2+Y)M}{h} |f|_{C^{s+2}} B_1^{N-1} + \frac{M^2(1+Y)}{h^2} |f|^2_{C^{s+2}} B_1^{N-2} \Big)$$
$$\leq \tilde{C}_1 \tilde{K}_1 \Big((|f|_{C^{s+2}} + |w|_{C^{s+2}} B_0) B_1^{N-1} + (|f|^2_{C^{s+2}} + |f|_{C^{s+2}} |w|_{C^{s+2}} B_0) B_1^{N-2} \Big),$$

provided that $B_0 < B_1$, $\tilde{K}_0 < \tilde{K}_1$, and $\tilde{C}_1$ is chosen appropriately; the proof is now complete. □

We are now in a position to prove the parametric analyticity of the first variation of the field, $u^{(1)}$.

Proof. (Theorem 4.2) We utilize an inductive method, therefore at order $n = 0$ we recall that we must solve (4.4) with

$$F^{(1)}_{x,0} = -\frac{2}{h} w \nabla_x u_0 + \frac{h+y}{h} \nabla_x w \partial_y u_0,$$
$$F^{(1)}_{y,0} = \frac{h+y}{h} \nabla_x w \cdot \nabla_x u_0,$$
$$F^{(1)}_{h,0} = \frac{1}{h} \nabla_x w \cdot \nabla_x u_0.$$

Using Lemma 4.5 we find that

$$\left\| u^{(1)}_0 \right\|_{H^{s+2}} \leq C_e \left\{ \frac{2M}{h} |w|_{C^{s+1}} \|u_0\|_{H^{s+2}} + \frac{YM}{h} |w|_{C^{s+2}} \|u_0\|_{H^{s+2}} \right.$$

$$+\frac{YM}{h}|w|_{C^{s+2}}\|u_0\|_{H^{s+2}}+\frac{M}{h}|w|_{C^{s+1}}\|u_0\|_{H^{s+1}}\Bigg\}$$
$$\le \frac{C_eM}{h}(3+2Y)|w|_{C^{s+2}}\tilde{K}_0B_0 \ .$$

We set

$$\tilde{K}_1=\max\left\{\tilde{K}_0,\frac{C_eM}{h}(3+2Y)|w|_{C^{s+2}}\tilde{K}_0B_0\right\},$$

and the case $n=0$ is established. We now assume (4.6) for all $n<N$ and use (4.4) and Lemma 4.5 to realize

$$\left\|u_N^{(1)}\right\|_{H^{s+2}}\le C_e\left\|F_N^{(1)}\right\|_{H^s}.$$

Since the u_n satisfy the estimate of Theorem 3.1, we can use Lemma 4.6 to imply that

$$\left\|u_N^{(1)}\right\|_{H^{s+2}}\le C_e\tilde{C}_1\tilde{K}_1\left\{D_1B_1^{N-1}+\tilde{D}_1B_1^{N-2}\right\}\le \tilde{K}_1B_1^N,$$

if we choose

$$B_1>\max\left\{2C_e\tilde{C}_1D_1,\sqrt{2C_e\tilde{C}_1\tilde{D}_1}\right\}.$$

□

Finally, we can show the parametric analyticity of $G^{(1)}$.

Proof. (Theorem 4.3) Again we work by induction and begin with $G_0^{(1)}$. An important realization to make is that our hypotheses guarantee that Theorem 3.2 holds together with its estimates on G_n. From (4.5a), we see at order zero that

$$G_0^{(1)}[\xi]\{w\}=\partial_y u_0^{(1)}(x,0)-\frac{1}{h}wG_0[\xi]-\nabla_x w\cdot\nabla_x u_0(x,0).$$

We now estimate

$$\begin{aligned}\left\|G_0^{(1)}[\xi]\{w\}\right\|_{H^{s+1/2}}&\le\left\|u_0^{(1)}(x,0)\right\|_{H^{s+3/2}}+\frac{M}{h}|w|_{C^{s+1/2+\varepsilon}}\|G_0[\xi]\|_{H^{s+1/2}}\\&\quad+M|w|_{C^{s+3/2+\varepsilon}}\|u_0(x,0)\|_{H^{s+3/2}}\\&\le\tilde{K}_1+\frac{M}{h}|w|_{C^{s+2}}K_0+M|w|_{C^{s+2}}\tilde{K}_0.\end{aligned}$$

If we set

$$K_1=\tilde{K}_1+\frac{M}{h}|w|_{C^{s+2}}K_0+M|w|_{C^{s+2}}\tilde{K}_0,$$

then the case $n = 0$ is resolved. We now suppose that (4.7) holds for $n < N$ and examine $G_N^{(1)}$:

$$
\begin{aligned}
\left\| G_N^{(1)}(f)[\xi]\{w\} \right\|_{H^{s+\frac{1}{2}}} &\leq \left\| u_N^{(1)}(x,0) \right\|_{H^{s+3/2}} + \frac{M}{h} |w|_{C^{s+1/2+\varepsilon}} \| G_N(f)[\xi] \|_{H^{s+\frac{1}{2}}} \\
&+ \frac{M}{h} |f|_{C^{s+\frac{1}{2}+\varepsilon}} \left\| G_{N-1}^{(1)}(f)[\xi]\{w\} \right\|_{H^{s+\frac{1}{2}}} + M |w|_{C^{s+\frac{3}{2}+\varepsilon}} \| u_N(x,0) \|_{H^{s+\frac{3}{2}}} \\
&+ M |f|_{C^{s+3/2+\varepsilon}} \left\| u_{N-1}^{(1)}(x,0) \right\|_{H^{s+3/2}} \\
&+ \frac{M^2}{h} |w|_{C^{s+1/2+\varepsilon}} |f|_{C^{s+3/2+\varepsilon}} \| u_{N-1}(x,0) \|_{H^{s+3/2}} \\
&+ \frac{M^2}{h} |f|_{C^{s+1/2+\varepsilon}} |w|_{C^{s+3/2+\varepsilon}} \| u_{N-1}(x,0) \|_{H^{s+3/2}} \\
&+ \frac{M^2}{h} |f|_{C^{s+1/2+\varepsilon}} |f|_{C^{s+3/2+\varepsilon}} \left\| u_{N-2}^{(1)}(x,0) \right\|_{H^{s+3/2}} \\
&+ 2M^2 |w|_{C^{s+3/2+\varepsilon}} |f|_{C^{s+3/2+\varepsilon}} \| u_{N-1}(x,0) \|_{H^{s+3/2}} \\
&+ M^2 |f|^2_{C^{s+3/2+\varepsilon}} \left\| u_{N-2}^{(1)}(x,0) \right\|_{H^{s+3/2}}.
\end{aligned}
$$

