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EXISTENCE AND CONTROLLABILITY RESULTS FOR NON-DENSELY DEFINED IMPULSIVE SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Nadjet Abada
Département de Mathématiques, Université Mentouri de Constantine, Algérie

Mouffak Benchohra
Laboratoire de Mathématiques, Université de Sidi Bel Abbès
BP 89, 22000 Sidi Bel Abbès, Algérie

Hadda Hammouche
Département de Mathématiques, Université Kasdi Merbah de Ouargla, Algérie

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Abstract. In this paper, we shall establish sufficient conditions for the existence of integral solutions and extremal integral solutions for some non-densely defined impulsive semilinear functional differential equations in separable Banach spaces. We shall rely on a fixed point theorem for the sum of completely continuous and contraction operators. The question of controllability of these equations are also considered.

1. Introduction

In this paper, we shall be concerned with existence of integral solutions and extremal integral solutions defined on a compact real interval for first order impulsive semilinear functional equations in a separable Banach space. In Section 3, we will consider the following first order impulsive semilinear differential equations of the form:

$$y'(t) - Ay(t) = f(t, y_t), \ a.e. \ t \in J = [0, b], \ t \neq t_k, \ k = 1, \dots, m \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{1.3}$$

where $f : J \times D \to E$ is a given function, $D = \{\psi : [-r, 0] \to E, \psi$ is continuous every where except for a finite number of points s at which $\psi(s^-), \psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in D$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \ldots < t_m <$

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$t_{m+1} = b$, $I_k : E \to E$ $(k = 1, 2, \ldots, m)$, $A : D(A) \subset E \to E$ is a non-densely defined closed linear operator on E, and E a real separable Banach space with norm $|.|$. For any function y defined on $[-r, b]\backslash\{t_1, t_2, ..., t_m\}$ and any $t \in J$, we denote by y_t the element of D defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here, $y_t(\cdot)$ represents the history of the state from $t - r$, up to the present time t.

Functional differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for functional differential equations is the books by Hale [30] and Hale and Verduyn Lunel [32] and Kolmanovskii and Myshkis [37] and the references therein.

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monoghraphs by Bainov and Simeonov [9], Benchohra *et al* [13], Lakshmikantham *et al* [38], and Samoilenko and Perestyuk [44]. In the case where the impulses are absent (i.e $I_k = 0, k = 1, \ldots, m$) and F is a single or multivalued map and A is a densely defined linear operator generating a C_0-semigroup of bounded linear operators, the problem (1.1)–(1.3) has been investigated on compact intervals in, for instance, the monographs by Ahmed [3], Hu and Papageorgiou [34], Kamenskii *et al* [35], Pazy [41] and Wu [46] and the papers of Benchohra and Ntouyas [14, 15, 16] and Cardinali and Rubbioni [22]. Controllability of differential inclusions with different conditions has been considered in the monograph of Benchohra *et al* [11] and in the papers of Balachandran and Dauer [8], Benchohra *et al* [10], Benchohra and Ntouyas [14], Fu [28], and Li and Xue [39] and the references cited therein. During the last decades problems of the form (1.1)–(1.3) in the case where A is densely defined have received much attention. Some existence and controllability results were given in the monograph by Ahmed [4] and Benchohra *et al* [13] and the papers by Ahmed [5, 6], Cardinali and Rubbioni [23], Liu [40] and Rogovchenko [42, 43], and the references therein. However, as indicated in [25], we sometimes need to deal with non-densely defined operators (see Section 3 for some examples of such operators). Recently evolution functional differential equations with non-densely defined linear operators have received much attention (see for instance the papers by Adimy *et al.* [1], Adimy and Ezzinbi [2], Benchohra *et al* [12], Ezzinbi and Liu [27]).

This paper is organized as follows: in Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we give some examples of operators with non-dense domain. In Section 4, we prove existence of integral solutions for problem (1.1)–(1.3). Our approach will be based for the existence of integral solutions, on a fixed point theorem of Burton and Kirk [17] for the sum of a contraction map and a completely continuous map. In Section 5, we shall prove the existence of extremal integral solutions of the problem (1.1)–(1.3), and our approach here is based on the concept of upper and lower solutions combined with a fixed point theorem on ordered Banach spaces established recently by Dhage [26]. In Section 6, we study the impulsive functional differential equations with nonlocal initial conditions, of the form

$$y'(t) - Ay(t) = f(t, y_t), \quad a.e.\ t \in J = [0, b]\,,\ t \neq t_k,\ k = 1, \dots, m \tag{1.4}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m \tag{1.5}$$

$$y(t) + h_t\,(y) = \phi(t), \quad t \in [-r, 0]\,, \tag{1.6}$$

where $h_t : PC([-r, b], \overline{D(A)}) \to \overline{D(A)}$ is given function (see Section 2 for the definition of $PC([-r, b], D(A))$. The non-local condition can be applied in physics with better effect than the classical initial condition $y\,(0) = y_0$. For example, $h_t\,(y)$ may be given by

$$h_t\,(y) = \sum_{i=1}^{p} c_i y(t_i + t), \quad t \in [-r, 0]\,, \tag{1.7}$$

where $c_i, i = 1, ..., p$, are given constants and $0 < t_1 < ... < t_p \leq p$. At time $t = 0$, we have

$$h_0\,(y) = \sum_{i=1}^{p} c_i y\,(t_i)\,.$$

Non-local conditions were initiated by Byszewski [18] (see also [19, 20, 21]) in which we refer for motivation and other references. Section 6 is devoted to an application to control theory. Finally in Section 7 we give an example to illustrate the abstract theory presented in the previous sections.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. For $\psi \in D$ the norm of ψ is defined by $\|\psi\|_D = \sup\{|\psi(\theta)| : \theta \in [-r, 0]\}$. Also $B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

$L^1(J, E)$ denotes the Banach space of measurable functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

Definition 2.1. [7]. *Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:*

(i) $S(0) = 0$;
(ii) $t \to S(t)$ *is strongly continuous;*
(iii) $S(s)S(t) = \int_0^s (S(t + r) - S(r))dr,$ *for all* $t, s \geq 0$.

Definition 2.2. *An integrated semigroup $(S(t))_{t\geq 0}$ is called exponential bounded, if there exists constant $M \geq 0$ and $\omega \in \mathbb{R}$ such that*

$$|S(t)| \leq Me^{\omega t}, \qquad \textit{for } t \geq 0.$$

Moreover, $(S(t))_{t\geq 0}$ is called nondegenerate if $S(t)x = 0$, for all $t \geq 0$, implies $x = 0$.

Definition 2.3. *An operator A is called a generator of an integrated semigroup, if there exists $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$, and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0) = 0$ and*

$$(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt \quad \textit{for all } \lambda > \omega.$$

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [7], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$. In particular, $S'(t)x := \frac{d}{dt}S(t)x$ defines a bounded operator on the set

$$E_1 := \{x \in E : t \to S(t)x \text{ is continuously differentiable on } [0, \infty)\},$$

and $(S'(t))_{t\geq 0}$ is a C_0 semigroup on $\overline{D(A)}$. Here and hereafter, we assume that A satisfies the Hille-Yosida condition, that is, there exists $M > 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and

$$\sup\{(\lambda I - \omega)^n |(\lambda I - A)^{-n}| : \lambda > \omega,\ n \in \mathbb{N}\} \leq M.$$

Note that, since A satisfies the Hille-Yosida condition,

$$\|S'(t)\|_{B(E)} \leq Me^{\omega t},\ t \geq 0,$$

where M and ω are the constants considered in the Hille-Yosida condition (see [36]). Let $(S(t))_{t\geq 0}$, be the integrated semigroup generated by A. Consider the Cauchy Problem

$$y'(t) = Ay(t) + f(t),\ t \in [0,b],\ y(0) = y_0 \in E. \tag{2.1}$$

Then we have the following.

