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LOCAL PROPERTY OF THE MOUNTAIN-PASS CRITICAL POINT AND THE MEAN FIELD EQUATION

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Abstract. A local property of the mountain-pass critical point obtained by Struwe's monotonicity trick is shown. Given one parameter family of functionals $\{I_\lambda\}$ provided with the structural assumption of monotonicity, we assume that each I_λ satisfies the bounded Palais-Smale condition, its mountain-pass critical value c_λ is differentiable at $\lambda = \lambda_0$, and $\mathrm{Cr}(I, c_{\lambda_0}) = \{v \mid I'_{\lambda_0}(v) = 0,\ I_{\lambda_0}(v) = c_{\lambda_0}\}$ is compact. Then, there is $v \in \mathrm{Cr}(I, c_{\lambda_0})$, either a local minimum or of mountain pass type. Application to the mean field equation is provided.

1. Introduction

The purpose of this present paper is to study the local property of the critical point obtained by the mountain-pass theorem without the Palais-Smale condition. First, we state the standard mountain-pass theorem, given a real Banach space $(X, \|\cdot\|)$, a C^1 functional $I : X \to \mathbf{R}$, and $u_0, u_1 \in X$ with $u_0 \neq u_1$. Thus, this (I, u_0, u_1) is a triplet satisfying the mountain-pass structure,

$$c_I > \max\{I(u_0), I(u_1)\}. \tag{1.1}$$

Then, taking the path space

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\},$$

joining u_0 and u_1, we define the mountain-pass value of I by

$$c_I := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)). \tag{1.2}$$

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We call $\{u_k\} \subset X$ a Palais-Smale sequence if

$$I(u_k) \to c \qquad \text{and} \qquad I'(u_k) \to 0 \text{ in } X^*,$$

for some $c \in \mathbf{R}$, and such a sequence is called the $(\mathrm{PS})_c$ sequence in short. The Palais-Smale condition, denoted by the (PS) condition, indicates that any $(\mathrm{PS})_c$ sequence admits a subsequence converging strongly in X, where $c \in \mathbf{R}$ is arbitrary.

A form of the mountain-pass theorem originated by Ambrosetti and Rabinowitz [1] is stated as follows:

Theorem 1 ([5]). *Suppose the mountain-pass structure* (1.1) *and the (PS) condition. Then, the mountain-pass value c_I defined by* (1.2) *is a critical value of I, i.e., there is $v \in X$ satisfying $I'(v) = 0$ and $I(v) = c_I$.*

We can weaken the above required (PS) condition to the local Palais-Smale condition denoted by $(\mathrm{PS})_{c_I}$; any $(\mathrm{PS})_{c_I}$ sequence has a strongly converging subsequence, see, e.g., [19].

Even this $(\mathrm{PS})_{c_I}$ condition, however, is not valid in several cases. Among them is the functional

$$I_\lambda(u) := \frac{1}{2}\int_M |\nabla u|^2 - \lambda \log\left(\frac{1}{|M|}\int_M e^u\right), \tag{1.3}$$

defined for $\lambda > 0$ and $u \in E$, where (M, g) is a two-dimensional compact orientable Riemannian manifold without boundary and

$$E := \left\{u \in H^1(M) : \int_M u = 0\right\}.$$

The Euler-Lagrange equation of this functional,

$$-\Delta_g v = \lambda\left(\frac{e^v}{\int_M e^v} - \frac{1}{|M|}\right), \qquad \int_M v = 0, \tag{1.4}$$

is called the mean field equation. In this example, the lack of the (PS) condition arises because the (PS) sequence for $\lambda \geq 8\pi$ is not necessarily bounded in E. There can occur, actually, the“bubbling” to a (PS) sequence, see [15, 16] and the references therein.

Still, we have a mountain pass structure for this functional. In fact, the trivial solution $v = 0$ to (1.4) is linearly stable if $\lambda < \nu_2^* |M|$, where ν_2^* is the first positive eigenvalue of $-\Delta_g$. (Its first eigenvalue $\nu_1^* = 0$ and E is orthogonal to its eigenspace.) The functional I_λ of (1.3), on the other hand, is not bounded from below if $\lambda > 8\pi$. Therefore, there is a triplet

(I_λ, u_0, u_1) satisfying (1.1), in the range of $8\pi < \lambda < \nu_2^* |M|$, where $u_0 = 0$ and $\|u_1\|_E \gg 1$.