Using the fact that $B_1 > B_0$,

$$
\begin{aligned}
\left\| G_N^{(1)} \right\|_{H^{s+1/2}} &\leq \tilde{K}_1 B_1^N + M\Big(\frac{K_1}{h} + \tilde{K}_1 \Big) |f|_{C^{s+2}} B_1^{N-1} \\
&\quad + M^2 \tilde{K}_1 \Big(\frac{1}{h^2} + 1 \Big) |f|^2_{C^{s+2}} B_1^{N-2} + M\Big(\frac{K_0}{h} + \tilde{K}_0 \Big) |w|_{C^{s+2}} B_0^N \\
&\quad + 2M^2 \tilde{K}_0 \Big(\frac{1}{h} + 1 \Big) |f|_{C^{s+2}} |w|_{C^{s+2}} B_0^{N-1} \\
&\leq \tilde{K}_1 B_1^N + M\Big[\Big(\frac{K_1}{h} + \tilde{K}_1 \Big) |f|_{C^{s+2}} + \Big(\frac{K_0}{h} + \tilde{K}_0 \Big) (B_0 |w|_{C^{s+2}}) \Big] B_1^{N-1} \\
&\quad + M^2 \Big[\tilde{K}_1 \Big(\frac{1}{h^2} + 1 \Big) |f|^2_{C^{s+2}} + 2\tilde{K}_0 \Big(\frac{1}{h} + 1 \Big) |f|_{C^{s+2}} (B_0 |w|_{C^{s+2}}) \Big] B_1^{N-2}.
\end{aligned}
$$

By the bound $2C_e \tilde{C}_1 D_1 < B_1$ we are done provided K_1 is chosen sufficiently large. □

Remark 4.7. In § 2.1 we explained in great detail that to consider the spectral stability problem with respect to general L^2 perturbations, then "Bloch boundary conditions" must be satisfied. In the preceding analysis we have, somewhat implicitly, enforced *periodic* boundary conditions on the perturbation w. We point out, however, that the entire analysis can be carried out for the more general Bloch conditions in precisely the same manner provided that a generalized version of the elliptic estimate, Lemma 4.5

with these quasi–periodic boundary conditions, is utilized. Such results are straightforward extensions of the existing theory and have been presented in, e.g., [33].

Remark 4.8. To conclude this section, we note an important point regarding the flexibility and utility of the TFE approach. The proofs of Theorems 4.2 and 4.3 used the infrastructure originally devised in [35, 36, 37] to handle Dirichlet data in the classical L^2–based Sobolev spaces. We find this machinery not only the simplest to present, but also appropriate in light of the quasi–periodic functions we must consider for the linear stability analysis we have in mind. However, one of the principal contributions of [15] was to demonstrate how the methods of [35, 36, 37] could be extended to quite general boundary conditions, the Schauder spaces $C^{k+\alpha}$, and domain shapes $g(x)$ in the Lipschitz class, provided that the Dirichlet data resides in somewhat more complicated spaces. Of course, these generalizations could also be made in the current setting resulting in the analogues of Theorems 4.2 and 4.3 for deformations f *and* perturbations w in these Schauder or Lipschitz spaces.

Here, we see how our new results improve upon those of Lannes [19] as a careful study of his Theorem 3.20 shows that for $d = 2$ (Lannes' $d = 1$) f and w must lie in $H^{5/2}$, while for $d = 3$ (Lannes' $d = 2$) these functions must be in the class $H^{7/2}$. Of course Lannes had a different purpose in mind (the Cauchy problem for water waves) and his Newton's iteration required *all* functions to reside in the same Sobolev spaces, necessitating the use of Algebra properties which demand indices greater than $d/2$. The same remarks hold for Theorems 5.3 and 5.4, again improving upon the results of Lannes for higher variations.

5. Higher Variations

Though the impact of higher variations of the DNO on a spectral stability analysis is not immediately apparent, we record in this section parametric analyticity results for these higher derivatives. However, we do restrict ourselves to the case of periodic perturbations as products of these functions appear in the relevant formulas, but the space of Bloch periodic functions is *not* closed under multiplication. At this point, the key role that the transformation (3.1) plays is particularly evident as the proof of the relevant analyticity theorem is no more difficult than that of the first variation case.

To begin, we record a helpful Proposition regarding variations of products which is easily established using induction.

Proposition 5.1. *Suppose that A and B are linear operators and U is a non-linear function of g, then if*

$$R(g) = A[g]U(g), \quad S(g) = A[g]B[g]U(g),$$

and $U^{(k)}$, $R^{(k)}$, and $S^{(k)}$ denote the k–th variations of U, R, and S, respectively, then

$$R^{(m)}\{w\} = A[g]U^{(m)}\{w\} + \sum_{j=1}^{m} A[w_j]U^{(m-1)}\{\tilde{w}_j\} \tag{5.1a}$$

$$\begin{aligned} S^{(m)}[w] = {} & A[g]B[g]U^{(m)}\{w\} + A[g]\sum_{j=1}^{m} B[w_j]U^{(m-1)}\{\tilde{w}_j\} \\ & + B[g]\sum_{j=1}^{m} A[w_j]U^{(m-1)}\{\tilde{w}_j\} \\ & + \sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} A[w_j]B[w_k]U^{(m-2)}\{\tilde{w}_{j,k}\}, \end{aligned} \tag{5.1b}$$

where

$$\begin{aligned} w &= (w_1, \dots, w_m) \\ \tilde{w}_j &= (w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_m) \\ \tilde{w}_{j,k} &= (w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_{k-1}, w_{k+1}, \dots, w_m). \end{aligned}$$

