Differential and Integral Equations Volume 21, Numbers 5-6 (2008), 497–512

HIGHER ORDER ORDINARY DIFFERENTIAL-OPERATOR EQUATIONS ON THE WHOLE AXIS IN UMD BANACH SPACES

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(Submitted by: J.A. Goldstein)

Abstract. We use a direct approach for proving an isomorphism result for a general higher order abstract ordinary differential equation in a *UMD* Banach space. In fact, it gives maximal L_p-regularity property. As a consequence, we get some interpolation theorem (about intermediate derivatives). A situation of a higher order equation generated by one operator is also treated. Finally, we present some application to elliptic PDEs.

1. Introduction

The striking result of L. Weis in 2001 [12] about Mikhlin type operator-valued Fourier multipliers in *UMD* Banach spaces stimulated a series of works and studies by many mathematicians for abstract parabolic and elliptic equations in *UMD* Banach spaces (previously, the corresponding results have been carried out only for Hilbert spaces). First and second order abstract equations have been studied.

In this paper, we present very general abstract higher order ordinary differential equations on $\mathbb{R}$ in *UMD* Banach spaces. An isomorphism has been proved in appropriate abstract W_p^n-Sobolev spaces. This becomes possible due to Weis' result [12]. In particular, it gives us maximal L_p-regularity

Accepted for publication: February 2008.
AMS Subject Classifications: 34G10, 35J40.

[1]The first author is a member of G.N.A.M.P.A. and the paper fits the RFO research program of the University of Bologna and of G.N.A.M.P.A.-I.N.D.A.M.

[2]The second author was supported by the Israel Ministry of Absorption.

property for the equation. We also get some interpolation results about intermediate derivatives. The corresponding theorems (Theorems 1 and 7) are formulated in such a way that they completely cover previous results for abstract second order elliptic equations on the whole axis in *UMD* Banach spaces (see Remark 9). The theorems also generalize previous higher order results in Hilbert spaces (in appropriate abstract W_p^n-Sobolev spaces with a weight) in the book by S. Yakubov and Ya. Yakubov [13, Section 5.2.2] to *UMD* Banach spaces. We also consider a situation when the equation is generated by one operator.

There is also an isomorphism higher order result in Hilbert spaces (in appropriate abstract W_2^n-Sobolev spaces with a weight) in the book by V. Kozlov and V. Maz'ya [6, Theorem 2.4.1]. They also considered a situation of Banach spaces but in some kind of local Sobolev spaces. Moreover, the solvability and uniqueness theorems (and not isomorphism theorems) have been proved in some special classes of functions (see [6, Theorems 10.9.7 and 10.9.8] for equations with constant operators and [6, Theorems 10.9.9 and 10.9.10] for equations with variable operators). Therefore, these theorems in Banach spaces do not give maximal L_p-regularity property of the equation.

Here we just remind some necessary definitions and notations.

If E and F are Banach spaces, $B(E,F)$ denotes the Banach space of all bounded, linear operators from E into F with the norm equal to the operator norm; moreover, $B(E) := B(E,E)$. The spectrum of a linear operator A in E is denoted by $\sigma(A)$, its resolvent set by $\rho(A)$. The domain and range of an operator A are denoted by $D(A)$ and $R(A)$, respectively. The resolvent of an operator A is denoted by $R(\lambda, A) := (\lambda I - A)^{-1}$.

We use the notation Ff or $\widehat{f}$ for the Fourier transform of a function f belonging to a vector-valued L_p-space, i.e., $L_p(\mathbb{R}; E)$,

$$Ff := (Ff)(\sigma) := \widehat{f}(\sigma) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\sigma x} f(x) dx,$$

and the inverse Fourier transform

$$F^{-1}f := (F^{-1}f)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\sigma x} f(\sigma) d\sigma.$$

The function $\sigma \to T(\sigma) : \mathbb{R} \to B(E)$ is called a *Fourier multiplier in the space* $L_p(\mathbb{R}; E)$, if

$$\|F^{-1}TFf\|_{L_p(\mathbb{R};E)} \leq C\|f\|_{L_p(\mathbb{R};E)}, \quad f \in L_p(\mathbb{R}; E).$$

A Banach space E is said to be of class HT if the Hilbert transform is bounded on $L_p(\mathbb{R}; E)$ for some (and then all) $p > 1$. Here the Hilbert transform H of a function $f \in S(\mathbb{R}; E)$, the Schwartz space of rapidly decreasing E-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f, \quad \text{i.e.,}$$
$$(Hf)(t) := \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\tau| > \varepsilon} \frac{f(t-\tau)}{\tau} d\tau.$$

These spaces are often also called *UMD* Banach spaces, where the *UMD* stands for the property of *unconditional martingale differences*. We prefer the notion *UMD* in the framework of this paper.

For $0 < \theta \le \pi$ we define the sector Σ_θ in the complex plane by

$$\Sigma_\theta := \{\lambda \in \mathbb{C} \backslash \{0\} : |\arg \lambda| < \theta\}.$$

Definition. *Let E be a complex Banach space, and A is a closed linear operator in E. The operator A is called* **sectorial** *if the following conditions are satisfied:*

(1) $\overline{D(A)} = E$, $\overline{R(A)} = E$, $(-\infty, 0) \subset \rho(A)$;
(2) $\|\lambda(\lambda + A)^{-1}\| \le M$ *for all* $\lambda > 0$, *and some* $M < \infty$.