The case $\nu_2^* |M| > 8\pi$ needed for this mountain pass structure actually arises when M is a flat torus with the fundamental cell domain $[0,1] \times [0,1]$, i.e., $\nu_2^*|M| = 4\pi^2$, and henceforth, we are always concerned with such (M, g). There is, however, the other case of $\nu_2^*|M| \le 8\pi$, e.g., the dumbbell surface attributed to Calabi, homeomorphic to S^2 with a slender pipe, see [2].

Fortunately, the other property of this functional compensates the lack of the (PS) condition. First,

$$\log\left(\frac{1}{|M|}\int_M e^u\right) \ge \log e^{\frac{1}{|M|}\int_M u} = 0, \tag{1.5}$$

holds by Jensen's inequality, and hence $\lambda \mapsto I_\lambda(u)$ is non-increasing for each u. Inequality (1.5) implies, also, the uniform mountain pass structure, i.e., we obtain (1.1) for any $\lambda \in [\lambda_0, \lambda_1]$ with fixed $u_1 \in E$ and $u_0 = 0$, where $8\pi < \lambda_0 < \lambda_1 < \nu_2^*|M|$ are arbitrary. Consequently, $\lambda \mapsto c(\lambda) \equiv c_{I_\lambda}$ is non-increasing, and $c'(\lambda) \equiv \frac{d}{d\lambda}c(\lambda)$ exists for a.e. λ.

The existence of $c'(\lambda)$ induces the existence of a bounded $(\mathrm{PS})_{c(\lambda)}$ sequence, see [18, Lemma 3.5]. Then, we can use the *bounded* Palais-Smale condition denoted by (BPS) of I_λ; every bounded (PS) sequence to I_λ has a convergence subsequence. (This (BPS) condition to (1.3) is a consequence of the Trudinger-Moser inequality and the elliptic estimate.) In this way, we obtain the following theorem.

Theorem 2 ([18, Lemma 3.3])**.** *If $\lambda \mapsto c(\lambda)(\equiv c_{I_\lambda})$ is differentiable at $\lambda \in (8\pi, \nu_2^* |M|)$, then this $c(\lambda)$ is a critical value of I_λ defined by* (1.3).

These arguments are sometimes called Struwe's monotonicity trick [9]. Similarly to Theorem 1, the proof of Theorem 2 is reduced to using the $(\mathrm{BPS})_{c(\lambda)}$ condition only, which means that any bounded $(\mathrm{PS})_{c(\lambda)}$ sequence has a strongly converging subsequence.

The following facts are to be noted in connection with Theorems 1 and 2. First, concerning the existence of the non-trivial solution to (1.4), the residual set of λ is compensated by the blowup analysis [12], and consequently, any $\lambda \in (8\pi, \nu_2^* |M|) \setminus 8\pi\mathbf{N}$ admits a non-trivial solution to (1.4). Second, there is an abstract local structure of the mountain-pass critical point. To state this, we put

$$\mathrm{Cr}(I, c) := \{v \in X | I(v) = c, I'(v) = 0\},$$
$$I^c := \{u \in X : I(u) \le c\}, \qquad \dot{I}^c := \{u \in X : I(u) < c\},$$

where $c \in \mathbf{R}$.

Definition 3 ([6], see also [5, 7]). *Given $I \in C^1(X, \mathbf{R})$ and $v \in \mathrm{Cr}(I, c)$, we say the following:*

(i) *v is a local minimum if there is an open neighbourhood V of v, such that $I(u) \geq I(v)$ for any $u \in V$.*

(ii) *v is of mountain-pass type if any open neighbourhood U of v has the properties that $U \cap \dot{I}^c \neq \emptyset$ (that is, v is not a local minimum) and $U \cap \dot{I}^c$ is not path-connected.*

Theorem 4 ([6]). *Let c_I be the mountain-pass value in Theorem 1. Then, there exists a critical point in $\mathrm{Cr}(I, c_I)$, either a local minimum or of mountain - pass type. If all the critical points in $\mathrm{Cr}(I, c_I)$ are isolated in X, furthermore, the set $\mathrm{Cr}(I, c_I)$ contains a critical point of mountain-pass type.*

Although the original proof of Theorem 4 uses the (PS) (or $(\mathrm{PS})_{c_I}$) condition, Theorems 2 and 4 are compatible. Thus, we shall show the following theorem in this paper.

