ON THE REGULARITY CRITERIA FOR THE GENERALIZED NAVIER-STOKES EQUATIONS AND LAGRANGIAN AVERAGED EULER EQUATIONS

Jishan Fan
Department of Applied Mathematics, Nanjing Forestry University
Nanjing 210037, P.R.China

Tohru Ozawa
Department of Mathematics, Hokkaido University
Sapporo, 060-0810, Japan

(Submitted by: Reza Aftabizadeh)

Abstract. We obtain some regularity conditions for solutions of the 3D generalized Navier-Stokes equations with fractional powers of the Laplacian, in terms of the velocity, the vorticity, and the pressure in Besov space, Triebel-Lizorkin space, and Lorentz space, respectively. We also present a regularity condition for the 3D Lagrangian averaged Euler equations.

1. Introduction

We study the following generalized Navier-Stokes equations:

$$\partial_t u + (u \cdot \nabla)u + \nabla\pi + (-\Delta)^{\ell/2} u = 0, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$u|_{t=0} = u_0(x),\ \operatorname{div} u_0 = 0,\ x \in \mathbb{R}^3, \tag{1.3}$$

where u is the velocity field, π is the scalar pressure, and ℓ is a positive constant. The case $\ell = 2$ corresponds to the usual Navier-Stokes equations.

The system (1.1)-(1.3) was first considered by J.L. Lions in [23], and the global regularity for $\ell \geq 5/2$ is shown there. When $\ell > 2$, existence and uniqueness of solutions for (1.1)-(1.3) has been studied by S. Tourville [31]. Y. Zhou [35] obtained the regularity condition:

$$u \in L^r(0,T;L^p(\mathbb{R}^3)), \quad \text{with} \quad \frac{\ell}{r} + \frac{3}{p} = \ell - 1, \frac{3}{\ell - 1} < p \leq \infty, \tag{1.4}$$

Accepted for publication: February 2008.
AMS Subject Classifications: 35Q35, 35M10, 76B03, 76D05.
Supported by NSFC (Grant No. 10301014).

or

$$\Lambda^{\ell/2}u \in L^r(0,T;L^p(\mathbb{R}^3)), \text{ with } \frac{\ell}{r}+\frac{3}{p}=\frac{3\ell}{2}-1,\ \frac{6}{3\ell-2}<p<\frac{6}{\ell-2}. \quad (1.5)$$

Here, $\Lambda := (-\Delta)^{1/2}$. When $0 < \ell < 2$, there were studies on the small data global well-posedness for (1.1)-(1.3) by D. Chae [4], M. Cannone and G. Karch [3], D. Chae and J. Lee [5], and J. Wu [33]. D. Chae [4] proved the regularity criterion:

$$\omega := \operatorname{curl} u \in L^r(0,T;L^p(\mathbb{R}^3)), \quad \text{with } \frac{3}{p}+\frac{\ell}{r}\le \ell, \quad (1.6)$$

where $\frac{6}{\ell} < p \le \infty$. This covered the case $\ell = 2$ due to H.B. da Veiga [1]. In this note, we are concerned with the regularity condition of the strong solutions to (1.1)-(1.3). The first result for Navier-Stokes equations in this direction is obtained independently by Serrin [26] and Ohyama [25] which states that if weak solution u satisfies

$$u \in L^r(0,T;L^p(\mathbb{R}^3)), \text{ with } \frac{2}{r}+\frac{3}{p}\le 1,\ 3<p\le\infty, \quad (1.7)$$

then u is smooth. After that there are further developments and refinements by Fabes, Jones, and Riviere [12], Giga [14], Sohr and Von Wahl [27], and Struwe [29]. In the important case $p = 3, r = \infty$ in (1.7), a smallness condition was required at first and removed by Escauriaza, Sverak, and Seregin [11]. Kozono, Ogawa, and Taniuchi [21] proved the following regularity condition:

$$\omega := \operatorname{curl} u \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)), \quad (1.8)$$

where $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov spaces. Kozono and Shimada [22] refined (1.7) by the following condition:

$$u \in L^{\frac{2}{1-s}}(0,T;\dot{F}^{-s}_{\infty,\infty}) \text{ for } 0<s<1, \quad (1.9)$$

where $\dot{F}^{-s}_{\infty,\infty}$ denotes the homogeneous Triebel-Lizorkin space. Y. Zhou [34] and M. Struwe [30] proved similar regularity conditions on the pressure:

$$\pi \in L^r(0,T;L^p(\mathbb{R}^3)) \text{ with } \frac{2}{r}+\frac{3}{p}=2 \text{ for } \frac{3}{2}<p\le\infty, \quad (1.10)$$

or

$$\nabla\pi \in L^r(0,T;L^p(\mathbb{R}^3)) \text{ with } \frac{2}{r}+\frac{3}{p}=3 \text{ for } 1<p\le\infty. \quad (1.11)$$

See also Cai and Zhai [2]. The purpose of this paper is to generalize all those available results for (1.1) in terms of homogeneous Besov space $\dot{B}^0_{p,\infty}$, homogeneous Triebel-Lizorkin space $\dot{F}^{-s}_{\infty,\infty}$, and Lorentz space $L^{p,\infty}$. We now state our main results in this paper.

Theorem 1.1. *Let u_0 be smooth and* $\operatorname{div} u_0 = 0$ *in $\mathbb{R}^3$ and $\ell < \frac{5}{2}$. Assume that one of the following conditions is satisfied:*

$$\ell > 1, u \in L^r(0,T;\dot{B}^0_{p,\infty}) \quad with \quad \frac{\ell}{r} + \frac{3}{p} = \ell - 1, \; \frac{3}{\ell - 1} < p < \infty; \tag{1.12}$$

$$\ell > 0, \omega \in L^r(0,T;\dot{B}^0_{p,\infty}) \quad with \quad \frac{\ell}{r} + \frac{3}{p} = \ell, \; \frac{3}{\ell} < p \leq \infty; \tag{1.13}$$

$$\ell > 1, u \in L^r(0,T;\dot{F}^{-s}_{\infty,\infty}) \quad with \quad \frac{\ell}{r} = \ell - 1 - s, \; 0 < s < 1, \; \frac{\ell}{2} - 1 < s < \ell - 1; \tag{1.14}$$

$$\ell > 0, \omega \in L^r(0,T;\dot{F}^{-s}_{\infty,\infty}) \quad with \quad \frac{\ell}{r} = \ell - s, \; 0 < s < 1, \; 0 < s < \frac{\ell}{2}; \tag{1.15}$$

$$\ell \geq 2, \pi \in L^r(0,T;L^{p,\infty}) \quad with \quad \frac{\ell}{r} + \frac{3}{p} = 2\ell - 2, \; \frac{3}{2\ell - 2} < p \leq \frac{3}{\ell - 2}. \tag{1.16}$$

Then there is no singularity up to T.

