Differential and Integral Equations Volume 21, Numbers 5-6 (2008), 401–419

ANISOTROPIC EQUATIONS: UNIQUENESS AND EXISTENCE RESULTS

Stanislav Antontsev
CMAF, Universidade de Lisboa
Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal

Michel Chipot
Institut für Mathematik, Abt. Angewandte Mathematik, Universität Zürich
Winterthurerstrasse 190, CH–8057 Zürich, Switzerland

(Submitted by: Reza Aftabizadeh)

Abstract. We study uniqueness of weak solutions for elliptic equations of the following type

$$-\partial_{x_i}\left(a_i(x,u)\,|\partial_{x_i}u|^{p_i-2}\,\partial_{x_i}u\right)+b(x,u)=f(x),$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ with Lipschitz continuous boundary $\Gamma=\partial\Omega$. We consider in particular mixed boundary conditions, i.e., Dirichlet condition on one part of the boundary and Neumann condition on the other part. We study also uniqueness of weak solutions for the parabolic equations

$$\begin{cases} \partial_t u=\partial_{x_i}\left(a_i(x,t,u)\,|\partial_{x_i}u|^{p_i-2}\,\partial_{x_i}u\right)+f & \text{in } \Omega\times(0,T),\\ u=0 & \text{on } \Gamma\times(0,T)=\partial\Omega\times(0,T),\\ u(x,0)=u_0 & x\in\Omega. \end{cases}$$

It is assumed that the constant exponents p_i satisfy $1<p_i<\infty$ and the coefficients a_i are such that $0<\lambda\le\lambda_i\le a_i(x,u)<\infty,\ \forall i, a.e.\ x\in\Omega$, (a.e. $t\in(0,T)$), $\forall u\in\mathbb{R}$. We indicate also conditions which guarantee existence of solutions.

1. Introduction

In recent years, an increasing interest has turned towards anisotropic elliptic and parabolic equations. A special interest in the study of such equations is motivated by their applications to the mathematical modeling of physical and mechanical processes in anisotropic continuous medium. We refer to the

Accepted for publication: February 2008.

AMS Subject Classifications: 35K65, 35J70, 35K60, 35K65, 46E35.

The first author was partially supported by the research project POCI/MAT/61576/2004, FCT/MCES, Portugal. The second author was supported by the Swiss National Science Foundation under the contract # 20-113287/1.

recent works [2, 3], [6]-[13], [17]-[19], [24, 25] where it is possible to find some references. In these papers, various types of non-linear anisotropic elliptic and parabolic equations are studied from the point of view of existence and non-existence, regularity, and qualitative properties with respect to the data.

The main aim of this note is to prove uniqueness of weak solutions to the above mentioned equations under weak, feeble restrictions on the coefficients a_i. To prove it, we will follow some of the ideas of [1, 15, 16].

We do not discuss the existence and the regularity of such solutions under generic conditions, we just underline some scheme to prove it. In fact, existence results are technical improvements of classical results stated in [21]-[23]. For more exotic equations, existence results can be found in [2]-[8], [10]-[13], [17]-[20],[24, 25].

The paper is organized as follows: In Section 2, we consider elliptic equations and prove uniqueness results when at least one $p_i \leq 2$, and when a lower order monotone term $b(x,u)$ exists and $p_i > 2, \forall i$. We give also an existence theorem and consider more general equations.

The similar scheme is developed in Section 3 for parabolic equations.

2. Elliptic equations

2.1. **Uniqueness result when at least one $p_i \leq 2$.** We consider the Dirichlet problem

$$\begin{cases} -\partial_{x_i}\left(a_i(x,u)\,|\partial_{x_i}u|^{p_i-2}\,\partial_{x_i}u\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{2.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$.

We assume that the components of the constant vector $\vec{p} = (p_1, p_2, ..., p_n)$ satisfy

$$1 < p_1 \leq p_2 \leq ... \leq p_n < \infty. \tag{2.2}$$

We define

$$W_0^{1,\,\vec{p}}(\Omega) = \left\{u \in W_0^{1,\,p_1}(\Omega) : \partial_{x_i}u \in L^{p_i}(\Omega)\,,\ i = 1, ..., n\right\},$$

and we equip this space with the norm

$$\|u\|_{W_0^{1,\,\vec{p}}(\Omega)} = \sum_{i=1}^{n} \|\partial_{x_i}u\|_{L^{p_i}(\Omega)}\,.$$

By $W_0^{-1,\,\vec{p}\,'}(\Omega)$, we denote the dual space of $W_0^{1,\,\vec{p}}(\Omega)$. A weak solution $u \in W_0^{1,\,\vec{p}}(\Omega)$ of problem (2.1) is a function u such that

$$\int_\Omega a_i(x,u)\,|\partial_{x_i}u|^{p_i-2}\,\partial_{x_i}u\;\partial_{x_i}v = <f,v>,\;\;\forall v\in W_0^{1,\,\vec{p}}(\Omega)\,, \tag{2.3}$$

$$<f,v> = \int_\Omega fv.$$

We are going to prove uniqueness and existence of a solution $u \in W_0^{1,\,\vec{p}}(\Omega)$ to (2.3). We will suppose

$$0<\lambda\le\lambda_i\le a_i(x,u)\le\Lambda<\infty,\;\;\forall i, a.e.\, x\in\Omega, \forall u\in\mathbb{R}, \tag{2.4}$$

$$|a_i(x,u)-a_i(x,v)|\le\omega(|u-v|),\;\;\forall i, a.e.\, x\in\Omega, \forall u,v\in\mathbb{R}, \tag{2.5}$$

where

$$\int_{0^+}\frac{\mathrm{d}s}{\omega(s)} = +\infty. \tag{2.6}$$

We will use the Young inequality ($q\in(1,\infty),\;0\le a,b,\delta<\infty$)

$$ab\le\delta a^q+\frac{(q\delta)^{-\frac{q'}{q}}}{q'}b^{q'}\;,\;\;\frac{1}{q}+\frac{1}{q'}=1, \tag{2.7}$$

and the coerciveness inequality

$$\gamma\,|\xi_i-\eta_i|^{2+\sigma}\,\{|\xi_i|+|\eta_i|\}^{p_i-2-\sigma}\le\left\{|\xi_i|^{p_i-2}\,\xi_i-|\eta_i|^{p_i-2}\,\eta_i\right\}(\xi_i-\eta_i), \tag{2.8}$$

which holds for some positive constant γ and any non-negative constant σ, for every $\xi_i,\eta_i, i=1,...,n$, (see Lemma 2.2, [5]).

