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TOPOLOGICAL ARGUMENTS IN PRESCRIBING THE SCALAR CURVATURE UNDER MINIMAL BOUNDARY MEAN CURVATURE CONDITION ON S^n_+

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Abstract. This paper is devoted to the prescribed scalar curvature problem under minimal boundary mean curvature condition on the standard n-dimensional half Sphere, with $n \geq 3$. Using tools related to the theory of critical points at infinity, we provide some topological conditions, on the level sets of a given positive function on S^n_+, under which we prove some existence results.

1. Introduction, notations and results

In this paper, we study a problem arising from a geometric context. Namely, let

$$S^n_+ = \left\{ x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : |x| = 1, \text{ and } x_{n+1} > 0 \right\},$$

be the standard half sphere endowed with its standard Riemannian metric g, $n \geq 3$, and given a function $K : S^n_+ \longrightarrow \mathbb{R}$, we consider the following problem: does there exist a metric $\tilde{g}$ conformally equivalent to g such that $\mathcal{R}_{\tilde{g}} = K$, and $h_{\tilde{g}} = 0$? Here, $\mathcal{R}_{\tilde{g}}$ is the scalar curvature of S^n_+ and $h_{\tilde{g}}$ is the Mean Curvature of ∂S^n_+, with respect to $\tilde{g}$. Setting $\tilde{g} = u^{\frac{4}{n-2}} g$, a "conformal" metric to g, the above problem is equivalent to find a smooth positive solution to the following equation

$$\begin{cases} -\Delta_g u + \frac{n(n-2)}{4} u = \frac{(n-2)}{4(n-1)} K u^{\frac{(n+2)}{n-2}} & \text{in} \quad S^n_+ \\ u > 0 \quad \text{in} \quad S^n_+ \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial S^n_+, \end{cases} \tag{1.1}$$

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where ν is the outward normal vector with respect to the metric g. Problems related to (1.1) attracted much attention (see [10], [11], [16], [12], [17], [18], [19], [20], [21], [22]).

It is easy to see that a necessary condition for solving (1.1) is that K has to be positive somewhere. Moreover, there are topological obstructions, very close to Kazdan-Warner obstructions for the scalar curvature problem on S^n (see [14]), so it is not expectable to solve problem (1.1) for all functions K, and then a natural question arises: which conditions on K do guarantee the existence of a positive solution to (1.1)?

In [22], [17] and [12], the authors studied problem (1.1) on the three dimensional standard half sphere. The method of [22] and [17] involves a fine blow up analysis of some subcritical approximations and the use of topological degree tools. However the method of [12] uses algebraic topological tools, from the theory of critical points at infinity (see [5]).

In the first part of this article, we focus on the three dimensional case to give more existence results, precisely we borrow some of the ideas developed in [17] and [12]. In order to state our results, we introduce the following notations and assumptions.

Through this paper, we assume that $K_1 = K/\partial S^n_+$ has only nondegenerate critical points $y_0, y_1, ..., y_h$ such that $K_1(y_0) \geqslant K_1(y_1) \geqslant ... \geqslant K_1(y_h)$. Let $k_i = (n-1) - \text{ind}(K_1, y_i)$, for $i = 0, 1, ..., h$, where $\text{ind}(K_1, y_i)$ denotes the Morse index of K_1 at y_i.

We now introduce the following sets:

$$F_+ = \Big\{ y_i \in \partial S^n_+ \ , \ DK_1(y_i) = 0 \quad \text{and} \quad \frac{\partial K}{\partial \nu}(y_i) > 0 \Big\} \tag{1.2}$$

and

$$F^0_+ = \Big\{ y_i \in \partial S^n_+, DK_1(y_i) = 0, \frac{\partial K}{\partial \nu}(y_i) = 0 \text{ and } -\Delta_g K_1(y_i) > 0 \Big\}. \tag{1.3}$$

Let $\mathcal{Z}$ be a pseudogradient for K_1 of Morse-Smale type (that is, the intersections of the unstable and the stable manifolds of critical points of K_1 are transverse). In what follows we will formulate some basic assumptions that will be used in the hypothesis of our main Theorems.

($\mathbf{A_1}$) Assume that $W_u(y_i) \cap W_s(y_j) = \emptyset$ for critical points y_i and y_j of K_1 such that $y_i \in (F^0_+ \cup F_+)^{\mathcal{C}}$ and $y_j \in (F^0_+ \cup F_+)$, where $W_u(y_i)$ is the unstable manifold of y_i and $W_s(y_j)$ is the stable manifold of y_j for the pseudogradient $\mathcal{Z}$.

($\mathbf{A_2}$) Assume that for each critical point y of K_1 such that $\frac{\partial K}{\partial \nu}(y) = 0$ we have $\Delta_g K_1(y) \neq 0$, furthermore, there is a constant $\bar{r} > 0$ such that $-\Delta_g K(y)\frac{\partial K}{\partial \nu}(a) \geqslant 0 \quad \forall a \in B(y, \bar{r}) \cap \partial S^n_+$.

($\mathbf{A_3}$) Assume that for all $y_i \neq y_j$ in F_+ we have

$$\frac{\frac{\partial K}{\partial \nu}(y_i)\frac{\partial K}{\partial \nu}(y_j)}{K(y_i)K(y_j)} < 32 G_1^2(y_i, y_j),$$

where G_1 is the function defined, for $x \neq y$ by $G_1(x,y) = \big(1 - \cos \mathrm{d}(x,y)\big)^{-\frac{1}{2}}$, where d is the geodesic distance on S^n_+.

Now, for each index i, $0 \leqslant i \leqslant h$, we define the set

$$X_i = \bigcup_{0 \leqslant j \leqslant i} \overline{W_s}(y_j), \quad y_j \in (F^0_+ \cup F_+). \tag{1.4}$$

Our first main result is the following:

Theorem 1.1. *Let $n = 3$. Under the assumptions ($\mathbf{A_1}$), ($\mathbf{A_2}$) and ($\mathbf{A_3}$), if there exists an index i, $0 \leqslant i \leqslant h$, satisfying the two following conditions:*

($\mathbf{A_4}$) *X_i is not contractible, and we denote by m_i the dimension of its first nontrivial reduced homology group.*

($\mathbf{A_5}$) *For each $y_j \in (F^0_+ \cup F_+)$ such that $j \in \{i+1, ..., h\}$ we have*

$$\frac{1}{K(y_j)^{\frac{1}{2}}} > \frac{1}{K(y_i)^{\frac{1}{2}}} + \frac{1}{K(y_0)^{\frac{1}{2}}},$$

then problem (1.1) has a solution of morse index $\geqslant m_i$.