Definition. *Let E and F be Banach spaces. A family of operators $\mathcal{T} \subset B(E, F)$ is called R-**bounded**, if there is a constant $C > 0$ and $p \ge 1$ such that for each natural number n, $T_j \in \mathcal{T}$, $u_j \in E$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables ε_j on $[0, 1]$ (e.g., the Rademacher functions $\varepsilon_j(t) = \operatorname{sign} \sin(2^j \pi t)$) the inequality*

$$\Big\| \sum_{j=1}^{n} \varepsilon_j T_j u_j \Big\|_{L_p((0,1);F)} \le C \Big\| \sum_{j=1}^{n} \varepsilon_j u_j \Big\|_{L_p((0,1);E)}$$

*is valid. The smallest such C is called R-**bound** of $\mathcal{T}$ and is denoted by $R\{\mathcal{T}\}$.*

From the definition of R-boundedness it follows that every R-bounded family of operators is (uniformly) bounded (it is enough to take $n = 1$). On the other hand, in a Hilbert space H every bounded set is R-bounded (see, e.g., [8, p. 75]). Therefore, in a Hilbert space, the notion of R-boundedness is equivalent to boundedness of a family of operators (see also [5, p. 26]).

Definition. *A sectorial operator A is called R-**sectorial** if*

$$R_A(0) := R\{\lambda(\lambda + A)^{-1} \; : \; \lambda > 0\} < \infty.$$

The number

$$\phi_A^R := \inf\{\theta \in (0,\pi) \ : \ R_A(\pi - \theta) < \infty\},$$

where $R_A(\theta) := R\{\lambda(\lambda + A)^{-1} \ : \ |\arg \lambda| \le \theta\}$, *is called the* R-**angle** *of the operator* A.

For the operator A closed in E, the domain of definition $D(A^n)$ of the operator A^n is a Banach space $E(A^n)$ with respect to the norm

$$\|u\|_{E(A^n)} := \Big(\sum_{k=0}^{n} \|A^k u\|^2\Big)^{\frac{1}{2}}.$$

The operator A^n from $E(A^n)$ into E is bounded.

We will, in particular, use the following proposition which was proved by A. Venni and was announced to us in private communication. In fact, the proposition is a generalization of well-known moment (norm-)inequality (see, e.g., [5, p. 19]) to moment (R-)inequality (even with a weight function). We believe that the proposition can be useful also for other future studies on the subject.

Let X and Y be complex Banach spaces and let S be an arbitrary set. We are given a sectorial operator A acting in Y, a family $(B(\sigma))_{\sigma\in S}$ of bounded linear operators from X into Y and a non-negative function $f : S \to \mathbb{R}$. Let α, β, γ be such real numbers that $0 \le \alpha < \beta < \gamma$ and assume that $\forall \sigma \in S$ the range of $B(\sigma)$ is contained in $D(A^\gamma)$.

Proposition. (*by* **A. Venni**) *If* $\{f(\sigma)^{\frac{\gamma-\alpha}{\gamma-\beta}} A^\alpha B(\sigma) \ : \sigma \in S\}$ *and* $\{A^\gamma B(\sigma) \ : \sigma \in S\}$ *are* R*-bounded sets of operators then* $\{f(\sigma)\, A^\beta B(\sigma) \ : \sigma \in S\}$ *is also* R*-bounded and*

$$\begin{aligned} &R\{f(\sigma)A^\beta B(\sigma) : \sigma \in S\} \\ &\le CR\{f(\sigma)^{\frac{\gamma-\alpha}{\gamma-\beta}} A^\alpha B(\sigma) \ : \sigma \in S\}^{\frac{\gamma-\beta}{\gamma-\alpha}} R\{A^\gamma B(\sigma) \ : \sigma \in S\}^{\frac{\beta-\alpha}{\gamma-\alpha}}, \end{aligned}$$

where C *is a positive constant which depends only on* α, β, γ, *and the constant of sectoriality of* A, *i.e., on* $M := \sup_{t>0} \|t(t+A)^{-1}\|$.

Proof. From, e.g., [9, formula (5.17)] it follows that

$$A^\beta x = A^{\beta-\alpha} A^\alpha x = C_0(\beta,\gamma) \int_0^\infty t^{\beta-\alpha-1} A^\gamma (t+A)^{-\gamma} A^\alpha x \, dt,$$

where $x \in D(A^\gamma)$ and $C_0(\beta,\gamma) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}$. Let I be a finite set of indices, and let $(\sigma_i)_{i\in I}$ and $(x_i)_{i\in I}$ be families of elements of S and X, respectively.

Set $K := \max_{i\in I} f(\sigma_i)$. Then, for every $\delta > 0$, we have

$$\begin{aligned}
&\sum_{\varepsilon\in\{-1,1\}^I} \Big\| \sum_{i\in I} \varepsilon_i f(\sigma_i)\, A^\beta B(\sigma_i) x_i \Big\| \\
&= C_0(\beta,\gamma) \sum_\varepsilon \Big\| \sum_i \varepsilon_i f(\sigma_i) \int_0^\infty t^{\beta-\alpha-1} A^\gamma (t+A)^{-\gamma} A^\alpha B(\sigma_i) x_i \, dt \Big\| \\
&\le C_0(\beta,\gamma) \int_0^\infty t^{\beta-\alpha-1} \sum_\varepsilon \Big\| \sum_i \varepsilon_i f(\sigma_i) A^\gamma (t+A)^{-\gamma} A^\alpha B(\sigma_i) x_i \Big\| \, dt \\
&= C_0(\beta,\gamma) \int_0^\delta t^{\beta-\alpha-1} \sum_\varepsilon \Big\| \sum_i \varepsilon_i f(\sigma_i) A^\gamma (t+A)^{-\gamma} A^\alpha B(\sigma_i) x_i \Big\| \, dt \\
&+ C_0(\beta,\gamma) \int_\delta^\infty t^{\beta-\alpha-1} \sum_\varepsilon \Big\| \sum_i \varepsilon_i f(\sigma_i) A^\alpha (t+A)^{-\gamma} A^\gamma B(\sigma_i) x_i \Big\| \, dt. \qquad (1.1)
\end{aligned}$$