Theorem 2.1. *Under the conditions* (2.4), (2.5), *any weak solution* $u\in W_0^{1,\,\vec{p}}(\Omega)$ *to* (2.1) (*or* (2.3)) *is unique if at least one* $p_i\le 2$.

Proof. We define

$$F_\varepsilon(x)=\begin{cases}\displaystyle\int_\varepsilon^x\frac{\mathrm{d}s}{\omega^2(s)} & x\ge\varepsilon,\\ 0 & x\le\varepsilon.\end{cases} \tag{2.9}$$

Let u,v be two solutions to (2.3). Set $w=u-v$. We take as test function in (2.3) $F_\varepsilon(w)$ (for u,v). We get by subtraction

$$\int_{\Omega_\varepsilon}a_i(x,u)\left\{|\partial_{x_i}u|^{p_i-2}\,\partial_{x_i}u-|\partial_{x_i}v|^{p_i-2}\,\partial_{x_i}v\right\}\frac{\partial_{x_i}w}{\omega^2} \tag{2.10}$$

$$=\int_{\Omega_\varepsilon}\{a_i(x,v)-a_i(x,u)\}\,|\partial_{x_i}v|^{p_i-2}\,\partial_{x_i}v\frac{\partial_{x_i}w}{\omega^2},$$

where

$$\Omega_\varepsilon = \{x \in \Omega : w(x) > \varepsilon\}\,. \tag{2.11}$$

(In the above we have a summation in i). Using the ellipticity conditions (2.4), (2.5) and (2.8), we get with summation in i and for $\mu = \gamma\lambda$

$$\mu \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} \le \int_{\Omega_\varepsilon} |\partial_{x_i} v|^{p_i-1} \left|\frac{\partial_{x_i} w}{\omega}\right|. \tag{2.12}$$

Suppose that $p_i \ge 2$. Then

$$\begin{aligned} I_i := {}& \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right| |\partial_{x_i} v|^{p_i-1} \\ \le{}& \delta \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^2 |\partial_{x_i} v|^{p_i-2} + C(\delta, p_i) \int_{\Omega_\varepsilon} |\partial_{x_i} v|^{p_i} \\ \le{}& \delta \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} + C(\delta, p_i) \int_{\Omega_\varepsilon} |\partial_{x_i} v|^{p_i}, \end{aligned} \tag{2.13}$$

for every $\delta > 0$ by the Young inequality (2.7) with $q = 2$.

Suppose that $p_i \le 2$. Then

$$I_i = \int_{\Omega_\varepsilon} |\partial_{x_i} v|^{p_i-1} \left|\frac{\partial_{x_i} w}{\omega}\right| \le \delta \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^{p_i} + C(\delta, p_i) \int_{\Omega_\varepsilon} |\partial_{x_i} v|^{p_i}, \tag{2.14}$$

for any $\delta > 0$ we used (2.7) with $q = p_i$. Moreover we have

$$\begin{aligned} \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^{p_i} ={}& \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^{p_i} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{\frac{p_i-2}{2}p_i} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{-\frac{p_i-2}{2}p_i} \\ \le{}& \delta \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} + C(\delta, p_i) \int_{\Omega_\varepsilon} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}\,, \end{aligned} \tag{2.15}$$

for any $\delta > 0$ (we used (2.7) with $q = 2/p_i$ with a slight modification if $p_i = 2$). Combining (2.14), (2.15), we come to the same estimate which was obtained for $p_i \ge 2$ (compare with (2.13))

$$I_i \le \delta \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} + C(\delta, p_i) \int_{\Omega_\varepsilon} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}\,. \tag{2.16}$$

Finally, we get for every i

$$I_i \le \delta \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} + C(\delta, p_i) \int_{\Omega_\varepsilon} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}\,, \tag{2.17}$$

for every $\delta > 0$. We collect now (2.12)-(2.16) with a suitable choice of $\delta > 0$. Then we obtain

$$\int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} \le C \int_{\Omega_\varepsilon} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i} \le C', \quad (2.18)$$

with some constant $C' = C'(\delta, p_i)$ independent of ε. If there is one p_i, say $p_k < 2$, applying the Hölder inequality with $q = 2/p_k, q' = 2/(2-p_k)$ we obtain

$$\int_{\Omega_\varepsilon} \left|\frac{\partial_{x_k} w}{\omega}\right|^{p_k} = \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^{p_k} (|\partial_{x_k} u| + |\partial_{x_k} v|)^{\frac{p_k-2}{2}p_k} (|\partial_{x_k} u| + |\partial_{x_k} v|)^{\frac{2-p_k}{2}p_k} \quad (2.19)$$

$$\le \left(\int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^2 (|\partial_{x_k} u| + |\partial_{x_k} v|)^{p_k-2}\right)^{\frac{p_k}{2}} \left(\int_{\Omega_\varepsilon} (|\partial_{x_k} u| + |\partial_{x_k} v|)^{p_k}\right)^{\frac{2-p_k}{2}}.$$

By (2.18), (2.19) we deduce

$$\int_{\Omega_\varepsilon} \left|\frac{\partial_{x_k} w}{\omega}\right|^{p_k} \le C'. \quad (2.20)$$

We introduce the function G_ε defined by

$$G_\epsilon(x) = \begin{cases} \displaystyle\int_\epsilon^x \frac{\mathrm{d}s}{\omega(s)} & x > \epsilon, \\ 0 & x \le \epsilon, \end{cases}$$

which satisfies from above

$$\int_{\Omega_\varepsilon} |\partial_{x_k} G_\epsilon(w)|^{p_k} \le C^{'}. \quad (2.21)$$

By the Poincaré inequality it follows that

$$\int_{\Omega_\varepsilon} |G_\epsilon(w)|^{p_k} \le C^{''}. \quad (2.22)$$

If $\{w = u - v > \epsilon\}$ has a positive measure, letting $\epsilon \to 0$, we obtain in accord to (2.6) that

$$\int_\Omega |G_\epsilon(w)|^{p_k} \to \infty,$$

which contradicts the above inequality. Hence, we have $u \ge v$. By changing the rôle of u and v this completes the proof. □

Remark 2.1. With the same proof as above one could allow, in (2.1), a monotone lower order term $b(x,u)$, where $u \mapsto b(x,u)$ is non-increasing (see also below).