Theorem 5. *If $c'(\lambda)$ exists and $\lambda \notin 8\pi\mathbf{N}$ in Theorem 2, then there is a critical point in $\mathrm{Cr}(I_\lambda, c(\lambda))$, either a local minimum or of mountain-pass type. If all the critical points in $\mathrm{Cr}(I_\lambda, c(\lambda))$ are isolated, furthermore, $\mathrm{Cr}(I_\lambda, c(\lambda))$ contains a critical point of mountain-pass type.*

We use the argument [9] for the proof, namely, we have an abstract theorem that guarantees Theorem 5. (This is Theorem 10.)

Motivated by the mean field equation (1.4), several studies have been done concerning the variational problem provided with the monotonicity property and the (BPS) condition [13, 14]. By a recent result [4], particularly, we obtain a solution to

$$-\Delta_g v = \lambda \left(\frac{K(x)e^v}{\int_M K(x)e^v} - \frac{1}{|M|} \right), \qquad \int_M v = 0, \tag{1.6}$$

as the mini-max critical point of I_λ, provided that $\lambda \in (0, +\infty) \setminus 8\pi\mathbf{N}$ and $0 < K = K(x)$ is a smooth function. Combining this with the earlier work on the total topological degree [3], in particular, we can infer that there is at least two solutions to (1.6) if $\lambda > 16\pi$ and the genus of M is zero.

This paper, however, is focused on the local property around the critical point of I_λ. To our knowledge, such a property has not been studied for (1.4). We refer to [17] and the encyclopedic [8], for further study on the critical points obtained by the mountain-pass theorem under the presence of the (PS) condition.

The local degree of an isolated critical point of mountain-pass type has been discussed in [5, 7] in the context of the abstract Theorem 4. The fundamental assumption posed in [7] is the simpleness of the first linearized eigenvalue in case that it is zero. This condition, unfortunately, is hard to examine to I_λ, and consequently, the local degree of the solution obtained in Theorem 5 is not definite. In spite of this difficulty, however, we can control the Morse index of the isolated mountain pass solution to (1.4). More precisely, besides the assumption of Theorem 1, if X is a real Hilbert space, $I : X \to \mathbf{R}$ is C^2, and I'' has the form of the identity minus a compact operator, then any isolated critical point of mountain pass type has the Morse index less than or equal to 1, see [7]. This result is obviously applicable to (1.4), and the solution obtained in Theorem 5 has the Morse index less than or equal to 1 if it is isolated.

This paper is composed of three sections. The next section is the preliminary and the abstract Theorem 10 is stated at the end. Then this theorem is proved in the final section.

2. Preliminaries

The (PS) condition is twofold in the proof of Theorem 4, that is, the compactness of $\mathrm{Cr}(I_\lambda, c(\lambda))$ and the construction of a deformation. We can avoid the first issue by the quantized blowup mechanism of the solution set [12, 16], under the cost of $\lambda \notin 8\pi\mathbf{N}$. The second issue is compensated by the monotonicity trick, adopting the abstract setting of [9]:

(**H1**) $(X, \|\cdot\|)$ is a real Banach space and $\Lambda \subset (0, \infty)$ is a non-void interval,

(**H2**) $\{I_\lambda\}_{\lambda\in\Lambda}$ is a family of C^1 functionals on X with the form

$$I_\lambda(u) = A(u) - \lambda B(u),$$

for $\lambda \in \Lambda$, where $B(u) \geq 0$ for any $u \in X$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to +\infty$,

(**H3**) The mountain-pass structure holds uniformly in $\lambda \in \Lambda$:

$$c(\lambda) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\left\{I_\lambda(u_0), I_\lambda(u_1)\right\},$$

where $u_0 \neq u_1$.

As we have seen in Section 1, the functional associated with the mean field equation satisfies the above assumptions, where $X = E$,

$$A(u) = \frac{1}{2}\|\nabla u\|_2^2, \qquad B(u) = \log\left(\frac{1}{|M|}\int_M e^u\right),$$

$u_0 = 0$, and $\|u_1\|_E \gg 1$.