Now we use the contraction principle (see, e.g., [5, Lemma 3.5] or [11, Proposition 1(c)]) and the inequalities $\|t^\xi A^\eta (t+A)^{-\xi-\eta}\| \le C(\xi,\eta,M)$ (for $\xi \ge 0$ and $\eta \ge 0$) proved in [11, Theorem 2]. Hence, continuing (1.1) we get

$$\begin{aligned}
&\sum_{\varepsilon\in\{-1,1\}^I} \Big\| \sum_{i\in I} \varepsilon_i f(\sigma_i)\, A^\beta B(\sigma_i) x_i \Big\| \\
&\le C_1(\beta,\gamma,M)\, K^{\frac{\alpha-\beta}{\gamma-\beta}} \int_0^\delta t^{\beta-\alpha-1} \, dt \sum_\varepsilon \Big\| \sum_i \varepsilon_i f(\sigma_i)^{\frac{\gamma-\alpha}{\gamma-\beta}} A^\alpha B(\sigma_i) x_i \Big\| \\
&+ C_1(\beta,\gamma,M)\, K \int_\delta^\infty t^{\beta-\gamma-1} \, dt \sum_\varepsilon \Big\| \sum_i \varepsilon_i A^\gamma B(\sigma_i) x_i \Big\| \\
&\le C_2(\alpha,\beta,\gamma,M) \sum_\varepsilon \Big\| \sum_i \varepsilon_i \, x_i \Big\| \Big(\delta^{\beta-\alpha} K^{\frac{\alpha-\beta}{\gamma-\beta}} R\{ f(\sigma)^{\frac{\gamma-\alpha}{\gamma-\beta}} A^\alpha B(\sigma) \ : \sigma \in S\} \\
&\qquad + \delta^{\beta-\gamma} K \, R\{A^\gamma B(\sigma) \ : \sigma \in S\} \Big). \qquad (1.2)
\end{aligned}$$

We now minimize the last expression with respect to δ, i.e., we set

$$\begin{aligned}
\delta &= \Big(\frac{\beta-\alpha}{\gamma-\beta}\Big)^{\frac{1}{\alpha-\gamma}} K^{\frac{1}{\gamma-\beta}} R\{A^\gamma B(\sigma) \ : \sigma \in S\}^{\frac{1}{\gamma-\alpha}} \\
&\times R\{ f(\sigma)^{\frac{\gamma-\alpha}{\gamma-\beta}} A^\alpha B(\sigma) \ : \sigma \in S\}^{\frac{1}{\alpha-\gamma}}
\end{aligned}$$

and we get from (1.2)

$$\sum_{\varepsilon\in\{-1,1\}^I}\Big\|\sum_{i\in I}\varepsilon_i f(\sigma_i)\,A^\beta B(\sigma_i)x_i\Big\|\le C_2(\alpha,\beta,\gamma,M)$$
$$\times\, R\{f(\sigma)^{\frac{\gamma-\alpha}{\gamma-\beta}}\,A^\alpha B(\sigma)\ :\sigma\in S\}^{\frac{\gamma-\beta}{\gamma-\alpha}}\,R\{A^\gamma B(\sigma)\ :\sigma\in S\}^{\frac{\beta-\alpha}{\gamma-\alpha}}\sum_\varepsilon\Big\|\sum_i\varepsilon_i\,x_i\Big\|.$$

2. General coercive equations of higher order

Consider the equation

$$L(D)u := u^{(n)}(x)+A_1u^{(n-1)}(x)+\cdots+A_nu(x)=f(x),\quad x\in\mathbb{R}, \tag{2.1}$$

with, in general, unbounded operators A_k acting in a UMD Banach space E, where $D^ju(x):=u^{(j)}(x):=\frac{d^ju(x)}{dx^j}$ is a generalized derivative of the function $u(x)$ with values from E.

Let there exist Banach spaces E_k, $k=0,\dots,n$, for which continuous embeddings $E_n\subset E_{n-1}\subset\cdots\subset E_0=E$ take place.

Introduce the space $W_p^n(\mathbb{R};E_n,\dots,E_0)$, $1<p<\infty$, of functions with the norm

$$\|u\|_{W_p^n(\mathbb{R};E_n,\dots,E_0)}:=\Big(\sum_{k=0}^n\int_{-\infty}^{\infty}\|D^{n-k}u(x)\|_{E_k}^p\,dx\Big)^{\frac1p}.$$

Denote $L_p(\mathbb{R};E):=W_p^0(\mathbb{R};E)$. The operator pencil

$$L(\lambda):=\lambda^nI+\lambda^{n-1}A_1+\cdots+A_n \tag{2.2}$$

is called a *characteristic operator pencil* of equation (2.1).

The following theorem is a generalization of [13, Theorem 5.2.2/3 for $\gamma=0$] from Hilbert spaces to *UMD* Banach spaces.

Theorem 1. *Let the following conditions be satisfied*:

(1) *an operator A_n is closed, densely defined and invertible in a UMD Banach space E and*
$$R\{\lambda R(\lambda,A_n)\ :\ \arg\lambda=\pi\}<\infty;$$
(2) *operators A_k from $E(A_n^{\frac kn})$ into E are bounded, $k=1,\dots,n$;*[3]
(3) *for $\lambda=i\sigma,\sigma\in\mathbb{R}$, the operator pencil $L(\lambda)$ is invertible in E and*
$$R\{\sigma^nL(i\sigma)^{-1}\ :\sigma\in\mathbb{R}\}<\infty\quad\textit{and}\quad R\{A_nL(i\sigma)^{-1}\ :\sigma\in\mathbb{R}\}<\infty.$$

[3]From condition (1) it follows that there exist fractional powers A_n^α for $\alpha\in\mathbb{R}$ (see, e.g., [5, Section 2.2]).