Remark 1.1. We observe that the system (1.1) is invariant under scaling transform $(u,\pi) \mapsto (u_\lambda, \pi_\lambda)$, where

$$u_\lambda(t,x) := \lambda^{\ell-1} u(\lambda^\ell t, \lambda x), \; \pi_\lambda(t,x) := \lambda^{2\ell-2}\pi(\lambda^\ell t, \lambda x), \; \lambda > 0,$$

which induces the scaling for the vorticity, $\omega \mapsto \omega_\lambda, \; \omega_\lambda(t,x) := \lambda^\ell \omega(\lambda^\ell t, \lambda x)$. Furthermore, we note that

$$\|u\|_{L^r(0,T;L^p(\mathbb{R}^3))} = \|u_\lambda\|_{L^r(0,\lambda^\ell T;L^p(\mathbb{R}^3))}, \quad if \quad \frac{\ell}{r} + \frac{3}{p} = \ell - 1;$$

$$\|\pi\|_{L^r(0,T;L^p(\mathbb{R}^3))} = \|\pi_\lambda\|_{L^r(0,\lambda^\ell T;L^p(\mathbb{R}^3))}, \quad if \quad \frac{\ell}{r} + \frac{3}{p} = 2\ell - 2;$$

$$\|\omega\|_{L^r(0,T;L^p(\mathbb{R}^3))} = \|\omega_\lambda\|_{L^r(0,\lambda^\ell T;L^p(\mathbb{R}^3))}, \quad if \quad \frac{\ell}{r} + \frac{3}{p} = \ell.$$

In this sense, our conditions (1.12)-(1.16) are optimal.

Remark 1.2. Since $L^p \subset \dot{B}^0_{p,\infty}, \; L^{3/s} \subset \dot{F}^{-s}_{\infty,\infty}$, and $L^\infty \subset BMO \subset \dot{B}^0_{\infty,\infty}$, our conditions (1.13) and (1.15) refines (1.6) in D.Chae [4]. Even when $\ell = 2$, our condition (1.15) is new.

Remark 1.3. It is a simple matter to generalize the conditions in Theorem 1.1 to the n-dimensional case. However, for simplicity we omit the details here.

2. Preliminaries

We first recall the definition of the homogeneous Littlewood-Paley decomposition which will be used to define function spaces. We follow [32]. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing smooth functions. Given $f \in \mathcal{S}$, its Fourier transform $\mathcal{F}(f) = \hat{f}$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx.$$

We consider $\varphi \in \mathcal{S}$ satisfying *supp* $\hat{\varphi} \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}$, and $\hat{\varphi} > 0$ if $\frac{2}{3} < |\xi| < \frac{3}{2}$. Setting $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$ (in other words, $\varphi_j(x) = 2^{jn}\varphi(2^j x)$), we can adjust the normalization constant in front of $\hat{\varphi}$ so that

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1,\ \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given $k \in \mathbb{Z}$, we define the function $S_k \in \mathcal{S}$ by its Fourier transform

$$\hat{S}_k(\xi) := 1 - \sum_{j\geq k+1} \hat{\varphi}_j(\xi).$$

We observe

$$\text{supp } \hat{\varphi}_j \cap \text{ supp } \hat{\varphi_{j'}} = \emptyset \ \text{ if } \ |j - j'| \geq 2.$$

Let $s \in \mathbb{R}, (p, q) \in [1, \infty) \times [1, \infty]$. Given $f \in \mathcal{S}'$, we denote $\Delta_j f := \varphi_j * f$, and then the homogeneous Triebel-Lizorkin semi-norm $\|f\|_{\dot{F}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{F}^s_{p,q}} := \begin{cases} \left\| \left(\sum\limits_{j\in\mathbb{Z}} 2^{jqs} |\Delta_j f(\cdot)|^q \right)^{1/q} \right\|_{L^p} & if \quad q \in [1, \infty), \\ \left\| \sup\limits_{j\in\mathbb{Z}} \left(2^{js} |\Delta_j f(\cdot)| \right) \right\|_{L^p} & if \quad q = \infty. \end{cases}$$

The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is a quasi-normed space with the quasi-norm given by $\|\cdot\|_{\dot{F}^s_{p,q}}$. For $s > 0, (p, q) \in [1, \infty) \times [1, \infty]$. We define the inhomogeneous Triebel-Lizorkin space norm $\|f\|_{F^s_{p,q}}$ of $f \in \mathcal{S}'$ as

$$\|f\|_{F^s_{p,q}} := \|f\|_{L^p} + \|f\|_{\dot{F}^s_{p,q}}.$$

The inhomogeneous Triebel-Lizorkin space is a Banach space equipped with the norm, $\|\cdot\|_{F^s_{p,q}}$. Similarly, for $s \in \mathbb{R}, (p,q) \in [1,\infty]^2$, the homogeneous Besov norm $\|f\|_{\dot{B}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{B}^s_{p,q}} := \begin{cases} \Big(\sum\limits_{-\infty}^{+\infty} 2^{jqs}\|\varphi_j * f\|^q_{L^p}\Big)^{1/q} & \text{if } \ q \in [1,\infty), \\ \sup\limits_j (2^{js}\|\varphi_j * f\|_{L^p}) & \text{if } \ q = \infty. \end{cases}$$

The homogeneous Besov space $\dot{B}^s_{p,q}$ is a quasi-normed space with the quasi-norm given by $\|\cdot\|_{\dot{B}^s_{p,q}}$. For $s > 0$ we define the inhomogeneous Besov space norm $\|f\|_{B^s_{p,q}}$ of $f \in \mathcal{S}'$ as $\|f\|_{B^s_{p,q}} := \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}$.