Corollary 1.1. *Let $n = 3$. Under the assumptions ($\mathbf{A_1}$), ($\mathbf{A_2}$) and ($\mathbf{A_3}$), if*

$$\sum_{y_i \in (F^0_+ \cup F_+)} \big(-1\big)^{\mathrm{ind}(K_1, y_i)} \neq 1$$

then, problem (1.1) has a solution.

Here $\mathrm{ind}(K_1, y_i)$ is the Morse index of K_1 at y_i.

Remark 1.1. Corollary 1.1 was proved in [17] and [12]. In the Remark 1.2 below, we will exhibit a situation where Corollary 1.1 does not provide a solution to (1.1) while Theorem 1.1 does.

To state our next result, we need to introduce the following notations. Let $k \in \mathbb{N}$ be such that

$$(n-1) - k = \min\big\{\mathrm{ind}(K_1, y)\ ,\ y \in F^0_+ \cup F_+\big\} \tag{1.5}$$

and, let X be the following set

$$X = \bigcup \overline{W_s}(y), \quad y \in B_k \tag{1.6}$$

where

$$B_k = \left\{ \ y \in F_+^0 \cup F_+ : \operatorname{ind}(K_1, y) = (n-1) - k \ \right\}. \tag{1.7}$$

We then have:

Theorem 1.2. *Let $n = 3$. Under the assumptions* $(\mathbf{A_1})$, $(\mathbf{A_2})$ *and* $(\mathbf{A_3})$, *if* $H_k(X, \mathbb{Z}_2) \neq 0$, *then* (1.1) *has a solution.*

Here, $H_k(X, \mathbb{Z}_2)$ denotes the k-dimensional Homology group of X with $\mathbb{Z}_2$-coefficients.

Remark 1.2. Here, we will give an example where Corollary 1.1 does not permit to guarantee existence (see also [17] and [12]), while by Theorem 1.1 or Theorem 1.2 we derive the existence of a solution to problem (1.1).

For this, let $K_1 : \partial S_+^n \longrightarrow \mathbb{R}$ be a positive function such that $F_+^0 = \{y_0, \, y_1\}$ and $F_+ = \{y_2\}$, satisfying $K_1(y_0) \geqslant K_1(y_1) \geqslant K_1(y_2)$, $\operatorname{ind}(K_1, y_0) = 2$, $\operatorname{ind}(K_1, y_1) = 1$, $\operatorname{ind}(K_1, y_2) = 0$ or 2, and any critical point y of K_1 which is not in $F_+^0 \cup F_+$ satisfies $K_1(y) < K_1(y_2)$. It is easy to see that

$$\sum_{y_i \in (F_+^0 \cup F_+)} (-1)^{\operatorname{ind}(K_1, y_i)} = 1$$

and thus, Corollary 1.1 does not work here. We distinguish two cases:

First case: $\operatorname{ind}(K_1, y_2) = 0$. If we assume that

$$\frac{1}{K(y_2)^{\frac{1}{2}}} > \frac{1}{K(y_1)^{\frac{1}{2}}} + \frac{1}{K(y_0)^{\frac{1}{2}}},$$

then using Theorem 1.1, problem (1.1) admits a solution. Indeed, taking $i = 1$, then we have $X_1 = W_s(y_1) \cup W_s(y_0)$, and thus X_1 defines a one-dimensional stratified set, without boundary, so X_1 is not contractible (it is easy to see that K_1 satisfies $(\mathbf{A_2})$ and $(\mathbf{A_3})$ by construction of F_+^0 and F_+ in this example).

Second case: $\operatorname{ind}(K_1, y_2) = 2$. Since $X = \overline{W_s}(y_1) = W_s(y_1) \cup W_s(y_0)$ satisfies $H_1(X, \mathbb{Z}_2) \neq 0$, we derive that (1.1) admits a solution by Theorem 1.2.

In the second part of this work, we propose to handle problem (1.1) in all dimensions $n \geqslant 3$. We give a contribution in the same direction as Aubin-Bahri paper's [4] concerning the Scalar curvature problem on closed Riemannian manifolds. We will introduce some notations and hypothesis needed in this second part:

Let $\ell, \ell' \in \mathbb{N}$, such that $\ell' \leqslant \ell \leqslant h$, ℓ is chosen so that for each $s \geqslant \ell + 1$, $y_s \notin F_+$. We now introduce the following sets:

$$X' = \bigcup_{0 \leqslant i \leqslant \ell'} W_s(y_i), \quad X = \bigcup_{0 \leqslant i \leqslant \ell} \overline{W_s}(y_i), \quad y_i \in F_+, \quad T = \bigcup_{0 \leqslant i \leqslant \ell} \overline{W_s}(y_i),$$

$$\mathcal{I}_- = \Big\{ z : z \notin F_+, \text{ and } \exists y \in F_+ \text{dominating} z : W_u(y) \cap W_s(z) \neq \emptyset \Big\}$$

$$\Gamma = \bigcup W_s(z), \quad z \in \mathcal{I}_-$$

and let $g : X' \hookrightarrow X$ be the natural embedding.

Now, we introduce the following assumptions which will be used in the hypothesis of our main Theorem:

($\mathbf{H_1}$) Assume that $\frac{\partial K}{\partial \nu}(y) \neq 0$ for each critical point y of K_1 in ∂S^n_+.

($\mathbf{H_2}$) Assume that $K(z) < K(y_\ell)$ for each critical point z of K in $\overset{\circ}{S^n_+}$, the interior of S^n_+.

($\mathbf{H_3}$) Assume that X' is contractible in T.

($\mathbf{H_4}$) Assume that $K(y_0) < 2^{\frac{2}{n-2}} K(y_\ell)$.

($\mathbf{H_5}$) Assume that g_* is nontrivial in dimension m and the critical points of Γ of index $n-1-m$ have a zero intersection number (modulo 2) with the critical points of index $n-1-(m+1)$ in X.