Theorem 2.1. [36]. *Let $f : [0,b] \to E$ be a continuous function. Then for $y_0 \in \overline{D(A)}$, there exists a unique continuous function $y : [0,b] \to E$ of the Cauchy Problem* (2.1) *such that*

(i) $\int_0^t y(s)ds \in D(A)$ *for* $t \in [0,b]$,
(ii) $y(t) = y_0 + A\int_0^t y(s)ds + \int_0^t f(s)ds, \quad t \in [0,b]$,
(iii) $|y(t)| \leq Me^{\omega t}\Big(|y_0| + \int_0^t e^{-\omega s}|f(s)|ds\Big), \quad t \in [0,b]$.

Moreover, y satisfies the variation of constants formula,

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \quad t \geq 0. \tag{2.2}$$

Let $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$. Then ([36]) for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. Also from the Hille-Yosida condition (with $n = 1$) it easy to see that $\lim_{\lambda\to\infty} |B_\lambda x| \leq M|x|$, since

$$|B_\lambda| = |\lambda(\lambda I - A)^{-1}| \leq \frac{M\lambda}{\lambda - \omega}.$$

Thus, $\lim_{\lambda\to\infty} |B_\lambda| \leq M$. Also if y is given by (2.2), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s)ds, \quad t \geq 0. \tag{2.3}$$

In order to define a integral solution of problems (1.1)–(1.3) and (1.4)–(1.6), we shall consider the space $PC\left([-r,b], D(A)\right) = \{y : [-r,b] \to D(A) : y(t)$ is continuous everywhere except for some t_k at which $y\left(t_k^-\right)$ and $y\left(t_k^+\right)$, $k = 1, 2, \ldots, m$ exist and $y\left(t_k^-\right) = y\left(t_k\right)\}$.

For any $t \in [0,b]$ and $y \in PC\left([-r,b], D(A)\right)$, we have $y_t \in D$ and $PC\left([-r,b], D(A)\right)$ is a Banach space with the norm

$$\|y\| = \sup\left\{|y(t)| : t \in [-r,b]\right\}.$$

Definition 2.4. *The map $f : J \times D \to E$ is said to be Carathéodory if:*

(i) *the function* $t \longmapsto f(t,u)$ *is measurable for each* $u \in D$;
(ii) *the function* $u \longmapsto f(t,u)$ *is continuous for almost all* $t \in J$.

3. Examples of Operators with Nondense Domain

In this section, we shall present examples of linear operators with nondense domain satisfying the Hille-Yosida estimate. More details can be found in the paper by Da Prato and Sinestrari [25].

Example 3.1. Let $E = C([0,1],\mathbb{R})$ and the operator $A : D(A) \to E$ defined by $Ay = y'$, where

$$D(A) = \{y \in C^1((0,1),\mathbb{R}) : y(0) = 0\}.$$

Then

$$\overline{D(A)} = \{y \in C((0,1),\mathbb{R}) : y(0) = 0\} \neq E.$$

Example 3.2. Let $E = C([0,1],\mathbb{R})$ and the operator $A : D(A) \to E$ defined by $Ay = y''$, where

$$D(A) = \{y \in C^2((0,1),\mathbb{R}) : y(0) = y(1) = 0\}.$$

Then

$$\overline{D(A)} = \{y \in C((0,1),\mathbb{R}) : y(0) = y(1) = 0\} \neq E.$$

Example 3.3. Let us set for some $\alpha \in (0,1)$

$$E = C_0^\alpha([0,1],\mathbb{R}) = \{y : [0,1] \to \mathbb{R} : y(0) = 0 \text{ and } \sup_{0 \le t < s \le 1} \frac{|y(t) - y(s)|}{|t-s|^\alpha} < \infty\},$$

and the operator $A : D(A) \to E$ defined by $Ay = -y'$, where

$$D(A) = \{y \in C^{1+\alpha}((0,1),\mathbb{R}) : y(0) = y'(0) = 0\}.$$

Then

$$\overline{D(A)} = h_0^\alpha(0,1),\mathbb{R}) = \{y : [0,1] \to \mathbb{R} : \lim_{\delta \to 0} \sup_{0 < |t-s| \le \delta} \frac{|y(t) - y(s)|}{|t-s|^\alpha} = 0\} \neq E.$$

Here, $C^{1+\alpha}([0,1],\mathbb{R}) = \{y : [0,1] \to \mathbb{R} : y' \in C^\alpha([0,1],\mathbb{R})\}$. The elements of $h^\alpha((0,1),\mathbb{R})$ are called little Holder functions and it can be proved that the closure of $C^1((0,1),\mathbb{R})$ in $C^\alpha((0,1),\mathbb{R})$ is $h^\alpha((0,1),\mathbb{R})$ (see [45] Theorem 5.3).

Example 3.4. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with regular boundary* Γ *and define* $E = C(\overline{\Omega},\mathbb{R})$ *and the operator* $A : D(A) \to E$ *defined by* $Ay = \Delta y$, *where*

$$D(A) = \{y \in C(\overline{\Omega},\mathbb{R}) : y = 0 \text{ on } \Gamma;\ \Delta y \in C(\overline{\Omega},\mathbb{R})\}.$$

Here, Δ *is the Laplacian in the sense of distributions on* Ω. *In this case, we have* $\overline{D(A)} = \{y \in C(\overline{\Omega}, \mathbb{R}) : y = 0 \text{ on } \Gamma\} \neq E$.

4. Existence of Integral Solutions

Now, we are able to state and prove our main theorem for the initial value problem (1.1)–(1.3). Before starting and proving this one, we give the definition of its integral solution.

Definition 4.1. *We say that* $y : [-r, T] \to E$ *is an integral solution of* (1.1)–(1.3) *if*

(i) $y(t) = \phi(0) + A\int_0^t y(s)ds + \int_0^t f(s, y_s)ds + \sum_{0<t_k<t} I_k\left(y\left(t_k^-\right)\right), \quad t \in J.$

(ii) $\int_0^t y(s)ds \in D(A)$ *for* $t \in J$, *and* $y(t) = \phi(t)$, $t \in [-r, 0]$.

From the definition it follows that $y(t) \in \overline{D(A)}$, for each $t \geq 0$, in particular $\phi(0) \in \overline{D(A)}$. Moreover, y satisfies the following variation of constants formula:

$$y(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)f(s, y_s)ds + \sum_{0<t_k<t} S'(t - t_k) I_k\left(y\left(t_k^-\right)\right), \quad t \geq 0. \tag{4.1}$$

We notice also that, if y satisfies (4.1), then for $t \geq 0$,

$$y(t) = S'(t)\phi(0) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s, y_s)ds + \sum_{0<t_k<t} S'(t - t_k) I_k\left(y\left(t_k^-\right)\right).$$

Our main result in this section is based upon the following fixed point theorem due to Burton and Kirk [17].

Theorem 4.1. *Let* X *be a Banach space, and* A, $B : X \to X$ *two operators satisfying:*

(i) A *is a contraction, and*
(ii) B *is completely continuous.*

Then either

(a) *the operator equation* $y = A(y) + B(y)$ *has a solution, or*
(b) *the set* $\mathcal{E} = \left\{u \in X : \lambda A\left(\frac{u}{\lambda}\right) + \lambda B(u) = u\right\}$ *is unbounded for* $\lambda \in (0, 1)$.

Let us introduce the following hypotheses:

(H1) A satisfies Hille-Yosida condition;

(H2) There exist constants $d_k > 0$, $k = 1, \ldots, m$ such that for each $y,\ x \in \overline{D(A)}$

$$|I_k(y) - I_k(x)| \le d_k |y - x|,$$

(H3) The function $f : J \times D \to E$ is Carathéodory;

(H4) the operator $S'(t)$ is compact in $\overline{D(A)}$ wherever $t > 0$;

(H5) There exists a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous non-decreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$|f(t,x)| \le p(t)\psi(\|x\|_D), \quad a.e.\ \ t \in J, \quad \text{for all} \ \ x \in D,$$

with $\int_0^b e^{-\omega s} p(s) ds < \infty$,

$$\int_{c_0}^{\infty} \frac{du}{\psi(u)} > c_1 \int_0^b e^{-\omega s} p(s)\, ds. \tag{4.2}$$

where

$$c_0 = \frac{e^{\omega b} M \left(\|\phi\| + \sum_{k=1}^{m} |I_k(0)| \right)}{1 - M e^{\omega b} \sum_{k=1}^{m} d_k}, \tag{4.3}$$

and

$$c_1 = \frac{M e^{\omega b}}{1 - M e^{\omega b} \sum_{k=1}^{m} d_k}. \tag{4.4}$$

Theorem 4.2. *Assume that (H1)-(H5) hold. If*

$$M e^{\omega b} \sum_{k=1}^{m} d_k < 1, \tag{4.5}$$

then the problem (1.1)–(1.3) *has at least one integral solution on* $[-r, b]$.