Below we recall lemmas that will be used in the proof of our results.

Lemma 2.1. (Bernstein's lemma). *Let $1 \le p \le \infty$. Then there exists a constant C_k such that for any $f \in L^p$ with* supp $\hat{f} \subset \{2^{j-2} \le |\xi| < 2^j\}$

$$C_k^{-1} 2^{jk}\|f\|_{L^p} \le \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p} \le C_k 2^{jk}\|f\|_{L^p}. \tag{2.1}$$

For the proof see J.Y. Chemin [6]. The following lemma is also well known (see, e.g., [32]).

Lemma 2.2. *For any $k \in \mathbb{Z}_+$, there exists a constant C_k such that the following inequality holds:*

$$C_k^{-1} \sum_{|\alpha|=h} \|\partial^\alpha f\|_{\dot{F}^s_{p,q}} \le \|f\|_{\dot{F}^{s+k}_{p,q}} \le C_k \sum_{|\alpha|=k} \|\partial^\alpha f\|_{\dot{F}^s_{p,q}}. \tag{2.2}$$

Lemma 2.3. *Let $\sigma > 0, 1 \le p_1, p_2 \le \infty, \sigma_1, \sigma_2 \in \mathbb{R}$ such that $\frac{3}{p_i} - \sigma_i > 0$ $i = 1,2$, then the following inequality holds:*

$$\begin{aligned} &\Big(\sum_{j\in\mathbb{Z}} 2^{2j\sigma}\|[f,\Delta_j]\nabla g\|^2_{L^2}\Big)^{1/2} \\ &\le C\Big(\|\nabla f\|_{\dot{B}^{\sigma_1}_{p_1,\infty}}\|g\|_{\dot{B}^{\sigma-\sigma_1+\frac{3}{p_1}}_{2,2}} + \|\nabla g\|_{\dot{B}^{\sigma_2}_{p_2,\infty}}\|f\|_{\dot{B}^{\sigma-\sigma_2+\frac{3}{p_2}}_{2,2}}\Big). \end{aligned} \tag{2.3}$$

If $\sigma_1 = 0$, $p_1 = \infty, \|\nabla f\|_{\dot{B}^{\sigma_1}_{p_1,\infty}}$ has to be replaced by $\|\nabla f\|_{L^\infty}$, or if $\sigma_2 = 0, p_2 = \infty, \|\nabla g\|_{\dot{B}^{\sigma_2}_{p_2,\infty}}$ has to be replaced by $\|\nabla g\|_{L^\infty}$. Here $[f,\Delta_j]\nabla g := f\Delta_j(\nabla g) - \Delta_j(f\nabla g)$.

For the proof, see [8].

Lemma 2.4. (1) *Let* $1 < p < \infty, 1 < q < \infty$ *and let* $s > 0, \alpha > 0, \beta > 0$. *Let* $1 < p_1 < \infty, 1 < p_2 \le \infty$ *and* $1 < r_1 \le \infty, 1 < r_2 < \infty$ *so that* $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$. *Then for any* $f \in \dot{F}^{s+\alpha}_{p_1,q} \cap \dot{F}^{-\beta}_{r_1,\infty}$ *and* $g \in \dot{F}^{-\alpha}_{p_2,\infty} \cap \dot{F}^{s+\beta}_{r_2,q}$ *we have* $fg \in \dot{F}^s_{p,q}$ *with the estimate*

$$\|fg\|_{\dot{F}^s_{p,q}} \le C\left(\|f\|_{\dot{F}^{s+\alpha}_{p_1,q}}\|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}}\|g\|_{\dot{F}^{s+\beta}_{r_2,q}}\right). \tag{2.4}$$

(2) *Let* $1 < p \le \infty$ *and let* $s > 0, \alpha > 0, \beta > 0$. *Let* $1 < p_1, p_2, r_1, r_2 \le \infty$ *satisfy* $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$. *Then for any* $f \in \dot{F}^{s+\alpha}_{p_1,\infty} \cap \dot{F}^{-\beta}_{r_1,\infty}$ *and* $g \in \dot{F}^{-\beta}_{p_2,\infty} \cap \dot{F}^{s+\alpha}_{r_2,\infty}$ *we have* $fg \in \dot{F}^s_{p,\infty}$ *with the estimate*

$$\|fg\|_{\dot{F}^s_{p,\infty}} \le C\left(\|f\|_{\dot{F}^{s+\alpha}_{p_1,\infty}}\|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}}\|g\|_{\dot{F}^{s+\beta}_{r_2,\infty}}\right). \tag{2.5}$$

For the proof see [22].

3. Proof of Theorem 1.1

The proof is based on the establishment of a priori estimates for u which can then be used to extend a smooth local solution globally in time.

We first show that Theorem 1.1. holds under the condition (1.12).

Let u be a smooth solution of (1.1)-(1.3). Applying the operator Δ_k to (1.1), then multiplying it by $\Delta_k u$ and integrating by parts and taking into account $\operatorname{div} u = 0$, we find that

$$\frac{1}{2}\frac{d}{dt}\|\Delta_k u\|^2_{L^2} + \|\Lambda^{\ell/2}\Delta_k u\|^2_{L^2} = \int_{\mathbb{R}^3}[u,\Delta_k]\nabla u\cdot\Delta_k u dx. \tag{3.1}$$

Integrating (3.1) over $(0,t)$ and multiplying by $2^{2km}(m \ge 3)$, then summing over $k \in \mathbb{Z}$, we obtain

$$\begin{aligned} & \frac{1}{2}\|u\|^2_{\dot{H}^m} + \int_0^t \|u\|^2_{\dot{H}^{m+\ell/2}} d\tau \\ \le\ & \frac{1}{2}\|u_0\|^2_{\dot{H}^m} + \int_0^t \sum_{k\in\mathbb{Z}} 2^{2km}\|[u,\Delta_k]\nabla u\|_{L^2}\|\Delta_k u\|_{L^2} d\tau \\ =\ & : \frac{1}{2}\|u_0\|^2_{\dot{H}^m} + \int_0^t I(\tau)d\tau. \end{aligned} \tag{3.2}$$