Thanks to $B(u) \geq 0$, the function $\lambda \in \Lambda \mapsto c(\lambda)$ is non-increasing and $c'(\lambda)$ exists for a.e λ. Then, there is a *mini-maximizing* sequence accompanied with paths of which tops are contained in a bounded set. We obtain, more precisely, the following lemma.

Lemma 6 ([9, Proposition 2.1]). *If $c'(\lambda)$ exists, then any $\lambda_k \uparrow \lambda$ takes $\{\gamma_k\} \subset \Gamma$ and $K = K(c'(\lambda)) > 0$ such that*

(i) $\|\gamma_k(t)\| \leq K$ *if* $I_\lambda(\gamma_k(t)) \geq c(\lambda) - (\lambda - \lambda_k)$, *where* $t \in (0, 1)$.

(ii) $\max_{t\in[0,1]} I_\lambda(\gamma_k(t)) \leq c(\lambda) + (-c'(\lambda) + 2)(\lambda - \lambda_k)$.

Here, we confirm the difference between Lemma 6 and the other arguments. First, similarly to the original assertion [18], the above sequence $\{\gamma_k\} \subset \Gamma$ is taken by

$$\max_{t\in[0,1]} I_{\lambda_k}(\gamma_k(t)) \leq c(\lambda_k) + (\lambda - \lambda_k). \tag{2.1}$$

In Lemma 6, however, this mini-maximizing sequence $\{\gamma_k\} \subset \Gamma$ is controlled in accordance with I_λ. It follows from (2.1) that $I_\lambda \leq I_{\lambda_k}$ and hence $c(\lambda) \leq c(\lambda_k)$, but Lemma 6 (ii) is more delicate. Actually, the derivation of Lemma 6 (ii) from (2.1) is not trivial. Second, the monotonicity assumption (H2) and the existence of $c'(\lambda)$ are not essential. These conditions can be replaced by the existence of a strict increasing sequence $\lambda_k \uparrow \lambda$ such that

$$\frac{c(\lambda_k) - c(\lambda)}{\lambda - \lambda_k} \leq M(\lambda),$$

with $M(\lambda) < \infty$ under the cost of an additional assumption to I_λ. Then, Denjoy's theorem is applicable to infer that the residual set of such λ is measure zero, see [11, Lemma 2.1].

Since the tops of $\{\gamma_k\}$ obtained by Lemma 6 are bounded, we are able to make a meaningful deformation of them, using the (BPS) condition for the (PS) condition. This is done by the quantitative deformation lemma of Willem [19] stated as follows.

Lemma 7 ([19, Lemma 2.3]). *Given a real Banach space $(X, \|\cdot\|)$ and $\varphi = \varphi(x) \in C^1(X, \mathbf{R})$, we suppose that $S \subset X$, $c \in \mathbf{R}$, $\varepsilon > 0$, and $\delta > 0$*

satisfy

$$\|\varphi'(u)\| \geq \frac{8\varepsilon}{\delta},$$

for every $u \in \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$, *where*

$$S_r := \{u \in X : \mathrm{dist}(u, S) \leq r\}.$$

Then, there exists $\eta \in C([0,1] \times X, X)$ *such that*

(i) $\eta(t,u) = u$ *if either* $t = 0$ *or* $u \notin \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$,
(ii) $\eta(1, \varphi^{c+\varepsilon} \cap S) \subset \varphi^{c-\varepsilon}$,
(iii) $\eta(t, \cdot)$ *is a homeomorphism of* X *for every* $t \in [0,1]$,
(iv) $\|\eta(t,u) - u\| \leq \delta$ *for every* $u \in X$ *and* $t \in [0,1]$,
(v) $\varphi(\eta(\cdot, u))$ *is non-increasing for every* $u \in X$,
(vi) $\varphi(\eta(t,u)) < c$ *for every* $u \in \varphi^c \cap S_\delta$ *and* $t \in (0,1]$.