Our main result in the second part is:

Theorem 1.3. *Let $n \geqslant 3$. Under assumptions ($\mathbf{H_1}$) to ($\mathbf{H_5}$), there exists a solution to problem* (1.1).

Remark 1.3. To see how one can construct a function K satisfying the assumptions of Theorem 1.3, we refer the interested reader to [4].

The rest of our paper is organized as follows. In Section 2, we introduce the general framework and recall some known facts. Section 3 is devoted to the proofs of Theorem 1.1, Corollary 1.1 and Theorem 1.2. Lastly, we prove Theorem 1.3 in Section 4.

2. Some known facts

In this Section we recall the variational formulation of problem (1.1), as well as some previous useful results. We introduce on $H^1(S^n_+, \mathbb{R})$ the norm

$$\|u\|^2 = \int_{S^n_+} (-\Delta_g u + \tfrac{n(n-2)}{4} u) u dv_g$$

associated to the Yamabe operator $-\Delta_g + \frac{n(n-2)}{4}$, where dv_g is the volume element of g on S^n_+. Now, we define $\sum = \{u \in H^1, \|u\| = 1\}$ to be the unit sphere of $H^1(S^n_+, \mathbb{R})$, and $\sum^+ = \{u \in \sum, u \geq 0\}$ the space of positive functions of $\sum$. The Euler functional on $H^1(S^n_+, \mathbb{R})$ corresponding to (1.1) is then

$$J(u) = \frac{\|u\|^2}{\left(\int_{S^n_+} K u^{\frac{2n}{(n-2)}} dv_g\right)^{\frac{n-2}{n}}}.$$

Indeed, to resolve problem (1.1) reverts to find a critical point of J under the constraint $u \in \sum^+$. The functional J is $\mathcal{C}^3$ but it fails to satisfy the Palais-Smale condition on $\sum^+$. This failure was studied by various authors, as H. Brezis and J.M. Coron in [15], M. Struwe in [24] and P.L. Lions in [23]. To characterize the sequences violating the Palais-Smale condition on $\sum^+$, we need to fix some notations. For $(a, \lambda) \in \overline{S^n_+} \times (0, +\infty)$ let

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda^{\frac{n-2}{2}}}{\left(\lambda^2 + 1 + (\lambda^2 - 1)\cos \mathrm{d}(a, x)\right)^{\frac{n-2}{2}}},$$

where d is the geodesic distance on (S^n_+, g), ($\delta_{(a,\lambda)}$ is known to be the solution of the Yamabe problem on the Sphere S^n), and c_0 is chosen so that

$$-\Delta_g \delta_{(a,\lambda)} + \tfrac{n(n-2)}{4}\delta_{(a,\lambda)} = \delta_{(a,\lambda)}^{\frac{n+2}{n-2}}$$

is satisfied on S^n_+. Observe that if $a \in \partial S^n_+$, we have $\frac{\partial \delta_{(a,\lambda)}}{\partial \nu} = 0$ on ∂S^n_+. However, $\frac{\partial \delta_{(a,\lambda)}}{\partial \nu} \neq 0$ if $a \notin \partial S^n_+$, then in this case we need to introduce another function $\varphi_{(a,\lambda)}$ which satisfies:

$$\begin{cases} -\Delta_g \varphi_{(a,\lambda)} + \frac{n(n-2)}{4}\varphi_{(a,\lambda)} = \delta_{(a,\lambda)}{}^{\frac{(n+2)}{n-2}} & \text{in} \quad S^n_+ \\ \dfrac{\partial \varphi_{(a,\lambda)}}{\partial \nu} = 0 & \text{on} \quad \partial S^n_+. \end{cases} \tag{2.1}$$

For $\varepsilon > 0$ and integers p, q such that $1 \leqslant q + p$, we define

$$\begin{aligned} V(p,q,\varepsilon) = \Big\{ & u \in \textstyle\sum^+ / \exists a_1, ..., a_{p+q} \in \overline{S^n_+},\ \exists \lambda_1, ..., \lambda_{p+q} > \tfrac{1}{\varepsilon} \ \text{ and} \\ & \exists \alpha_1, ..., \alpha_{p+q} > 0, \text{such that } \lambda_i d_i < \varepsilon \text{ for } 1 \leqslant i \leqslant p; \\ & \lambda_{p+j} d_{p+j} > \frac{1}{\varepsilon} \text{ for } 1 \leqslant j \leqslant q \ ; \ \varepsilon_{kl} < \varepsilon \ , \ \left| \frac{\alpha_k^{\frac{4}{n-2}} K(a_k)}{\alpha_l^{\frac{4}{n-2}} K(a_l)} - 1 \right| < \varepsilon \\ & \text{for } 1 \leqslant k \neq l \leqslant p+q, \quad \text{and} \end{aligned}$$

$$\left\|u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} - \sum_{j=1}^{q} \alpha_{p+j} \varphi_{(a_{p+j},\lambda_{p+j})}\right\| < \varepsilon \Bigg\},$$

where $d_k = \mathrm{d}(a_k, \partial S^n_+)$ and $\varepsilon_{kl} = \Big(\frac{\lambda_k}{\lambda_l} + \frac{\lambda_l}{\lambda_k} + \frac{1}{\varepsilon}\lambda_k\lambda_l\big(1 - \cos \mathrm{d}(a_k, a_l)\big)\Big)^{\frac{2-n}{2}}$.

The failure of the Palais-Smale condition can be described, (see [5], [7] and references therein), as follows:

Proposition 2.1. *Suppose that J has no critical point in $\sum^+$, and, let (u_k) be a sequence in $\sum^+$ such that $J(u_k)$ is bounded and $\partial J(u_k) \to 0$. Then, there exists integers p, q such that $1 \leqslant q+p$, a sequence $\varepsilon_k > 0$ $(\varepsilon_k \to 0)$, and an extracted subsequence of (u_k), again denoted by (u_k), such that $\forall k \in \mathbb{N}$, $u_k \in V(p, q, \varepsilon_k)$.*

Here, ∂J is the gradient of J with respect of the H^1-inner product

$$(u, v) = \int_{S^n_+} (-\Delta_g u + \tfrac{n(n-2)}{4} u) v dv_g.$$

We consider the following minimization problem for a function $u \in V(p, q, \varepsilon)$ with ε small:

$$\min\Bigg\{ \left\|u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} - \sum_{j=1}^{q} \alpha_{p+j} \varphi_{(a_{p+j},\lambda_{p+j})}\right\|, \quad \text{for} \tag{2.2}$$
$$\alpha_i > 0, \lambda_i > 0, a_i \in \partial S^n_+, 1 \leqslant i \leqslant p, \alpha_{p+j} > 0,$$
$$\lambda_{p+j} > 0, a_{p+j} \in S^n_+, 1 \leqslant j \leqslant q \Bigg\}.$$

We then have the next proposition which defines a parameterization of the set $V(p, q, \varepsilon)$. It follows from the corresponding statements in [5] and [7]:

Proposition 2.2. *For any integers p, q such that $1 \leqslant q+p$, there exists $\varepsilon(p, q) > 0$ such that if $\varepsilon < \varepsilon(p, q)$ and $u \in V(p, q, \varepsilon)$, the minimization problem* (2.2) *has a unique solution* (*up to permutation*).