Gateaux's definition [21] of the m–th variation of a functional F with respect to a function φ at φ_0 in the direction $\psi = (\psi_1, \dots, \psi_m)$ is

$$\begin{aligned} \delta_\varphi^m F(\varphi_0)\{\psi\} := \lim_{\tau_m \to 0} \frac{1}{\tau_m} \big[& \delta_\varphi^{m-1} F(\varphi_0 + \tau_m \psi_m)\{\psi_1, \dots \psi_{m-1}\} \\ & - \delta_\varphi^{m-1} F(\varphi_0)\{\psi_1, \dots, \psi_{m-1}\} \big]. \end{aligned}$$

As the DNO and its underlying elliptic BVP (in transformed coordinates) are given in (3.3) and (3.2), it is easy to derive equations for their m–th variations,

$$u^{(m)}(x, y; g)\{w\} := \delta_g^m u(x, y; g)\{w\}, \qquad G^{(m)}(g)[\xi]\{w\} := \delta_g^m G(g)[\xi]\{w\}.$$

First, for the m–th variation of the field, $u^{(m)}$ satisfies the following elliptic problem:

$$\Delta u^{(m)} = F^{(m)}(x, y) \qquad \text{in } S_{h,0} \tag{5.2a}$$

$$u^{(m)}(x, 0) = 0 \tag{5.2b}$$

$$\partial_y u^{(m)}(x,-h) = 0 \tag{5.2c}$$

$$u^{(m)}(x+\gamma,y) = u^{(m)}(x,y) \qquad \forall\ \gamma\in\Gamma, \tag{5.2d}$$

where

$$F^{(m)} = \mathrm{div}_x\left[F_x^{(m)}\right] + \partial_y F_y^{(m)} + F_h^{(m)}. \tag{5.2e}$$

To derive the forms of $F_x^{(m)}$, $F_y^{(m)}$, and $F_h^{(m)}$ we use Proposition 5.1 repeatedly. For instance, the first term in the expression for F_x is

$$R(u) = -\frac{2}{h} g\nabla_x u = -\frac{2}{h} A[g]U(g),$$

where $A = I$ and $U(g) = \nabla_x u(g)$. By Proposition 5.1,

$$\begin{aligned} R^{(m)}[w] &= -\frac{2}{h}\Big(A[g]U^{(m)}[w] + \sum_{j=1}^m A[w_j]U^{(m-1)}[\tilde{w}_j]\Big) \\ &= -\frac{2}{h}\Big(g\nabla_x u^{(m)}\{w\} + \sum_{j=1}^m w_j\nabla_x u^{(m-1)}\{\tilde{w}_j\}\Big). \end{aligned}$$

Proceeding in this way we can derive the following expressions:

$$\begin{aligned} F_x^{(m)} = &-\frac{2}{h}\Big(g\nabla_x u^{(m)}\{w\} + \sum_{j=1}^m w_j\nabla_x u^{(m-1)}\{\tilde{w}_j\}\Big) \\ &- \frac{1}{h^2}\Big(g^2\nabla_x u^{(m)}\{w\} + 2g\sum_{j=1}^m w_j\nabla_x u^{(m-1)}\{\tilde{w}_j\} \\ &+ \sum_{j=1}^m \sum_{k=1,k\neq j}^m w_j w_k \nabla_x u^{(m-2)}\{\tilde{w}_{j,k}\}\Big) \\ &+ \frac{h+y}{h}\Big(\nabla_x g\partial_y u^{(m)}\{w\} + \sum_{j=1}^m \nabla_x w_j \partial_y u^{(m-1)}\{\tilde{w}_j\}\Big) \\ &+ \frac{h+y}{h^2}\Big(g\nabla_x g\partial_y u^{(m)}\{w\} + g\sum_{j=1}^m \nabla_x w_j\partial_y u^{(m-1)}\{\tilde{w}_j\} \\ &+ \nabla_x g\sum_{j=1}^m w_j\partial_y u^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^m\sum_{k=1,k\neq j}^m w_j\nabla_x w_k\partial_y u^{(m-2)}\{\tilde{w}_{j,k}\}\Big), \end{aligned} \tag{5.2f}$$

$$F_y^{(m)} = \frac{h+y}{h}\Big(\nabla_x g\cdot\nabla_x u^{(m)}\{w\} + \sum_{j=1}^m \nabla_x w_j\cdot\nabla_x u^{(m-1)}\{\tilde{w}_j\}\Big)$$

$$
\begin{aligned}
&+ \frac{h+y}{h^2}\Big(g\nabla_x g \cdot \nabla_x u^{(m)}\{w\} + g\sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}\{\tilde{w}_j\} \\
&+ \nabla_x g \cdot \sum_{j=1}^{m} w_j \nabla_x u^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m} \sum_{k=1, k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u^{(m-2)}\{\tilde{w}_{j,k}\}\Big) \\
&- \frac{(h+y)^2}{h^2}\Big(|\nabla_x g|^2 \partial_y u^{(m)}\{w\} + 2\nabla_x g \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u^{(m-1)}\{\tilde{w}_j\} \\
&+ \sum_{j=1}^{m} \sum_{k=1, k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u^{(m-2)}\{\tilde{w}_{j,k}\}\Big),
\end{aligned}
\tag{5.2g}
$$

and

$$
\begin{aligned}
F_h^{(m)} &= \frac{1}{h}\Big(\nabla_x g \cdot \nabla_x u^{(m)}\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}\{\tilde{w}_j\}\Big) \\
&+ \frac{1}{h^2}\Big(g\nabla_x g \cdot \nabla_x u^{(m)}\{w\} + g\sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}\{\tilde{w}_j\} \\
&+ \nabla_x g \cdot \sum_{j=1}^{m} w_j \cdot \nabla_x u^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m} \sum_{k=1, k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u^{(m-2)}\{\tilde{w}_{j,k}\}\Big) \\
&- \frac{h+y}{h^2}\Big(|\nabla_x g|^2 \partial_y u^{(m)}\{w\} + 2\nabla_x g \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u^{(m-1)}\{\tilde{w}_j\} \\
&+ \sum_{j=1}^{m} \sum_{k=1, k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u^{(m-2)}\{\tilde{w}_{j,k}\}\Big).
\end{aligned}
\tag{5.2h}
$$