Under these preparations, we can show the following deformation lemma à la Hofer [6, Lemma 2] (or [5, Lemma 1], [7, Lemma 1]) suitable for our case:

Lemma 8. *Let* $I \in C^1(X, \mathbf{R})$ *satisfy (BPS)*$_c$ *for* $c \in \mathbf{R}$. *Suppose that* $\mathrm{Cr}(I,c)$ *is bounded and contained in an open neighbourhood* $W \subset B_R(0)$, *where* $R > 0$ *and* $2\overline{\delta} \equiv \mathrm{dist}(\partial W, \mathrm{Cr}(I,c)) > 0$. *Then, each* $\overline{\varepsilon} > 0$ *and* $\delta \in (0, \overline{\delta})$ *admit* $\varepsilon \in (0, \overline{\varepsilon}]$ *and* $\eta \in C([0,1] \times X, X)$ *such that*

(i) $\eta(0,u) = u$ *and* $I(\eta(\cdot,u))$ *is non-increasing for every* $u \in X$,
(ii) $\eta(1, (I^{c+\varepsilon} \backslash W) \cap B_R(0)) \subset I^{c-\varepsilon}$,
(iii) $\|\eta(t,u) - u\| \leq \delta$ *for every* $u \in \overline{W}$ *and* $t \in [0,1]$,
(iv) $\eta(t,u) = u$ *for every* $t \in [0,1]$ *and* $u \in I^{-1}((-\infty, c-\overline{\varepsilon}]) \cup I^{-1}([c+\overline{\varepsilon}, \infty)) \cup B_{R+2\delta}(0)^c$.

Proof. Putting $S = B_R(0) \backslash W$, we have $\overline{S_{2\delta}} \cap \mathrm{Cr}(I,c) = \emptyset$ and $S_{2\delta} \subset B_{R+2\overline{\delta}}(0)$ for $\delta \in (0, \overline{\delta})$. By (BPS)$_c$, on the other hand, there are $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $\|I'(u)\| \geq \delta_0$ for every $u \in I^{-1}([c-2\varepsilon_0, c+2\varepsilon_0]) \cap S_{2\delta}$. Taking $\varepsilon \in (0, \min(\varepsilon_0, \delta_0\delta/8, \overline{\varepsilon}/3))$, therefore, the conclusion is obtained by Lemma 7 with these S, c, ε, and δ. □

If the (PS)$_c$ condition arises to $I \in C^1(X, \mathbf{R})$, then the (BPS)$_c$ condition holds and $\mathrm{Cr}(I,c)$ is compact. This compactness of $\mathrm{Cr}(I,c)$ implies its boundedness, and also the positivity of $2\overline{\delta}$. Lemma 8 has thus decomposed the (PS)$_c$ condition into the (BPS)$_c$ condition, the boundedness of $\mathrm{Cr}(I,c)$, and $2\overline{\delta} > 0$. If $\mathrm{Cr}(I,c) = \emptyset$, we are able to take $\overline{\delta} = \infty$.

Now, we shall state the topological device used for the proof of Theorem 5.

Lemma 9 ([6, Lemma 1])**.** *Let (X,d) be a metric space and $\Sigma, K \subset X$ be non-empty subsets such that K is compact and $K \subset \overline{\Sigma}$. We assume that there is an open cover $\{U_\kappa\}_{\kappa\in K}$ of K such that $\kappa \in U_\kappa$ and $U_\kappa \cap \Sigma$ is path-connected. Then, there is a finite disjoint open cover $\{V_i\}_{i=1,2,\dots,m}$ of K in X such that $V_i \cap \Sigma$ is contained in a path-connected component of $U \cap \Sigma$, where $U = \bigcup_{\kappa\in K} U_\kappa$.*

Finally, Theorem 5 is a consequence of the following abstract result, because $\mathrm{Cr}(I_\lambda, c(\lambda))$ is compact in (1.4) if $\lambda \notin 8\pi\mathbf{N}$, see [18]:

Theorem 10. *Suppose (H1)-(H3) and the existence of $c'(\lambda)$. Then, the $(BPS)_{c(\lambda)}$ condition implies $\mathrm{Cr}(I_\lambda, c(\lambda)) \neq \emptyset$. If $\mathrm{Cr}(I_\lambda, c(\lambda))$ is compact, moreover, there is an element in $\mathrm{Cr}(I_\lambda, c(\lambda))$, either a local minimum or a mountain-pass type. If all the critical points in $\mathrm{Cr}(I_\lambda, c(\lambda))$ are isolated, finally, then $\mathrm{Cr}(I_\lambda, c(\lambda))$ contains a critical point of mountain-pass type.*

3. Proof of Theorem 10

First, a bounded $(\mathrm{PS})_{c(\lambda)}$ sequence is obtained from the assumption of this theorem, see [9, Theorem1.1] and also [10, Lemma 4.2]. This implies $\mathrm{Cr}(I_\lambda, c(\lambda)) \neq \emptyset$ by the $(\mathrm{BPS})_{c(\lambda)}$ condition.