2.2. **Uniqueness result when a lower order term exists and** $p_i > 2, \forall i$**.** Let us consider the problem

$$\begin{cases} -\partial_{x_i}\left(a_i(x,u)|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u\right) + b(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{2.23}$$

That is to say $u \in W_0^{1,\,\vec{p}}(\Omega)$ is a weak solution in the sense that

$$\int_\Omega a_i(x,u)|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u\ \partial_{x_i}v + b(x,u)v =< f, v > \ \ \forall v \in W_0^{1,\,\vec{p}}(\Omega)\,. \tag{2.24}$$

Let us denote by $m(t)$ the uniform modulus of continuity of the a_i's, i.e.,

$$m(t) = \max_i \sup_{x\in\Omega,\ |u-v|\le t} |a_i(x,u) - a_i(x,v)|. \tag{2.25}$$

If the $a_i(x,u)$ are Carathéodory functions uniformly continuous with respect to x , we have

$$m(t) \to 0 \text{ as } t \to 0. \tag{2.26}$$

Moreover, $m(t)$ is non-decreasing with t. We denote by ω the function

$$\omega(t) = \min(m(t), 1).$$

We assume that (2.4) holds, and, since the a_i's are bounded for some constant C, we have

$$|a_i(x,u) - a_i(x,v)| \le C\omega(|u-v|)\ \forall i,\ \text{a.e. } x \in \Omega,\ \forall u, v \in \mathbf{R}. \tag{2.27}$$

We suppose that there exists α such that

$$\int_{0^+} \frac{\mathrm{d}s}{\omega^\alpha(s)} = +\infty, \quad 1 < \alpha \le \frac{p_n}{p_n - 1} \le \frac{p_i}{p_i - 1},\ \forall i. \tag{2.28}$$

Then, for every $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ such that

$$\int_{\delta(\varepsilon)}^{\varepsilon} \frac{\mathrm{d}s}{\omega^\alpha(s)} = 1. \tag{2.29}$$

We then define $F_\varepsilon(x)$as

$$F_\varepsilon(x) = \begin{cases} 0 & x \le \delta(\varepsilon), \\ \displaystyle\int_{\delta(\varepsilon)}^{x} \frac{\mathrm{d}s}{\omega^\alpha(s)} & \delta(\varepsilon) \le x \le \varepsilon, \\ 1 & \varepsilon \le x. \end{cases} \tag{2.30}$$

Clearly, this function is Lipschitz continuous. We assume also

$$u \to b(x,u) \text{ is increasing.} \tag{2.31}$$

Theorem 2.2. *Under the conditions* (2.4), (2.27), (2.28), (2.31) *the weak solution* $u \in W_0^{1,\vec{p}}(\Omega)$ *to* (2.24) *is unique.*

Proof. Let u, v be two solutions to (2.23). Setting $w = u - v$, we take as test function in (2.24) $F_\varepsilon(w)$. Thus, we have, dropping x in $b(x,u)$

$$\begin{aligned} \int_{\Omega_\varepsilon} a_i(x,u)\left\{|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u - |\partial_{x_i}v|^{p_i-2}\partial_{x_i}v\right\}\partial_{x_i}w\ F'_\varepsilon(w) \\ + \int_\Omega (b(u)-b(v))\,F_\varepsilon(w) \\ = \int_{\Omega_\varepsilon}\{a_i(x,v)-a_i(x,u)\}\,|\partial_{x_i}v|^{p_i-2}\ \partial_{x_i}v\ \partial_{x_i}w\ F'_\varepsilon(w). \end{aligned} \tag{2.32}$$

The integration in the integral containing F'_ε is taking place only on the set

$$\Omega_\varepsilon = \{x : \delta(\varepsilon) < w(x) < \varepsilon\}, \tag{2.33}$$

when some derivative occurs. Using (2.4), (2.27), we derive for some constant $\mu > 0$

$$\begin{aligned} &\mu\sum_i \int_{\Omega_\varepsilon} \frac{|\partial_{x_i}w|^{p_i}}{\omega^\alpha}dx + \int_\Omega (b(u)-b(v))\,F_\varepsilon(w) \\ &\le \sum_i \int_{\Omega_\varepsilon} |a_i(x,v)-a_i(x,u)||\partial_{x_i}v|^{p_i-1}\frac{|\partial_{x_i}w|}{\omega^\alpha} \\ &\le C\sum_i \int_{\Omega_\varepsilon} \frac{|\partial_{x_i}w|}{\omega^{(\alpha-1)}}|\partial_{x_i}v|^{p_i-1}. \end{aligned} \tag{2.34}$$

Next, we use the Young inequality (2.7) with $q = p_i$ to evaluate the left hand side. From (2.34) we get

$$\begin{aligned} &\mu\sum_i \int_{\Omega_\varepsilon} \frac{|\partial_{x_i}w|^{p_i}}{\omega^\alpha}dx + \int_\Omega (b(u)-b(v))\,F_\varepsilon(w) \\ &\le \delta\sum_i \int_{\Omega_\varepsilon} \frac{|\partial_{x_i}w|^{p_i}}{\omega^{(\alpha-1)p_i}} + C\sum_i \int_{\Omega_\varepsilon} |\partial_{x_i}v|^{p_i}, \end{aligned} \tag{2.35}$$

for any $\delta > 0$ and for some constant $C = C(\delta, p_i)$. We know that $\omega \le 1$ and thus $\omega^{(\alpha-1)p_i} \ge \omega^\alpha$ since

$$\alpha \ge (\alpha-1)p_i \Leftrightarrow \alpha \le \frac{p_i}{p_i-1}. \tag{2.36}$$

Then, from (2.35), we obtain

$$\mu \sum_i \int_{\Omega_\varepsilon} \frac{|\partial_{x_i} w|^{p_i}}{\omega^\alpha} dx + \int_\Omega (b(u) - b(v))\, F_\varepsilon(w)$$
$$\leq \delta \sum_i \int_{\Omega_\varepsilon} \frac{|\partial_{x_i} w|^{p_i}}{\omega^\alpha} + C \sum_i \int_{\Omega_\varepsilon} |\partial_{x_i} v|^{p_i}.$$

Let us choose δ small enough such that $\delta < \mu$, (δ is now fixed). We obtain

$$\int_\Omega (b(x,u) - b(x,v))\, F_\varepsilon(w) \leq C' \sum_i \int_{\Omega_\varepsilon} |\partial_{x_i} v|^{p_i},$$

for some constant C'. Note that $\chi_{\Omega_\varepsilon} \to 0$ a.e., $F_\varepsilon(w) \to 1$ on $w > 0$. Passing to the limit we get

$$\int_{u-v>0} (b(x,u) - b(x,v)) \leq 0,$$

and thus, $u - v \leq 0$, if b is monotone increasing in u. Exchanging the rôle of u, v, we get $u = v$ provided α is chosen such that $\alpha \leq \frac{p_i}{p_i - 1}$ $\forall i$. $\square$

Remark 2.2. Since

$$\beta = \inf_i \frac{p_i}{p_i - 1} = \frac{p_n}{p_n - 1} > 1,$$

one can take $\alpha = \beta$ and (2.28) holds for $m(t) \leq Ct^{\frac{1}{\beta}}$, i.e., for a_i's Hölder continuous in u of exponent $\frac{1}{\beta}$.