Thus, we can write any $u \in V(p, q, \varepsilon)$ uniquely as follows:

$$u = \sum_{i=1}^{p} \bar{\alpha}_i \delta_{(\bar{a}_i,\bar{\lambda}_i)} + \sum_{j=1}^{q} \bar{\alpha}_{p+j} \varphi_{(\bar{a}_{p+j},\bar{\lambda}_{p+j})} + v,$$

where $(\bar{\alpha}_1, ..., \bar{\alpha}_{p+q}, \bar{a}_1, ..., \bar{a}_{p+q}, \bar{\lambda}_1, ..., \bar{\lambda}_{p+q})$ is the solution of the minimization problem (2.2), and $v \in H^1(S^n_+)$ is such that

$$\|v\| < \varepsilon \text{ and } (v, \psi) = 0 \text{ for } \psi \in \Big\{\delta_i, \frac{\partial \delta_i}{\partial a_i}, \frac{\partial \delta_i}{\partial \lambda_i}, \varphi_j, \frac{\partial \varphi_j}{\partial a_j}, \frac{\partial \varphi_j}{\partial \lambda_j}; {}_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q}\Big\}, \tag{2.3}$$

where $\delta_i = \delta_{(a_i,\lambda_i)}$ and $\varphi_j = \varphi_{(a_{p+j},\lambda_{p+j})}$.

We also have the following proposition whose proof is similar, up to minor modifications, to the corresponding statements in [5] and [6].

Proposition 2.3. *There exists a $\mathcal{C}^1$ map which, to each $(\alpha, a, \lambda) = (\alpha_1, ..., \alpha_{p+q}, a_1, ..., a_{p+q}, \lambda_1, ..., \lambda_{p+q})$ such that*

$$\sum_{i=1}^{p} \alpha_i \delta_i + \sum_{j=1}^{q} \alpha_{p+j} \varphi_j \in V(p, q, \varepsilon),$$

with ε small, associates $\bar{v} = \bar{v}(\alpha, a, \lambda)$, satisfying

$$J\Big(\sum_{i=1}^{p} \alpha_i\delta_i + \sum_{j=1}^{q} \alpha_{p+j}\varphi_j + \bar{v}\Big) = \min_{v \text{ verifying}(2.3)} J\Big(\sum_{i=1}^{p} \alpha_i\delta_i + \sum_{j=1}^{q} \alpha_{p+j}\varphi_j + v\Big).$$

Moreover, there exists $c > 0$ such that

$$\|\bar{v}\| \leqslant c\Big(\sum_{i=1}^{p} \frac{|DK(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{j=1}^{q} \frac{1}{\lambda_{p+j} d_{p+j}} + \sum_{k\neq l} \varepsilon_{kl} (\ln \varepsilon_{kl}^{-1})^{\frac{1}{2}}\Big).$$

The next propositions, whose proofs are given by Corollary 3.2 in [12] and Corollary 4.1 in [10], characterize the critical points at infinity of the associated variational problem. We recall that critical points at infinity are the orbits of the gradient flow of J which remain in $V(p, q, \varepsilon(s))$ where $\varepsilon(s)$ is a function tending to 0 when $s \to +\infty$ (see [5]).

Proposition 2.4. *Let $n = 3$. Suppose that J has no critical point in $\sum^+$. Under the assumptions $(\mathbf{A_2})$ and $(\mathbf{A_3})$, the only critical points at infinity of J in $V(1, 0, \varepsilon)$, correspond to $\delta_{(y,+\infty)}$, where $y \in (F_+^0 \cup F_+)$.*

For a proof one can see Corollary 3.2 in [12].

Proposition 2.5. *Let $n \geqslant 3$. Suppose that J has no critical point in $\sum^+$. Under the assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$, the only critical points at infinity of J in $V(1, 0, \varepsilon)$, correspond to $\delta_{(y,+\infty)}$, where $y \in F_+$.*

For a proof see Corollary 4.1 in [10].

3. Proofs of Theorems 1.1 and 1.2.

We start the proofs by recalling the following results:

Lemma 3.1. *Let $a_1, a_2 \in \partial S^3_+$, and let $\alpha_1, \alpha_2 > 0$. For λ large enough positive parameter, let $u = \alpha_1 \delta_{(a_1,\lambda)} + \alpha_2 \delta_{(a_2,\lambda)}$. Then*

$$J(u) \leqslant \big(\tfrac{S_3}{2}\big)^{\frac{2}{3}} \Big(\frac{1}{K(a_1)^{\frac{1}{2}}} + \frac{1}{K(a_2)^{\frac{1}{2}}}\Big)^{\frac{2}{3}} \big(1 + o(1)\big) = c_\infty(a_1, a_2)\big(1 + o(1)\big)$$

where S_3 is the Sobolev constant for S^3.

For a proof we refer the reader to Lemma 4.1 in [12].