Then, the operator $\mathbb{L}: \ u \to \mathbb{L}u := u^{(n)}(x) + A_1 u^{(n-1)}(x) + \cdots + A_n u(x)$ *from* $W_p^n(\mathbb{R}; E(A_n), E(A_n^{1-\frac{1}{n}}), \ldots, E)$ *onto* $L_p(\mathbb{R}; E)$ *is an isomorphism and a solution of* (2.1) *is given by the formula*

$$u(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{i\sigma x} L(i\sigma)^{-1} \widehat{f}(\sigma) d\sigma, \tag{2.3}$$

where $\widehat{f}(\sigma) := (Ff)(\sigma)$ *is the Fourier transform.*

Proof. Let $u(x)$ be a solution of (2.1) that belongs to $W_p^n(\mathbb{R}; E(A_n), E(A_n^{1-\frac{1}{n}}), \ldots, E)$. Apply the Fourier transform on (2.1),

$$(i\sigma)^n \widehat{u}(\sigma) + \sum_{k=1}^{n} (i\sigma)^{n-k} A_k \widehat{u}(\sigma) = \widehat{f}(\sigma), \quad \sigma \in \mathbb{R}. \tag{2.4}$$

Since the operator $L(\lambda)$ in E is invertible on the line $\Re\lambda = 0$, then it follows from (2.4) that

$$\widehat{u}(\sigma) = L(i\sigma)^{-1} \widehat{f}(\sigma).$$

Hence,

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{i\sigma x} \widehat{u}(\sigma) d\sigma = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{i\sigma x} L(i\sigma)^{-1} \widehat{f}(\sigma) d\sigma.$$

Thus, formula (2.3) is established.

The boundedness of the operator $\mathbb{L}$ is obvious. We show now that if $f \in L_p(\mathbb{R}; E)$, then the function $u(x)$ given by (2.3) belongs to $W_p^n(\mathbb{R}; E(A_n), E(A_n^{1-\frac{1}{n}}), \ldots, E)$ and is a solution of (2.1). We have

$$\begin{aligned}
\|u\|^p_{W_p^n(\mathbb{R}; E(A_n), E(A_n^{1-\frac{1}{n}}), \ldots, E)} &= \sum_{k=0}^{n} \int_{-\infty}^{\infty} \left\| F^{-1} F D^{n-k} u(x) \right\|^p_{E(A_n^{\frac{k}{n}})} dx \\
&= \sum_{k=0}^{n} \int_{-\infty}^{\infty} \left\| (F^{-1} (i\sigma)^{n-k} \widehat{u}(\sigma))(x) \right\|^p_{E(A_n^{\frac{k}{n}})} dx \\
&= \sum_{k=0}^{n} \int_{-\infty}^{\infty} \left\| A_n^{\frac{k}{n}} (F^{-1} (i\sigma)^{n-k} L(i\sigma)^{-1} \widehat{f}(\sigma))(x) \right\|^p dx \\
&= \sum_{k=0}^{n} \int_{-\infty}^{\infty} \left\| (F^{-1} (i\sigma)^{n-k} A_n^{\frac{k}{n}} L(i\sigma)^{-1} (Ff)(\sigma))(x) \right\|^p dx.
\end{aligned} \tag{2.5}$$

It is enough now to show that functions $T_k(\sigma) := (i\sigma)^{n-k} A_n^{\frac{k}{n}} L(i\sigma)^{-1}$, $k = 0, \ldots, n$, are Fourier multipliers in the space $L_p(\mathbb{R}; E)$. From condition (3) and Venni's proposition from the Introduction (with $\alpha = 0, \beta = \frac{k}{n}, \gamma = 1$, $f(\sigma) = \sigma^{n-k}$, $B(\sigma) = L(i\sigma)^{-1}$, $k = 1, \ldots, n-1$) it follows that

$$R\{T_k(\sigma) \ : \ \sigma \in \mathbb{R}\backslash\{0\}\} < \infty, \quad k = 0, \ldots, n. \tag{2.6}$$

Since

$$\begin{aligned} T_k'(\sigma) = (n-k)i^{n-k}\sigma^{n-k-1}A_n^{\frac{k}{n}}L(i\sigma)^{-1} - i^{n-k+1}\sigma^{n-k} \\ \times A_n^{\frac{k}{n}}L(i\sigma)^{-1}L'(i\sigma)L(i\sigma)^{-1}, \end{aligned}$$

where

$$L'(\lambda) = n\lambda^{n-1}I + \sum_{j=1}^{n-1}(n-j)\lambda^{n-j-1}A_j,$$

then

$$\begin{aligned} \sigma T_k'(\sigma) &= (n-k)T_k(\sigma) - i\sigma T_k(\sigma) \\ &\quad \times \Big(n(i\sigma)^{n-1}I + \sum_{j=1}^{n-1}(n-j)(i\sigma)^{n-j-1}A_j\Big)L(i\sigma)^{-1} \\ &= (n-k)T_k(\sigma) - T_k(\sigma)\Big(nT_0(\sigma) + \sum_{j=1}^{n-1}(n-j)A_jA_n^{-\frac{j}{n}}T_j(\sigma)\Big). \end{aligned} \tag{2.7}$$

From condition (2) it follows that operators $A_jA_n^{-\frac{j}{n}}$, $j = 1, \ldots, n-1$, are bounded in E. Then, using (2.6) and, e.g., [5, Proposition 3.4], we get from (2.7) that

$$R\{\sigma T_k'(\sigma) \ : \ \sigma \in \mathbb{R}\backslash\{0\}\} < \infty, \quad k = 0, \ldots, n. \tag{2.8}$$

By virtue of, e.g., [5, Theorem 3.19], which is, in fact, a theorem of L. Weis [12, Theorem 3.4], from (2.6) and (2.8) it follows that the functions $T_k(\sigma)$, $k = 0, \ldots, n$, are Fourier multipliers in the space $L_p(\mathbb{R}; E)$. Then (2.5) implies

$$\|u\|_{W_p^n(\mathbb{R};E(A_n),E(A_n^{1-\frac{1}{n}}),\ldots,E)} \leq C\|f\|_{L_p(\mathbb{R};E)}.$$

On the other hand, from (2.3) we have

$$L(D)u = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{i\sigma x}\widehat{f}(\sigma)d\sigma = f(x).$$

The theorem has been proved. □

Remark 2. Theorem 1 is still true for the operator

$$u \to \mathbb{L}u := u^{(n)}(x) + A_1u^{(n-1)}(x) + \cdots + A_{n-1}u'(x) - A_nu(x).$$

Proof is the same.