Proof. Trans form the problem (1.1)–(1.3) into a fixed point problem. Consider the two operators:

$$\mathcal{A}, \mathcal{B} : PC\left([-r,b], \overline{D(A)}\right) \to PC\left([-r,b], \overline{D(A)}\right),$$

defined by

$$\mathcal{A}(y)(t) := \begin{cases} 0, & \text{if } t \in [-r, 0]; \\ \sum\limits_{0 < t_k < t} S'(t - t_k)\, I_k\left(y\left(t_k^-\right)\right), & \text{if } t \in J, \end{cases}$$

and

$$\mathcal{B}(y)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r, 0]; \\ S'(t)\phi(0) \\ +\frac{d}{dt}\int_0^t S(t-s) f(s, y_s)\, ds, & \text{if } t \in J. \end{cases}$$

The problem of finding the solution of problem (1.1)–(1.3) is reduced to finding the solution of the operator equation $\mathcal{A}(y)(t) + B(y)(t) = y(t), t \in [-r, b]$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfies all conditions of Theorem 4.1. For better readability, we break the proof into a sequence of steps.

Step 1: $\mathcal{B}$ is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $PC([-r, b], \overline{D(A)})$. Then for $\omega > 0$ (if $\omega < 0$ one has $e^{\omega t} < 1$)

$$\begin{aligned} |\mathcal{B}(y_n)(t) - \mathcal{B}(y)(t)| &= \left| \frac{d}{dt} \int_0^t S(t-s)[f(s, y_{n_s}) - f(s, y_s)] ds \right| \\ &\leq M e^{\omega b} \int_0^b e^{-\omega s} |f(s, y_{n_s}) - f(s, y_s)|\, ds. \end{aligned}$$

Since $f(s, \cdot)$ is continuous for a.e. $s \in J$, we have by the Lebesgue dominated convergence theorem

$$|\mathcal{B}(y_n)(t) - \mathcal{B}(y)(t)| \to 0 \ as \ n \to \infty.$$

Thus, $\mathcal{B}$ is continuous.

Step 2: $\mathcal{B}$ maps bounded sets into bounded sets in $PC([-r, b], \overline{D(A)})$. It is enough to show that for any $q > 0$ there exists a positive constant l such that for each $y \in B_q = \{y \in PC([-r, b], \overline{D(A)}) : \|y\| \leq q\}$ we have $\|\mathcal{B}(y)\| \leq l$. So choose $y \in B_q$, then we have for each $t \in J$

$$\begin{aligned} |\mathcal{B}(y)(t)| &= \left| S'(t)\phi(0) + \frac{d}{dt} \int_0^t S(t-s) f(s, y_s) ds \right| \\ &\leq M e^{\omega b} |\phi(0)| + M e^{\omega b} \psi(q) \int_0^b e^{-\omega s} p(s)\, ds. \end{aligned}$$

Then we have

$$|\mathcal{B}(y)(t)| \leq M e^{\omega b} \|\phi\| + M e^{\omega b} \psi(q) \int_0^b e^{-\omega s} p(s)\, ds := l.$$

Step 3: $\mathcal{B}$ maps bounded sets into equicontinuous sets of $PC([-r, b], \overline{D(A)})$. We consider B_q as in Step 2 and let $\tau_1, \tau_2 \in J \backslash \{t_1, ..., t_m\}$, $\tau_1 < \tau_2$.Thus if

$\epsilon > 0$ and $\epsilon \leq \tau_1 < \tau_2$, we have

$$|\mathcal{B}(y)(\tau_2) - \mathcal{B}(y)(\tau_1)| \leq |S'(\tau_2)\phi(0) - S'(\tau_1)\phi(0)|$$
$$+ \Big| \lim_{\lambda\to\infty} \int_0^{\tau_1-\epsilon} [S'(\tau_2 - s) - S'(\tau_1 - s)]B_\lambda f(s, y_s)ds\Big|$$
$$+ \Big| \lim_{\lambda\to\infty} \int_{\tau_1-\epsilon}^{\tau_1} [S'(\tau_2 - s) - S'(\tau_1 - s)]B_\lambda f(s, y_s)\, ds\Big|$$
$$+ \Big| \lim_{\lambda\to\infty} \int_{\tau_1}^{\tau_2} S'(\tau_2 - s)B_\lambda f(s, y_s)\, ds\Big|.$$

As $\tau_1 \to \tau_2$ and ϵ become sufficiently small, the right-hand side of the above inequality tends to zero, since $S'(t)$ is a strongly continuous operator and the compactness of $S'(t)$ for $t > 0$ implies the continuity in the uniform operator topology. This proves the equicontinuity for the case where $t \neq t_i, k = 1, 2, \ldots, m + 1$. It remains to examine the equicontinuity at $t = t_i$.

First, we prove equicontinuity at $t = t_i^-$. Fix $\delta_1 > 0$ such that

$$\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset.$$

For $0 < h < \delta_1$ we have

$$|\mathcal{B}(y)(t_i - h) - \mathcal{B}(y)(t_i)| \leq | \left(S'(t_i - h) - S'(t_i)\right) \phi(0)|$$
$$+ \lim_{\lambda\to\infty} \int_0^{t_i-h} \| \left(S'(t_i - h - s) - S'(t_i - s)\right) B_\lambda f(s, y_s)\|ds$$
$$+ Me^{\omega b}\psi(q) \int_{t_i-h}^{t_i} e^{-\omega s}p(s)\, ds;$$

which tends to zero as $h \to 0$. Define

$$\widehat{\mathcal{B}}_0(y)(t) = \mathcal{B}(y)(t), \quad t \in [0, t_1]\,,$$

and

$$\widehat{\mathcal{B}}_i(y)(t) = \begin{cases} \mathcal{B}(y)(t), & \text{if } t \in (t_i, t_{i+1}] \\ \mathcal{B}(y)(t_i^+), & \text{if } t = t_i. \end{cases}$$

Next, we prove equicontinuity at $t = t_i^+$. Fix $\delta_2 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. For $0 < h < \delta_2$, we have

$$|\widehat{\mathcal{B}}(y)(t_i + h) - \widehat{\mathcal{B}}(y)(t_i)| \leq | \left(S'(t_i + h) - S'(t_i)\right) \phi(0)|$$
$$+ \lim_{\lambda\to\infty} \int_0^{t_i} \| \left(S'(t_i + h - s) - S'(t_i - s)\right) B_\lambda f(s, y_s)\|ds$$
$$+ Me^{\omega b}\psi(q) \int_{t_i}^{t_i+h} e^{-\omega s}p(s)\, ds;$$

The right-hand side tends to zero as $h \to 0$. The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ follows from the uniform continuity of ϕ on the interval $[-r, 0]$. As consequence of steps 1 to 3 together with Arzelá-Ascoli theorem it suffices to show that $\mathcal{B}$ maps B_q into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$ we define

$$\mathcal{B}_\epsilon(y)(t) = S'(t)\phi(0) + S'(\epsilon) \lim_{\lambda\to\infty} \int_0^{t-\epsilon} S'(t-s-\epsilon)B_\lambda f(s, y_s)ds.$$

Note

$$\left\{ \lim_{\lambda\to\infty} \int_0^{t-\epsilon} S'(t-s-\epsilon)B_\lambda f(s, y_s)\, ds : y \in B_q \right\},$$

is a bounded set since

$$\Big| \lim_{\lambda\to\infty} \int_0^{t-\epsilon} S'(t-s-\epsilon)B_\lambda f(s, y_s)\, ds \Big| \leq Me^{\omega b}\psi(q) \int_0^{t-\epsilon} e^{-\omega s}p(s)ds.$$

Since $S'(t)$ is a compact operator, the set

$$Y_\epsilon(t) = \{\mathcal{B}_\epsilon(y)(t) : \quad y \in B_q\},$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover, for every $y \in B_q$ we have

$$|\mathcal{B}(y)(t) - \mathcal{B}_\epsilon(y)(t)| \quad \leq \quad Me^{\omega b}\psi(q) \int_t^{t-\epsilon} e^{-\omega s}p(s)ds.$$

Therefore, there are precompact sets arbitrarily close to the set $Y_\epsilon(t) = \{\mathcal{B}_\epsilon(y)(t) : \quad y \in B_q\}$. Hence, the set $Y(t) = \{\mathcal{B}(y)(t) : \quad y \in B_q\}$ is precompact in E. Hence, the operator $\mathcal{B} : PC([-r, b], \overline{D(A)}) \to PC([-r, b], \overline{D(A)})$ is completely continuous.