Lemma 3.2. *Let $n = 3$. For $p + q \geqslant 1$, there exists a pseudogradient V for J, and a constant $c > 0$ independent of*

$$u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \sum_{j=1}^{q} \alpha_j \varphi_{(a_j,\lambda_j)} \in V(p, q, \varepsilon),$$

such that the following properties hold

(**i**)

$$(-\partial J(u), V) \geqslant \sum_{i=1}^{p} \frac{1}{\lambda_i}\Big|\frac{\partial K(a_i)}{\partial \nu}\Big| + \frac{|\nabla_T K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \big|1 - J(u)^3 \alpha_i^4 K(a_i)\big| + \sum_{j=1}^{q} \frac{c}{\lambda_{p+j} d_{p+j}} + c \sum_{k \neq l} \varepsilon_{kl}.$$

(**ii**)

$$(-\partial J(u + \bar{v}), V + \frac{\partial \bar{v}}{\partial(\alpha, a, \lambda)}(V)) \geqslant \sum_{i=1}^{p} \frac{1}{\lambda_i}\Big|\frac{\partial K(a_i)}{\partial \nu}\Big| + \frac{|\nabla_T K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \big|1 - J(u)^3 \alpha_i^4 K(a_i)\big| + \sum_{j=1}^{q} \frac{c}{\lambda_{p+j} d_{p+j}} + c \sum_{k \neq l} \varepsilon_{kl}.$$

(**iii**) *V is bounded.*

For a proof one can see Proposition 3.1 in [12].

Lemma 3.3. *For $u = \alpha \delta_{(a,\lambda)} \in V(1, 0, \varepsilon)$ such that $a \in \mathcal{V}(y, \gamma)$, $y \in F^0_+ \cup F_+$ is a critical point of K_1, and $\gamma > 0$, there is a change of variables $(\tilde{a}, \tilde{\lambda})$ such that, if $y \in F_+$, then*

$$J(u) = \psi_1(\tilde{a}, \tilde{\lambda}) = \big(\tfrac{S_n}{2}\big)^{\frac{2}{n}} \frac{1}{K(\tilde{a})^{\frac{n-2}{n}}} \Big(1 + (1 - \eta)\frac{c}{\tilde{\lambda}} \frac{\partial K}{\partial \nu}(y)\Big)$$

and if $y \in F_+^0$, then

$$J(u) = \psi_2(\tilde{a}, \tilde{\lambda}) = \big(\tfrac{S_n}{2}\big)^{\frac{2}{n}} \frac{1}{K(\tilde{a})^{\frac{n-2}{n}}} \Big(1 + \frac{c}{\tilde{\lambda}} \frac{\partial K}{\partial \nu}(\tilde{a}) + (1-\eta)\tilde{c} \frac{\Delta_g K(\tilde{a})}{\tilde{\lambda}^2}\Big),$$

where η is a small positive real and $c, \tilde{c}$ are positive constants. $\mathcal{V}(y,\gamma)$ denotes a neighborhood of y.

For a proof one can see Proposition 2.8 in [12].

Proof of Theorem 1.1. We argue by contradiction. We assume that J has no critical points in $V_\eta(\sum^+)$ where,

$$V_\eta(\textstyle\sum^+) = \big\{u \in \sum \ / \ \|u^-\| \leqslant \eta\big\}$$

where η is a small positive constant and $u^- = \max(0, -u)$ denotes the negative part of u. $V_\eta(\sum^+)$ is a neighborhood of $\sum^+$ in $\sum$. Let

$$c_\infty(y_0, y_1) = \big(\tfrac{S_3}{2}\big)^{\frac{2}{3}} \Big(\frac{1}{K(y_0)^{\frac{1}{2}}} + \frac{1}{K(y_1)^{\frac{1}{2}}}\Big)^{\frac{2}{3}}.$$

We observe that under the assumption $(\mathbf{A_3})$, the flow lines of the pseudogradient V defined by the preceding Lemma 3.2, satisfy the Palais-Smale condition in $V(p,q,\varepsilon)$ for $p+q \geqslant 2$, since we have no critical points at infinity containing multiple masses under the assumption $(\mathbf{A_3})$, (see Proposition 3.1 in [12]). Thus, the critical points at infinity of our variational problem lie in $V(p,q,\varepsilon)$ with $p+q=1$, that is to say, they are made up of a single mass, and hence, they lie in $V(1,0,\varepsilon) \cup V(0,1,\varepsilon)$. Using Proposition 2.4, and the assumption $(\mathbf{A_5})$ of Theorem 1.1, it follows that the only critical points at infinity of J under the level $c_i = c_\infty(y_0, y_i) + \varepsilon$, for ε small enough, are $\delta_{(y_j,+\infty)}$, where $y_j \in (F_+^0 \cup F_+)$, for $0 \leqslant j \leqslant i$. We let in the sequel $\delta_{(y_j,+\infty)} = y_{j\infty}$.

The unstable manifolds at infinity, for the pseudogradient V, of such critical points at infinity, which we denote by $W_u(y_{j\infty})$, can be described using the expansion given by Proposition 2.8 of [12], as the product of $W_s(y_j)$, for the pseudogradient $\mathcal{Z}$ of K_1, by $[A, +\infty)$ which is the domain of the variable λ, for some large enough positive real A. Since J has no critical point, it follows that

$$J_{c_i} = \big\{u \in \textstyle\sum^+ / J(u) \leqslant c_i\big\}$$

retracts by deformation onto,

$$X_{i\infty} = \bigcup_{0 \leqslant j \leqslant i} \overline{W_u}(y_{j\infty}), \quad y_j \in (F_+^0 \cup F_+)$$

(see Proposition 7.24 and Theorem 8.2 of [9]). Using assumption $(\mathbf{A_1})$, $X_{i\infty}$ can be parameterized by $X_i \times [A, +\infty)$ where X_i is given by (1.4), (we recall that the stable manifold for K is the unstable manifold for $\frac{1}{K}$). We now claim that $X_{i\infty}$ is contractible in J_{c_i}. Indeed, let $a_1, a_2 \in \partial S^3_+$, $\alpha_1, \alpha_2 > 0$, and λ large enough. For $u = \alpha_1 \delta_{(a_1,\lambda)} + \alpha_2 \delta_{(a_2,\lambda)}$, we have, by Lemma 3.1, the following expansion:

$$J(u) \leqslant \left(\tfrac{S_3}{2}\right)^{\frac{2}{3}} \left(\frac{1}{K(a_1)^{\frac{1}{2}}} + \frac{1}{K(a_2)^{\frac{1}{2}}}\right)^{\frac{2}{3}} \big(1 + o(1)\big) = c_\infty(a_1, a_2) + o(1). \quad (3.1)$$

