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ASYMPTOTIC ANALYSIS OF THE ABSTRACT TELEGRAPH EQUATION

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Abstract. It is known that each solution of the telegraph equation

$$u''(t) + 2au'(t) + A^2u(t) = 0, \tag{0.1}$$

($A = A^*$on $\mathcal{H}, a > 0$) is approximately equal to some solution of the abstract heat equation,

$$2av'(t) + A^2v(t) = 0. \tag{0.2}$$

It is shown how to find $v(0)$, in terms of $u(0)$ and $u'(0)$, so that one can say that a given solution of (0.1) is like a specific solution of (0.2).

1. Introduction

Let A be a selfadjoint operator on a complex Hilbert space. For $a > 0$, the initial value problem of the abstract telegraph equation,

$$u''(t) + 2au'(t) + A^2u(t) = 0 \quad (t \geq 0), \tag{1.1}$$

is well posed. By writing this as a first order system, the asymptotics of (1.1) follows from the ergodic theorem (cf. [3]) when A has a non-trivial null space. The case of A being injective is harder to analyze. [2] showed that if u satisfies (1.1), then

$$u(t) = v(t)(1 + o(1)),$$

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as $t \to \infty$ for some solution $v(t)$ of the abstract heat equation

$$2av'(t) + A^2 v(t) = 0 \quad (t \geq 0). \tag{1.2}$$

The unsatisfying part of [2] is that $h = v(0)$ can be chosen in many ways, and one would like to define h in a canonical way from $u(0)$ and $u'(0)$. This is precisely what we do in this paper, but at the expense of a technical assumption connecting $u(0)$, $u'(0)$ and a. But this seems a quite reasonable hypothesis, and our Theorem 2.2 seems close to being an optimal result.

The problem was motivated by the applications. The telegraph equation gives a better fit to the data than the heat equation does for certain diffusion experiments involving blood flow; see [2], [7], and the references therein. The summary formula for the telegraph equation, which relates the expected distance increments of elapsed times in the dissipative flow fields, fitted to data with statistical p-values of 0.001 or less, (see [6]). Fits to the forms for convective diffusion were not significant. One does not normally expect a hyperbolic equation to describe diffusion. Our result justifies this in a precise way in some cases.

2. Preliminaries

Let A be a non-negative injective operator on a complex Hilbert space $\mathcal{H}$. By the spectral theorem (see, for example, [3], [5], [8]), A can be represented as a multiplication operator on an L^2 space, as follows. There is a unitary operator

$$U : \mathcal{H} \to L^2(\Omega, \Sigma, \mu),$$

for some measure space (Ω, Σ, μ), such that

$$A = U^{-1} M_m U,$$

where M_m is the multiplication operator defined by

$$(M_m f)(\omega) = m(\omega) f(\omega),$$

for $\omega \in \Omega$ and $f \in \mathcal{D}(M_m) = \{g \in L^2(\Omega, \Sigma, \mu) \colon mg \in L^2(\Omega, \Sigma, \mu)\}$, and $m \colon \Omega \to [0, \infty)$ is Σ measurable.

Alternatively,

$$A = \int_{[0,\infty)} \lambda dE(\lambda),$$

where $\{E(\lambda) \colon \lambda \in \mathbb{R}\}$ is the resolution of the identity associated with A. We have

$$\sigma(A) \subset [0, \infty) \quad \sigma(A) = \text{ess Range}\,(m),$$

where

$$\text{ess Range}\,(m) = \bigcap\{\overline{m(\Omega\backslash N)} : N \in \Sigma, \mu(N) = 0\}.$$

Let $F\colon \sigma(A) \to \mathbb{C}$ be Borel measurable. Then

$$F(A) = \int_{[0,\infty)} F(\lambda)dE(\lambda) = U^{-1}M_{F(m)}U,$$

according to the operational calculus for selfadjoint operators. In particular, if $\mathcal{X}_\Gamma$ is the characteristic function of $\Gamma \subset [0,\infty)$, then $\mathcal{X}_\Gamma(\omega) = 1$ or 0, accordingly as $\omega \in \Gamma$ or $\omega \notin \Gamma$; and $E(\lambda) = E_{[0,\lambda]}$ for $\lambda \geq 0$. We write $E(\Gamma)$ in place of $\mathcal{X}_\Gamma(A)$.