Now, the variation of the DNO satisfies the formula

$$
G^{(m)}(g)[\xi]\{w\} = \partial_y u^{(m)}(x,0)\{w\} + H^{(m)}(x)\{w\}, \tag{5.3a}
$$

where

$$
\begin{aligned}
H^{(m)} &= -\frac{1}{h}\Big(gG^{(m)}(g)[\xi]\{w\} + \sum_{j=1}^{m} w_j G^{(m-1)}(g)[\xi]\{\tilde{w}_j\}\Big) \\
&- \Big(\nabla_x g \cdot \nabla_x u^{(m)}(x,0)\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}(x,0)\{\tilde{w}_j\}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{h}\Big(g \nabla_x g \cdot \nabla_x u^{(m)}(x,0)\{w\} + g \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}(x,0)\{\tilde{w}_j\} \\
&\qquad + \nabla_x g \cdot \sum_{j=1}^{m} w_j \nabla_x u^{(m-1)}(x,0)\{\tilde{w}_j\} \\
&\qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u^{(m-2)}(x,0)\{\tilde{w}_{j,k}\} \Big) \\
&+ \Big(|\nabla_x g|^2 \partial_y u^{(m)}(x,0)\{w\} + 2 \nabla_x g \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u^{(m-1)}(x,0)\{\tilde{w}_j\} \\
&\qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u^{(m-2)}(x,0)\{\tilde{w}_{j,k}\} \Big). \qquad (5.3b)
\end{aligned}
$$

5.1. **Analyticity of Higher Variations.** Following the development of § 4.1 we can now establish the analyticity of the m–th variations of the field and the DNO. Again, if $g = \varepsilon f$ is sufficiently smooth then both

$$
u^{(m)}(x,y,\varepsilon)\{w\} = \sum_{n=0}^{\infty} u_n^{(m)}(x,y)\{w\}\ \varepsilon^n, \qquad (5.4a)
$$

$$
G^{(m)}(\eta)[\xi]\{w\} = \sum_{n=0}^{\infty} G_n^{(m)}[\xi]\{w\}\ \varepsilon^n, \qquad (5.4b)
$$

will converge strongly; see Theorems 5.3 and 5.4. Given these expansions, it is not difficult to see that the $u_n^{(m)}$ must satisfy

$$
\begin{aligned}
&\Delta u_n^{(m)} = F_n^{(m)}(x,y) && \text{in } S_{h,0} && (5.5a)\\
&u_n^{(m)}(x,0) = 0 && && (5.5b)\\
&\partial_y u_n^{(m)}(x,-h) = 0 && && (5.5c)\\
&u_n^{(m)}(x+\gamma,y) = u_n^{(m)}(x,y) && \forall\ \gamma \in \Gamma, && (5.5d)
\end{aligned}
$$

where

$$
F_n^{(m)} = \mathrm{div}_x \left[F_{x,n}^{(m)} \right] + \partial_y F_{y,n}^{(m)} + F_{h,n}^{(m)}, \qquad (5.5e)
$$

and

$$
F_{x,n}^{(m)} = -\frac{2}{h}\Big(f \nabla_x u_{n-1}^{(m)}\{w\} + \sum_{j=1}^{m} w_j \nabla_x u_n^{(m-1)}\{\tilde{w}_j\} \Big)
$$

$$
\begin{aligned}
&- \frac{1}{h^2}\Big(f^2 \nabla_x u_{n-2}^{(m)}\{w\} + 2f \sum_{j=1}^{m} w_j \nabla_x u_{n-1}^{(m-1)}\{\tilde{w}_j\} \\
&\qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j w_k \nabla_x u_n^{(m-2)}\{\tilde{w}_{j,k}\}\Big) \\
&+ \frac{h+y}{h}\Big(\nabla_x f \partial_y u_{n-1}^{(m)}\{w\} + \sum_{j=1}^{m} \nabla_x w_j \partial_y u_n^{(m-1)}\{\tilde{w}_j\}\Big) \\
&+ \frac{h+y}{h^2}\Big(f \nabla_x f \partial_y u_{n-2}^{(m)}\{w\} + f \sum_{j=1}^{m} \nabla_x w_j \partial_y u_{n-1}^{(m-1)}\{\tilde{w}_j\} \\
&+ \nabla_x f \sum_{j=1}^{m} w_j \partial_y u_{n-1}^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \partial_y u_n^{(m-2)}\{\tilde{w}_{j,k}\}\Big),
\end{aligned} \tag{5.5f}
$$

$$
\begin{aligned}
F_{y,n}^{(m)} &= \frac{h+y}{h}\Big(\nabla_x f \cdot \nabla_x u_{n-1}^{(m)}\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_n^{(m-1)}\{\tilde{w}_j\}\Big) \\
&+ \frac{h+y}{h^2}\Big(f \nabla_x f \cdot \nabla_x u_{n-2}^{(m)}\{w\} + f \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_{n-1}^{(m-1)}\{\tilde{w}_j\} \\
&+ \nabla_x f \cdot \sum_{j=1}^{m} w_j \nabla_x u_{n-1}^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_n^{(m-2)}\{\tilde{w}_{j,k}\}\Big) \\
&- \frac{(h+y)^2}{h^2}\Big(|\nabla_x f|^2 \partial_y u_{n-2}^{(m)}\{w\} + 2\nabla_x f \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u_{n-1}^{(m-1)}\{\tilde{w}_j\} \\
&\qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_n^{(m-2)}\{\tilde{w}_{j,k}\}\Big),
\end{aligned} \tag{5.5g}
$$