3. Coercive equations of higher order generated by one operator

Consider now the equation on $\mathbb{R}$

$$L(D)u := u^{(n)}(x) + a_1 A u^{(n-1)}(x) + \cdots + a_n A^n u(x) = f(x), \tag{3.1}$$

with a closed operator A, in a UMD Banach space E, where a_k are complex numbers, $a_n \neq 0$. The roots of the equation

$$\omega^n + a_1\omega^{n-1} + \cdots + a_n = 0 \tag{3.2}$$

will be denoted by ω_k, $k = 1, \ldots, n$. Denote $\ell(a, \varphi) := \{z \;:\; z \in \mathbb{C},\; z = a + r\mathrm{e}^{i\varphi}, r \geq 0\}$ and $\ell(\varphi) := \ell(0, \varphi)$.

Lemma 3. *Let A be a closed operator in a Banach space E and, for some $d \geq 0$ and $M > 0$,*

$$R\{\lambda^{1-d}R(\lambda, A) : \lambda \in \ell(a\omega_j^{-1}, \varphi - \arg\omega_j),\; |\lambda| \geq M\} < \infty,\; j = 1, \ldots, n.$$

Then for the operator pencil

$$L_0(\lambda) := \lambda^n I + \lambda^{n-1}a_1 A + \cdots + a_n A^n, \tag{3.3}$$

the following estimates, for $k = 0, \ldots, n$, hold

$$R\{\lambda^{n-k-nd}A^k L_0(\lambda)^{-1} : \lambda \in \ell(a, \varphi),\; |\lambda| \geq M \max_{1\leq j\leq n} |\omega_j|\} < \infty. \tag{3.4}$$

Proof. From (3.2) and (3.3) it follows that

$$L_0(\lambda) = \prod_{j=1}^{n}(\lambda I - \omega_j A) = \prod_{j=1}^{n}\omega_j(\lambda\omega_j^{-1}I - A) \tag{3.5}$$

which is true for any $\lambda \in \mathbb{C}$. Since, for $j = 1, \ldots, n$,

$$R\{\lambda^{1-d}R(\lambda\omega_j^{-1}, A) : \lambda \in \ell(a, \varphi),\; |\lambda| \geq M|\omega_j|\} < \infty \tag{3.6}$$

then, from

$$(\lambda\omega_j^{-1}I - A)R(\lambda\omega_j^{-1}, A) = I$$

and from, e.g., [5, Proposition 3.4] it follows that, for $j = 1, \ldots, n$,

$$\begin{aligned} &R\{\lambda^{-d}AR(\lambda\omega_j^{-1}, A) : \lambda \in \ell(a, \varphi),\; |\lambda| \geq M|\omega_j|\} \\ &\leq R\{\lambda^{-d}I : \lambda \in \ell(a, \varphi),\; |\lambda| \geq M|\omega_j|\} \\ &+ R\{\lambda^{1-d}R(\lambda\omega_j^{-1}, A) : \lambda \in \ell(a, \varphi),\; |\lambda| \geq M|\omega_j|\} < \infty. \end{aligned} \tag{3.7}$$

From (3.5) it follows that, for $k = 0, \dots, n$,

$$\lambda^{n-k-nd} A^k L_0(\lambda)^{-1} = \prod_{j=1}^{k} \omega_j^{-1} \lambda^{-d} A R(\lambda \omega_j^{-1}, A) \prod_{j=k+1}^{n} \omega_j^{-1} \lambda^{1-d} R(\lambda \omega_j^{-1}, A). \tag{3.8}$$

Then, from (3.6)-(3.8) and, e.g., [5, Proposition 3.4] it follows that (3.4) is satisfied. □

Corollary 4. *Let A be a closed operator in a Banach space E and, for some $\gamma \in \mathbb{R}$ and $M > 0$,*

$$R\{\lambda R(\lambda, A) : \lambda \in \ell(\gamma\omega_j^{-1}, \pm\frac{\pi}{2} - \arg \omega_j),\ |\lambda| \geq M\} < \infty, \quad j = 1, \dots, n.$$

Then for the operator pencil $L_0(\lambda) := \lambda^n I + \lambda^{n-1} a_1 A + \cdots + a_n A^n$ the following estimates, for $k = 0, \dots, n$, hold

$$R\{\lambda^{n-k} A^k L_0(\lambda)^{-1} : \Re\lambda = \gamma, \quad |\lambda| \geq M \max_{1 \leq j \leq n} |\omega_j|\} < \infty.$$

Proof. The proof follows from Lemma 3 under $d = 0$, $a = \gamma$, $\varphi = \pm\frac{\pi}{2}$. □

Theorem 5. *Let the following conditions be satisfied*:

(1) *an operator A is closed, densely defined and invertible in a UMD Banach space E and, for some $M > 0$ and all $j = 1, \dots, n$,*

$$R\{\lambda R(\lambda, A) : \lambda \in \ell(0, \pm\frac{\pi}{2} - \arg \omega_j),\ |\lambda| \geq M\} < \infty;$$

(2) $R\{\lambda R(\lambda, a_n A^n) : \arg \lambda = \pi\} < \infty$;

(3) *for $\lambda = i\sigma$, $\sigma \in \mathbb{R}$, the operator pencil $L_0(\lambda) := \lambda^n I + \lambda^{n-1} a_1 A + \cdots + a_n A^n$ is invertible in E.*[4]

Then the operator $\mathbb{L} : u \to \mathbb{L}u := u^{(n)}(x) + a_1 A u^{(n-1)}(x) + \cdots + a_n A^n u(x)$ from $W_p^n(\mathbb{R}; E(A^n), E(A^{n-1}), \dots, E)$ onto $L_p(\mathbb{R}; E)$, $1 < p < \infty$, is an isomorphism.