Let

$$\begin{aligned} \psi : [0,1] \times X_i \times [A, +\infty) &\longrightarrow \textstyle\sum^+ \\ (t, x, \lambda) &\longmapsto \frac{t\delta_{(y_0,\lambda)} + (1-t)\delta_{(x,\lambda)}}{\|t\delta_{(y_0,\lambda)} + (1-t)\delta_{(x,\lambda)}\|} \end{aligned}$$

ψ is continuous and satisfies: $\psi(0, x, \lambda) = \frac{\delta_{(x,\lambda)}}{\|\delta_{(x,\lambda)}\|}$ and $\psi(1, x, \lambda) = \frac{\delta_{(y_0,\lambda)}}{\|\delta_{(y_0,\lambda)}\|}$. Now, since $K_1(x) \geqslant K_1(y_i)$ for any $x \in X_i$, it follows from inequality (3.1) that $J\big(\psi(t, x, \lambda)\big) < c_i$ for all $(t, x, \lambda) \in [0,1] \times X_i \times [A, +\infty)$. Thus, the contraction ψ is performed under the level c_i. We derive that $X_{i\infty}$ is contractible in J_{c_i}, which (in turn) retracts by deformation onto $X_{i\infty}$. Therefore, $X_{i\infty}$ is contractible, leading to the contractibility of X_i, which is in contradiction with the assumption $(\mathbf{A_4})$. Hence, there exists a critical point of J in $V_\eta(\sum^+)$. By standard regularity results, we derive that this critical point of J is a solution for (1.1).

Next, we will show that this critical point has a Morse index $\geqslant m_i$.

Arguing by contradiction, we assume that this Morse index is $\leqslant m_i - 1$. Then, up to a perturbation of J, we may assume that all the critical points of J are nondegenerate and have their Morse indexes $\leqslant m_i - 1$. Such critical points do not change the homological group in dimension m_i, of the level sets of J. Now, let

$$c_\infty(y_i) = \left(\tfrac{S_3}{2}\right)^{\frac{2}{3}} \frac{1}{K(y_i)^{\frac{1}{3}}}$$

and let ε be a small positive real. Since $X_{i\infty}$ defines a homological class in dimension m_i which is trivial in J_{c_i}, but not trivial in $J_{c_\infty(y_i)+\varepsilon}$, our result follows.

Proof of Corollary 1.1. Recall that K_1 has only nondegenerate critical points $y_0, y_1, ..., y_h$. Thus, for $i = h$, we have by (1.4),

$$X_h = \bigcup_{y_i \in (F^0_+ \cup F_+)} \overline{W_s}(y_i).$$

Let $\chi(X_h)$ be the Euler-Poincaré characteristic of X_h. Then, we see that:

$$\chi(X_h) = \sum_{y_i \in (F_+^0 \cup F_+)} (-1)^{2-\mathrm{ind}(K_1, y_i)} = \sum_{y_i \in (F_+^0 \cup F_+)} (-1)^{\mathrm{ind}(K_1, y_i)}.$$

Now, the hypothesis of Corollary 1.1 implies that $\chi(X_h) \neq 1$, from which we derive that X_h is not contractible, and thus we have the result by Theorem 1.1.

Proof of Theorem 1.2. We recall that here $n = 3$. Let

$$X_\infty = \bigcup_{y_i \in B_k} \overline{W_u}(y_{i\infty}) \tag{3.2}$$

where B_k, given by (1.7), is $B_k = \{\ y \in F_+^0 \cup F_+ : \mathrm{ind}(K_1, y) = 2 - k\ \}$ and where $W_u(y_{i\infty})$ is the unstable manifold at infinity of the critical point at infinity $\delta_{(y_i,\infty)}$, for a decreasing pseudogradient V for J (see [6], [8]). Then, as we saw in the proof of Theorem 1.1 above, for each $y \in (F_+^0 \cup F_+)$ the unstable manifold $W_u(y)_\infty$ can be described as the product $W_s(y) \times [A, +\infty)$, where A is a large enough positive real. Thus, X_∞ can be parameterized by $X \times [A, +\infty)$.

Arguing by contradiction, we suppose that J has no critical point in $\sum^+$. It follows from Proposition 7.24 and Theorem 8.2 of [9], that $\sum^+$ retracts by deformation onto

$$\bigcup_{y_i \in (F_+^0 \cup F_+)} W_u(y_{i\infty}). \tag{3.3}$$

More precisely, $\sum^+$ retracts by deformation onto $X_\infty \cup D_\infty$ where

$$D_\infty = \bigcup_{y_i \in D} W_u(y_{i\infty})$$

with $D = \{y \in (F_+^0 \cup F_+),\ \mathrm{ind}(K_1, y) > 2 - k\}$. For each $y \in D$, the critical point at infinity $\delta_{(y,\infty)}$ has a Morse index $\leqslant k - 1$. Recall that a critical point at infinity can be treated as a usual critical point, once around it a Morse Lemma at infinity is performed, from which we can derive, just as in the classical Morse Theory, the difference of topology induced by this noncompact orbit, and compute its Morse index. A Morse Lemma at infinity has been established for problem (1.1) in [12], from which we derive that, $\mathrm{ind}(J, \delta_{(y,\infty)}) = 2 - \mathrm{ind}(K_1, y)$. Thus, D_∞ is a stratified set of dimension at most $k - 1$. Since $\sum^+$ is contractible, then $H_s(X_\infty \cup D_\infty) = 0$ for each $s \in \mathbb{N}^*$. Writing then, the exact homology sequence of the pair $(X_\infty \cup D_\infty, X_\infty)$, we have $\cdots \longrightarrow H_{k+1}(X_\infty \cup D_\infty) \xrightarrow{\pi} H_{k+1}(X_\infty \cup D_\infty, X_\infty) \xrightarrow{\partial}$

$H_k(X_\infty) \xrightarrow{i} H_k(X_\infty \cup D_\infty) \longrightarrow \cdots$. Since $H_s(X_\infty \cup D_\infty) = 0$ for each $s \in \mathbb{N}^*$, then we derive that $H_k(X_\infty) = H_{k+1}(X_\infty \cup D_\infty, X_\infty)$. Observe now that $(X_\infty \cup D_\infty, X_\infty)$ is a stratified set of dimension at most k, so we have $H_{k+1}(X_\infty \cup D_\infty, X_\infty) = 0$. Therefore, $H_k(X_\infty) = 0$, and thus $H_k(X) = 0$ (since $X_\infty = X \times [A, +\infty)$), which is in contradiction with the hypothesis $H_k(X) \neq 0$ of our Theorem 1.2. Hence (1.1) has a solution. Theorem 1.2 is thereby proved.