Let $E[\delta_1, \delta](\mathcal{H})$ be the range of $E([\delta_1, \delta])$, and let

$$\mathcal{H}_\delta := \bigcup_{0<\delta_1<\delta} E[\delta_1, \delta](\mathcal{H}),$$

for each $\delta > 0$. We assume

(Hyp 1) $$\mathcal{H}_\delta \neq \{0\} \quad \text{for each } \delta > 0.$$

This is equivalent to $\sigma(A) \cap (0,\delta) \neq \emptyset$ for each $\delta > 0$, which is equivalent to $\dim(\mathcal{H}_\delta) = \infty$, for each $\delta > 0$.

(Hyp 2) $$a^2I - A^2 \quad \text{is injective.}$$

This means $a \notin \sigma_p(A)$, i.e., a is not an eigenvalue of A.

Consider the abstract telegraph equation

$$u''(t) + 2au'(t) + A^2u(t) = 0, \qquad t \geq 0, \tag{2.1}$$

with initial conditions

$$u(0) = f, \quad u'(0) = g. \tag{2.2}$$

There exists a unique, strong C^2 solution u of (2.1) provided $f \in \mathcal{D}(A^2)$ and $g \in \mathcal{D}(A)$. There exists a unique mild solution of (2.1) if $f \in \mathcal{D}(A)$ and $g \in \mathcal{H}$. (see [3]).

A concrete example of this, corresponding to $A = (-\Delta)^{\frac{1}{2}}$ on $\mathcal{H} = L^2(\mathbb{R}^n)$, is the telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + 2a\frac{\partial u}{\partial t} - \Delta u = 0, \qquad (t \geq 0, \quad x \in \mathbb{R}^n)$$

$$u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x) \qquad (x \in \mathbb{R}^n).$$

(Hyp 3) $$(g + af) \in \mathcal{R}\big((a^2I - A^2)^{\frac{1}{2}}\big).$$

This is a condition on the initial conditions (see (2.2)) together with a (see (2.1)).

Theorem 2.1. *Assume* (Hyp 1) – (Hyp 3) *and let* $u(t)$ *satisfy* (2.1), (2.2). *Then*

$$u(t) = e^{-\frac{tA^2}{2a}} h(1 + o(1)) \quad as \quad t \to \infty,$$

where $h = \frac{f}{2} + (a^2 - A^2)^{-\frac{1}{2}}(\frac{g+af}{2})$ *provided that* $0 \neq h \in \mathcal{H}_\delta$ *for some* δ *satisfying* $0 < \delta < a$.

Theorem 2.2. *Let the hypotheses of Theorem* 2.1 *hold, except consider* h *defined in the same way so that* $h = h_1 + h_2$ *where* $0 \neq h_1 \in \mathcal{H}_\delta$ *for some* $\delta \in (0, a)$ *and* $h_2 \in E[a, \infty](\mathcal{H})$. *Then,*

$$u(t) = e^{-\frac{tA^2}{2a}} h_1(1 + o(1)) \quad as \quad t \to \infty.$$

In other words, the relevant part of h in the multiplicative representation of A is the part where $m < a$, provided that the essential range of Uh misses $[a - \epsilon, a)$ for some $\epsilon > 0$.

3. Proofs

Let $T_j(t) = e^{tA_j}, \quad j = 1, 2$, where

$$A_1 = A_+ = -aI + (a^2 I - A^2)^{\frac{1}{2}}$$
$$A_2 = A_- = -aI - (a^2 I - A^2)^{\frac{1}{2}}.$$

Then,

$$u(t) = \sum_{j=1}^{2} T_j(t) f_j \qquad f_j \in \mathcal{H},$$

represents the general solution of equation (2.1) (see [4]).

From the initial conditions, $f = f_1 + f_2$ and $A_1 f_1 + A_2 f_2 = g$, an expression for f_1 can be obtained in terms of f and g, namely

$$f_1 = \frac{f}{2} + \left(a^2 I - A^2\right)^{-\frac{1}{2}} \left(\frac{g + af}{2}\right).$$

This expression for f_1 is well-defined by (Hyp 3) and defines the h of Theorems 2.1 and 2.2.

The asymptotics of $T_2(t) f_2$ is now investigated.

$$T_2(t) f_2 = \exp\left[- ta\left(I + \left(I - \frac{A^2}{a^2}\right)^{\frac{1}{2}}\right)\right] f_2.$$

This implies

$$\|T_2(t)f_2\| \leq e^{-ta}\|f_2\|,$$

for all $t \geq 0$. The above inequality holds, since

$$\Big\| \exp\Big[-ta\Big(I - \frac{A^2}{a^2}\Big)^{\frac{1}{2}}\Big] \Big\| \leq 1,$$

by the spectral theorem. This is because

$$Re\big(1 - \frac{m(\omega)}{a^2}\big)^{\frac{1}{2}} \geq 0,$$

whenever $0 \leq m(\omega)$.