2.3. **Existence of solutions.** In this section, we discuss the existence of weak solutions for the problem (2.24). We do not consider this question under the weakest possible assumptions, our aim in this note being uniqueness.

To prove the desired existence results, we follow [1, 2, 3], in which the Dirichlet problem was studied.

Considering the equation

$$\int_\Omega a_i(x,u)|\partial_{x_i} u|^{p_i - 2} \partial_{x_i} u \ \partial_{x_i} v + b(x,u)v =< f, v > \ \forall v \in W_0^{1,\vec{p}}(\Omega), \quad (2.37)$$

we assume that

$$a_i, b : \Omega \times \mathbb{R} \to \mathbb{R} \text{ are Carathéodory functions,} \quad (2.38)$$

and that

$$0 < \lambda \leq \lambda_i \leq a_i(x,u) \leq \Lambda < \infty, \ \forall i, a.e.\ x \in \Omega, \forall u \in \mathbb{R}, \quad (2.39)$$

$$|b(x,u)| \le C_0|u|^{\beta-1} + h(x),\ 0 \le h(x) \in L^{\frac{\beta}{\beta-1}}(\Omega), 1 \le \beta \le p_n, \tag{2.40}$$

(recall that p_n is the largest p_i)

$$f(x) \in W_0^{-1,\,\vec{p}\,'}(\Omega)\,. \tag{2.41}$$

Theorem 2.3. *Under the conditions* (2.2), (2.38)-(2.41) *the problem* (2.37) *has at least one weak solution* $u \in W_0^{1,\,\vec{p}}\,(\Omega)$.

Proof. Let $v(x)$ be any given function such that $v(x) \in L^{p_n}(\Omega)$. We define functions A_i by

$$A_i(x) = a_i(x, v(x)), \qquad \forall i.$$

We then introduce the operator $\Lambda : W_0^{1,\,\vec{p}}(\Omega) \mapsto W_0^{-1,\,\vec{p}\,'}(\Omega)$,

$$(\Lambda u, \varphi) = \int_\Omega A_i(x)|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u\ \partial_{x_i}\varphi + b(x,u)\varphi.$$

It is obvious that the mapping $\Lambda : W_0^{1,\,\vec{p}}(\Omega) \mapsto W_0^{-1,\,\vec{p}\,'}(\Omega)$ is continuous and monotone. Let us verify that it is coercive. Using (2.4), (2.40) we get

$$\lambda \sum_i \int_\Omega |\partial_{x_i} u|^{p_i} - \int_\Omega \Big(C_0|u|^\beta + h|u|\Big) \le (\Lambda u, u)\,. \tag{2.42}$$

Next, applying the Young, Hölder and Poincaré inequalities, we obtain

$$\int_\Omega \Big(C_0|u|^\beta + h|u|\Big) \le C' \int_\Omega \Big(|u|^\beta + h^{\frac{\beta}{\beta-1}}\Big) \tag{2.43}$$

$$\le C'' \left(\left(\int_\Omega |\partial_{x_n} u|^{p_n} \right)^{\frac{\beta}{p_n}} + \int_\Omega h^{\frac{\beta}{\beta-1}} \right) \le \delta \int_\Omega |\partial_{x_n} u|^{p_n} + C''' \left(1 + \int_\Omega h^{\frac{\beta}{\beta-1}}\right).$$

Combining (2.42), (2.43) with $2\delta = \lambda$, we come to

$$\frac{\lambda}{2} \sum_i \int_\Omega |\partial_{x_i} u|^{p_i} - C\left(1 + \int_\Omega h^{\frac{\beta}{\beta-1}}\right) \le (\Lambda u, u)\,, \tag{2.44}$$

which guarantee that the operator Λ is coercive. The space $W_0^{1,\,\vec{p}}(\Omega)$ is reflexive. By the Browder-Minty theorem ([14], Theorem 7.3.2) the equation

$$\Lambda u = f, \tag{2.45}$$

has at least one weak solution $u \in W_0^{1,\,\vec{p}}(\Omega)$ for every $f \in W_0^{-1,\,\vec{p}\,'}(\Omega)$. Thus, the mapping

$$\Phi : v \to u, \tag{2.46}$$

defines an operator from $L^{p_n}(\Omega)$ into itself. Moreover, if u is solution to (2.45) by (2.44) we have

$$\frac{\lambda}{2}\sum_i |\partial_{x_i} u|^{p_i}_{L^{p_i}(\Omega)} \le C\left(1+\int_\Omega h^{\frac{\beta}{\beta-1}}\right)+|f|_{W_0^{-1,\,\vec{p}'}(\Omega)}\sum_i |\partial_{x_i} u|_{L^{p_i}(\Omega)}, \quad (2.47)$$

where $|f|_{W_0^{-1,\,\vec{p}'}(\Omega)}$ denotes the strong dual norm of f. It follows by Young's inequality that

$$\frac{\lambda}{2}\sum_i |\partial_{x_i} u|^{p_i}_{L^{p_i}(\Omega)} \le \delta\sum_i |\partial_{x_i} u|^{p_i}_{L^{p_i}(\Omega)} + C(h,f). \quad (2.48)$$

Choosing $\delta = \frac{\lambda}{4}$ we derive that if u is solution to (2.45)

$$\frac{\lambda}{2}\sum_i |\partial_{x_i} u|_{L^{p_i}(\Omega)} \le C(\lambda,h,f). \quad (2.49)$$

Due to the Poincaré inequality

$$|u|_{L^{p_n}(\Omega)} \le C|\partial_{x_n} u|_{L^{p_n}(\Omega)}, \quad (2.50)$$

we obtain for some new constant

$$|u|_{L^{p_n}(\Omega)} \le C(\lambda,h,f) = R. \quad (2.51)$$

It is easy to verify that the operator Φ is compact and continuous and transforms the ball $\{u \in L^{p_n}(\Omega) : |u|_{L^{p_n}(\Omega)} \le R\}$ into itself. According to the Schauder fixed point theorem the operator Φ has at least one fixed point, which defines a weak solution of problem (2.37). This completes the proof of the theorem. □

2.4. **Generalizations.** Let us consider more generally the problem

$$\begin{cases} -\partial_{x_i}\left(a_i(x,u,\nabla u)\right) + b(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega. \end{cases} \quad (2.52)$$

A solution u is an element $u \in W_0^{1,\,\vec{p}}(\Omega)$ such that

$$\int_\Omega a_i(x,u,\nabla u)\ \partial_{x_i} v + \int_\Omega b(x,u)v = \langle f, v\rangle \quad \forall v \in W_0^{1,\,\vec{p}}(\Omega). \quad (2.53)$$