Step 4: $\mathcal{A}$ is a contraction. Let $x, y \in PC([-r, b], \overline{D(A)})$. Then for $t \in J$

$$|\mathcal{A}(y)(t) - \mathcal{A}(x)(t)| = \Big| \sum_{0<t_k<t} S'\left(t - t_k\right) \left(I_k\left(y\left(t_k^-\right)\right) - I_k\left(x\left(t_k^-\right)\right)\right) \Big|$$

$$\leq Me^{\omega b} \sum_{0<t_k<t} \left|I_k\left(y\left(t_k^-\right)\right) - I_k\left(x\left(t_k^-\right)\right)\right|$$

$$\leq Me^{\omega b} \sum_{k=1}^{m} d_k \left|y\left(t_k^-\right) - x\left(t_k^-\right)\right| \leq Me^{\omega b} \sum_{k=1}^{m} d_k \left\|y - x\right\|.$$

Then

$$\|\mathcal{A}(y) - \mathcal{A}(x)\| \leq Me^{\omega b} \sum_{k=1}^{m} d_k \left\|y - x\right\|,$$

which is a contraction from (4.5).

Step 5: A priori bounds. Now, it remains to show that the set

$$\mathcal{E} = \left\{ y \in PC([-r,b], \overline{D(A)}) : y = \lambda \mathcal{B}(y) + \lambda \mathcal{A}\left(\frac{y}{\lambda}\right) \quad \text{for some} \quad 0 < \lambda < 1 \right\},$$

is bounded. Let $y \in \mathcal{E}$. Then $y = \lambda \mathcal{B}(y) + \lambda \mathcal{A}\left(\frac{y}{\lambda}\right)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$,

$$y(t) = \lambda S'(t)\phi(0) + \lambda \frac{d}{dt} \int_0^t S(t-s) f(s, y_s) ds + \lambda \sum_{0<t_k<t} S'(t-t_k) I_k\Big(\frac{y}{\lambda}(t_k^-)\Big).$$

This implies by (H2), (H5) that, for each $t \in J$, we have

$$\begin{aligned}
|y(t)| &\le \lambda M e^{\omega t}|\phi(0)| + \lambda M e^{\omega t} \int_0^t e^{-\omega s} p(s)\psi(\|y_s\|) ds \\
&\quad + \lambda M e^{\omega t} \sum_{k=1}^m \left| I_k\left(\frac{y}{\lambda}(t_k^-)\right)\right| \\
&\le \lambda M e^{\omega t}\|\phi\| + \lambda M e^{\omega t} \int_0^t e^{-\omega s} p(s)\psi(\|y_s\|) ds \\
&\quad + \lambda M e^{\omega t} \sum_{k=1}^m \left| I_k\left(\frac{y}{\lambda}(t_k^-)\right) - I_k(0)\right| + \lambda M e^{\omega t} \sum_{k=1}^m |I_k(0)| \\
&\le \lambda M e^{\omega t}\Big(\|\phi\| + \sum_{k=1}^m |I_k(0)|\Big) + \lambda M e^{\omega t} \int_0^t e^{-\omega s} p(s)\psi(\|y_s\|) ds \\
&\quad + \lambda M e^{\omega t} \sum_{k=1}^m d_k \left|\frac{y}{\lambda}(t_k^-)\right| \\
&\le c e^{\omega t} + M e^{\omega t}\Big[\int_0^t e^{-\omega s} p(s)\psi(\|y_s\|) ds + \sum_{k=1}^m d_k \left|y(t_k^-)\right|\Big],
\end{aligned}$$

where

$$c = M\Big(\|\phi\| + \sum_{k=1}^m |I_k(0)|\Big). \tag{4.6}$$

Now, we consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : \quad -r \le s \le t\}, \quad 0 \le t \le b.$$

Then $\|y_s\| \le \mu(t)$ for all $t \in J$ and there is a point $t^* \in [-r, t]$ such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, b]$, by the previous inequality and (4.6) we have for

$t \in [0, b]$ (note $t^* \leq t$)

$$\mu(t) \leq ce^{\omega b} + Me^{\omega b} \int_0^t e^{-\omega s} p(s)\psi(\mu(s))ds + Me^{\omega b} \sum_{k=1}^{m} d_k \mu(t).$$

Then

$$\Big(1 - Me^{\omega b} \sum_{k=1}^{m} d_k\Big)\mu(t) \leq ce^{\omega b} + Me^{\omega b} \int_0^t e^{-\omega s} p(s)\psi(\mu(s))ds.$$

Thus, from (4.3) and (4.4) we have

$$\mu(t) \leq c_0 + c_1 \int_0^t e^{-\omega s} p(s)\psi(\mu(s))ds. \tag{4.7}$$

Let us take the right hand-side of (4.7) as $v(t)$. Then we have

$$\mu(t) \leq v(t) \quad \text{for all } \ t \in J, \qquad v(0) = c_0,$$

and

$$v'(t) = c_1 e^{-\omega t} p(t)\psi(\mu(t)), \quad a.e. \ \ t \in J.$$

Using the non-decreasing character of ψ we get

$$v'(t) \leq c_1 e^{-\omega t} p(t)\psi(v(t)), \quad a.e. \ \ t \in J.$$

That is,

$$\frac{v'(t)}{\psi(v(t))} \leq c_1 e^{-\omega t} p\left(t\right), \quad a.e. \ \ t \in J.$$

Integrating from 0 to t we get

$$\int_0^t \frac{v'(s)}{\psi(v(s))} ds \leq c_1 \int_0^t e^{-\omega s} p\left(s\right) ds.$$

By a change of variable and (4.2) we get

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq c_1 \int_0^b e^{-\omega s} p\left(s\right) ds < \int_{c_0}^{\infty} \frac{du}{\psi(u)}.$$

Hence, there exist a constant N such that

$$\mu(t) \leq v(t) \leq N \quad \text{for all } \ t \in J.$$

Now from the definition of μ it follows that

$$\|y\| = \sup_{t\in[-r,b]} |y(t)| \leq \max(\|\phi\|, N) \text{ for all } \ y \in \mathcal{E}.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 4.1 we deduce that $\mathcal{A}+\mathcal{B}$ has a fixed point which is a integral solution of problem (1.1)–(1.3). □

Now, we give a result where f Lipschitz with respect to y.