$I(\tau)$ can be bounded as follows:

$$I \le \Big(\sum_{k\in\mathbb{Z}} 2^{2k(m-\ell/2)}\|[u,\Delta_k]\nabla u\|^2_{L^2}\Big)^{1/2}\Big(\sum_{k\in\mathbb{Z}} 2^{2k(m+\ell/2)}\|\Delta_k u\|^2_{L^2}\Big)^{1/2}$$

$$\le C\|u\|_{\dot{B}^0_{p,\infty}}\|u\|_{\dot{B}_{2,2}^{m-\ell/2+1+\frac{3}{p}}}\|u\|_{\dot{H}^{m+\ell/2}}$$

$$\le C\|u\|_{\dot{B}^0_{p,\infty}}\|u\|^{\theta}_{\dot{H}^m}\|u\|^{2-\theta}_{\dot{H}^{m+\ell/2}} \le \frac{1}{2}\|u\|^2_{\dot{H}^{m+\ell/2}} + C\|u\|^r_{\dot{B}^0_{p,\infty}}\|u\|^2_{\dot{H}^m}, \quad (3.3)$$

by (2.3) (taking $\sigma_1 = \sigma_2 = -1, p_1 = p_2 = p, \sigma = m - \ell/2$) and the interpolation inequality [32]:

$$\|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,q}} \le \|f\|^{\theta}_{\dot{B}^{s_1}_{p,q}}\|f\|^{1-\theta}_{\dot{B}^{s_2}_{p,q}}, \quad (3.4)$$

with $s_1 = m, s_2 = m + \ell/2, \theta = \frac{\ell-(1+3/p)}{\ell/2}$ and $r = \frac{2}{\theta}$.

Inserting (3.3) into (3.2) and using the Gronwall's inequality gives the result.

Assume that (1.13) holds with $p < \infty$. Similarly, we have (3.2).

Using (2.3) with $\sigma_1 = \sigma_2 = 0, p_1 = p_2 = p > \frac{3}{\ell}$ and $\sigma = m - \frac{3}{2p}$ and the interpolation inequality (3.4) with $s_1 = m, s_2 = m + \ell, \theta = \frac{p\ell-3}{2p}$ and $r = \frac{1}{\theta}$, then $I(\tau)$ can be bounded as follows.

$$\begin{aligned} I &\le \Big(\sum_{k\in\mathbb{Z}} 2^{2k\left(m-\frac{3}{2p}\right)}\|[u,\Delta_k]\nabla u\|^2_{L^2}\Big)^{1/2}\Big(\sum_{k\in\mathbb{Z}} 2^{2k\left(m+\frac{3}{2p}\right)}\|\Delta_k u\|^2_{L^2}\Big)^{1/2} \\ &\le C\|\nabla u\|_{\dot{B}^0_{p,\infty}}\|u\|^2_{\dot{B}_{2,2}^{m+\frac{3}{2p}}} \le C\|\nabla u\|_{\dot{B}^0_{p,\infty}}\|u\|^{2\theta}_{\dot{H}^m}\|u\|^{2(1-\theta)}_{\dot{H}^{m+\ell}} \\ &\le \frac{1}{2}\|u\|^2_{\dot{H}^{m+\ell/2}} + C\|\nabla u\|^r_{\dot{B}^0_{p,\infty}}\|u\|^2_{\dot{H}^m} \\ &\le \frac{1}{2}\|u\|^2_{\dot{H}^{m+\ell/2}} + C\|\omega\|^r_{\dot{B}^0_{p,\infty}}\|u\|^2_{\dot{H}^m}. \end{aligned} \quad (3.5)$$

Here, we have used the following inequality [28]:

$$\|\nabla u\|_{\dot{B}^0_{p,\infty}} \le C\|\omega\|_{\dot{B}^0_{p,\infty}}. \quad (3.6)$$

Inserting (3.5) into (3.2) and the Gronwall's inequality yields the result.

Assume that (1.13) holds with $p = \infty$. Similarly, we have (3.2). Using (2.3) with $\sigma_1 = \sigma_2 = 0, p_1 = p_2 = p = \infty$ and $\sigma = m$, then $I(\tau)$ can be estimated as follows.

$$\begin{aligned} I &\le \Big(\sum_{k\in\mathbb{Z}} 2^{2km}\|[u,\Delta_k]\nabla u\|^2_{L^2}\Big)^{1/2}\Big(\sum_{k\in\mathbb{Z}} 2^{2km}\|\Delta_k u\|^2_{L^2}\Big)^{1/2} \\ &\le C\|\nabla u\|_{L^\infty}\|u\|^2_{\dot{H}^m} \le C\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}\|u\|^2_{\dot{H}^m}\log\left(3 + \|u\|^2_{\dot{H}^m}\right) \\ &\le C\|\omega\|_{\dot{B}^0_{\infty,\infty}}\|u\|^2_{\dot{H}^m}\log(3 + \|u\|^2_{\dot{H}^m}), \end{aligned} \quad (3.7)$$

where we have used the logarithmic Sobolev inequality [21]:

$$\|\nabla u\|_{L^\infty} \le C\|\nabla u\|_{\dot B^0_{\infty,\infty}} \log(3+\|u\|^2_{\dot H^m}), \tag{3.8}$$

and the following inequality [28]:

$$\|\nabla u\|_{\dot B^0_{\infty,\infty}} \le C\|\omega\|_{\dot B^0_{\infty,\infty}}. \tag{3.9}$$

Inserting (3.7) into (3.2) and the Gronwall's inequality leads to the result.

Assume that (1.14) holds. Let $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index with $|\alpha| = \alpha_1+\alpha_2+\alpha_3 \ge 1$, and let $v_\alpha := \partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}$.