4. Proof of Theorem 1.3

We will argue by contradiction, so we assume that equation (1.1) has no solution, or equivalently, that J has no critical point.

Recall that $\ell' \leqslant \ell \leqslant h$, ℓ is chosen so that for each $s \geqslant \ell + 1$, $y_s \notin F_+$. We denote by $y_0, \dots, y_{\ell'}, \dots, y_\ell, \dots, y_h$ the critical points of K_1, ordered in such a way that

$$K(y_0) \geqslant \dots K(y_{\ell'}) \geqslant \dots K(y_\ell) \geqslant \dots K(y_h). \tag{4.1}$$

Thus, $F_+ \subset \{y_0, \dots, y_{\ell'}, \dots, y_\ell\}$. Let $F'_+ = F_+ \cap \{y_0, \dots, y_{\ell'}\}$. Since under our hypothesis $(\mathbf{H_1})$, $F^0_+ = \emptyset$. We may then write

$$X' = \bigcup_{y \in F'_+} W_s(y), \quad X = \bigcup_{y \in F_+} \overline{W_s}(y), \quad T = \bigcup_{0 \leqslant i \leqslant \ell} \overline{W_s}(y_i), \quad \Gamma = \bigcup_{z \in \mathcal{I}_-} W_s(z),$$

where $\mathcal{I}_- = \left\{ \ z : z \notin F_+, \text{ and } \exists y \in F_+ \text{dominating } z : W_u(y) \cap W_s(z) \neq \emptyset \right\}$.

Let us now introduce the following sets:

$$X'_\infty = \bigcup_{y \in F'_+} W_u(y_\infty), \qquad T_\infty = \bigcup_{0 \leqslant i \leqslant \ell} \overline{W_u}(y_{i\infty}),$$

where y_∞ (resp. $y_{i\infty}$) is the critical point at infinity of J generated by the critical point y (resp. y_i) of K. By hypothesis, X'_∞ is contractible in T_∞. Using the flow, one can bring back this deformation into T'_∞, where $T'_\infty = \bigcup_\phi \overline{W_u}(\phi)$, where ϕ runs over critical points or, critical points at infinity of J. Thus, we have a natural embedding $T_\infty \subset T'_\infty$. Since, arguing by contradiction, we supposed that J has no critical points, ϕ runs over critical points at infinity of J only. Since, by $(\mathbf{H_4})$, the contraction is performed under the first energy level of the critical points at infinity made up of two masses, then T'_∞ contains only the unstable manifolds at infinity of the critical points at infinity made up of a single mass. Moreover, by $(\mathbf{H_2})$, the critical points at infinity of J, generated by the critical points of K in $\overset{\circ}{S^n_+}$, have a larger energy level than the highest energy level reached in

T_∞ by the critical points at infinity of J generated by critical points of K_1, this level is $J\big(\delta_{(y_\ell,+\infty)}\big) = \big(\frac{S_n}{2}\big)^{\frac{2}{n}} \frac{1}{K(y_\ell)^{\frac{n-2}{n}}}$. Thus, T'_∞ contains only the unstable manifolds at infinity of the critical points at infinity generated by the critical points of K_1. Using then Proposition 2.5, we derive that $T'_\infty = \bigcup_{y \in F_+} \overline{W_u}(y_\infty)$. Hence, it follows that in the absence of critical points for J we have

$$T_\infty = T'_\infty = \bigcup_{y \in F_+} \overline{W_u}(y_\infty).$$

Let ω be the Homological class of X' which is nontrivial in X. Since ω is m-dimensional, there exists a subset $I'_m \subset F'_+$ of critical points x^m of K_1 in F'_+ with $\mathrm{ind}(K_1, x^m) = n-1-m$, such that

$$\omega = \sum_{x^m \in I'_m} W_s(x^m) \qquad (\text{for } K) = \sum_{x^m \in I'_m} W_u(x^m) \qquad (\text{for } \frac{1}{K}).$$

Now, consider the chain

$$\omega_\infty = \sum_{x^m \in I'_m} W_u(x^m_\infty),$$

where x^m_∞ is the critical point at infinity (for J), associated to the critical point x^m (for $\frac{1}{K}$), (that is $x^m_\infty = \delta_{(x^m,\infty)}$), and $W_u(x^m_\infty)$ is the unstable manifold at infinity of x^m_∞ for a pseudogradient for J. We denote in the sequel, by ∂ the boundary operator of the Floer-Milnor Homology for $\frac{1}{K}$, and by ∂_∞ the analogous for J. By definition, ∂ operates on critical points of $\frac{1}{K}$, in the following way: if x^k is a critical point of $\frac{1}{K}$ of Morse index k, then we have

$$\partial W_u(x^k) = \sum_{x^{k-1}} i(x^k, x^{k-1}) W_u(x^{k-1}),$$

where x^{k-1} runs over the critical points (of $\frac{1}{K}$) of Morse index $k-1$, and, $i(x^k, x^{k-1})$ is the intersection number of x^k and x^{k-1}, that is to say, of $W_u(x^k)$ and $W_s(x^{k-1})$, (see [25] for definitions and notations).

If x^{m-1}_∞ is a critical point at infinity of Morse index $m-1$, we claim that

$$i(x^m_\infty, x^{m-1}_\infty) = i(x^m, x^{m-1}) + \sum_{z^{m-1} \in \Gamma} i(x^m, z^{m-1}).i(z^{m-1}_\infty, x^{m-1}_\infty), \qquad (4.2)$$

where z^{m-1} runs over the critical points in Γ of Morse index $m-1$. According to the expansion of J provided by Lemma 3.3, we see that z^{m-1}_∞ is the false critical point at infinity of J generated by z^{m-1}. Thus, z^{m-1}_∞ possesses only a half-unstable manifold at infinity, which is formed by the deconcentration direction. More precisely, since $z^{m-1} \notin F_+$, Proposition 2.5 above shows that z_∞^{m-1} is not a critical point at infinity. We can construct in some neighborhood of z_∞^{m-1} a pseudogradient which has no asymptotes in this region. $W_u(z_\infty^{m-1})$ then is not really an unstable manifold at infinity, but a set of flow lines emanating from a neighborhood of the point at infinity z_∞^{m-1}. (see [6] for more precise details).