Proof. Let us apply Theorem 1. Denote $A_n := a_n A^n$, $A_k := a_k A^k = a_k a_n^{-\frac{k}{n}} A_n^{\frac{k}{n}}$, $k = 1, \dots, n-1$. Then, condition (1) of Theorem 1 follows from condition (2) and condition (2) of Theorem 1 is obvious. Let us check condition (3) of Theorem 1.

From condition (1), by virtue of Corollary 4 (take $\gamma = 0$), we have, for $k = 0, \dots, n$,

$$R\{\sigma^{n-k} A^k L_0(i\sigma)^{-1} : \sigma \in \mathbb{R},\ |\sigma| \geq M \max_{1 \leq j \leq n} |\omega_j|\} < \infty. \tag{3.9}$$

[4] In this case, $D(L_0(\lambda)) := \bigcap_{k=1}^{n} D(A^k) = D(A^n)$.

By virtue of condition (3), $\lambda^{n-k}A^k L_0(\lambda)^{-1}$ is a holomorphic function in some neighborhood of the imaginary axis. Then, by, e.g., [5, Proposition 3.10],

$$R\{\sigma^{n-k}A^k L_0(i\sigma)^{-1} : \sigma \in \mathbb{R},\ |\sigma| \le M \max_{1\le j\le n} |\omega_j|\} < \infty. \tag{3.10}$$

Combining (3.9) and (3.10) (it is enough to take only $k = 0, n$) we get condition (3) of Theorem 1. □

Theorem 6. *Let conditions of Theorem* 5 *be satisfied. Then the operator* $\mathbb{L} : u \to \mathbb{L}u := u^{(n)}(x) + a_1 A u^{(n-1)} u(x) + \cdots + a_n A^n u(x)$ *from* $W_p^{n+m}(\mathbb{R}; E(A^{n+m}), \ldots, E)$ *onto* $W_p^m(\mathbb{R}; E(A^m), \ldots, E)$*, where an integer* $m \ge 0$*,* $1 < p < \infty$*, is an isomorphism.*

Proof. Obviously, the operator $\mathbb{L}$ acts continuously from $W_p^{n+m}(\mathbb{R}; E(A^{n+m}), \ldots, E)$ into $W_p^m(\mathbb{R}; E(A^m), \ldots, E)$. Then, from Theorem 5 it follows that the mapping $\mathbb{L}$ is an injection. Let us prove that $\mathbb{L}$ is also surjective, i.e., for any $f \in W_p^m(\mathbb{R}; E(A^m), \ldots, E)$ the equation

$$u^{(n)}(x) + a_1 A u^{(n-1)} u(x) + \cdots + a_n A^n u(x) = f(x), \quad x \in \mathbb{R}, \tag{3.11}$$

has a solution that belongs to the space $W_p^{n+m}(\mathbb{R}; E(A^{n+m}), \ldots, E)$. Consider the problem

$$v^{(n)}(x) + a_1 A v^{(n-1)}(x) + \cdots + a_n A^n v(x) = (\omega I - A)^m f(x),\ x \in \mathbb{R}, \tag{3.12}$$

where $\omega \in \rho(A)$. Since $(\omega I - A)^m f \in L_p(\mathbb{R}; E)$, by virtue of Theorem 5, equation (3.12) has a unique solution $v \in W_p^n(\mathbb{R}; E(A^n), \ldots, E)$. Then, the function $u(x) = (\omega I - A)^{-m} v(x)$ is a solution of equation (3.11) such that $u^{(k)} = (\omega I - A)^{-m} v^{(k)} \in L_p(\mathbb{R}; E(A^{n+m-k}))$ for $k = 0, \ldots, n$. Multiplying equation (3.12) by $(\omega I - A)^{-1}$ we find

$$\begin{aligned}(\omega I - A)^{-1} v^{(n)}(x) = -a_1 A(\omega I - A)^{-1} v^{(n-1)}(x) - \cdots \\ - a_n A^n (\omega I - A)^{-1} v(x) + (\omega I - A)^{m-1} f(x).\end{aligned}$$

There exist first order derivatives of all terms on the right-hand side of this equation and the derivatives belong to the space $L_p(\mathbb{R}; E)$. Hence, $(\omega I - A)^{-1} v^{(n+1)}(x)$ exists and belongs to $L_p(\mathbb{R}; E)$, i.e., $(\omega I - A)^{m-1} u^{(n+1)}(x)$ exists and belongs to $L_p(\mathbb{R}; E)$. So, recurrently one can prove that, for $k = n+1, \ldots, n+m$, $u^{(k)} \in L_p(\mathbb{R}; E(A^{n+m-k}))$. □

Let E and E_n be Banach spaces for which the continuous embedding $E_n \subset E$ takes place. Let us introduce the space $W_p^n(\mathbb{R}; E_n, E)$, $1 < p < \infty$, of functions with the norm

$$\|u\|_{W_p^n(\mathbb{R}; E_n, E)} := \|u\|_{L_p(\mathbb{R}; E_n)} + \|u^{(n)}\|_{L_p(\mathbb{R}; E)}.$$

We now prove a theorem about intermediate derivatives (an interpolation result) which is a generalization of [13, Theorem 5.2.2/6] from Hilbert spaces to *UMD* Banach spaces.

Theorem 7. *Let A be a closed, densely defined and invertible operator in a UMD Banach space E and*

$$R\{\lambda R(\lambda, A) : \arg\lambda = \pi\} < \infty.$$

Then the operator $D^j : \ u \to D^j u := u^{(j)}(x)$ from $W_p^{2n}(\mathbb{R}; E(A), E)$ into $W_p^{2n-j}(\mathbb{R};\ E(A^{1-\frac{j}{2n}}), E(A^{1-\frac{j+1}{2n}}), \ldots, E)$, $1 < p < \infty$, $j = 0, \ldots, 2n$, is bounded.