Assume that $\forall u \in W_0^{1,\,\vec{p}}(\Omega)$; $f \in W_0^{-1,\,\vec{p}'}(\Omega)$,

$$a_i(x,u,\nabla u) \in L^{p_i}(\Omega),\ b(x,u) \in W_0^{-1,\,\vec{p}'}(\Omega). \quad (2.54)$$

First, we consider the case

$$2 < p_i < \infty, \quad i = 1,...,n. \quad (2.55)$$

We assume that for some positive constant $\mu > 0$, a.e. $x \in \Omega, \forall u, v \in \mathbb{R}\ \forall \eta, \xi \in \mathbb{R}^n$ we have

$$\mu|\eta_i - \xi_i|^{p_i} - \omega(|u-v|)\left(|\eta_i| + |\xi_i|\right)^{p_i - 1}|\eta_i - \xi_i| \leq \left(a_i(x,u,\eta) - a_i(x,v,\xi), \eta_i - \xi_i\right), \tag{2.56}$$

where the function $\omega(s), \alpha$ satisfy (2.28), (2.36). Let $u,\ v$ be two solutions of (2.52) and $w = u - v$. Then, taking as test function in (2.53) $F_\varepsilon(w)$ defined by (2.30), we come to an analog to (2.32). Indeed, we have

$$\int_{\Omega_\varepsilon} \{a_i(x,u,\nabla u) - a_i(x,v,\nabla v)\}\,\partial_{x_i} w\ F'_\varepsilon(w) + \int_\Omega (b(u) - b(v))\, F_\varepsilon(w) = 0. \tag{2.57}$$

Using (2.56), we get

$$\mu \sum_i \int_{\Omega_\varepsilon} \frac{|\partial_{x_i} w|^{p_i}}{\omega^\alpha} dx + \int_\Omega (b(u) - b(v))\, F_\varepsilon(w) \leq \sum_i \int_{\Omega_\varepsilon} \frac{|\partial_{x_i} w|}{\omega^{\alpha-1}} \left(|\partial_{x_i} u| + |\partial_{x_i} v|\right)^{p_i - 1}, \tag{2.58}$$

(compare with (2.34)). Next, repeating the arguments of the proof of the Theorem 2.2, we have

Theorem 2.4. *Under the conditions* (2.54)–(2.56), *there exists at most one solution* $u \in W_0^{1,\vec{p}}(\Omega)$ *to* (2.53).

Now, we consider the case when at least one p_i is less or equal to 2. We assume that for some positive constant $\mu > 0$, *a.e.* $x \in \Omega, \forall u, v \in \mathbb{R}\ \forall \eta, \xi \in \mathbb{R}^n$

$$\mu|\eta_i - \xi_i|^2\left(|\eta_i| + |\xi_i|\right)^{p_i - 2} - \omega(|u-v|)\left(|\eta_i| + |\xi_i|\right)^{p_i - 1}|\eta_i - \xi_i| \leq \left(a_i(x,u,\eta) - a_i(x,v,\xi), \eta_i - \xi_i\right), \tag{2.59}$$

where the function $\omega(s)$ defined by (2.9). Repeating the arguments of the Theorem 2.1, we obtain

Theorem 2.5. *Under the conditions* (2.54), (2.59) *there exists at most one solution* $u \in W_0^{1,\vec{p}}(\Omega)$ *to* (2.53) *if at least one* p_i *is less or equal to* 2.

Remark 2.3. The theorems 2.1-2.5 remain valid for mixed boundary conditions

$$u = 0 \text{ on } \Gamma_D,\ \ a_i(x,u,\nabla u)\nu_i = 0 \text{ on } \Gamma_N, \tag{2.60}$$

where $\vec{\nu} = (\nu_1, ..., \nu_1)$ is the unit normal vector to Γ_N, $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$ and $mes\,\Gamma_D > 0$. In this case we consider weak solutions $u \in W_0^{1,\,\vec{p}}(\Omega, \Gamma_D)$ where

$$W_0^{1,\,\vec{p}}(\Omega, \Gamma_D) = \left\{ u \in W_0^{1,\,p_1}(\Omega, \Gamma_D) : \partial_{x_i} u \in L^{p_i}(\Omega),\; i = 1, ..., n \right\}.$$

3. Parabolic equations

3.1. **Uniqueness of solution.** We consider the problem

$$\begin{cases} \partial_t u = \partial_{x_i}\left(a_i(x,t,u)|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u\right) + f & \text{in } Q_T = \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0 & x \in \Omega. \end{cases} \tag{3.1}$$

We define

$$V_0^{\vec{p}}(Q_T) = L^\infty(0,T; L^2(\Omega)) \cap L^{\vec{p}}(0,T; W_0^{1,\,\vec{p}}(\Omega)),$$

and we equip it with the norm

$$|u|_{V_0^{\vec{p}}(Q_T)} = \sup_{0 \le t \le T} |u|_{L^2(\Omega)} + \sum_{i=1}^n \left(\int_0^T |\partial_{x_i} u|^{p_i}_{L^{p_i}(\Omega)} \right)^{\frac{1}{p_i}}.$$

(Recall that $W_0^{1,\,\vec{p}}(\Omega)$ was defined in the section (2.1).) By $V_0^{-1,\vec{p}'}(Q_T)$ we denote the dual space of $V_0^{1,\,\vec{p}}(Q_T)$. We define

$$V(Q_T) = \left\{ u \in V_0^{1,\,\vec{p}}(Q_T),\; u_t \in V_0^{-1,\,\vec{p}'}(Q_T) \right\}.$$

By a solution to (3.1) we mean a $u \in V(Q_T)$ such that

$$\begin{cases} (u_t, \varphi)_{Q_T} + \sum_{i=1}^n \left(a_i(x,t,u)|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u, \partial_{x_i}\varphi\right)_{Q_T} = (f, \varphi)_{Q_T}, \\ u(x,0) = u_0, \quad x \in \Omega, \\ \forall \varphi \in V_0^{\vec{p}}(Q_T), \quad a.e.\ t \in (0,T),\; (u,v)_{Q_t} = \int_{Q_t} uv. \end{cases} \tag{3.2}$$

Note that, according to (3.2), u satisfies $u_t \in V_0^{-1,\,\vec{p}'}(\Omega)$ and

$$|(u_t, \varphi)_{Q_T}| \le C|\varphi|_{V_0^{\vec{p}}(Q_T)} \left(|u|_{V_0^{\vec{p}}(Q_T)} + |f|_{V_0^{-1,\,\vec{p}'}(Q_T)} \right). \tag{3.3}$$