Consider δ_1 such that $0 < \delta_1 < a$ and let $\Phi = \mathcal{X}_J$ for any interval J. Then, $E_J = U^{-1}M_\Phi U$ for all intervals J. We have

$$\begin{aligned}\|T_1(t)f_1\|^2 &\geq \int_{m^{-1}((0,\delta_1])} \Big| \exp\Big[at\Big(-I + \Big(I - \frac{m^2}{a^2}\Big)^{\frac{1}{2}}\Big)\Big]\hat{f}_1\Big|^2 d\mu, \\ &\text{where} \quad \hat{f}_1 = Uf_1, \\ &\geq \int_{m^{-1}((0,\delta_1])} \Big| \exp\Big[2at\Big(-I + \Big(I - \frac{\delta_1^2}{a^2}\Big)^{\frac{1}{2}}\Big)\Big]\Big|\Big|\hat{f}_1\Big|^2 d\mu \\ &\geq \exp[-2at]\exp\Big[2at\sqrt{1 - \frac{\delta_1^2}{a^2}}\Big]\|E_{(0,\delta_1]}f_1\|^2.\end{aligned}$$

This implies

$$\|T_1(t)f_1\|^2 \geq \exp[-2at]\exp\Big[2at\sqrt{1 - \frac{\delta_1^2}{a^2}}\Big]\|E_{(0,\delta_1]}f_1\|^2.$$

Taking the square root of both sides of the inequality yields

$$\|T_1(t)f_1\| \geq \exp[-at]\exp\Big[at\sqrt{1 - \frac{\delta_1^2}{a^2}}\Big]\|E_{(0,\delta_1]}f_1\|. \tag{3.1}$$

From earlier, we have

$$\|T_2(t)f_2\| \leq e^{-ta}\|f_2\|. \tag{3.2}$$

Combining inequalities (3.1) and (3.2) gives

$$\frac{\|T_2(t)f_2\|}{\|T_1(t)f_1\|} \leq \frac{e^{-at}\|f_2\|}{\|T_1(t)f_1\|} = \frac{e^{-at}\|f_2\|}{\exp[-2at]\exp\Big[2at\sqrt{1 - \frac{\delta_1^2}{a^2}}\Big]\|E_{(0,\delta_1]}f_1\|}. \tag{3.3}$$

The right hand side of (3.3) goes to zero as $t \to \infty$, assuming that

$$\|E_{(0,\delta_1]}f_1\| > 0.$$

We now state a lemma that will be useful for the rest of the proof.

Lemma 3.1.

$$\lim_{\delta\downarrow 0}\|E_{[\delta_1,\delta]}q\| = 0 \qquad \textit{for all} \quad q\in\mathcal{H} \quad \textit{where} \quad 0<\delta_1<\delta.$$

Proof. Letting $\hat{f}_1 = Uf_1$ gives

$$\lim_{\delta\downarrow 0}\|E_{[\delta_1,\delta]}q\|^2 = \lim_{\delta\downarrow 0}\int_{\delta_1\le\hat{f}_1\le\delta}|\hat{f}_1|^2 d\mu = 0,$$

by the dominated convergence theorem. □

We now break up $T_1(t)f_1$ as follows,

$$T_1(t)f_1 = T_1(t)E_{(0,\delta)}f_1 + T_1(t)E_{[\delta_1,\delta]}f_1 + T_1(t)E_{(\delta,\infty)}f_1, \tag{3.4}$$

and treat each part of the right hand side of (3.4) separately.

First, we look at $T_1(t)E_{(\delta,\infty)}f_1$.

$$\begin{aligned}\|T_1(t)E_{(\delta,\infty)}f_1\|^2 &= \int_{[m(\omega)>\delta]}\Big|\exp\Big[at\Big(-I+\Big(I-\frac{m(\omega)^2}{a^2}\Big)^{\frac{1}{2}}\Big)\Big]\hat{f}_1\Big|^2 d\mu\\ &\le \exp\Big[2at\Big(-I+\Big(1-\frac{\delta^2}{a^2}\Big)^{\frac{1}{2}}\Big)\Big]\|\hat{f}_1\|^2. \end{aligned}\tag{3.5}$$

It follows that

$$\|T_1(t)E_{(\delta,\infty)}f_1\| \le e^{-\epsilon t}\|\hat{f}_1\|, \tag{3.6}$$

where $\epsilon = a - a(I-\frac{\delta^2}{a^2})^{\frac{1}{2}} > 0$.