Theorem 4.3. *Assume that (H1)-(H4) hold and the condition*

(H5') *There exists a function $l \in L^1(J,\mathbb{R}_+)$ such that*

$$|f(t,x)-f(t,y)| \leq l(t)\|x-y\|_D \ \text{a.e. } t\in J, \ \text{and for all } x,y\in D,$$

with $\int_0^b e^{-\omega s} l(s)ds < \infty$,

$$c_0^* = \frac{Me^{\omega b}\left(\|\phi\| + \sum\limits_{k=1}^{m} |I_k(0)| + \int_0^b e^{-\omega s}|f(s,0)|ds\right)}{1 - Me^{\omega b}\sum\limits_{k=1}^{m} d_k}, \tag{4.8}$$

and

$$c_1^* = \frac{Me^{\omega b}}{1 - Me^{\omega b}\sum\limits_{k=1}^{m} d_k}. \tag{4.9}$$

If

$$Me^{\omega b}\sum_{k=1}^{m} d_k < 1,$$

then the problem (1.1)–(1.3) *has at least one integral solution on* $[-r,b]$.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ the operator defined in Theorem 4.2. It can be shown, as in the proof of Theorem 4.2 that $\mathcal{B}$ is completely continuous and $\mathcal{A}$ is a contraction. Now, we prove that

$$\mathcal{E} = \left\{ y \in PC([-r,b], \overline{D(A)}) : y = \lambda\mathcal{B}(y) + \lambda\mathcal{A}\left(\frac{y}{\lambda}\right) \ \text{ for some } \ 0<\lambda<1 \right\},$$

is bounded.

Let $y \in \mathcal{E}$. Then $y = \lambda\mathcal{B}(y) + \lambda\mathcal{A}\left(\frac{y}{\lambda}\right)$ for some $0<\lambda<1$. Thus, for each $t \in J$,

$$y(t) = \lambda S'(t)\phi(0) + \lambda\frac{d}{dt}\int_0^t S(t-s)f(s,y_s)ds + \lambda\sum_{0<t_k<t} S'(t-t_k)\, I_k\Big(\frac{y}{\lambda}(t_k^-)\Big).$$

This implies by (H2), (H5') that, for each $t\in J$, we have

$$|y(t)| \leq \lambda Me^{\omega t}|\phi(0)| + \lambda Me^{\omega t}\int_0^t e^{-\omega s}|f(s,y_s) - f(s,0)|ds$$

$$
\begin{aligned}
&+\lambda M e^{\omega t}\int_0^t e^{-\omega s}|f(s,0)|ds+\lambda M e^{\omega t}\sum_{k=1}^m \left|I_k\left(\frac{y}{\lambda}\left(t_k^-\right)\right)\right| \\
&\leq \lambda M e^{\omega t}\|\phi\|+\lambda M e^{\omega t}\int_0^t e^{-\omega s}l(s)\|y_s\|ds+\lambda M e^{\omega t}\int_0^t e^{-\omega s}|f(s,0)|ds \\
&+\lambda M e^{\omega t}\sum_{k=1}^m \left|I_k\left(\frac{y}{\lambda}\left(t_k^-\right)\right)-I_k(0)\right|+\lambda M e^{\omega t}\sum_{k=1}^m |I_k(0)| \\
&\leq M e^{\omega t}\Big(\|\phi\|+\int_0^t e^{-\omega s}|f(s,0)|ds+\sum_{k=1}^m |I_k(0)|\Big) \\
&+M e^{\omega t}\int_0^t e^{-\omega s}l(s)\|y_s\|ds+M e^{\omega t}\sum_{k=1}^m d_k y\left(t_k^-\right).
\end{aligned}
$$

Now, we consider the function μ defined by

$$\mu(t)=\sup\{|y(s)|:\quad -r\leq s\leq t\},\quad 0\leq t\leq b.$$

Then $\|y_s\|\leq\mu(t)$ for all $t\in J$ and there is a point $t^*\in[-r,t]$ such that $\mu(t)=|y(t^*)|$. If $t^*\in[0,b]$, by the previous inequality we have for $t\in[0,b]$ (note $t^*\leq t$)

$$
\begin{aligned}
\mu(t) &\leq M e^{\omega t}\Big(\|\phi\|+\int_0^t e^{-\omega s}|f(s,0)|ds+\sum_{k=1}^m |I_k(0)|\Big) \\
&+ M e^{\omega t}\int_0^t e^{-\omega s}l(s)\mu(s)ds+M e^{\omega t}\sum_{k=1}^m d_k\mu(t).
\end{aligned}
$$

Then

$$
\begin{aligned}
\Big(1-M e^{\omega b}\sum_{k=1}^m d_k\Big)\mu(t) &\leq M e^{\omega t}\Big(\|\phi\|+\int_0^t e^{-\omega s}|f(s,0)|ds+\sum_{k=1}^m |I_k(0)|\Big) \\
&+ M e^{\omega b}\int_0^t e^{-\omega s}l(s)\mu(s)ds.
\end{aligned}
$$

Thus, by (4.8) and (4.9) we have

$$\mu(t)\leq c_0^*+c_1^*\int_0^t e^{-\omega s}l(s)\mu(s)ds.$$

By Gronwall inequality ([31]), we get for each $t\in J$

$$\mu(t)\leq c_0^* exp\Big(c_1^*\int_0^t e^{-\omega s}l(s)ds\Big).$$

Thus,

$$\|y\| \le c_0^* \exp\left(c_1^* \int_0^b e^{-\omega s} l(s) ds\right) := M^*.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 4.1 we deduce that $\mathcal{A}+\mathcal{B}$ has a fixed point which is a integral solution of problem (1.1)–(1.3). □

5. Existence of extremal integral solutions

In this section, we shall prove the existence of maximal and minimal integral solutions of problem (1.1)–(1.3) under suitable monotonicity conditions on the functions involved in it.

Definition 5.1. *A non-empty closed subset C of a Banach space $(X, \|\cdot\|)$ is said to be a cone if*

(i) $C + C \subset C$,
(ii) $\lambda C \subset C$
(iii) $\{-C\} \bigcap \{C\} = \{0\}$.

A cone C is called normal if the norm $\|\cdot\|$ is semi-monotone on C, i.e., there exists a constant $N > 0$ such that $\|x\| \le N\|y\|$, whenever $x \le y$. We equip the space $X = C(J, E)$ with the order relation $\le$ induced by a regular cone $\mathcal{C}$ in E, that is for all $y, \overline{y} \in X : y \le \overline{y}$ if and only if $\overline{y}(t) - y(t) \in \mathcal{C}, \quad \forall t \in J$. In what follows, we will assume that the cone C is normal. Cones and their properties are detailed in [29, 33]. Let $a, b \in X$ be such that $a \le b$. Then, by an order interval $[a, b]$ we mean a set of points in X given by

$$[a, b] = \{x \in X \mid a \le x \le b\}.$$

Definition 5.2. *Let X be an ordered Banach space. A mapping $T : X \to X$ is called isotone increasing if $T(x) \le T(y)$ for any $x, y \in X$ with $x < y$. Similarly, T is called isotone decreasing if $T(x) \ge T(y)$ whenever $x < y$.*

Definition 5.3. *We say that $x \in X$ is the least fixed point of G in X if $x = Gx$ and $x \le y$ whenever $y \in X$ and $y = Gy$. The greatest fixed point of G in X is defined similarly by reversing the inequality. If both least and greatest fixed point of G in X exist, we call them extremal fixed point of G in X.*

Very recently Dhage has proved the following.

Theorem 5.1. [26]. *Let $[a, b]$ be an order interval in a Banach space and let $B_1, B_2 : [a, b] \to X$ be two functions satisfying:*

(a) *B_1 is a contraction,*
(b) *B_2 is completely continuous,*
(c) *B_1 and B_2 are strictly monotone increasing, and*
(d) $B_1(x) + B_2(x) \in [a, b],\ \forall\, x \in [a, b]$.
Further if the cone C in X is normal, then the equation $x = B_1(x) + B_2(x)$ has a least fixed point x_* and a greatest fixed point $x^* \in [a, b]$. Moreover, $x_* = \lim_{n\to\infty} x_n$ and $x^* = \lim_{n\to\infty} y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a, b]$ defined by

$$x_{n+1} = B_1(x_n) + B_2(x_n),\ x_0 = a \ \text{ and } \ y_{n+1} = B_1(y_n) + B_2(y_n),\ y_0 = b.$$

We need the following definitions in the sequel.