Proof. Consider the operator pencil $L(\lambda) := \lambda^{2n} I + A$, where $n = 2k$. Then, using the condition of the theorem, we get that

$$R\{\sigma^{2n} L(i\sigma)^{-1} : \sigma \in \mathbb{R}\} < \infty. \tag{3.13}$$

Hence, obviously,

$$R\{A L(i\sigma)^{-1} : \sigma \in \mathbb{R}\} < \infty. \tag{3.14}$$

Therefore, (3.13) and (3.14) imply condition (3) of Theorem 1 (conditions (1) and (2) of Theorem 1 are obvious).

Let now $u \in W_p^{2n}(\mathbb{R}; E(A), E)$. Then, $f := u^{(2n)} + Au \in L_p(\mathbb{R}; E)$. Hence, by virtue of Theorem 1, a solution of the equation

$$u^{(2n)}(x) + Au(x) = f(x), \quad x \in \mathbb{R}$$

belongs to the space $W_p^{2n}(\mathbb{R}; E(A), E(A^{1-\frac{1}{2n}}), \ldots, E)$ and

$$\|u\|_{W_p^{2n}(\mathbb{R}; E(A), E(A^{1-\frac{1}{2n}}), \ldots, E)} \le C\|f\|_{L_p(\mathbb{R}; E)},$$

i.e., for $j = 0, \ldots, 2n$,

$$\|u^{(j)}\|_{W_p^{2n-j}(\mathbb{R}; E(A^{1-\frac{j}{2n}}), \ldots, E)} \le C\|f\|_{L_p(\mathbb{R}; E)} \le C\|u\|_{W_p^{2n}(\mathbb{R}; E(A), E)}.$$

If $n = 2k+1$ then we consider the operator pencil $L(\lambda) := \lambda^{2n} I - A$ and use Remark 2. □

Corollary 8. *Theorem 7 is also true if we consider $(0, 1)$ instead of $\mathbb{R}$. In this case, we have to use a bounded continuation operator from $(0, 1)$ onto $\mathbb{R}$.*

Remark 9. W. Arendt and M. Duelli [2], in particular, considered the following abstract elliptic equation $u''(t) = Au(t) + f(t)$ on $\mathbb{R}$. Their solvability result, Theorem 3.4, is a particular case of Theorem 1 together with Remark 2 (under $n = 2$, $A_1 = 0$, and $A_2 = -A$; in this case, condition (2) of Theorem 1 is trivial and condition (3) of Theorem 1 is unnecessary because it follows directly from condition (1) – see the beginning of the proof of Theorem 7). Their interpolation result, Corollary 3.6, is a particular case of Theorem 7 (under $n = 1$ and $j = 0$). Note also that our Theorems 1 and 5, for $n = 1$, are not "interesting" because, in this case, there is a more strong result with necessary and sufficient conditions in the same paper [2, Theorem 2.4]. So, our theorems are "designed" for cases of $n \geq 2$.

4. Application to elliptic PDEs

Our main goal in this paper was to give a generalization of maximal L_p-regularity property for second order abstract elliptic equations to arbitrary order abstract elliptic equations in *UMD* Banach spaces. The systematic investigation of possible application of the obtained abstract results to general elliptic PDEs is out of the framework of this paper. Let us give only some first application to fourth order elliptic PDEs (since our abstract results cover known second order abstract elliptic equations then some relevant application to second order elliptic PDEs can be found, e. g., in [2]).

Consider a boundary value problem for the biharmonic equation in an infinite strip

$$\Delta^2 u(x,y) = \frac{\partial^4 u(x,y)}{\partial x^4} + 2\frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x,y)}{\partial y^4} = f(x,y), \; x \in \mathbb{R}, \; y \in (0,1), \tag{4.1}$$

$$u(x,0) = u(x,1) = 0, \quad x \in \mathbb{R}, \tag{4.2}$$

$$\frac{\partial^2 u(x,0)}{\partial y^2} = \frac{\partial^2 u(x,1)}{\partial y^2} = 0, \quad x \in \mathbb{R}. \tag{4.3}$$

Theorem 10. *Let $f \in L_p(\mathbb{R}; L_q(0,1))$, where $1 < p, q < \infty$. Then problem (4.1)–(4.3) has a unique solution $u(x,y)$ belonging to the space $W_p^4(\mathbb{R}; W_q^4(0,1), L_q(0,1)) \cap W_p^2(\mathbb{R}; W_q^4(0,1), W_q^2(0,1))$ and for the solution the following estimate is hold:*

$$\|u(x,y)\|_{L_p(\mathbb{R};W_q^4(0,1))} + \left\|\frac{\partial^2 u(x,y)}{\partial x^2}\right\|_{L_p(\mathbb{R};W_q^2(0,1))} + \left\|\frac{\partial^4 u(x,y)}{\partial x^4}\right\|_{L_p(\mathbb{R};L_q(0,1))} \leq C\|f(x,y)\|_{L_p(\mathbb{R};L_q(0,1))}.$$

Proof. In the space $E = L_q(0,1)$, consider an operator A such that

$$A^2 u := -u''(y),$$
$$D(A^2) = W_q^2((0,1); u(0) = u(1) = 0).$$

Then

$$A^4u := u''''(y),$$
$$D(A^4) = W_q^4((0,1); u(0) = u(1) = u''(0) = u''(1) = 0).$$

