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SHARP ANALYTIC–GEVREY REGULARITY ESTIMATES DOWN TO $t = 0$ FOR SOLUTIONS TO SEMILINEAR HEAT EQUATIONS

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Abstract. We study the Gevrey regularity down to $t = 0$ of solutions to the initial-value problem for the semilinear heat equation $\partial_t u - \Delta u + F[u] = 0$ with polynomial non-linearities. The approach is based on suitable iterative fixed point methods in L^p-based Banach spaces with anisotropic Gevrey norms with respect to the time and space variables. We also construct explicit solutions uniformly analytic in $t \geq 0$ and $x \in \mathbb{R}^n$ for some conservative non-linear terms with symmetries.

1. Introduction and organization of the paper

We consider the initial-value problem for the semilinear heat equation

$$\partial_t u = \Delta u + F[u], \quad u|_{t=0} = u_0, \tag{1.1}$$

where Δ is the Laplace operator on $\mathbb{R}^n$ and $F[u]$ stands for a semilinear perturbation of the type

$$F[u] = f(u, \nabla u), \tag{1.2}$$

with f being typically a polynomial, or more general, analytic non-linearity. The initial data will be supposed analytic or, more generally, belonging to some Gevrey space G^σ. Broadly speaking, the main aim of the present paper is to study in detail *simultaneously* the analytic-Gevrey regularity

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with respect to $t \geq 0$ and $x \in \mathbb{R}^n$. The paper is the extension of [13], where the special case of $f(u) = u^M$ was treated.

We construct a new, as far as we know, functional framework consisting of scales of Banach spaces of functions having anisotropic Gevrey regularity $G^{\tau,\sigma}$ with respect to $t \geq 0$, $x \in \mathbb{R}^n$ and derive precise estimates in such scales for the solutions of the IVP down to $t = 0$. It is well known that for the heat equation one has in general $\tau \geq 2\sigma$ (in fact, if $\sigma = 1$ it was M. Gevrey who derived in 1918, what we will call nowadays $G^{1,2}([0, +\infty[\times\mathbb{R}^n_x)$, estimates for the heat kernel cf. [12]).

In the linear case, we derive sharp estimates on the uniform anisotropic regularity in $t \geq 0$, $x \in \mathbb{R}^n$ down to $t = 0$ and we allow initial data in subspaces of entire functions of exponential type $1/(1-\sigma)$, $\sigma \in]0,1[$. Next, we dwell upon the Banach algebra properties of such scales when the Gevrey index $\sigma \geq 1$ and provide an insight into the loss of the Banach algebra properties when the Gevrey index $\sigma \in]0,1[$. The Banach algebra properties allows us to show local existence and uniqueness by means of standard Picard iteration schemes combined with the contraction principle. Finally, inspired by the celebrated Oseen solution for the 2-D Navier–Stokes equation, we exhibit classes of semilinear equations with conservative nonlinear terms with symmetries admitting uniformly analytic solutions with respect to $t \geq 0$ and $x \in R^n$, provided the initial conditions admit suitable symmetries.

We are motivated by two different approaches dealing with the analytical regularity for parabolic equations.

We recall that by the classical theory of the linear non-Kowalevskian equations, since the linear part of (1.1) is characteristic, in general the solution need not be analytic in time at $t = 0$ even if the initial data is globally analytic. The first motivation for our work comes from investigations of divergent in t solutions of linear non-Kowalevskian equations by means of the Borel summability. The existence of formal power series solutions to the characteristic Cauchy problem for linear equations was studied by S. Ōuchi in [31] and for some non-linear equations in [32]. He established the asymptotic relation between formal and genuine solutions. In the case of the one-dimensional linear heat equation, D. A. Lutz, M. Miyake and S. Schäfke in [22] gave conditions on the initial data under which the formal solutions are Borel summable to the exact ones. The result was extended to the multidimensional heat equation by S. Michalik in [27] and to other linear equations by M. Miyake in [28], by W. Balser in [2], [3] and by S. Ōuchi in [33].

It was shown in [23] that in the case $f(u) = u^2$ the solution to (1.1) need not be analytic in time at zero even if the initial data is globally analytic. The result can be extended to other cases. Similar results for the Korteweg-de Vries equation were obtained in [8] and [24].