Definition 5.4. *We say that a continuous function $v : [-r, b] \to E$ is a lower integral solution of problem* (1.1)–(1.3) *if $v(t) = \phi(t)$, $t \in [-r, 0]$, and*

$$v(t) \leq S'(t)\phi(0) + A\int_0^t v(s)ds + \int_0^t f(s, v_s)ds + \sum_{0<t_k<t} S'(t - t_k)\, I_k\left(v\left(t_k^-\right)\right),$$

$t \in J$, $t \neq t_k$, and $v(t_k^+) - v(t_k^-) \leq I_k(v(t_k))$, $t = t_k$, $k = 1, \ldots, m$. Similarly, an upper integral solution w of problem (1.1)–(1.3) is defined by reversing the order.

Definition 5.5. *A solution x_M of problem* (1.1)–(1.3) *is said to be maximal if for any other solution x of problem* (1.1)–(1.3) *on J, we have that $x(t) \leq x_M(t)$ for each $t \in J$.*
Similarly, a minimal solution of problem (1.1)–(1.3) *is defined by reversing the order of the inequalities.*

Definition 5.6. *A function $f(t, x)$ is called strictly monotone increasing in x almost everywhere for $t \in J$, if $f(t, x) \leq f(t, y)$ a.e. $t \in J$ for all $x,\ y \in D$ with $x < y$. Similarly $f(t, x)$ is called strictly monotone decreasing in x almost everywhere for $t \in J$, if $f(t, x) \geq f(t, y)$ a.e. $t \in J$ for all $x,\ y \in D$ with $x < y$.*

We consider the following assumptions in the sequel.

(H6) The function $f(t, y)$ is strictly monotone increasing in y for almost each $t \in J$.
(H7) $S'(t)$ is preserving the order, that is $S'(t)v \geq 0$ whenever $v \geq 0$.
(H8) The functions I_k, $k = 1, ...m$ are continuous and nondecreasing.
(H9) The problem (1.1)–(1.3) has a lower integral solution v and an upper integral solution w with $v \leq w$.

Theorem 5.2. *Assume that assumptions* (H1)-(H9) *hold. Then problem* (1.1)–(1.3) *has a minimal and a maximal integral solutions on* $[-r,b]$.

Proof. It can be shown, as in the proof of Theorem 4.2 that $\mathcal{B}$ is completely continuous and $\mathcal{A}$ is a contraction on $[v,w]$. We shall show that $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on $[v,w]$. Let $y,\overline{y}\in[a,b]$ be such that $y\le\overline{y}$, $y\ne\overline{y}$. Then by (H6), (H7), we have for each $t\in J$

$$\begin{aligned}\mathcal{B}(y)(t) &= S'(t)\phi(0)+\frac{d}{dt}\int_0^t S(t-s)f(s,y_s)\,ds\\ &\le S'(t)\phi(0)+\frac{d}{dt}\int_0^t S(t-s)f(s,\overline{y}_s)\,ds=\mathcal{B}(\overline{y})(t)\,.\end{aligned}$$

and by (H8), we have for each $t\in J$

$$\begin{aligned}\mathcal{A}(y)(t) &= \sum_{0<t_k<t} S'(t-t_k)\,I_k\left(y\left(t_k^-\right)\right)\\ &\le \sum_{0<t_k<t} S'(t-t_k)\,I_k\left(\overline{y}\left(t_k^-\right)\right)=\mathcal{A}(\overline{y})(t)\,.\end{aligned}$$

Therefore, $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on $[v,w]$. Finally, let $x\in[v,w]$ be any element. By (H9) we deduce that

$$v\le\mathcal{A}(v)+\mathcal{B}(v)\le\mathcal{A}(x)+\mathcal{B}(x)\le\mathcal{A}(w)+\mathcal{B}(w)\le w,$$

which shows that $\mathcal{A}(x)+\mathcal{B}(x)\in[v,w]$ for all $x\in[v,w]$. Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all conditions of Theorem 5.1, hence problem (1.1)–(1.3) has a maximal and a minimal integral solutions on $[-r,b]$. This completes the proof. □

6. IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NON-LOCAL CONDITIONS

In this section, we shall prove existence results for problem (1.4)–(1.6). Non-local conditions were initiated by Byszewski [18] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [19, 21], the non-local condition can be more useful than the standard initial condition to describe some physical phenomena.

Definition 6.1. *A function* $y\in PC\big([-r,b],\overline{D(A)}\big)$ *is said to be a integral solution of problem* (1.4)–(1.6) *if* $y(t)=\phi(t)-h_t(y)$, $t\in[-r,0]$, *and*

$$y(t)=S'(t)\left(\phi(0)-h_0(y)\right)+\int_0^t T(t-s)f(s,y_s)\,ds$$

$$+\sum_{0<t_k<t} T(t-t_k)\, I_k\left(y\left(t_k^-\right)\right), \quad t \in J.$$

Theorem 6.1. *Assume that hypotheses (H1)-(H4) hold and moreover*

(A1) *The function h is continuous with respect to t, and there exists a constant $\alpha > 0$ such that*

$$|h_t(u)| \leq \alpha, \qquad u \in PC([-r,b],\overline{D(A)}),$$

and for each $k > 0$ the set

$$\{\phi(0) - h_0(y),\ y \in PC([-r,b],\overline{D(A)}),\ \|y\| \leq k\},$$

is precompact in E

(A2) *There exists a function $p \in L^1(J,\mathbb{R}_+)$ and a continuous non-decreasing function $\psi : [0,\infty) \to (0,\infty)$ such that*

$$|f(t,x)| \leq p(t)\psi(\|x\|_D), \quad a.e.\ \ t \in J, \quad \text{for all } \ x \in D,$$

with

$$\int_{\tilde{c}_0}^{\infty} \frac{du}{\psi(u)} > \tilde{c}_1 \int_0^b e^{-\omega s} p(s)ds,$$

where

$$\tilde{c}_0 = \frac{Me^{\omega b}[\|\phi\|_D + \alpha + \sum_{k=1}^{m} |I_k(0)|]}{1 - Me^{\omega b}\sum_{k=1}^{m} d_k}, \qquad \tilde{c}_1 = \frac{Me^{\omega b}}{1 - Me^{\omega b}\sum_{k=1}^{m} d_k}.$$

Moreover, we suppose that

$$Me^{\omega b}\sum_{k=1}^{m} d_k < 1,$$

then the problem (1.4)–(1.6) has at least one integral solution on $[-r,b]$.

Proof. Transform the problem (1.4)–(1.6) into a fixed point problem. Consider the two operators : $\mathcal{B}_1 : PC\big([-r,b],\overline{D(A)}\big) \to PC\big([-r,b],\overline{D(A)}\big)$

$$\mathcal{B}_1(y)(t) := \begin{cases} \phi(t) - h_t(y), & \text{if } t \in [-r,0]; \\ S'(t)\left(\phi(0) - h_0(y)\right) + \frac{d}{dt}\int_0^t S(t-s) f\left(s, y_s\right) ds & \text{if } t \in J, \end{cases}$$

$$\mathcal{A}_1(y)(t) = \begin{cases} 0, & \text{if } t \in [-r,0]; \\ \displaystyle\sum_{0<t_k<t} S'(t-t_k) I_k(y(t_k^-)), & \text{if } t \in J. \end{cases}$$

Then the problem of finding the solution of problem (1.4)–(1.6) is reduced to finding the solution of the operator equation $\mathcal{A}_1(y)(t)+\mathcal{B}_2(y)(t) = y(t)$, $t \in [-r, b]$. As in Section 3, we can show that the operators $\mathcal{A}_1$ and $\mathcal{B}_1$ satisfy all conditions of Theorem 4.1. □

7. Applications to Control Theory

This section is devoted to an application of the argument used in the previous sections to the controllability of impulsive functional differential equations. More precisely, we will consider the following problem:

$$y'(t) - Ay(t) = f(t, y_t) + Bu(t), \ a.e.\ t \in J = [0, b]\,, \ t \neq t_k, \ k = 1, \dots, m \tag{7.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, \dots, m \tag{7.2}$$

$$y(t) = \phi(t), \ t \in [-r, 0]\,, \tag{7.3}$$

where A, f and I_k are as in Section 3, the control function $u(\cdot)$ is given in $L^2(J, U)$ a Banach space of admissible control functions with U as a Banach. Finally, B is a bounded linear operator from U to $\overline{D(A)}$.