and

$$
\begin{aligned}
F_{h,n}^{(m)} &= \frac{1}{h}\Big(\nabla_x f \cdot \nabla_x u_{n-1}^{(m)}\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_n^{(m-1)}\{\tilde{w}_j\}\Big) \\
&+ \frac{1}{h^2}\Big(f \nabla_x f \cdot \nabla_x u_{n-2}^{(m)}\{w\} + f \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_{n-1}^{(m-1)}\{\tilde{w}_j\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \nabla_x f \cdot \sum_{j=1}^{m} w_j \cdot \nabla_x u_{n-1}^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_n^{(m-2)}\{\tilde{w}_{j,k}\}\Big) \\
&- \frac{h+y}{h^2}\Big(|\nabla_x f|^2 \, \partial_y u_{n-2}^{(m)}\{w\} + 2\nabla_x f \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u_{n-1}^{(m-1)}\{\tilde{w}_j\} \\
&\qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_n^{(m-2)}\{\tilde{w}_{j,k}\}\Big). \qquad (5.5\text{h})
\end{aligned}
$$

The $G_n^{(m)}$ can be computed via

$$
G_n^{(m)}(f)[\xi]\{w\} = \partial_y u_n^{(m)}(x,0) + H_n^{(m)}(x), \qquad (5.6\text{a})
$$

where

$$
\begin{aligned}
H_n^{(m)} = &-\frac{1}{h}\Big(f G_{n-1}^{(m)}(f)[\xi]\{w\} + \sum_{j=1}^{m} w_j G_n^{(m-1)}(f)[\xi]\{\tilde{w}_j\}\Big) \\
&- \Big(\nabla_x f \cdot \nabla_x u_{n-1}^{(m)}(x,0)\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_n^{(m-1)}(x,0)\{\tilde{w}_j\}\Big) \\
&- \frac{1}{h}\Big(f \nabla_x f \cdot \nabla_x u_{n-2}^{(m)}(x,0)\{w\} + f \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_{n-1}^{(m-1)}(x,0)\{\tilde{w}_j\} \\
&\qquad + \nabla_x f \cdot \sum_{j=1}^{m} w_j \nabla_x u_{n-1}^{(m-1)}(x,0)\{\tilde{w}_j\} \\
&\qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_n^{(m-2)}(x,0)\{\tilde{w}_{j,k}\}\Big) \\
&+ \Big(|\nabla_x f|^2 \, \partial_y u_{n-2}^{(m)}(x,0)\{w\} + 2\nabla_x f \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u_{n-1}^{(m-1)}(x,0)\{\tilde{w}_j\} \\
&\qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_n^{(m-2)}(x,0)\{\tilde{w}_{j,k}\}\Big). \qquad (5.6\text{b})
\end{aligned}
$$

We are now in a position to prove our final results, the parametric analyticity of the m–th variation of the field and DNO with respect to ε. Again, for precision, we define quantities D_m and $\tilde{D}_m$ which quantify the radius of convergence of the Taylor series, (5.4).

Definition 5.2. *For any integer $m \geq 2$, positive real numbers B_{m-1} and B_{m-2}, and functions $f, w_1, \ldots, w_m \in C^{s+2}$, let*

$$D_m := |f|_{C^{s+2}} + B_{m-1} \sum_{j=1}^{m} |w_j|_{C^{s+2}}$$

$$\tilde{D}_m := |f|_{C^{s+2}}^2 + 2B_{m-1} |f|_{C^{s+2}} \sum_{j=1}^{m} |w_j|_{C^{s+2}} + B_{m-2}^2 \sum_{j=1}^{m} \sum_{k=1, k\neq j}^{m} |w_j|_{C^{s+2}} |w_k|_{C^{s+2}} .$$

Theorem 5.3. *Given an integer $s \geq 0$, if*

$$f \in C^{s+2}, \quad \xi \in H^{s+3/2}, \quad w_1, \ldots, w_m \in C^{s+2},$$

and the series for $u^{(p)}$ $(0 \leq p \leq m-1)$ in (5.4) are strongly convergent, then the series for $u^{(m)}$ in (5.4) converges strongly. In other words, there exist constants $\tilde{C}_m$ and $\tilde{K}_m$ such that

$$\left\| u_n^{(m)} \right\|_{H^{s+2}} \leq \tilde{K}_m B_m^n, \tag{5.7}$$

for any

$$B_m > \max \left\{ B_0, \ldots, B_{m-1}, 2C_e \tilde{C}_m D_m, \sqrt{2C_e \tilde{C}_m \tilde{D}_m} \right\},$$

where $B_0, B_1, \ldots, B_{m-1}$ are given by the analyticity of u, $u^{(1)}$, $\ldots$, $u^{(m-1)}$.

The parametric analyticity of $G^{(m)}$ now follows.

Theorem 5.4. *Given an integer $s \geq 0$, if*

$$f \in C^{s+2}, \quad \xi \in H^{s+3/2}, \quad w_1, \ldots, w_m \in C^{s+2},$$

and the series for $G^{(p)}$ $(0 \leq p \leq m-1)$ in (5.4) are strongly convergent, then the series for $G^{(m)}$ in (5.4) converges strongly as an operator from $H^{s+3/2}$ to $H^{s+1/2}$. In other words, there exist constants C_m and K_m such that

$$\left\| G_n^{(m)}(f)[\xi]\{w\} \right\|_{H^{s+1/2}} \leq K_m B_m^n, \tag{5.8}$$

for any

$$B_m > \max \left\{ B_0, \ldots, B_{m-1}, C_m D_m, C_m \sqrt{D_m} \right\}.$$

Remark 5.5. These results would easily lead to an inductive proof for the parametric analyticity of *all* variations of the field and DNO provided one had control over the growth of the B_m as $m \to \infty$. At present, it is not clear whether such a bound can be found so we make no such claim.

Our inductive proof again requires a recursive estimate.