Applying ∂^α to (1.1), we have for v_α the equation

$$\partial_t v_\alpha + \nabla q_\alpha + (-\Delta)^{\ell/2} v_\alpha = F_\alpha, \tag{3.10}$$

where $q_\alpha := \partial^\alpha \pi$ and

$$F_\alpha := -\partial^\alpha (u\cdot\nabla u) = -\partial^\alpha \nabla\cdot(u\otimes u).$$

Multiplying (3.10) by v_α and integrating by parts, we see that

$$\frac{1}{2}\|v_\alpha(t)\|^2_{L^2} + \int_0^t \|\Lambda^{\ell/2} v_\alpha(\tau)\|^2_{L^2} d\tau \le \frac{1}{2}\|v_\alpha(0)\|^2_{L^2} + \int_0^t |(F_\alpha, v_\alpha)| d\tau. \tag{3.11}$$

By the Schwarz inequality, there holds

$$|(F_\alpha, v_\alpha)| = |(\Lambda^{-\ell/2}\partial^\alpha \nabla\cdot u\otimes u, \Lambda^{\ell/2} v_\alpha)| \le \|u\otimes u\|_{\dot H^{1+|\alpha|-\ell/2}} \|\Lambda^{\ell/2} v_\alpha\|_{L^2},$$

it follows from Lemma 2.4 and the interpolation inequality that

$$\begin{aligned} |(F_\alpha, v_\alpha)| &\le C\|u\otimes u\|_{\dot F^{1+|\alpha|-\ell/2}_{2,2}} \|\Lambda^{\ell/2} v_\alpha\|_{L^2} \\ &\le C\|u\|_{\dot F^{-s}_{\infty,\infty}} \|u\|_{\dot F^{1+|\alpha|-\ell/2+s}_{2,2}} \|\Lambda^{\ell/2} v_\alpha\|_{L^2} \\ &\le C\|u\|_{\dot F^{-s}_{\infty,\infty}} \|v_\alpha\|^\theta_{L^2} \|\Lambda^{\ell/2} v_\alpha\|^{2-\theta}_{L^2} \\ &\le \frac{1}{2}\|\Lambda^{\ell/2} v_\alpha\|^2_{L^2} + C\|u\|^r_{\dot F^{-s}_{\infty,\infty}} \|v_\alpha\|^2_{L^2}, \end{aligned} \tag{3.12}$$

where $\theta = \frac{\ell-1-s}{\ell/2}$ and $r = \frac{2}{\theta}$.

Inserting (3.12) into (3.11) and the result follows from the Gronwall's inequality.

Assume that (1.15) holds. Applying curl to (1.1), we easily get for $\omega := \operatorname{curl} u$ the equation

$$\partial_t \omega + u\cdot\nabla\omega + (-\Delta)^{\ell/2}\omega = \omega\cdot\nabla u. \tag{3.13}$$

Multiplying (3.13) by $-\Delta\omega$ and integrating by parts, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 + \|\Lambda^{1+\ell/2}\omega\|_{L^2}^2 \\ = & -\int_{\mathbb{R}^3} \omega\cdot\nabla u\cdot\Delta\omega dx + \int_{\mathbb{R}^3} u\cdot\nabla\omega\cdot\Delta\omega dx =: I_1 + I_2. \end{aligned} \tag{3.14}$$

Using Lemma 2.4 and the interpolation inequality, we bound I_1 as follows.

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^3} \omega\cdot\nabla u\cdot\Lambda^2\omega dx = \int_{\mathbb{R}^3} \Lambda^{1-s}(\omega\cdot\nabla u)\cdot\Lambda^{1+s}\omega dx \\ &\leq \|\Lambda^{1-s}(\omega\cdot\nabla u)\|_{L^2}\|\Lambda^{1+s}\omega\|_{L^2} \leq C\|\omega\cdot\nabla u\|_{\dot F^{1-s}_{2,2}}\|\Lambda^{1+s}\omega\|_{L^2} \\ &\leq C\left(\|\omega\|_{\dot F^{-s}_{\infty,\infty}}\|\nabla u\|_{\dot F^1_{2,2}} + \|\nabla u\|_{\dot F^{-s}_{\infty,\infty}}\|\omega\|_{\dot F^1_{2,2}}\right)\|\Lambda^{1+s}\omega\|_{L^2} \\ &\leq C\|\omega\|_{\dot F^{-s}_{\infty,\infty}}\|\nabla\omega\|_{L^2}\|\Lambda^{1+s}\omega\|_{L^2} \\ &\leq C\|\omega\|_{\dot F^{-s}_{\infty,\infty}}\|\nabla\omega\|_{L^2}^{1+\theta}\|\Lambda^{1+\ell/2}\omega\|_{L^2}^{1-\theta} \\ &\leq \frac{1}{4}\|\Lambda^{1+\ell/2}\omega\|_{L^2}^2 + C\|\omega\|_{\dot F^{-s}_{\infty,\infty}}^r\|\nabla\omega\|_{L^2}^2, \end{aligned} \tag{3.15}$$

with $r = \frac{2}{1+\theta}$ and $\theta = \frac{\ell-2s}{\ell}$. Here, we have used the following inequalities [28]:

$$\|\nabla u\|_{\dot F^{-s}_{\infty,\infty}} \leq C\|\omega\|_{\dot F^{-s}_{\infty,\infty}}, \tag{3.16}$$

and

$$\|\nabla u\|_{\dot F^1_{2,2}} \leq C\|\omega\|_{\dot F^1_{2,2}}. \tag{3.17}$$

By integration by parts, we rewrite I_2 as

$$\begin{aligned} I_2 &= \sum_{i,k}\int_{\mathbb{R}^3} u_i\partial_i\omega\cdot\partial_k^2\omega dx = -\sum_{i,k}\int_{\mathbb{R}^3} u_i\omega\partial_i\partial_k^2\omega dx \\ &= \sum_{i,k}\int_{\mathbb{R}^3} \partial_k u_i\cdot\omega\partial_i\partial_k\omega dx \\ &= \sum_{i,k}\int_{\mathbb{R}^3} \Lambda^{1-s}(\partial_k u_i\cdot\omega)\cdot\partial_i\partial_k(-\Delta)^{-1}\cdot\Lambda^{1+s}\omega dx \\ &\leq \sum_{i,k}\|\Lambda^{1-s}(\omega\partial_k u_i)\|_{L^2}\|\partial_i\partial_k(-\Delta)^{-1}\cdot\Lambda^{1+s}\omega\|_{L^2} \\ &\leq C\|\omega\cdot\nabla u\|_{\dot F^{1-s}_{2,2}}\|\Lambda^{1+s}\omega\|_{L^2}, \end{aligned}$$

and we obtain, in the same as that of I_1,

$$I_2 \le \frac{1}{4}\|\Lambda^{1+\ell/2}\omega\|_{L^2}^2 + C\|\omega\|_{\dot F^{-s}_{\infty,\infty}}^r \|\nabla\omega\|_{L^2}^2. \tag{3.18}$$

Inserting (3.15) and (3.18) into (3.14) and we apply the Gronwall's inequality to get the result.