Therefore, problem (4.1)–(4.3) can be rewritten in the operator form

$$u''''(x) - 2A^2u''(x) + A^4u(x) = f(x), \quad x \in \mathbb{R}. \tag{4.4}$$

Let us use Theorem 5. The operator A^2 (and thus, by, e. g., [5, Theorem 2.3 and Definition 2.9], operators $A = (A^2)^{\frac{1}{2}}$ and $A^4 = (A^2)^2$ too) admits a bounded $\mathcal{H}^\infty$-functional calculus on Σ_θ for every $\theta > 0$ by W. Arendt and A.F.M. ter Elst [3, Theorem 5.7]. Since $L_q(0,1)$ is a UMD Banach space with property (α), operators A and A^4 admit a bounded $R\mathcal{H}^\infty$-functional calculus on Σ_θ for every $\theta > 0$, i.e., $\phi_A^{R_\infty} = 0$ and $\phi_{A^4}^{R_\infty} = 0$, by N. Kalton and L. Weis [7, Theorem 5.3]. Then, since (see, e.g., [5, pp. 50-51]) $R\mathcal{H}^\infty(E) \subset RS(E)$ with $\phi_A^{R_\infty} \geq \phi_A^R$ and $\phi_{A^4}^{R_\infty} \geq \phi_{A^4}^R$, we get that A and A^4 are R-sectorial operators with their R-angles equal to 0. Therefore, since $\omega_j = \pm 1$, i.e., $\arg \omega_j = 0, \pi$, conditions (1) and (2) of Theorem 5 are fulfilled for (4.4).

Let us now check the last condition (3) of Theorem 5. From the proof of [13, Theorem 5.2.4] it follows that the pencil $L_0(\lambda) := \lambda^4 - 2\lambda^2A^2 + A^4$ is invertible for $\lambda = i\sigma$, $\sigma \in \mathbb{R}$, in $L_q(0,1)$ for $q = 2$. On the other side, the spectrum of the pencil $L_0(\lambda)$ and, therefore, the resolvent set, does not depend on $q \in (1,\infty)$ (cf. [1]) . Therefore, the pencil $L_0(\lambda)$ is invertible for $\lambda = i\sigma$, $\sigma \in \mathbb{R}$, in $E = L_q(0,1)$ for $q \in (1,\infty)$. □

A multidimensional analog of problem (4.1)–(4.3) from an application point of view is the following problem in a cylinder

$$\frac{\partial^4 u(x,y)}{\partial x^4} + 2\frac{\partial^2 \Delta_y u(x,y)}{\partial x^2} + \Delta_y^2 u(x,y) = f(x,y), \ x \in \mathbb{R}, \ y \in G, \tag{4.5}$$

$$u(x,y') = 0, \quad x \in \mathbb{R}, \ y' \in \partial G, \tag{4.6}$$
$$\Delta_y u(x,y') = 0, \quad x \in \mathbb{R}, \ y' \in \partial G, \tag{4.7}$$

where a bounded domain $G \subset \mathbb{R}^m$, $m \geq 2$, is given with a sufficiently smooth boundary ∂G.

Theorem 11. *Let $f \in L_p(\mathbb{R}; L_q(G))$, where $1 < p,q < \infty$. Then problem (4.5)–(4.7) has a unique solution $u(x,y)$ belonging to the space $W_p^4(\mathbb{R}; W_q^4(G), L_q(G)) \cap W_p^2(\mathbb{R}; W_q^4(G), W_q^2(G))$ and for the solution the following* *estimate is valid:*

$$\|u(x,y)\|_{L_p(\mathbb{R};W_q^4(G))} + \left\|\frac{\partial^2 u(x,y)}{\partial x^2}\right\|_{L_p(\mathbb{R};W_q^2(G))} + \left\|\frac{\partial^4 u(x,y)}{\partial x^4}\right\|_{L_p(\mathbb{R};L_q(G))} \le C\|f(x,y)\|_{L_p(\mathbb{R};L_q(G))}.$$

Proof. In the space $E = L_q(G)$, consider an operator A such that

$$A^2 u := -\Delta u(y),$$
$$D(A^2) = W_q^2(G; u|_{\partial G} = 0).$$

Then

$$A^4 u := \Delta^2 u(y),$$
$$D(A^4) = W_q^4(G; u|_{\partial G} = \Delta u|_{\partial G} = 0).$$

Therefore, problem (4.5)–(4.7) can be rewritten in the operator form

$$u''''(x) - 2A^2 u''(x) + A^4 u(x) = f(x), \quad x \in \mathbb{R}.$$

The rest part of the proof is similar to that of Theorem 10. □

Remark 12. Two problems considered in this section are only model examples. Using [4, Theorem 2.3], one can take for the operator $A^2 := \mu_\phi + A_B$, where μ_ϕ is some non-negative number and the operator A_B is the realization of a general $2m$-order positive definite selfadjoint elliptic operator $A(y,D)$ in $L_q(G)$ with the domain $W_q^{2m}(G)$ and general boundary value conditions $B_j(y,D),\ j = 1,\dots,m$ (for all smoothness conditions on coefficients and Shapiro-Lopatinskii condition on $A(y,D)$ and $B_j(y,D),\ j = 1,\dots,m$, see [4, pp. 550-552]). Then, one can consider the following $4m$-order boundary value problem, for $x \in \mathbb{R}$, $y \in G$,

$$\frac{\partial^{4m} u(x,y)}{\partial x^{4m}} - 2\frac{\partial^{2m}(\mu_\phi + A(y,D))u(x,y)}{\partial x^{2m}} + (\mu_\phi + A(y,D))^2 u(x,y) = f(x,y),$$

$$B_j(y,D)u(x,y) = 0, \quad x \in \mathbb{R},\ y = y' \in \partial G,\ j = 1,\dots,m,$$
$$B_j(y,D)(\mu_\phi + A(y,D))u(x,y) = 0, \quad x \in \mathbb{R},\ y = y' \in \partial G,\ j = 1,\dots,m.$$

Remark 13. If $1 < p = q < \infty$, then more general elliptic boundary value problems in cylinders have been considered in the book by S.A. Nazarov and B.A. Plamenevsky [10]. One can find, e.g., a very general an isomorphism theorem in weighted Sobolev spaces in [10, Theorem 6.4, p.78].

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