Lemma 5.6. *Let $s \geq 0$ be an integer and let $f, w_1, \dots w_m \in C^{s+2}$. Assume*

$$\begin{aligned} \|u_n\|_{H^{s+2}} &\leq \tilde{K}_0 B_0^n && \forall n \\ \left\| u_n^{(p)} \right\|_{H^{s+2}} &\leq \tilde{K}_p B_p^n && 0 < p < m, \forall n \\ \left\| u_n^{(m)} \right\|_{H^{s+2}} &\leq \tilde{K}_m B_m^n && n < N, \end{aligned}$$

and constants $\tilde{K}_0, \dots \tilde{K}_m, B_0, \dots, B_m > 0$. Then, if

$$B_m > \max\{B_0, \dots, B_{m-1}\}, \quad \tilde{K}_m > \max\{\tilde{K}_0, \dots, \tilde{K}_{m-1}\},$$

there exists a constant $\tilde{C}_m$ such that

$$\left\| F_N^{(m)} \right\|_{H^s} \leq \tilde{C}_m \tilde{K}_m \left\{ D_m B_m^{N-1} + \tilde{D}_m B_m^{N-2} \right\}.$$

Proof. Again, we focus our attention on one term in $F_N^{(m)}$ as the others can be handled in a similar fashion; consider $F_{y,N}^{(m)}$ and recall that since it is $\partial_y F_{y,N}^{(m)}$ which appears in $F_N^{(m)}$ we measure in the H^{s+1} norm.

$$\begin{aligned} \left\| F_{y,N}^{(m)} \right\|_{H^{s+1}} \leq \frac{Y}{h} \Big(& M \left| f \right|_{C^{s+2}} \left\| u_{N-1}^{(m)} \right\|_{H^{s+2}} \\ & + \sum_{j=1}^{m} M \left| w_j \right|_{C^{s+2}} \left\| u_N^{(m-1)} \right\|_{H^{s+2}} \Big) \\ & + \frac{Y}{h^2} \Big(M^2 \left| f \right|_{C^{s+1}} \left| f \right|_{C^{s+2}} \left\| u_{N-2}^{(m)} \right\|_{H^{s+2}} \\ & + M \left| f \right|_{C^{s+1}} \sum_{j=1}^{m} M \left| w_j \right|_{C^{s+2}} \left\| u_{N-1}^{(m-1)} \right\|_{H^{s+2}} \\ & + M \left| f \right|_{C^{s+2}} \sum_{j=1}^{m} M \left| w_j \right|_{C^{s+1}} \left\| u_{N-1}^{(m-1)} \right\|_{H^{s+2}} \\ & + \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} M^2 \left| w_j \right|_{C^{s+1}} \left| w_k \right|_{C^{s+2}} \left\| u_N^{(m-2)} \right\|_{H^{s+2}} \Big) \end{aligned}$$

$$+ \frac{Y^2}{h^2}\Big(M^2 \left|f\right|^2_{C^{s+2}} \left\| u^{(m)}_{N-2} \right\|_{H^{s+2}} + 2M \left|f\right|_{C^{s+2}} \sum_{j=1}^{m} M \left|w_j\right|_{C^{s+2}} \left\| u^{(m-1)}_{N-1} \right\|_{H^{s+2}} + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} M^2 \left|w_j\right|_{C^{s+2}} \left|w_k\right|_{C^{s+2}} \left\| u^{(m-2)}_{N} \right\|_{H^{s+2}} \Big).$$

Using the inductive bounds, we now conclude the following:

$$\begin{aligned} \left\| F^{(m)}_{y,N} \right\|_{H^{s+1}} \leq{}& \frac{MY}{h}\Big(\left|f\right|_{C^{s+2}} \tilde{K}_m B_m^{N-1} + \sum_{j=1}^{m} \left|w_j\right|_{C^{s+2}} \tilde{K}_{m-1} B_{m-1}^{N} \Big) \\ &+ \frac{M^2 Y}{h^2}\Big(\left|f\right|^2_{C^{s+2}} \tilde{K}_m B_m^{N-2} + 2 \left|f\right|_{C^{s+2}} \sum_{j=1}^{m} \left|w_j\right|_{C^{s+2}} \tilde{K}_{m-1} B_{m-1}^{N-1} \\ &\qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \left|w_j\right|_{C^{s+2}} \left|w_k\right|_{C^{s+2}} \tilde{K}_{m-2} B_{m-2}^{N} \Big) \\ &+ \frac{M^2 Y^2}{h^2}\Big(\left|f\right|^2_{C^{s+2}} \tilde{K}_m B_m^{N-2} + 2 \left|f\right|_{C^{s+2}} \sum_{j=1}^{m} \left|w_j\right|_{C^{s+2}} \tilde{K}_{m-1} B_{m-1}^{N-1} \\ &\qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \left|w_j\right|_{C^{s+2}} \left|w_k\right|_{C^{s+2}} \tilde{K}_{m-2} B_{m-2}^{N} \Big). \end{aligned}$$

By rearranging and using

$$B_m > \max\{B_0, \dots, B_{m-1}\}, \quad \tilde{K}_m > \max\{\tilde{K}_0, \dots, \tilde{K}_{m-1}\}$$

we obtain:

$$\begin{aligned} \left\| F^{(m)}_{y,N} \right\|_{H^{s+1}} \leq{}& \frac{MY}{h}\tilde{K}_m\Big(\left|f\right|_{C^{s+2}} + \sum_{j=1}^{m} \left|w_j\right|_{C^{s+2}} B_{m-1} \Big) B_m^{N-1} \\ &+ \frac{M^2 Y}{h^2}\tilde{K}_m\Big(\left|f\right|^2_{C^{s+2}} + 2 \left|f\right|_{C^{s+2}} \sum_{j=1}^{m} \left|w_j\right|_{C^{s+2}} B_{m-1} \\ &\qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \left|w_j\right|_{C^{s+2}} \left|w_k\right|_{C^{s+2}} B_{m-2}^{2} \Big) B_m^{N-2} \\ &+ \frac{M^2 Y^2}{h^2}\tilde{K}_m\Big(\left|f\right|^2_{C^{s+2}} + 2 \left|f\right|_{C^{s+2}} \sum_{j=1}^{m} \left|w_j\right|_{C^{s+2}} B_{m-1} \end{aligned}$$

$$+\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}\left|w_j\right|_{C^{s+2}}\left|w_k\right|_{C^{s+2}}B_{m-2}^2\Big)B_m^{N-2},$$

and we are done if $\tilde{C}_m$ is chosen appropriately. □

We are now in a position to prove the parametric analyticity of the m–th variation of the field, $u^{(m)}$.