A survey of other results on the analytic smoothing effect for solutions to dispersive non-linear equations such as the KdV and non-linear Schrödinger equation can be found in [18].

The second motivation is based on functional–analytic methods and it deals primarily with the analytic regularization of the heat operator with respect to the space variables for positive times for various classes of semilinear parabolic equations, the Burgers' equation, the Navier-Stokes equation, etc. We refer to the pioneering paper of Foias and Temam [10] for the Navier–Stokes equation with periodic data, while concerning semilinear parabolic equations, H. Aikawa and N. Hayashi proved in [1] that if $u_0 \in L^p(\mathbb{R}^n)$ and f is a polynomial of degree $\leq 1 + 2p/n$ then for $t > 0$ the solution $u(t, \cdot)$ extends analytically to a strip in $\mathbb{C}^n$ with width proportional to $\sqrt{t}$. Later on, results for more general semilinear parabolic equations have been proved (cf. [35], [9], [5], [6], [16] and the references therein). We stress that the aforementioned papers do not deal with the Gevrey type regularity in t down to $t = 0$. One of the main technical difficulties is more pertinent to the non-linear analysis, namely suitable nonlinear superposition estimates for Gevrey anisotropic spaces.

There are also results for analyticity with respect to the time variable, but only in a conic neighborhood of $t = 0$. The first result in this direction was obtained by S. Ōuchi [30] who proved analyticity in time in a sector $\{t = re^{i\theta} : 0 < r < \infty, |\theta| < \alpha\}$ under the assumption that $f = f(u)$ is a monotone non-increasing polynomial and the initial function is bounded and continuous. We mention also the paper of Z. Grujić and I. Kukavica [17] for similar results for the Kuramoto–Sivashinsky equation.

We can summarize our approach as follows: in order to address the issues of the uniform regularity in $t \geq 0$ and $x \in \mathbb{R}^n$, treated so far in the first direction by means of Borel type transformations, we borrow ideas from the second direction and develop a new functional analytic framework which enables us to deal simultaneously with the regularity with respect to $t \geq 0$ and $x \in \mathbb{R}^n$.

The organization of the paper is the following. In Section 2, we introduce the Banach spaces G of uniformly Gevrey functions and state some of their properties including the Banach algebra property. In Section 3, we investigate the Cauchy problem for the linear heat equation in spaces G. Since in the spaces G one observes the loss of regularity (Propositions 3.1 and 3.2) in the next section we introduce new Gevrey spaces $\widetilde{G}$. We study the properties of $\widetilde{G}$ and the relations between G and $\widetilde{G}$. In Section 5, we study the linear heat equation in spaces $\widetilde{G}$ and in Section 6 we give non-linear estimates in $\widetilde{G}$. Section 7 contains the main result on local existence and regularity in terms of the spaces $\widetilde{G}$ of solutions to (1.1). The approach is based on suitable iterative fixed point methods in L^p-based Banach spaces with anisotropic Gevrey norms with respect to the time and space variables. The crucial novelty is the introduction of global anisotropic $\widetilde{G}^{\tau,\sigma}$ spaces which, combined with novel non-linear estimates, leads to stability-type Gevrey estimates with respect to $t \geq 0$, $x \in \mathbb{R}^n$ for solutions to (1.1). In the final section, we give examples of analyticity down to $t = 0$ of (Oseen type) solutions to (1.1) provided the non-linear term consists of conservative terms having invariance properties (symmetries) on the Euclidean spheres $S^{n-1}(r)$, $r > 0$.