Next, we prove the last part; assuming that all the critical points in $\mathrm{Cr}(I_\lambda, c(\lambda))$ are isolated, we shall show that there is an element in $\mathrm{Cr}(I_\lambda, c(\lambda))$ of mountain-pass type. For this purpose we need not use the topological Lemma 9.

Suppose the contrary; $\mathrm{Cr}(I_\lambda, c(\lambda))$ does not contain any critical point of mountain-pass type. Since $\mathrm{Cr}(I_\lambda, c(\lambda))$ is compact, it contains only a finite number of critical points $\{v_1, \dots, v_m\}$ and we can find a corresponding open neighbourhood U_i of v_i such that $U_i \cap \dot{I}_\lambda^{c(\lambda)}$ is either empty or path-connected for each $i = 1, \dots, m$. Putting $U := \bigcup_{i=1}^m U_i (\supset \mathrm{Cr}(I_\lambda, c(\lambda)))$, we set, as in [5, Theorem 1] (or [7, Theorem 1]),

$$\overline{\varepsilon} := \frac{1}{2}(c(\lambda) - \max\{I_\lambda(u_0), I_\lambda(u_1)\}) \tag{3.1}$$

$$\overline{\delta} := \frac{1}{8} \min \{\mathrm{dist}((\partial U) \cup \{u_0, u_1\}, \mathrm{Cr}(I_\lambda, c(\lambda))) \\ \min\{\mathrm{dist}(v_i, \mathrm{Cr}(I_\lambda, c(\lambda)) \backslash \{v_i\}) : i = 1, \dots, m\}\}$$

$$W := \{u \in X : \mathrm{dist}(u, \mathrm{Cr}(I_\lambda, c(\lambda))) < \overline{\delta}\}. \tag{3.2}$$

Given $\lambda_k \uparrow \lambda$, now we take $\{\gamma_k\}$ and $K = K(c'(\lambda))$ of Lemma 6. Since $\mathrm{Cr}(I_\lambda, c(\lambda))$ is compact, we may assume $W \subset B_R(0)$ for some $R \geq K(c'(\lambda))$.

Applying Lemma 8 with these $\overline{\varepsilon}$, $c = c(\lambda)$, and $W \subset B_R(0)$, we obtain $\varepsilon \subset (0, \overline{\varepsilon}]$ and $\eta \in C([0,1] \times X, X)$ for each $\delta \in (0, \overline{\delta}/2)$. This η satisfies

$$\eta(1, \overline{W_i}) \subset (\overline{W_i})_\delta \subset U_i, \tag{3.3}$$

for $i = 1, \dots, m$ by Lemma 8 (iii), where $W_i = W \cap U_i$.

It holds that

$$\max_{t \in [0,1]} I_\lambda(\gamma_k(t)) \leq c(\lambda) + \varepsilon, \tag{3.4}$$

for $k \gg 1$. Now, we derive a contradiction by deforming this γ_k into a path in $\dot{I}_\lambda^{c(\lambda)}$, taking regards that W is a residual set of η in Lemma 8 (ii). Thus, we define

$$M := \{t \in [0,1] \mid \gamma_k(t) \notin W\} \tag{3.5}$$

$$B := \left(U \cap \dot{I}_\lambda^{c(\lambda)}\right) \bigcup \eta(1, \gamma_k(M)). \tag{3.6}$$

First, we confirm $B \subset \dot{I}_\lambda^{c(\lambda)}$. In fact,

$$\eta(1, \gamma_k(M) \cap B_R(0)) \subset I_\lambda^{c(\lambda)-\varepsilon} \subset \dot{I}_\lambda^{c(\lambda)},$$

by (3.4) and Lemma 8 (ii), while $\gamma_k(t) \in B_R(0)^c \subset B_K(0)^c$ implies

$$I_\lambda(\gamma_k(t)) < c(\lambda) - (\lambda - \lambda_k) < c(\lambda),$$

by Lemma 6 (i) and

$$\eta(1, \gamma_k(M) \backslash B_R(0)) \subset \dot{I}_\lambda^{c(\lambda)},$$

from the monotonicity of $I_\lambda(\eta(\cdot, u))$. This proves $B \subset \dot{I}_\lambda^{c(\lambda)}$.