Definition 7.1. *A function $y \in PC\big([-r, b]\,, \overline{D(A)}\big)$ is said to be a integral solution of problem* (7.1)–(7.3) *if $y(t) = \phi(t)$, $t \in [-r, 0]$, and y is a solution of impulsive integral equation*

$$y(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s) f(s, y_s)\, ds + \frac{d}{dt}\int_0^t S(t-s) Bu(s)\, ds$$
$$+ \sum_{0<t_k<t} S'(t - t_k)\, I_k\left(y\left(t_k^-\right)\right), \quad t \in J.$$

Definition 7.2. *The system* (7.1)–(7.3) *is said to be controllable on the interval $[-r, b]$ if for every initial function $\phi \in D$ and every $y_1 \in \overline{D(A)}$, there exists a control $u \in L^2(J, U)$, such that the mild solution $y(t)$ of system* (7.1)–(7.3) *satisfies $y(b) = y_1$.*

Our main result in this section is the following.

Theorem 7.1. *Assume that hypotheses (H1)-(H4) hold. Moreover we suppose that*

(B1) *The linear operator $W : L^2(J, U) \to \overline{D(A)}$, defined by*

$$Wu = \int_0^b T(b-s)\, Bu(s)\, ds,$$

has an invertible operator W^{-1} *which takes values in* $L^2(J,U)\backslash KerW$ *and there exist positive constants* $\overline{M}$, $\overline{M_1}$, *such that* $\|B\| \leq \overline{M}$ *and* $\left\|W^{-1}\right\| \leq \overline{M_1}$.

(B2) *There exists a function* $l \in L^1(J, \mathbb{R}_+)$ *such that*

$$|f(t,x) - f(t,y)| \leq l(t)\|x-y\|_D \text{ for a.e. } t \in J, \quad \text{and for all } x, y \in D,$$

with

$$M^2 e^{2\omega b}\overline{MM_1}b\int_0^b e^{-\omega s} l(s)ds + Me^{\omega b}(1 + Me^{\omega b}\overline{MM_1}b)\sum_{k=1}^m d_k < 1.$$

(B3) *There exists a function* $p \in L^1(J, \mathbb{R}_+)$ *and a continuous non-decreasing function* $\psi : [0,\infty) \to (0,\infty)$ *such that*

$$|f(t,x)| \leq p(t)\psi(\|x\|_D), \quad a.e. \ \ t \in J, \quad \text{for all } \ x \in D,$$

with $\displaystyle\int_0^b e^{-\omega s} p(s)ds < \infty$,

$$\int_{c_3}^{\infty} \frac{ds}{s + \psi\,(s)} > \|\widehat{m}\|_{L^1}\,, \tag{7.4}$$

where

$$c_3 = \frac{c_2}{1 - Me^{\omega b}(1 + M\overline{MM_1}be^{\omega b})\sum\limits_{k=1}^m d_k}, \tag{7.5}$$

$$c_2 = M(1+M\overline{MM_1}e^{\omega b}b)\|\phi\| + M\overline{MM_1}b|y_1| + M(1+M\overline{MM_1}e^{\omega b}b)\sum_{k=1}^m |I_k(0)|, \tag{7.6}$$

$$\widehat{m}\,(s) = \max\{\omega, c_4 p(s)\}, \tag{7.7}$$

and

$$c_4 = \frac{M + M^2\overline{MM_1}e^{\omega b}b}{1 - Me^{\omega b}(1 + M\overline{MM_1}be^{\omega b})\sum\limits_{k=1}^m d_k}. \tag{7.8}$$

Then the problem (7.1)–(7.3) *is controllable on* $[-r, b]$.

Remark 7.1. The construction of operator W^{-1} and its properties are discussed in [24].

Proof. Using hypothesis (B1) for an arbitrary function $y\,(.)$ we define the control

$$\begin{aligned} u_y\,(t) &= W^{-1}\Big[y_1 - S'\,(b)\,\phi\,(0) - \lim_{\lambda\to\infty}\int_0^b S'(b-s)B_\lambda f(s,y_s)ds \\ &- \sum_{0<t_k<t} S'\,(b-t_k)\,I_k\left(y\left(t_k^-\right)\right)\Big]\,(t)\,. \end{aligned}$$

Consider the two operators:

$$\overline{\mathcal{A}},\overline{\mathcal{B}} : PC\left(\left[-r,b\right],\overline{D(A)}\right) \to PC\left(\left[-r,b\right],\overline{D(A)}\right),$$

defined by

$$\overline{\mathcal{A}}(y)\,(t) := \begin{cases} 0, & \text{if } t\in[-r,0]; \\ \frac{d}{dt}\int_0^t S(t-s)Bu(s)ds \\ +\sum\limits_{0<t_{k<t}} S'\,(t-t_k)\,I_k\left(y\left(t_k^-\right)\right), & \text{if } t\in J, \end{cases}$$

and

$$\overline{\mathcal{B}}(y)\,(t) := \begin{cases} \phi(t), & \text{if } t\in[-r,0]; \\ S'(t)\phi(0) \\ +\frac{d}{dt}\int_0^t S(t-s)f\,(s,y_s)\,ds, & \text{if } t\in J. \end{cases}$$

As in Section 3, we can prove that the operators $\overline{\mathcal{A}}$ is a contraction operator and $\overline{\mathcal{B}}$ is completely continuous operator. Now, we prove that

$$\mathcal{E} = \left\{ y\in PC([-r,b],\overline{D(A)}) : y = \lambda\overline{\mathcal{B}}(y) + \lambda\overline{\mathcal{A}}\left(\frac{y}{\lambda}\right) \ \text{ for some } \ 0<\lambda<1\right\},$$

is bounded. Let $y\in\mathcal{E}$. Then $y = \lambda\overline{\mathcal{B}}(y) + \lambda\overline{\mathcal{A}}\left(\frac{y}{\lambda}\right)$ for some $0<\lambda<1$. Thus, for each $t\in J$,

$$\begin{aligned} y(t) &= \lambda S'(t)\phi(0) + \lambda\frac{d}{dt}\int_0^t S(t-s)f(s,y_s)ds \\ &+ \lambda\frac{d}{dt}\int_0^t S(t-s)Bu_y\,(s)\,ds + \lambda\sum_{0<t_{k<t}} S'\,(t-t_k)\,I_k\left(\frac{y}{\lambda}\left(t_k^-\right)\right). \end{aligned}$$

This implies by (B1)-(B3) that, for each $t\in J$, we have

$$|y(t)| \le \lambda M e^{\omega t}|\phi(0)| + \lambda M e^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|)ds$$

$$+\lambda M e^{\omega t}\int_0^t e^{-\omega s}\left|Bu_y(s)\right|ds+\lambda M e^{\omega t}\sum_{k=0}^{m}\left|I_k\left(\frac{y}{\lambda}\left(t_k^-\right)\right)\right|$$

$$\leq \lambda M e^{\omega t}\left[(1+M\overline{MM_1}be^{\omega b})\|\phi\|+\overline{MM_1}b|y_1|\right]$$

$$+\lambda M e^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|)ds+\lambda M^2\overline{M_1M}be^{\omega t}e^{\omega b}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|)ds$$

$$+\lambda M^2\overline{M_1M}be^{\omega t}e^{\omega b}\sum_{k=1}^{m}|I_k(\frac{y}{\lambda}(t_k^-))|+\lambda M e^{\omega t}\sum_{k=1}^{m}\left|I_k\left(\frac{y}{\lambda}\left(t_k^-\right)\right)\right|$$

$$\leq \lambda M e^{\omega t}[(1+M\overline{MM_1}be^{\omega b})\|\phi\|$$

$$+\overline{MM_1}b|y_1|+Me^{\omega t}(1+M\overline{MM_1}be^{\omega b})\sum_{k=1}^{m}|I_k(0)|]$$

$$+\lambda M e^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|)ds+\lambda M^2e^{\omega t}\overline{MM_1}be^{\omega b}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|)ds$$

$$+\lambda M e^{\omega t}(1+M\overline{MM_1}be^{\omega b})\sum_{k=1}^{m}|y(t_k^-))|.$$