Equality (4.2) signifies that the intersection number of $W_u(x_\infty^m)$ and $W_s(x_\infty^{m-1})$ is written as the intersection number of $W_u(x^m)$ and $W_s(x^{m-1})$ for $\frac{1}{K}$, (some kind of intersection number "at infinity", i.e., a part from any deconcentration direction), to which is added the deconcentration intersection number: every time where x^m dominates $i(x^m, z^{m-1})$ times some z^{m-1}, one has, (see [9], [25]), that $W_u(x^m)$ is locally fibrated $i(x^m, z^{m-1})$ times over $W_u(z^{m-1})$. On the other hand, if x^{m-1} is a critical point in X, $W_s(x_\infty^{m-1})$ may intersect $W_u(z_\infty^{m-1})$. By a dimensional argument, this intersection is made up of a finite number of points. These points are not in $\partial W_u(z_\infty^{m-1})$ the boundary of $W_u(z_\infty^{m-1})$, here again by a dimensional argument. Thus, we have a well defined intersection number $i(z_\infty^{m-1}, x_\infty^{m-1})$: equation (4.2) then, only expresses the fact that if $W_u(x^m)$ covers $i(x^m, z^{m-1})$ times $W_u(z^{m-1})$, then $W_u(x_\infty^m)$ covers $i(x^m, z^{m-1})$ times $W_u(z_\infty^{m-1})$. Now, since $W_u(z_\infty^{m-1})$ intersects $i(z_\infty^{m-1}, x_\infty^{m-1})$ times $W_s(x_\infty^{m-1})$, then $W_u(x_\infty^m)$ intersects $i(x^m, z^{m-1}).i(z_\infty^{m-1}, x_\infty^{m-1})$ times $W_s(x_\infty^{m-1})$ by deconcentration, that is to say, independently of the intersection at infinity which has been counted in the first term $i(x^m, x^{m-1})$ of the right hand side of equation (4.2).

If now, x_∞^{m+1} is a critical point at infinity of Morse index $m+1$, we prove in a same way, that

$$i(x_\infty^{m+1}, x_\infty^m) = i(x^{m+1}, x^m) + \sum_{z^m \in \Gamma} i(x^{m+1}, z^m).i(z_\infty^m, x_\infty^m). \tag{4.3}$$

Using then our hypothesis ($\mathbf{H_5}$), we have $i(x^{m+1}, z^m) = 0$, and (4.3) becomes

$$i(x_\infty^{m+1}, x_\infty^m) = i(x^{m+1}, x^m). \tag{4.4}$$

On the other side,

$$\omega = \sum_{x^m \in I'_m} W_u(x^m)$$

is a cycle. Thus, $\partial\omega = 0$, and therefore

$$\sum_{x^m \in I'_m} i(x^m, z^{m-1}) = 0$$

and, also for the same reason

$$\sum_{x^m \in I'_m} i(x^m, x^{m-1}) = 0.$$

Combining with equation (4.2) we derive that

$$\sum_{x^m \in I'_m} i(x^m_\infty, x^{m-1}_\infty) = 0. \tag{4.5}$$

Thus, the chain

$$\omega_\infty = \sum_{x^m \in I'_m} W_u(x^m_\infty)$$

is a cycle for ∂_∞, because all its intersection numbers with critical points of Morse index $m-1$ vanish. Since X'_∞ is contractible in T'_∞, this cycle ω_∞ is null in $H_m(T'_\infty)$. Therefore,

$$\omega_\infty = \partial_\infty\big(\sum_{x^{m+1}_\infty} W_u(x^{m+1}_\infty)\ \big). \tag{4.6}$$

for a family of critical points at infinity x^{m+1}_∞ of Morse index $m+1$. But, we have

$$\partial_\infty\big(\sum_{x^{m+1}_\infty} W_u(x^{m+1}_\infty)\ \big) = \sum_{x^{m+1}_\infty}\big(\sum_{x^m_\infty} i(x^{m+1}_\infty, x^m_\infty)\ \big)W_u(x^m_\infty),$$

where x^m_∞ runs over the critical points at infinity of Morse index m. Using then (4.2), we find

$$\partial_\infty\big(\sum_{x^{m+1}_\infty} W_u(x^{m+1}_\infty)\ \big) = \sum_{x^{m+1}_\infty}\big(\sum_{x^m_\infty} i(x^{m+1}, x^m)\ \big)W_u(x^m_\infty).$$

Since $\omega_\infty = \partial_\infty\big(\sum_{x^{m+1}_\infty} W_u(x^{m+1}_\infty)\ \big)$ we then have,

$$\sum_{x^m \in I'_m} W_u(x^m_\infty) = \sum_{x^m_\infty}\ \sum_{x^{m+1}_\infty} i(x^{m+1}, x^m) W_u(x^m_\infty).$$

Thus, if x^m is in X, (i.e. x^m generates a critical point at infinity x^m_∞), we have

$$\begin{cases} \displaystyle\sum_{x^{m+1}_\infty} i(x^{m+1}, x^m) & = 1 \quad \text{if } x^m \in I'_m, \\ & = 0 \quad \text{otherwise.} \end{cases}$$

If x^m is in Γ, then by hypothesis $i(x^{m+1}, x^m) = 0$. From which we derive that:

$$\sum_{x^m \in I'_m} W_u(x^m) = \sum_{x^m} \sum_{x^{m+1}} i(x^{m+1}, x^m) W_u(x^m),$$

where the sum in the right hand side is now extended to all critical points x^m of index m of $\frac{1}{K_1}$, that is, to the set

$$B_m = \{ \ y \in F_+ \quad \text{s.t.} \quad \text{ind}(K_1, y) = (n-1) - m \}.$$

Hence,

$$\omega = \partial \Big(\sum_{x^{m+1} \in I_{m+1}} W_u(x^{m+1}) \ \Big),$$

where I_{m+1} is a set of critical points x^{m+1} of K_1 of index $m+1$, which generate critical points at infinity x_∞^{m+1}. That is to say, using Proposition 2.5, $I_{m+1} \subset F_+$. Thus, ω is null in X, which is the researched contradiction. Theorem 1.3 is thereby proved. □

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