Next, noting that $B(\supset \eta(1, \gamma_k(M)))$ contains u_0 and u_1, we take the path-component of B containing u_0, denoted by $\tilde{B}$. We shall derive $u_1 \in \tilde{B}(\subset B \subset \dot{I}_\lambda^{c(\lambda)})$, which contradicts the definition of $c(\lambda)$. This proof is similar to [5, Theorem 1] (or [7, Theorem 1]), and we shall only sketch it.

It suffices to prove $t_0 = 1$, where

$$t_0 := \sup\{t \in M \mid \eta(1, \gamma_k(t)) \in \tilde{B}\}.$$

In fact, we may assume $M \neq [0,1]$, and therefore, if $t_0 = 1$, then it holds that $\eta(1, \gamma_k(t)) \in \tilde{B}$ for a family of $\{t\}$ converging to 1. We have $I_\lambda(\gamma_k(t)) < c(\lambda) - \overline{\varepsilon}$ for such t, and hence $\eta(1, \gamma_k(t)) = \gamma_k(t) \in \tilde{B}$. This fact implies the desired $u_1 \in \tilde{B}$, because $\tilde{B}$ is path-connected and $u_1 = \gamma_k(1) \in B$.

Suppose $t_0 < 1$. Since $0 \in \mathrm{int}(M, [0,1])$, the relative interior of M with respect to $[0,1]$, we obtain $t_0 \in (0,1)$. If $[t^-, t^+]$ denotes the component of the closed set M containing t_0, then it follows that $t_0 = t^+$. In fact, in the case of $t^- < t_0$, the equality $t_0 = t^+$ holds because the image of each component of M by the mapping $\eta(1, \gamma_k(\cdot))$ is path-connected in B. In the other case of $t_0 = t^- < t^+$ (note that $t^- = t^+$ implies $(t^- =)t^+ = t_0$), we obtain $\gamma_k(t^-) \in \partial W \subset B_R(0)$ and therefore, $\eta(1, \gamma_k(t^-)) \in I_\lambda^{c(\lambda)-\varepsilon} \subset \dot{I}_\lambda^{c(\lambda)}$ by Lemma 8 (ii). Since $\eta(1, \gamma_k(t^-)) \in \overline{W}_\delta \subset U$ holds by Lemma 8 (iii), this implies $\eta(1, \gamma_k(t^-)) \in U \cap \dot{I}_\lambda^{c(\lambda)} \subset B$, and therefore, there exists $\hat{\varepsilon} > 0$ such that $B_{\hat{\varepsilon}}(\eta(1, \gamma_k(t^-))) \subset B$. This means $t_0 (= t^-) \in \tilde{B}$ from the definition of t_0. We have, on the other hand, $t^+ \in \tilde{B}$, and hence $t_0 = t^- < t^+ \leq t_0$, a contradiction. Thus, it holds that $t_0 = t^+$.

Since $\gamma_k(t_0) \in \partial W$, there exists $i_0 \in \{1, \ldots, m\}$ such that $\|\gamma_k(t_0) - v_{i_0}\| = \delta$. Then, we obtain $\hat{t} \in (t_0, 1)$ by $t_0 = t^+$, where

$$\hat{t} = \sup\{t \in [0,1] \mid \gamma_k(t) \in \overline{W_{i_0}}\}. \tag{3.7}$$