Finally, we assume that (1.16) holds. Multiplying (1.1) by $|u|^2u$ and integrating by parts and using (1.2), we obtain

$$\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3}|u|^4dx + \int_{\mathbb{R}^3}\Lambda^\ell u\cdot|u|^2udx = -\int_{\mathbb{R}^3}|u|^2u\nabla\pi dx =: J(t). \tag{3.19}$$

The viscosity term on the left hand side is estimated by

$$\int_{\mathbb{R}^3}\Lambda^\ell u\cdot|u|^2udx \ge \frac{1}{4}\int_{\mathbb{R}^3}|\Lambda^{\ell/2}(|u|^2)|^2dx, \tag{3.20}$$

where we used Lemma 2.4 of [9] for the estimate of the fractional derivative. Next, we estimate J as

$$J \le 2\int_{\mathbb{R}^3}|\pi||u|^2\big|\nabla|u|\big|dx = \int_{\mathbb{R}^3}|\pi|\cdot|u|\big|\nabla|u|^2\big|dx. \tag{3.21}$$

For simplicity, denote $v = |u|^2$. Then we have by (3.19)-(3.21) that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}v^2dx + \int_{\mathbb{R}^3}|\Lambda^{\ell/2}v|^2dx\\
\le\ & 4\int_{\mathbb{R}^3}|\pi|\cdot|v|^{1/2}|\nabla v|dx = C\int_{\mathbb{R}^3}|\pi|^{1/2}\cdot|\pi|^{1/2}\cdot|v|^{1/2}\cdot|\nabla v|dx\\
\le\ & C\||\pi|^{1/2}\|_{L^{2p,\infty}}\||\pi|^{1/2}\|_{L^{2q,4}}\||v|^{1/2}\|_{L^{2q,4}}\|\nabla v\|_{L^{6/(5-\ell),2}}\\
\le\ & C\|\pi\|_{L^{p,\infty}}^{1/2}\|\pi\|_{L^{q,2}}^{1/2}\|v\|_{L^{q,2}}^{1/2}\|\nabla v\|_{L^{6/(5-\ell),2}},
\end{aligned}$$

where we have used the generalized Hölder inequality [32] with $1/2p+1/q+(5-\ell)/6 = 1$ and basic properties of the non-increasing rearrangement.

We use the representation formula of the pressure by means of the velocity and Sobolev and Gagliardo-Nirenberg inequalities involving Lorentz spaces to obtain

$$\begin{aligned}
&\|\pi\|_{L^{q,2}} \le C\|v\|_{L^{q,2}}, \qquad \|\nabla v\|_{L^{6/(5-\ell),2}} \le C\|\Lambda^{\ell/2}v\|_{L^2},\\
&\|v\|_{L^{q,2}} \le C\|\Lambda^{\ell/2}v\|_{L^2}^\theta\|v\|_{L^2}^{1-\theta},
\end{aligned}$$

where $\theta = 3/\ell - 6/\ell q$.

Combining those estimate above, we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} v^2 dx + \int_{\mathbb{R}^3} |\Lambda^{\ell/2} v|^2 dx \le C\|\pi\|_{L^{p,\infty}}^{1/2} \|v\|_{L^2}^{1-\theta} \|\Lambda^{\ell/2} v\|_{L^2}^{1+\theta}. \tag{3.22}$$

For θ with $0 \le \theta < 1$, which is equivalent to $3/(2\ell - 2) < p \le 3/(\ell - 2)$, the right hand side on (3.22) is estimated as

$$\frac{1}{2}\|\Lambda^{\ell/2} v\|_{L^2}^2 + C\|\pi\|_{L^{p,\infty}}^r \|v\|_{L^2}^2,$$

where $1/r = 1 - \theta = 2 - 2/\ell - 3/\ell p$.

Substituting this into (3.19) and using the Gronwall's inequality, we obtain

$$u \in L^\infty(0, T; L^4(\mathbb{R}^3)),$$

which implies the result from (1.12). This completes the proof. □

4. The Lagrangian averaged Euler equations

In this section, we consider the 3D Lagrangian averaged Euler equations in the following form [15, 16, 24]:

$$\partial_t u + (u_\alpha \cdot \nabla) u + (\nabla u_\alpha)^T \cdot u = -\nabla \pi, \tag{4.1}$$

$$u = \operatorname{curl} \psi, \tag{4.2}$$

$$u_\alpha = (1 - \alpha^2 \Delta)^{-1} u, \quad \alpha > 0, \tag{4.3}$$

$$u|_{t=0} = u_0(x), \quad x \in \mathbb{R}^3. \tag{4.4}$$

One of the important properties of the averaged Euler equations is the following identities:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u_\alpha|^2 + \alpha^2 |\nabla u_\alpha|^2 dx = 0.$$

This conservation property gives an a priori bound on the H^1 norm of u_α:

$$\|u_\alpha\|_{H^1} \le C_\alpha. \tag{4.5}$$

The averaged Euler models have been used to study the average behavior of the 3D Euler and Navier-Stokes equations and used as a turbulent closure model [7]. However, the global existence of the 3D Lagrangian averaged Euler equations is still open, although the Lagrangian averaged Navier-Stokes equations have been shown to have global existence [24, 13]. Very recently, Hou-Li [17] obtain the following regularity condition:

$$\psi \in L^1(0, T; BMO(\mathbb{R}^3)), \tag{4.6}$$

with the initial condition

$$u_0 \in H^1(\mathbb{R}^3). \tag{4.7}$$

We will prove

Theorem 4.1. *Let $\alpha > 0$ and let $u_0 \in H^s(\mathbb{R}^3)$ with $s > \frac{1}{2}$. Then the Lagrangian averaged 3D Euler equations* (4.1)-(4.4) *have a unique global solution $u \in L^\infty(0,T; H^s(\mathbb{R}^3))$ satisfying*

$$\|u(t)\|_{H^s} \le C\|u_0\|_{H^s} \quad \text{for} \quad 0 \le t \le T, \tag{4.8}$$

provided that

$$\psi \in L^1(0,T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3)). \tag{4.9}$$

Remark 4.1. Since $L^\infty \subset BMO \subset \dot{B}^0_{\infty,\infty}$, our result (4.9) generalizes (4.6) in [17]. It is interesting to note that our initial condition $u_0 \in H^s$ with $s > \frac{1}{2}$ is weaker than (4.7) in [17].