Secondly, we investigate $T_1(t)E_{[\delta_1,\delta]}f_1$.

$$\|T_1(t)E_{[\delta_1,\delta]}f_1\|^2 = \int_{[\delta_1\le m(\omega)\le\delta]}\exp\Big[-2at\Big(1+\Big(1-\frac{m(\omega)^2}{a^2}\Big)^{\frac{1}{2}}\Big)\Big]\Big|\hat{f}_1\Big|^2 d\mu. \tag{3.7}$$

Since $m(\omega)\in[\delta_1,\delta]$ we have

$$\sqrt{1-\frac{m(\omega)^2}{a^2}} \in \Big[1-\frac{\delta^2}{a^2}, 1-\frac{\delta_1^2}{a^2}\Big].$$

Therefore,

$$-1+\sqrt{1-\frac{m(\omega)^2}{a^2}} \in \Big[\frac{-\delta_1^2}{a^2}, \frac{-\delta^2}{a^2}\Big].$$

Combining the above with (3.7) gives

$$\|T_1(t)E_{[\delta_1,\delta]}f_1\|^2 \le \int_{[\delta_1\le m(\omega)\le\delta]}\exp\Big[-t\frac{2\delta_1^2}{a}\Big]\Big\|\hat{f}_1\Big\|^2 d\mu = O\Big(\exp\Big[-t\frac{2\delta_1^2}{a}\Big]\Big).$$

Lastly, we explore the interval $(0,\delta]$ and complete the proof of Theorem 2.1. We begin by considering

$$\begin{aligned}&\|T_1(t)f_1 - e^{-\frac{tA^2}{2a}}E_{(0,\delta]}f_1\| \\ &\le \|T_1(t)E_{(0,\delta]}f_1 - e^{-\frac{tA^2}{2a}}E_{(0,\delta]}f_1\| + \|T_1(t)E_{(\delta,\infty)}f_1\|.\end{aligned} \tag{3.8}$$

But,

$$\|T_1(t)E_{(\delta,\infty)}f_1\| \le e^{-\epsilon t}\|\hat{f}_1\|,$$

from (3.6). We have

$$\|T_1(t)E_{(0,\delta]}f_1 - e^{-\frac{tA^2}{2a}}E_{(0,\delta]}f_1\|. \tag{3.9}$$

But, $E_{(0,\delta]}f_1 = E_{[\delta_1,\delta]}f_1$ for some $\delta_1 \in (0,\delta)$ since $f_1 \in \mathcal{H}_\delta$. In light of (3.8) and (3.9), we now have

$$\|T_1(t)E_{(0,\delta]}f_1 - e^{-\frac{tA^2}{2a}}E_{(0,\delta]}f_1\| = \|T_1(t)E_{[\delta_1,\delta]}f_1 - e^{-\frac{tA^2}{2a}}E_{[\delta_1,\delta]}f_1\|. \tag{3.10}$$

Substituting for $T_1(t)$ yields

$$\begin{aligned}&\|T_1(t)E_{[\delta_1,\delta]}f_1 - e^{-\frac{tA^2}{2a}}E_{[\delta_1,\delta]}f_1\| \\ &= \Big\|\exp\Big[at\Big(-I+\Big(I-\frac{A^2}{a^2}\Big)^{\frac{1}{2}}\Big)\Big]h - e^{-\frac{tA^2}{2a}}h\Big\|,\end{aligned} \tag{3.11}$$

where $h = E_{[\delta_1,\delta]}f_1$. $T_1(t)$ may be written as

$$\exp\Big[-t\Big(\frac{A^2}{2a}+\frac{A^4}{8a^3}+F(A)\Big)\Big],$$

by expanding $(I-(I-\frac{A^2}{a^2})^{\frac{1}{2}})$ in a Taylor series and putting terms of order greater than 4 into the remainder term $F(A)$.