Proof. (Theorem 5.3) We utilize an induction in n; at order $n=0$ we recall that we must solve (5.5) with

$$\begin{aligned}
F_{x,0}^{(m)} &= -\frac{2}{h}\sum_{j=1}^{m} w_j \nabla_x u_0^{(m-1)}\{\tilde{w}_j\} - \frac{1}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} w_j w_k \nabla_x u_0^{(m-2)}\{\tilde{w}_{j,k}\} \\
&\quad + \frac{h+y}{h}\sum_{j=1}^{m} \nabla_x w_j \partial_y u_0^{(m-1)}\{\tilde{w}_j\} \\
&\quad + \frac{h+y}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \partial_y u_0^{(m-2)}\{\tilde{w}_{j,k}\},
\end{aligned}$$

$$\begin{aligned}
F_{y,0}^{(m)} &= \frac{h+y}{h}\sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_0^{(m-1)}\{\tilde{w}_j\} \\
&\quad + \frac{h+y}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_0^{(m-2)}\{\tilde{w}_{j,k}\} \\
&\quad - \frac{(h+y)^2}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_0^{(m-2)}\{\tilde{w}_{j,k}\},
\end{aligned}$$

$$\begin{aligned}
F_{h,0}^{(m)} &= \frac{1}{h}\sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_0^{(m-1)}\{\tilde{w}_j\} \\
&\quad + \frac{1}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_0^{(m-2)}\{\tilde{w}_{j,k}\} \\
&\quad - \frac{h+y}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_0^{(m-2)}\{\tilde{w}_{j,k}\}.
\end{aligned}$$

Using Lemmas 4.4 & 4.5 we find that

$$
\begin{aligned}
\left\|u_0^{(1)}\right\|_{H^{s+2}} \leq C_e\Big\{ & \frac{2M}{h}\sum_{j=1}^{m}|w_j|_{C^{s+1}}\left\|u_0^{(m-1)}\right\|_{H^{s+2}} \\
& + \frac{M^2}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+1}}|w_k|_{C^{s+1}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}} \\
& + 2\frac{MY}{h}\sum_{j=1}^{m}|w_j|_{C^{s+2}}\left\|u_0^{(m-1)}\right\|_{H^{s+2}} \\
& + 2\frac{M^2Y}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+1}}|w_k|_{C^{s+2}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}} \\
& + \frac{M^2Y^2}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+2}}|w_k|_{C^{s+2}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}} \\
& + \frac{M}{h}\sum_{j=1}^{m}|w_j|_{C^{s+2}}\left\|u_0^{(m-1)}\right\|_{H^{s+2}} \\
& + \frac{M^2}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+1}}|w_k|_{C^{s+2}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}} \\
& + \frac{M^2Y}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+2}}|w_k|_{C^{s+2}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}}\Big\}.
\end{aligned}
$$

We set

$$
\begin{aligned}
\tilde{K}_m = {} & (3+2Y)\frac{M}{h}\sum_{j=1}^{m}|w_j|_{C^{s+2}}\left\|u_0^{(m-1)}\right\|_{H^{s+2}} \\
& + (Y^2+3Y+2)\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+2}}|w_k|_{C^{s+2}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}},
\end{aligned}
$$

and the case $n = 0$ is established. We now assume (5.7) for all $n < N$ and use (5.5) and Lemma 4.5 to realize

$$
\left\|u_N^{(m)}\right\|_{H^{s+2}} \leq C_e\left\|F_N^{(m)}\right\|_{H^s}.
$$

By our hypotheses on the analyticity of u, $u^{(1)}$, $\dots u^{(m-1)}$, Lemma 5.6 holds which we now use to imply that

$$
\left\|u_N^{(m)}\right\|_{H^{s+2}} \leq C_e\tilde{C}_m\tilde{K}_m\left\{D_mB_m^{N-1}+\tilde{D}_mB_m^{N-2}\right\} \leq \tilde{K}_mB_m^N,
$$

provided we choose $B_m > \max\left\{2C_e\tilde{C}_m D_m, \sqrt{2C_e\tilde{C}_m\tilde{D}_m}\right\}$. □

Finally, we can show the parametric analyticity of $G^{(m)}$.

Proof. (Theorem 5.4) By our hypotheses of the analyticity of G, $G^{(1)}$, ..., $G^{(m-1)}$ we have estimates on the terms G_n, $G_n^{(1)}$, ..., $G_n^{(m-1)}$, which are used later in this proof. We proceed inductively in n and from (5.6a), we see at order zero that

$$\begin{aligned}
G_0^{(m)}[\xi]\{w\} = \partial_y u_0^{(m)}(x,0)\{w\} - \frac{1}{h}\sum_{j=1}^{m} w_j G_0^{(m-1)}(f)[\xi]\{\tilde{w}_j\} \\
- \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_0^{(m-1)}(x,0)\{\tilde{w}_j\} \\
- \frac{1}{h}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_0^{(m-2)}(x,0)\{\tilde{w}_{j,k}\} \\
+ \sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_0^{(m-2)}(x,0)\{\tilde{w}_{j,k}\}.
\end{aligned}$$