We suppose that

$$0 < \lambda \le \lambda_i \le a_i(x,t,u) \le \Lambda \quad \forall i, \forall u \in \mathbb{R}, a.e.\,(x,t) \in \Omega \times (0,T). \tag{3.4}$$

Denote by $m(t)$ the uniform modulus of continuity of the a_i in u, i.e., set

$$m(s) = \sup_{i,x,t,|u-v|\leq s} |a_i(x,t,u) - a_i(x,t,v)|. \tag{3.5}$$

Modifying eventually $m(t)$ for $t \geq 1$ one can find a function ω which coincides with $m(t)$ for $t \leq 1$ and which satisfies

$$|a_i(x,t,u) - a_i(x,t,v)| \leq \omega(|u-v|), \int_1^\infty \frac{\mathrm{d}s}{\omega^2(s)} < +\infty, \tag{3.6}$$
$$\forall i, \forall\, u, v \in \mathbb{R},\ a.e.\ (x,t) \in \Omega \times (0,T).$$

Suppose that

$$\int_{0^+} \frac{\mathrm{d}s}{\omega^2(s)} = +\infty. \tag{3.7}$$

Set

$$F_\varepsilon(x) = \begin{cases} \displaystyle\int_\varepsilon^x \frac{\mathrm{d}s}{\omega^2(s)} & x \geq \varepsilon, \\ 0 & x \leq \varepsilon, \end{cases} \tag{3.8}$$

$$I_\varepsilon = \int_\varepsilon^\infty \frac{\mathrm{d}s}{\omega^2(s)},\ H_\varepsilon(x) = \frac{F_\varepsilon(x)}{I_\varepsilon}. \tag{3.9}$$

Since for $x \geq \varepsilon$ one has

$$H_\varepsilon(x) = \frac{\int_\varepsilon^x \frac{\mathrm{d}s}{\omega^2(s)}}{I_\varepsilon} = \frac{I_\varepsilon - \int_x^\infty \frac{\mathrm{d}s}{\omega^2(s)}}{I_\varepsilon}. \tag{3.10}$$

It is clear that $I_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and thus

$$H_\varepsilon(x) \to H(x) \text{ the Heaviside function.} \tag{3.11}$$

Theorem 3.1. *Let u, v be two solutions of* (3.1) *corresponding to initial values u_0, v_0. Under the conditions* (2.2), (3.4), (3.6), (3.7), *we have*

$$\int_\Omega (u-v)^+ dx \leq \int_\Omega (u_0 - v_0)^+ dx,\ (u)^+ = \max(u, 0), \tag{3.12}$$

which implies in particular uniqueness of the solution to (3.1).

Proof. Let u, v be two solutions of (3.1) corresponding to initial values u_0, v_0. Set $w = u - v$. By difference of the equations (3.2) for u and v we have

$$(w_t, \varphi)_{Q_t} + \sum_{i=1}^n \left(a_i(x,t,u)\left\{|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u - |\partial_{x_i} v|^{p_i-2}\partial_{x_i} v\right\}, \partial_{x_i}\varphi\right)_{Q_t} \tag{3.13}$$

$$= \sum_{i=1}^{n} \left(\{a_i(x,t,v) - a_i(x,t,u)\} \, |\partial_{x_i} v|^{p_i-2} \partial_{x_i} v, \partial_{x_i} \varphi \right)_{Q_t}.$$

We take as test function in (3.13)

$$H_\varepsilon(w) = \frac{F_\varepsilon(w)}{I_\varepsilon}.$$

Then, we obtain

$$\begin{aligned} &\int_0^t \int_{\Omega_\varepsilon} \partial_t w \ H_\varepsilon(w) \\ &= -\frac{1}{I_\varepsilon} \int_0^t \int_{\Omega_\varepsilon} a_i(x,s,u) \left\{ |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u - |\partial_{x_i} v|^{p_i-2} \partial_{x_i} v \right\} \partial_{x_i} F_\varepsilon(w) \quad (3.14) \\ &\quad + \frac{1}{I_\varepsilon} \int_0^t \int_{\Omega_\varepsilon} \{a_i(x,s,v) - a_i(x,s,u)\} \, |\partial_{x_i} v|^{p_i-2} \partial_{x_i} v \ \partial_{x_i} F_\varepsilon(w), \end{aligned}$$

where Ω_ε is the set defined by

$$\Omega_\varepsilon = \{x \in \Omega : (u-v)(x,s) > \varepsilon\}, \tag{3.15}$$

(outside of Ω_ε, $F_\varepsilon(w)$ vanishes, see (3.8)). Using (3.4) and the definition of F_ε we have for some $\mu > 0$

$$\begin{aligned} &\mu \int_{\Omega_\varepsilon} \left| \frac{\partial_{x_i} w}{\omega} \right|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} \quad (3.16) \\ &\leq \int_{\Omega_\varepsilon} a_i(x,s,u) \left\{ |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u - |\partial_{x_i} v|^{p_i-2} \partial_{x_i} v \right\} \frac{\partial_{x_i} w}{\omega^2} \\ &= \int_{\Omega_\varepsilon} a_i(x,s,u) \left\{ |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u - |\partial_{x_i} v|^{p_i-2} \partial_{x_i} v \right\} \partial_{x_i} F_\varepsilon(w). \end{aligned}$$

By (3.6), we have also

$$\begin{aligned} &\left| \int_{\Omega_\varepsilon} \{a_i(x,s,v) - a_i(x,s,u)\} \, |\partial_{x_i} v|^{p_i-2} \partial_{x_i} v \partial_{x_i} F_\varepsilon(w) \right| \quad (3.17) \\ &\leq \int_{\Omega_\varepsilon} |a_i(x,s,v) - a_i(x,s,u)| |\partial_{x_i} v|^{p_i-1} \frac{|\partial_{x_i} w|}{\omega^2} \\ &\leq \int_{\Omega_\varepsilon} |\partial_{x_i} v|^{p_i-1} \frac{|\partial_{x_i} w|}{\omega}. \end{aligned}$$

From (3.14), we then derive

$$\int_0^t \int_{\Omega_\varepsilon} \partial_t w \ H_\varepsilon(w) \leq -\frac{\mu}{I_\varepsilon} \int_0^t \sum_i \int_{\Omega_\varepsilon} \left| \frac{\partial_{x_i} w}{\omega} \right|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} \tag{3.18}$$

$$+\frac{1}{I_\varepsilon}\int_0^t \sum_i \int_{\Omega_\varepsilon} |\partial_{x_i} v|^{p_i-1} \frac{|\partial_{x_i} w|}{\omega}.$$

(In the formulae before, we did not write the summation in i). Then, we proceed with the same estimates than in the elliptic case.