View F as a real function of x on $[0,\infty)$. The remainder, $F(x)$, is given as (see [1])

$$\frac{x^5}{5!}\frac{d^5}{dx^5}\Big|_\xi\Big(1-\big(1-\frac{x^2}{a^2}\big)^{\frac{1}{2}}\Big)$$

for some $\xi = \xi(x)$ satisfying $0 < \xi < x$. This yields

$$F(x) = \frac{x^5}{5!}\sum_{j=1}^{3} a_j\big(1-\frac{\xi^2}{a^2}\big)^{-\frac{3}{2}-j}\xi^{2j-1}, \tag{3.12}$$

where $a_1 = \frac{45}{a^6}$, $a_2 = \frac{150}{a^8}$, and $a_3 = \frac{105}{a^{10}}$ with $0 < \xi < x$. Also, it is clear that

$$\lim_{x\downarrow 0} F(x) = \Big(\lim_{x\downarrow 0} \frac{x^5}{5!}\Big)\Big(\lim_{x\downarrow 0} \sum_{j=1}^{3} a_j\Big(1 - \frac{\xi^2}{a^2}\Big)^{-\frac{3}{2}-j} \xi^{2j-1}\Big) = 0.$$

We now rewrite equation (3.11) in the following form:

$$\begin{aligned}\Big\| \exp\Big[-t\Big(\frac{A^2}{2a} + \frac{A^4}{8a^3} + F(A)\Big)\Big]h - e^{-\frac{tA^2}{2a}}h\Big\| \\ \leq \Big\|\Big(\exp\Big[-t\Big(\frac{A^4}{8a^3} + F(A)\Big)\Big] - I\Big)h\Big\|, \\ \text{since} \quad \|e^{-\frac{tA^2}{2a}}\| \leq 1.\end{aligned} \tag{3.13}$$

It will now be shown that

$$\Big\|\exp\Big[-t\Big(\frac{A^4}{8a^3} + F(A)\Big)\Big] - I\Big\|\Big\|h\Big\| = \|h\|(1 + o(1)) \quad \text{as} \quad t \to \infty.$$

This will complete the proof of Theorem 2.1. From equation (3.13), we write

$$\Big\|\exp\Big[-t\Big(\frac{A^4}{8a^3} + F(A)\Big)\Big] - I\Big\| \leq \Big\|\exp\Big[-t\Big(\frac{A^4}{8a^3} + F(A)\Big)\Big]\Big\| + 1. \tag{3.14}$$

Letting $G(A) = \frac{A^4}{8a^3} + F(A)$, it will be shown that

$$\|e^{-tG(A)}\| \leq e^{-t\epsilon}, \tag{3.15}$$

for some $\epsilon > 0$. As before, we replace A by the function $m(\omega)$, and here $0 < \delta_1 \leq m(\omega) \leq \delta < a$. Therefore, the remainder $F(m(\omega))$ is

$$F(m(\omega)) = \frac{m(\omega)^5}{5!} \sum_{j=1}^{3} a_j \left(1 - \frac{\xi_\omega^2}{a^2}\right)^{-\frac{3}{2}-j} \xi_\omega^{2j-1},$$

where $0 < \xi_\omega \leq \delta$ with a_1, a_2, and a_3 the same as in equation (3.12). It follows that $F(m(\omega)) > 0$ and $G(A) = G(m(\omega)) > \frac{\delta_1^4}{8a^3} > 0$. By letting $\frac{\delta_1^4}{8a^3} = \epsilon$ we have

$$\|e^{-tG(A)}\| \leq e^{-t\epsilon}.$$

This proves Theorem 2.1. □

Proof of Theorem 2.2. For u the unique solution of (2.1), (2.2), define

$$h = f_1 = \frac{f}{2} + \big(a^2 I - A^2\big)^{-\frac{1}{2}}\Big(\frac{g + af}{2}\Big),$$

as before. Suppose $h = h_1 + h_2$, as in the statement of Theorem 2.2. Then, since $h_1 \neq 0$,

$$\frac{\|T_2(t)f_2\|}{\|T_1(t)h_1\|} \quad \to \quad 0,$$

as $t \to \infty$ by our earlier argument. Next,

$$\|T_1(t)h_2\| = e^{-ta}\left\|e^{t\left(a^2-A^2\right)^{\frac{1}{2}}}h_2\right\| = e^{-ta}\|h_2\|,$$

since $h_2 \in E[a,\infty](\mathcal{H})$. Combining this with (3.1) yields

$$\frac{\|T_1(t)h_2\|}{\|T_1(t)h_1\|} \quad \to \quad 0,$$

as $t \to \infty$, since $E_{(0,\delta_1]}h_1 \neq 0$ by assumption. Thus,

$$\sum_{j=1}^{2} T_j(t)f_j = T_1(t)h_1(1 + o(1)),$$

as $t \to \infty$, and now Theorem 2.2 follows from Theorem 2.1. □

4. Concluding Remarks

The proofs given here do not work in the case when a is replaced by ia. In this case, (1.1), (1.2) become the Klein-Gordon equation (modulo a minor modification in the unknown) and the Schrödinger equation, respectively. A future project is to determine whether some version of the conclusion of Theorem 2.2 holds in this case.

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