Set

$$\begin{aligned}\alpha &= M(1+M\overline{MM_1}e^{\omega b}b)\|\phi\|+M\overline{MM_1}b|y_1|\\ &+ M(1+M\overline{MM_1}e^{\omega b}b)\sum_{k=1}^{m}|I_k(0)|.\end{aligned}$$

Consider the function μ defined by

$$\mu(t)=\sup\{|y(s)|:\quad -r\leq s\leq t\},\quad 0\leq t\leq b.$$

Then $\|y_s\|\leq\mu(t)$ for all $t\in J$ and there is a point $t^*\in[-r,t]$ such that $\mu(t)=|y(t^*)|$. If $t^*\in[0,b]$, by the previous inequality we have for $t\in[0,b]$ (note $t^*\leq t$)

$$\begin{aligned}\mu(t) &\leq \alpha e^{\omega t}+Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\mu(s))ds\\ &+ M^2e^{\omega t}\overline{MM_1}be^{\omega b}\int_0^t e^{-\omega s}p(s)\psi(\mu(s))ds\\ &+ \lambda M e^{\omega t}(1+M\overline{MM_1}be^{\omega b})\sum_{k=1}^{m}d_k\mu(t).\end{aligned}$$

Then

$$[1 - Me^{\omega b}(1 + M\overline{MM_1}be^{\omega b})\sum_{k=1}^{m} d_k]\mu(t) \leq \alpha e^{\omega t}$$
$$+Me^{\omega t}(1 + M\overline{MM_1}be^{\omega b})\int_0^t p(s)\psi(\mu(s))ds.$$

Thus by (7.5), (7.6), (7.8) we have

$$e^{-\omega t}\mu(t) \leq c_3 + c_4\int_0^t p(s)\psi(\mu(s))ds. \tag{7.9}$$

Let us take the right hand-side of (7.9) as $v(t)$. Then we have

$$\mu(t) \leq e^{\omega t}v(t) \quad \text{for all } t \in J, \qquad v(0) = c_3,$$

and

$$v'(t) = c_4 p(t)\psi(\mu(t)), \quad a.e.\ \ t \in J.$$

Using the non-decreasing character of ψ we get

$$v'(t) \leq c_4 p(t)\psi(e^{\omega t}v(t)), \quad a.e.\ \ t \in J.$$

Then by (7.7) for a.e. $t \in J$, we have

$$\begin{aligned}(e^{\omega t}v(t))' &= \omega e^{\omega t}v(t) + v'(t)e^{\omega t} \leq \omega e^{\omega t}v(t) + c_4 p(t)e^{\omega t}\psi(e^{\omega t}v(t)) \\ &\leq \widehat{m}(t)[e^{\omega t}v(t) + \psi(e^{\omega t}v(t))].\end{aligned}$$

Thus, by (7.4)

$$\int_{v(0)}^{e^{\omega t}v(t)} \frac{du}{u + \psi(u)} \leq \int_0^b \widehat{m}(s)ds = \|\widehat{m}\|_{L^1} < \int_{c_3}^{\infty} \frac{du}{u + \psi(u)}.$$

Consequently, by (B3), there exists a constant d such that $e^{\omega t}v(t) \leq d,\ \ t \in J$ and, hence, $\|y\| \leq d$. This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 4.2 we deduce that $\overline{\mathcal{A}} + \overline{\mathcal{B}}$ has a fixed point which is a integral solution of problem (7.1)–(7.3). Thus, the system (7.1)–(7.3) is controllable on $[-r, b]$. □

8. An example

As an application of our results we consider the following impulsive partial functional differential equation of the form

$$\frac{\partial}{\partial t}z(t,x) = \frac{\partial^2}{\partial x^2}z(t,x) \tag{8.1}$$
$$+ Q(t, z(t-r,x)) + Bu(t),\ \ x \in [0,\pi],\ \ t \in [0,b]\backslash\{t_1, t_2, \ldots, t_m\}.$$

$$z(t_k^+, x) - z(t_k^-, x) = b_k z(t_k^-, x),\ x \in [0,\pi],\ k = 1,\dots,m \tag{8.2}$$
$$z(t,0) = z(t,\pi) = 0,\ t \in [0,b] \tag{8.3}$$
$$z(t,x) = \phi(t,x),\ t \in [-r,0],\ x \in [0,\pi], \tag{8.4}$$

where $r > 0,\ b_k > 0,\ k = 1,\dots,m,\ \phi \in \mathcal{D} = \{\psi : [-r,0] \times [0,\pi] \to \mathbb{R}; \psi,$ is continuous everywhere except for a countable number of points at which $\psi(s^-), \psi(s^+)$ exist with $\psi(s^-) = \psi(s)\},\ 0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = b,\ z(t_k^+) = \lim_{(h,x)\to(0^+,x)} z(t_k + h, x), z(t_k^-) = \lim_{(h,x)\to(0^-,x)} z(t_k + h, x)$ and $Q : [0,b] \times \mathbb{R} \to \mathbb{R}$ is a given function. Let

$$y(t)(x) = z(t,x),\ t \in [0,b],\ x \in [0,\pi],$$
$$I_k(y(t_k^-))(x) = b_k z(t_k^-, x),\ x \in [0,\pi],\ k = 1,\dots,m$$
$$F(t,\phi)(x) = Q(t, \phi(\theta,x)),\ \theta \in [-r,0],\ x \in [0,\pi],$$
$$\phi(\theta)(x) = \phi(\theta,x),\ \theta \in [-r,0],\ x \in [0,\pi].$$

Consider $E = C(\overline{\Omega})$, the Banach space of continuous function on $\overline{\Omega}$ with values in $\mathbb{R}$. Define the linear operator A on E by

$$Az = \frac{\partial^2}{\partial x^2} z, \quad \text{in} \quad D(A) = \{z \in C(\overline{\Omega}) : z = 0 \text{ on } \ \partial\Omega,\ \frac{\partial^2}{\partial x^2} z \in C(\overline{\Omega}\},$$

Now, we have

$$\overline{D(A)} = C_0(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : v = 0 \ \text{ on } \ \partial\Omega\} \neq C(\overline{\Omega}).$$

It is well known from [25] that A is sectorial, $(0,+\infty) \subseteq \rho(A)$ and for $\lambda > 0$

$$\|R(\lambda, A)\|_{B(E)} \leq \frac{1}{\lambda}.$$

It follows that A generates an integrated semigroup $(S(t))_{t\geq 0}$ and that $\|S'(t)\|_{B(E)} \leq e^{-\mu t}$ for $t \geq 0$ for some constant $\mu > 0$ and A satisfied the Hille-Yosida condition. Assume that the operator $B : U \to Y, U \subset [0,\infty)$, is a bounded linear operator and the operator

$$Wu = \int_0^b T(b-s)Bu(s)ds,$$

has a bounded invertible operator W^{-1} which takes values in $L^2([0,b],U) \setminus kerW$. Also assume that there exists an integrable function $\sigma : [0,b] \to \mathbb{R}^+$ such that

$$|Q(t, w(t-r,x))| \leq \sigma(t)\Omega(|w|),$$

where $\Omega : [0,\infty) \to (0,\infty)$ is continuous and non-decreasing with

$$\int_1^\infty \frac{ds}{s + \Omega(s)} = +\infty.$$

Assume that there exists a function $\tilde{l} \in L^1([0,b],\mathbb{R}^+)$ such that

$$|Q(t,w(t-r,x)) - Q(t,\overline{w}(t-r,x))| \leq \tilde{l}(t)|w-\overline{w}|, \ t \in [0,b], \ w,\overline{w} \in \mathbb{R}.$$

We can show that problem (7.1)-(7.3) is an abstract formulation of problem (8.1)-(8.4). Since all the conditions of Theorem 7.1 are satisfied, the problem (8.1)-(8.2) has a solution z on $[-r,b] \times [0,\pi]$.

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