Proof Testing (4.1) by u, we see that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \le \|\nabla u_\alpha\|_{L^\infty}\|u\|_{L^2}^2. \tag{4.10}$$

Applying Λ^s to equation (4.1), we have

$$\partial_t \Lambda^s u + \Lambda^s(u_\alpha \cdot \nabla u) + \Lambda^s((\nabla u_\alpha)^T \cdot u) = -\nabla \Lambda^s \pi.$$

Testing this equation by $\Lambda^s u$ and using the divergence free property, we see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Lambda^s u\|_{L^2}^2 \le & \Big| \int_{\mathbb{R}^3} (\Lambda^s(u_\alpha \cdot \nabla u) - u_\alpha \cdot \nabla \Lambda^s u) \cdot \Lambda^s u dx \Big| \\ & + \Big| \int_{\mathbb{R}^3} \Lambda^s((\nabla u_\alpha)^T \cdot u) \cdot \Lambda^s u dx \Big| =: J_1 + J_2. \end{aligned} \tag{4.11}$$

We will use the following bilinear commutator and product estimates due to David-Journé [10], Kato-Ponce [18] and Kenig-Ponce-Vega [19]:

$$\|\Lambda^r(fg) - f\Lambda^r g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{r-1}g\|_{L^{q_1}} + \|\Lambda^r f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{4.12}$$

$$\|\Lambda^r(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|\Lambda^r g\|_{L^{q_1}} + \|\Lambda^r f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{4.13}$$

with $r > 0$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

We apply (4.12) to $f = (u_\alpha)_i, g = u$ and $p_1 = \infty, p = q_1 = 2$ to bound J_1 as

$$\begin{aligned} J_1 &= \Big| \sum_{i=1}^3 \int_{\mathbb{R}^3} (\Lambda^s \partial_i((u_\alpha)_i u) - (u_\alpha)_i \cdot \Lambda^s \partial_i u) \cdot \Lambda^s u dx \Big| \\ &\le \sum_{i=1}^3 \|\Lambda^s \partial_i((u_\alpha)_i u) - (u_\alpha)_i \cdot \Lambda^s \partial_i u\|_{L^2}\|\Lambda^s u\|_{L^2} \end{aligned}$$

$$\leq C(\|\nabla u_\alpha\|_{L^\infty}\|\Lambda^s u\|_{L^2} + \|\Lambda^{s+1}u_\alpha\|_{L^{p_2}}\|u\|_{L^{q_2}})\|\Lambda^s u\|_{L^2}, \qquad (4.14)$$

with $\frac{1}{2} = \frac{1}{p_2} + \frac{1}{q_2}$. We use (4.13) to $f = (\nabla u_\alpha)^T, g = u$ and $p_1 = \infty, p = q_1 = 2$ to bound J_2 as follows

$$J_2 \leq C(\|\nabla u_\alpha\|_{L^\infty}\|\Lambda^s u\|_{L^2} + \|\Lambda^{s+1}u_\alpha\|_{L^{p_2}}\|u\|_{L^{q_2}})\|\Lambda^s u\|_{L^2}, \qquad (4.15)$$

with $\frac{1}{2} = \frac{1}{p_2} + \frac{1}{q_2}$. We will use the following Gagliardo-Nirenberg inequalities:

$$\|\Lambda^{s+1}u_\alpha\|_{L^{p_2}} \leq C\|\nabla u_\alpha\|_{L^\infty}^{\theta}\|\Lambda^{s+2}u_\alpha\|_{L^2}^{1-\theta}, \qquad (4.16)$$

$$\|\Lambda^{2}u_\alpha\|_{L^{q_2}} \leq C\|\nabla u_\alpha\|_{L^\infty}^{1-\theta}\|\Lambda^{s+2}u_\alpha\|_{L^2}^{\theta}, \qquad (4.17)$$

where $\theta = \frac{\frac{3}{p_2} - \frac{1}{2}}{s - \frac{1}{2}} \in (0,1), 3 < p_2 < 6$, and $2 < q_2 < 6$.

Using (4.16),(4.17), (4.3) and (4.5), we see that

$$\begin{aligned}
&\|\Lambda^{s+1}u_\alpha\|_{L^{p_2}}\|u\|_{L^{q_2}} \leq C\|\Lambda^{s+1}u_\alpha\|_{L^{p_2}}(\|\Lambda^2 u_\alpha\|_{L^{q_2}} + \|u_\alpha\|_{L^{q_2}})\\
&\leq C\|\nabla u_\alpha\|_{L^\infty}\|\Lambda^{2+s}u_\alpha\|_{L^2} + C\|\nabla u_\alpha\|_{L^\infty}^{\theta}\|\Lambda^{2+s}u_\alpha\|_{L^2}^{1-\theta}\\
&\leq C(\|\nabla u_\alpha\|_{L^\infty} + 1)\|u\|_{H^s}. \qquad (4.18)
\end{aligned}$$

Inserting (4.18) into (4.14) and (4.15) and using the logarithimic Sobolev inequality (3.8), we find that

$$\begin{aligned}
\frac{d}{dt}\|u\|_{H^s}^2 &\leq C(\|\nabla u_\alpha\|_{L^\infty} + 1)\|u\|_{H^s}^2\\
&\leq C(\|\nabla u_\alpha\|_{\dot{B}^0_{\infty,\infty}} + 1)\log(e + \|u_\alpha\|_{H^{2+s}}^2)\|u\|_{H^s}^2\\
&\leq C(\|\psi\|_{\dot{B}^0_{\infty,\infty}} + 1)\log(e + \|u\|_{H^s}^2)\|u\|_{H^s}^2, \qquad (4.19)
\end{aligned}$$

where we have used the following inequality [28]:

$$\|\nabla u_\alpha\|_{\dot{B}^0_{\infty,\infty}} \leq C\|\psi\|_{\dot{B}^0_{\infty,\infty}}.$$

Thus, (4.8) follows from (4.19). The proof is complete. □

Acknowledgments. We are grateful to Professor D. Chae for information on [4] and to Professor Kozono for discussions on regularity criteria.