Suppose that $p_i < 2$. Combining the estimates (2.12), (2.15) of the elliptic case, we have

$$\int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right| \left|\partial_{x_i} v\right|^{p_i-1} \leq \delta \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^{p_i} + C(\delta, p_i) \int_{\Omega_\varepsilon} \left|\partial_{x_i} v\right|^{p_i} \tag{3.19}$$

$$\leq \delta^2 \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} + C(\delta, p_i) \int_{\Omega_\varepsilon} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}.$$

Suppose that $p_i \geq 2$. From the estimate (2.13) of the elliptic case, we have

$$\int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right| |\partial_{x_i} v|^{p_i-1} \tag{3.20}$$

$$\leq \delta \int_{\Omega_\varepsilon} \left|\frac{\partial_{x_i} w}{\omega}\right|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} + C(\delta, p_i) \int_{\Omega_\varepsilon} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}.$$

Choosing and fixing δ such that δ and $\delta^2 < \mu$. In (3.19) and (3.20), it is clear that from (3.18)-(3.20) we get for some constant $C = C(\delta, p_i)$

$$\int_0^t \int_{\Omega_\varepsilon} \partial_t w H_\varepsilon(w) \leq \frac{C}{I_\varepsilon} \int_0^t \sum_i \int_{\Omega_\varepsilon} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}. \tag{3.21}$$

Let us set

$$G_\varepsilon(x) = \int_{-\infty}^x H_\varepsilon(t) dt.$$

We have

$$\partial_t w H_\varepsilon(w) = \partial_t G_\varepsilon(w),$$

and the above inequality gives us

$$\int_{\Omega_\varepsilon} G_\varepsilon(w)(t) - \int_{\Omega_\varepsilon} G_\varepsilon(w)(0) \leq \frac{C}{I_\varepsilon} \int_0^t \sum_i \int_{\Omega_\varepsilon} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}.$$

Letting $\epsilon \to 0$, we get (3.12) since $G_\varepsilon(w) \to (w)^+,\ I_\varepsilon \to +\infty$. □

Remark 3.1. The condition (3.7) allows the $a_i(x,t,u)$ to be Hölder continuous with Hölder exponent greater or equal to 1/2.

3.2. **Existence of solutions.** In this section, we discuss the existence of weak solutions $u \in V(Q_T)$ for the problem (3.1). We show here only a sketch of the proof for a simple case.

Considering the problem

$$(u_t, \varphi)_{Q_t} + \sum_{i=1}^{n} \left(a_i(x,t,u)|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u, \partial_{x_i}\varphi\right)_{Q_t} = (f,\varphi)_{Q_t}\,,\ u(x,0) = u_0, \tag{3.22}$$

we assume that

$$a_i, b : Q_T \times \mathbb{R} \to \mathbb{R} \text{ are Carathéodory functions,} \tag{3.23}$$

such that $a.e.\,(x,t) \in Q_T, \forall u \in \mathbb{R}$,

$$0 < \lambda \leq \lambda_i \leq a_i(x,t,u) \leq \Lambda < \infty,\ \forall i. \tag{3.24}$$

We suppose

$$f(x,t) \in V_0^{-1,\,\vec{p}'}(Q_T)\,. \tag{3.25}$$

Then, we have

Theorem 3.2. *Under the conditions* (3.23)-(3.25), *the problem* (3.22) *has at least one weak solution* $u \in V_0^{\vec{p}}(Q_T)$.

Proof. Let $v(x,t)$ be a function such that $v(x,t) \in L^{p_n}(Q_T)$. We set

$$A_i(x,t) = a_i(x,t,v(x,t)), \quad \forall i.$$

Let us consider the problem

$$u_t - \Lambda(u) = f, \quad u(x,0) = u_0, \tag{3.26}$$

$$\Lambda(u) = \sum_{i=1}^{n} \partial_{x_i}\left(A_i(x,t)|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u\right).$$

It is easy to verify that the operator $\Lambda : V_0^{1,\,\vec{p}}(Q_T) \mapsto V_0^{-1,\,\vec{p}'}(Q_T)$ is continuous, monotone and coercive. Thus, according to ([23], Ch2.), problem (3.26) has a unique solution which defines an operator

$$\Phi : v \to u. \tag{3.27}$$

According to (3.24) for all solution u the following estimate

$$|u|_{V_0{}^{\vec{p}}(Q_T)} = \sup_{0\leq t\leq T} |u|_{L^2(\Omega)} + \sum_{i=1}^{n}\left(\int_0^T |\partial_{x_i} u|^{p_i}_{L^{p_i}(\Omega)}\right)^{\frac{1}{p_i}}$$
$$\leq C\left(|u_0|_{L^2(\Omega)} + |f|_{V_0^{-1,\,\vec{p}'}(Q_T)}\right) = C',$$

holds. It is easy to verify that the operator Φ is compact and continuous from $L^{p_n}(Q_T)$ into itself and transforms, for some R large enough, the ball of center 0 and radius R in $L^{p_n}(Q_T)$ into itself. According to the Schauder fixed point theorem the operator Φ has at least one fixed point, which is a weak solution of problem (3.22). The theorem is proved. □

3.3. **Generalizations.** Let us consider more generally $u \in V\,(Q_T)$ solution to

$$(u_t, \varphi)_{Q_T} + \sum_{i=1}^{n} (a_i(x,t,u,\nabla u), \partial_{x_i}\varphi)_{Q_T} = (f,\varphi)_{Q_T}\,,\ u(x,0) = u_0. \quad (3.28)$$

Assume that

$$a_i(x,u,t,\nabla u) \in L^{p_i'}(Q_T),\ \forall u \in V_0^{1,\,\vec{p}}(Q_T)\,;\ f \in V_0^{-1,\,\vec{p}'}(Q_T)\,. \quad (3.29)$$

We address here the case

$$1 < p_i < \infty,\ i = 1, ..., n. \quad (3.30)$$

We suppose that for some positive constant $\mu > 0, a.e.\, x \in \Omega, \forall t, u, v \in \mathbb{R}\ \forall \eta, \xi \in \mathbb{R}^n$

$$\begin{aligned} \mu|\eta_i - \xi_i|^2\,(|\eta_i| + |\xi_i|)^{p_i-2} - \omega(|u-v|)\,(|\eta_i| + |\xi_i|)^{p_i-1}\,|\eta_i - \xi_i| \quad (3.31)\\ \le (a_i(x,t,u,\eta) - a_i(x,t,v,\xi), \eta_i - \xi_i)\,, \end{aligned}$$

where the function $\omega(s)$ satisfies (3.7). Let u, v be two solutions of (3.28) and $w = u - v$. Then taking as test function in (3.28) $H_\varepsilon(w)$ for u and v, we come to an analog of (3.14)

$$\begin{aligned} &\int_0^t \int_{\Omega_\varepsilon} \partial_t w\ H_\varepsilon(w)\\ &= -\frac{1}{I_\varepsilon}\int_0^t \int_{\Omega_\varepsilon} \sum_{i=1}^{n} \{a_i(x,t,u,\nabla u) - a_i(x,t,v,\nabla v)\}\,\partial_{x_i} w\ F_\varepsilon'(w). \quad (3.32) \end{aligned}$$