We now estimate

$$\begin{aligned}
\left\| G_0^{(m)}[\xi]\{w\} \right\|_{H^{s+1/2}} \leq{} & \left\| u_0^{(m)}(x,0) \right\|_{H^{s+3/2}} \\
& + \frac{M}{h}\sum_{j=1}^{m} |w_j|_{C^{s+1/2+\varepsilon}} \left\| G_0^{(m-1)}(f)[\xi]\{\tilde{w}_j\} \right\|_{H^{s+1/2}} \\
& + M\sum_{j=1}^{m} |w_j|_{C^{s+3/2+\varepsilon}} \left\| u_0^{(m-1)} \right\|_{H^{s+3/2}} \\
& + \frac{M^2}{h}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+1/2+\varepsilon}} |w_k|_{C^{s+3/2+\varepsilon}} \\
& \times \left\| u_0^{(m-2)} \right\|_{H^{s+3/2}} \\
& + M^2\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} |w_j|^2_{C^{s+3/2+\varepsilon}} \left\| u_0^{(m-2)} \right\|_{H^{s+3/2}} \\
\leq{} & \tilde{K}_m + \frac{M}{h}\sum_{j=1}^{m} |w_j|_{C^{s+2}} K_{m-1} + M\sum_{j=1}^{m} |w_j|_{C^{s+2}} \tilde{K}_{m-1}
\end{aligned}$$

$$+\frac{M^2}{h}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+2}}\,|w_k|_{C^{s+2}}\,\tilde{K}_{m-2}$$
$$+M^2\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+2}}\,|w_k|_{C^{s+2}}\,\tilde{K}_{m-2}.$$

If we set

$$\begin{aligned}K_m=\tilde{K}_m+\frac{M}{h}\sum_{j=1}^{m}|w_j|_{C^{s+2}}\,K_{m-1}+M\sum_{j=1}^{m}|w_j|_{C^{s+2}}\,\tilde{K}_{m-1}\\
+\frac{M^2}{h}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+2}}\,|w_k|_{C^{s+2}}\,\tilde{K}_{m-2}\\
+M^2\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+2}}\,|w_k|_{C^{s+2}}\,\tilde{K}_{m-2},\end{aligned}$$

then the case $n=0$ is resolved. We now suppose that (5.8) holds for $n<N$ and examine $G_N^{(m)}$ in $H^{s+1/2}$:

$$\begin{aligned}\left\|G_N^{(m)}\right\|_{H^{s+1/2}}&\le\left\|u_N^{(m)}(x,0)\right\|_{H^{s+3/2}}\\
&+\frac{M}{h}\Big(|f|_{C^{s+1/2+\varepsilon}}\left\|G_{N-1}^{(m)}\right\|_{H^{s+1/2}}+\sum_{j=1}^{m}|w_j|_{C^{s+1/2+\varepsilon}}\left\|G_N^{(m-1)}\right\|_{H^{s+1/2}}\Big)\\
&\qquad+M\Big(|f|_{C^{s+3/2+\varepsilon}}\left\|u_{N-1}^{(m)}(x,0)\right\|_{H^{s+3/2}}\\
&\qquad+\sum_{j=1}^{m}|w_j|_{C^{s+3/2+\varepsilon}}\left\|u_N^{(m-1)}(x,0)\right\|_{H^{s+3/2}}\Big)\\
&\quad+\frac{M^2}{h}\Big(|f|_{C^{s+1/2+\varepsilon}}\,|f|_{C^{s+3/2+\varepsilon}}\left\|u_{N-2}^{(m)}(x,0)\right\|_{H^{s+3/2}}\\
&\quad+|f|_{C^{s+1/2+\varepsilon}}\sum_{j=1}^{m}|w_j|_{C^{s+3/2+\varepsilon}}\left\|u_{N-1}^{(m-1)}(x,0)\right\|_{H^{s+3/2}}\\
&\quad+|f|_{C^{s+3/2+\varepsilon}}\sum_{j=1}^{m}|w_j|_{C^{s+1/2+\varepsilon}}\left\|u_{N-1}^{(m-1)}(x,0)\right\|_{H^{s+3/2}}\\
&+\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+1/2+\varepsilon}}\,|w_k|_{C^{s+3/2+\varepsilon}}\left\|u_N^{(m-2)}(x,0)\right\|_{H^{s+3/2}}\Big)\end{aligned}$$

$$+ M^2\Big(|f|^2_{C^{s+3/2+\varepsilon}} \left\| u^{(m)}_{N-2}(x,0) \right\|_{H^{s+3/2}}$$
$$+ 2\,|f|_{C^{s+3/2+\varepsilon}} \sum_{j=1}^{m} |w_j|_{C^{s+3/2+\varepsilon}} \left\| u^{(m-1)}_{N-1}(x,0) \right\|_{H^{s+3/2}}$$
$$+ \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+3/2+\varepsilon}} \, |w_k|_{C^{s+3/2+\varepsilon}} \left\| u^{(m-2)}_{N}(x,0) \right\|_{H^{s+3/2}} \Big).$$

Now,

$$\left\| G^{(m)}_N \right\|_{H^{s+1/2}} \leq \tilde{K}_m B^N_m + \frac{M}{h}\Big(|f|_{C^{s+2}} \, K_m B^{N-1}_m + \sum_{j=1}^{m} |w_j|_{C^{s+2}} \, K_{m-1} B^N_{m-1} \Big)$$
$$+ M\Big(|f|_{C^{s+2}} \, \tilde{K}_m B^{N-1}_m + \sum_{j=1}^{m} |w_j|_{C^{s+2}} \, \tilde{K}_{m-1} B^N_{m-1} \Big)$$
$$+ \frac{M^2}{h}\Big(|f|^2_{C^{s+2}} \, \tilde{K}_m B^{N-2}_m + 2\,|f|_{C^{s+2}} \sum_{j=1}^{m} |w_j|_{C^{s+2}} \, \tilde{K}_{m-1} B^{N-1}_{m-1}$$
$$+ \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+2}} \, |w_k|_{C^{s+2}} \, \tilde{K}_{m-2} B^N_{m-2} \Big)$$
$$+ M^2\Big(|f|^2_{C^{s+2}} \, \tilde{K}_m B^{N-2}_m + 2\,|f|_{C^{s+2}} \sum_{j=1}^{m} |w_j|_{C^{s+2}} \, \tilde{K}_{m-1} B^{N-1}_{m-1}$$
$$+ \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+2}} \, |w_k|_{C^{s+2}} \, \tilde{K}_{m-2} B^N_{m-2} \Big),$$

which can be bounded above by $K_m B^N_m$ provided

$$B_m > \max\{B_0, \ldots, B_{m-1}\},$$

and K_m is chosen sufficiently large (see the proof of Theorem 4.3). $\square$

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