REFERENCES

[1] H. Beirão da Veiga, *A new regularity class for the Naivier-Stokes equations in $\mathbb{R}^n$*, Chinese Ann. Math., 16 (1995), 407-412.

[2] Z. Cai and J. Zhai, *Regularity criteria in weak spaces for Navier-Stokes equations in $\mathbb{R}^3$*, Preprint, 2006.

[3] M. Cannone and G. Karch, *Incompressible Navier-Stokes equations in abstract Banach spaces,* in "Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics," (2001), 27-41.

[4] D. Chae, *On the regularity conditions for the Navier-Stokes and related equations,* Revista Mat. Iberoamericana, 23 (2007), 373-386.

[5] D. Chae and J. Lee, *On the Global well-posedness and stability of the Navier-Stokes and the related equations,* Adv. Math. Fluid Mech., "Contributions to Current Challenges in Mathematical Fluid Mechanics", Birkhäuser, Basel, (2004), 31-51.

[6] J.-Y. Chemin, "Perfect Incompressible Fluids," Oxford Lecture Series 14, Oxford University, 1998.

[7] S.Y. Chen, C. Foias, D.D. Holm, E.J. Olson, E.S. Titi, and S. Wynne, *The Camassa-Holm equations as a closure model for turbulent channel and pipe flows,* Phys. Fluids, 11 (1999), 2343-2353.

[8] Q. Chen and C. Miao, *Blow-up criterion of smooth solutions to the two-fluid MHD equations,* preprint, 2006.

[9] A. Córdoba and D. Córdoba, *A maximum principle applied to quasi-geostrophic equations,* Comm. Math. Phys., 249 (2004), 511-528.

[10] G. David and J.L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators,* Ann. Math., 120 (1994), 371-397.

[11] L. Escauriaza, G. Seregin, and V. Sverak, *$L^{3,\infty}$-solutions of Navier-Stokes equations and backward Uniqueness,* Russian Math. Surveys, 58 (2003), 211-250.

[12] E. Fabes, B. Jones, and N. Riviere, *The initial value problem for the Navier-Stokes equations with data in L^p,* Arch. Rat. Mech. Anal., 45 (1972), 222-248.

[13] C. Foias, D.D. Holm, and E.S. Titi, *The three dimensional viscous Camassa-Holm equations, and their relation to the Navier-Stokes equations and turbulence theory,* J. Dynam. Differential Equations, 14 (2002), 1-35.

[14] Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes equations,* J. Diff. Eqns. 62 (1986), 186-212.

[15] D.D. Holm, J.E. Marsden, and T.S. Ratiu, *Euler-Poincaré models of ideal fluids with nonlinear dispersion,* Phys. Rev. Lett., 349 (1998), 4173-4177.

[16] D.D. Holm, J.E. Marsden, and T.S. Ratiu, *Euler-Poincaré equations and semidirect products with applications to continuum theories,* Adv. Math., 137 (1998), 1-81.

[17] T.Y. Hou and C. Li, *On global well-posedness of the Lagrangian averaged Euler equations,* SIAM J. Math. Anal., 38 (2006), 782-794.

[18] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations,* Comm. Pure Appl. Math., 41 (1988), 891-907.

[19] C. Kenig, G. Ponce, and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation,* J. Amer. Math. Soc., 4 (1991), 323-347.

[20] H. Kim, *A blow-up criterion for the nonhomogeneous incompressible Navier-Stokes equations,* SIAM J. Math. Anal., 37 (2006), 1417-1434.

[21] H. Kozono, T. Ogawa, and Y. Taniuchi, *The critical Sobolev inequalities in Besov spaces and regularity criterion to some semilinear evolution equations,* Math. Z., 242 (2002), 251-278.

[22] H. Kozono and Y. Shimada, *Bilinear estimates in homogeneous Triebel-Lizorkin spaces and the Navier-Stokes equations,* Math. Nachr., 276 (2004), 63-74.

[23] J.L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites Nonlinéaires," Dunod, Paris, (1969).

[24] J. Marsden and S. Shkoller, *Global well-posedness for the Lagrangian average Navier-Stokes (LANS-α) equations on bounded domains,* R. Soc. Lond. Philos. Trans. Ser. A, Math. Phys. Eng. Sci., 359 (2001), 1449-1468.
[25] T. Ohyama, *Interior regularity of weak solutions to the Navier-Stokes equations,* Proc. Japan Acad., 36 (1960), 273-277.
[26] J. Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations,* Arch. Rat. Mech. Anal., 9 (1962), 187-191.
[27] H. Sohr and W. von Wahl, *On the regularity of the pressure of weak solutions of Navier-Stokes equations,* Archiv. Math., 46 (1986), 428-439.
[28] E.M. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, NJ, (1970).
[29] M. Struwe, *On partial regularity results for the Navier-Stokes equations,* Comm. Pure Appl. Math., 41 (1988), 437-458.
[30] M. Struwe, *On a Serrin-type regularity criterion for the Navier-Stokes equations in terms of the pressure,* J. Math. Fluid Mech., online (2006).
[31] S. Tourville, *Existence and uniqueness of solutions for the Navier-Stokes equations with hyperdissipation,* J. Math. Anal. Appl., 281 (2003), 62-75.
[32] H. Triebel, "Theory of Function Spaces," Birkhäuser, Basel, (1983).
[33] J. Wu, *The generalized incompressible Navier-Stokes equations in Besov spaces,* Dynamics of PDE 1, (2004), 381-400.
[34] Y. Zhou, *On regularity criteria in terms of pressure for the Navier-Stokes equations in* $\mathbb{R}^3$*,* Proc. Amer. Math. Soc., 134 (2005), 149-156.
[35] Y. Zhou, *Regularity criteria for the generalized viscous MHD equations,* Ann. I. H. Poincaré-Analyse Nonlinéaire, 24 (2007), 491-505.