We have, on the other hand, $\gamma_k(\hat{t}) \in \partial W_{i_0} \subset B_R(0)$, and therefore,

$$\eta(1, \gamma_k(\hat{t})) \in (\overline{W_{i_0}})_\delta \cap \dot{I}_\lambda^{c(\lambda)} \subset U_{i_0} \cap \dot{I}_\lambda^{c(\lambda)},$$

by (3.3), (3.4), and Lemma 8 (ii). In particular, $U_{i_0} \cap \dot{I}_\lambda^{c(\lambda)}$ is path-connected because it it not empty. Similarly, it follows that $\eta(1, \gamma_k(t_0)) \in U_{i_0} \cap \dot{I}_\lambda^{c(\lambda)}$, and thus, $\eta(1, \gamma_k(\hat{t}))$ and $\eta(1, \gamma_k(t_0))$ are in the same path-component of $B \supset U_{i_0} \cap \dot{I}_\lambda^{c(\lambda)}$. This implies $t_0 < \hat{t} \leq t_0$, a contradiction. The proof of the last part of Theorem 10 is complete.

Now we turn to the general case that all the critical points in $\mathrm{Cr}(I_\lambda, c(\lambda))$ are not necessarily isolated, putting $K = \mathrm{Cr}(I_\lambda, c(\lambda))$ and $\Sigma = \dot{I}_\lambda^{c(\lambda)}$. We assume the existence of $c'(\lambda)$, suppose that any $\kappa \in K$ is neither a local minimum nor of mountain pass type, and derive a contradiction. In fact, in this case $K \subset \overline{\Sigma}$ and there is a neighbourhood U_κ of κ such that $U_\kappa \cap \Sigma$ is not empty and path-connected. Thus, Lemma 9 is applicable to these Σ, K, and $\{U_\kappa\}_{\kappa \in K}$, and we obtain a finite disjoint open cover $\{V_i\}_{i=1,\ldots,m}$ of K in X such that each $V_i \cap \Sigma$ is contained in a path-connected component of $U \cap \Sigma$. We put $V = \bigcup_{i=1}^m V_i$.

Since $K = \mathrm{Cr}(I_\lambda, c(\lambda))$ is compact, we have

$$\alpha = \mathrm{dist}\,(\partial V, K) = \min_{i=1,\ldots,m} \mathrm{dist}\,(\partial V_i, V_i \cap K) > 0.$$

Following [6, Theorem 1.1], we now define

$$\overline{\delta} = \frac{1}{8} \min\left\{\mathrm{dist}((\partial U) \cup \{u_0, u_1\}, \mathrm{Cr}(I_\lambda, c(\lambda))), \alpha\right\},$$

and take $\overline{\varepsilon}$ and W as in (3.1)-(3.2). Given $\lambda_k \uparrow \lambda$, we are provided with $\{\gamma_k\}$ and $K(c'(\lambda))$ in Lemma 6, similarly to the previous case. Taking $R \geq K(c'(\lambda))$ such that $W \subset B_R(0)$, then we apply Lemma 8 for these $\overline{\varepsilon}$, $c = c(\lambda)$, and $W \subset B_R(0)$. There are $\varepsilon \in (0, \overline{\varepsilon}]$ and $\eta \in C([0,1] \times X, X)$ satisfying the conclusions of this lemma for every $\delta \in (0, \overline{\delta}/2)$, and we put M and B as in (3.5) and (3.6), respectively. Let $\tilde{B}$ be the path-component of B containing u_0 as before. We shall show $u_1 \in \tilde{B}(\subset B \subset \Sigma)$, which contradicts the definition of $c(\lambda)$ as in the previous case.

For this purpose, we repeat the same argument and reach $t_0 = t^+$, if this is not the case. Putting $W_i = W \cap V_i$, we take $i_0 \in \{1, \ldots, m\}$ by $\operatorname{dist}(\gamma_k(t_0), W_{i_0} \cap K) = \delta$, and then obtain $\hat{t} \in (t_0, 1)$, where $\hat{t}$ is defined by (3.7). Since $\gamma_k(\hat{t}) \in \partial W_{i_0} \subset B_R(0)$, it holds that

$$\eta(1, \gamma_k(\hat{t})) \in (\overline{W_{i_0}})_\delta \cap \Sigma \subset V_{i_0} \cap \Sigma.$$

We obtain $\eta(1, \gamma_k(t_0)) \in V_{i_0} \cap \Sigma$, similarly. This implies $t_0 < \hat{t} \leq t_0$ because $V_{i_0} \cap \Sigma$ is in a path-component of $U \cap \Sigma$, a contradiction. $\square$

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