Using (3.31), we get

$$\begin{aligned} \sum_{i=1}^{n}\left(\mu\frac{|\partial_{x_i} w|^{p_i}}{\omega^2}\,(|\partial_{x_i} u| + |\partial_{x_i} v_i|)^{p_i-2} - \frac{|\partial_{x_i} w|}{\omega}\,(|\partial_{x_i} u| + |\partial_{x_i} v_i|)^{p_i-1}\right) \quad (3.33)\\ \le \sum_{i=1}^{n} \{a_i(x,t,u,\nabla u) - a_i(x,t,v,\nabla v)\}\,\partial_{x_i} w\ F_\varepsilon'(w). \end{aligned}$$

Thus, we arrive to an inequality like (3.18). Next, repeating the arguments of the proof of the Theorem 3.1, we obtain

Theorem 3.3. *Let u, v be two solutions of* (3.28) *corresponding to the initial values u_0, v_0. Under conditions* (3.29)-(3.31) *we have*

$$\int_\Omega (u-v)^+ dx \leq \int_\Omega (u_0-v_0)^+ dx,\ \ (u)^+ = \max(u,0), \tag{3.34}$$

which implies in particular uniqueness of the solution (3.28).

Remark 3.2. Assertions of Theorems 3.2, 3.3 remain valid for mixed boundary conditions

$$u = 0 \text{ on } \Gamma_D,\ \ a_i(x,t,u,\nabla u)\nu_i = 0 \text{ on } \Gamma_N, \tag{3.35}$$

where $\vec{\nu} = (\nu_1, ..., \nu_1)$ is the unit normal vector to $\Gamma_N,\ \Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$ and $mes\ \Gamma_D > 0$.

REFERENCES

[1] S. Antontsev, M. Chipot, and Y. Xie, *Uniqueness results for equations of the $p(x)$-Laplacian type*, Adv. Math. Sci. Appl., 17 (2007), 287–304.

[2] S. Antontsev and S. Shmarev, *Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness and localization properties of solutions*, Journal Nonlinear Analysis, 65 (2006), 722–755.

[3] ———, *Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions*, Elsevier, 2006. Handbook of Differential Equations. Stationary Partial Differential Equations, Elsevier, Vol. 3, Chapter 1, pp. 1-100.

[4] ———, *Parabolic equations with anisotropic nonstandard growth conditions*, in Internat. Ser. Numer. Math. 154, Birkhäuser, Verlag Basel/Switzerland, 2006, 33–44.

[5] J. Barrett and W. B. Liu, *Finite element approximation of the parabolic $p-$Laplacian*, SIAM Journal on Numerical Analysis, 31 (1994), 413–428.

[6] M. Bendahmane and K. H. Karlsen, *Renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations*, SIAM J. Math. Anal., 36 (2004), 405–422.

[7] ———, *Nonlinear anisotropic elliptic and parabolic equations in $\mathbb{R}^{\mathbb{N}}$ with advection and lower order terms and locally integrable data*, Potential Anal., 22 (2005), 207–227.

[8] ———, *Uniqueness of entropy solutions for doubly nonlinear anisotropic degenerate parabolic equations*, Contemporary Mathematics, 371 (2005), 1–27.

[9] ———, *Anisotropic doubly nonlinear degenerate parabolic equations*, in Numerical mathematics and advanced applications, Springer, Berlin, 2006, 381–386.

[10] ———, *Anisotropic nonlinear elliptic systems with measure data and anisotropic harmonic maps into spheres*, Electron. J. Differential Equations, No. 46, (2006), 1-30.

[11] ———, *Renormalized solutions of an anisotropic reaction-diffusion-advection system with L^1 data*, Commun. Pure Appl. Anal., 5 (2006), 733–762.

[12] M. Bendahmane, M. Langlais, and M. Saad, *On some anisotropic reaction-diffusion systems with L^1-data modeling the propagation of an epidemic disease*, Nonlinear Anal., 54 (2003), 617–636.

[13] M. Bendahmane and M. Saad, *Entropy solution for anisotropic reaction-diffusion-advection systems with L^1 data*, Rev. Mat. Complut., 18 (2005), 49–67.

[14] P. Blanchard and E. Brüning, *Variational methods in mathematical physics*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1992. A unified approach, Translated from the German by Gillian M. Hayes.

[15] M. Chipot and M. Michaille, *Uniqueness results and monotonicity properties for strongly nonlinear elliptic variational inequalities*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 16 (1989), 137–166.

[16] ———, *Uniqueness results and monotonicity properties for the solutions of some variational inequalities. Existence of a free boundary*, in Free boundary problems: theory and applications, Vol. I, (Irsee,1987), Pitman Research Notes in Mathematics Series,185, Longman Sci. Tech., Harlow (1990), 271–276.

[17] L. Feng-Quan, *Anisotropic Elliptic Equations in L^m*, Journal of Convex Analysis, 8 (2001), 417–422.

[18] I. Fragalà, F. Gazzola, and B. Kawohl, *Existence and nonexistence results for anisotropic quasilinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 21 (2004), 715–734.

[19] I. Fragalà, F. Gazzola, and G. Lieberman, *Regularity and nonexistence results for anisotropic quasilinear elliptic equations in convex domains*, Discrete Contin. Dyn. Syst., (2005), 280–286.

[20] E. Kim, *Existence results for singular anisotropic elliptic boundary-value problems*, Electronic Journal of Differential Equations, 17 (2000), 1–17.

[21] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and quasilinear equations of parabolic type*, American Mathematical Society, Providence, R.I., 1967. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23.

[22] O. A. Ladyzhenskaya and N. N. Ural′tseva, *Linear and quasilinear elliptic equations*, Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis, Academic Press, New York, 1968.

[23] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.

[24] B. Song, *Anisotropic diffusions with singular advections and absorptions. part 1. existence*, Applied Mathematics Letters, 14 (2001), 811–816.

[25] ———, *Anisotropic diffusions with singular advections and absorptions. part 2. uniqueness*, Applied Mathematics Letters, 14 (2001), 817–823.