2. Banach spaces of uniformly Gevrey functions

Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $\sigma > 0$. We define $G^{\sigma}_{un}(\Omega)$ - the spaces of uniformly Gevrey functions of index σ - as the set of all $f \in C^\infty(\Omega)$ such that there exists $C < \infty$ satisfying

$$\sup_{x\in\Omega} |\partial_x^\alpha f(x)| \leq C^{|\alpha|+1}\alpha!^\sigma, \qquad \alpha \in \mathbb{N}_0^n, \tag{2.1}$$

where $\alpha! = \alpha_1! \dots \alpha_n!$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$. Given $f \in G^{\sigma}_{un}(\mathbb{R}^n)$, in view of (2.1), we can define

$$\rho_\sigma[f] = \sup\{C^{-1} > 0 : \text{such that (2.1) holds}\}. \tag{2.2}$$

Clearly, if $\sigma = 1$ then every $f \in G^1_{un}(\mathbb{R}^n)$ extends to a holomorphic function in $\{z = (z_1, \dots, z_n) \in \mathbb{C}^n \ : \ \max_{j=1,\dots,n} |\mathrm{Im} z_j| < \rho_1[f]\}$. Furthermore, if $f \in G^{\sigma}_{un}(\mathbb{R}^n), \sigma > 1$ and $f \sim \hat{f}$ as $x \to 0$, where $\hat{f} = \sum_{\alpha\in\mathbb{N}_0^n} f_\alpha x^\alpha$ is a formal power series, then the radius of convergence of the Borel transform of order $\sigma - 1$ of $\hat{f}$ is not less then $\rho_\sigma(f)$, see [2].

Local Gevrey spaces $G^\sigma(\Omega)$ are defined in a natural way as the projective limit of $G^{\sigma}_{un}(\widetilde{\Omega})$ over an exhaustive sequence of open sets $\widetilde{\Omega}$ relatively compact in Ω. Thus $f \in G^\sigma(\Omega)$ if for every $\widetilde{\Omega} \subset\subset \Omega$ one can find $C > 0$ such that (2.1) holds with Ω replaced by $\widetilde{\Omega}$. In particular, if $\sigma = 1$ we recover the well-known set $G^1(\Omega) = A(\Omega)$ of the real analytic functions on Ω while for $\sigma > 1$ the $G^\sigma(\Omega)$ admit non-zero compactly supported functions. We refer to cf. [34], [26] for more details on Gevrey spaces with index $\sigma \geq 1$. We point out that the Gevrey spaces G^σ, $\sigma \geq 1$, are a natural framework for the study of PDEs with multiple characteristics and questions of regularity of solutions to evolution PDEs of mathematical physics, see [26] and the references therein.

The case of $0 < \sigma < 1$ has not been dealt with in the literature on PDEs. In that case the space $G^\sigma_{un}(\Omega)$ consists of restrictions to Ω of functions from the space $\mathcal{O}^{exp}(\mathbb{C}^n; \rho)$ of entire functions of exponential order $\rho = 1/(1-\sigma)$; i.e., satisfying the estimate

$$|F(z)| \leq C \exp\{L|z|^\rho\} \text{ for } z \in \mathbb{C}^n, \tag{2.3}$$

with some $C < \infty$ and $L < \infty$ (cf. [21] for the one-dimensional case, and Proposition 2.4 below for the general case). Typically, in the applications, one introduces scales of Banach spaces depending on a parameter $\rho > 0$:

$G^\sigma(\rho, X) = \{u \in X : \partial^\alpha u \in X$ for any $\alpha \in \mathbb{N}_0^n$ and $\|u\|_{\sigma,\rho;X} < \infty\}$, where

$$\|u\|_{\sigma,\rho;X} := \sum_{\alpha \in \mathbb{N}_0^n} \frac{\rho^{|\alpha|}}{(\alpha!)^\sigma} \|\partial^\alpha u\|_X, \tag{2.4}$$

and X is some Banach space of functions on Ω (e.g., X might be $C^k(\overline{\Omega})$ or a Sobolev space $H^k_p(\Omega)$, $1 \leq p \leq \infty$, $k \geq 0$). We set $\rho_\sigma(u) = \sup\{\rho > 0 : u \in G^\sigma(\rho, X)\}$, cf. [4] and the references therein.

Lemma 2.1. *Let $u \in G^\sigma(\rho, X)$ and $\beta \in \mathbb{N}_0^n$. Then $\partial^\beta u \in G^\sigma(\widetilde{\rho}, X)$ for any $\widetilde{\rho} < \rho$ and*

$$\|\partial^\beta u\|_{\sigma,\tilde{\rho};X} \leq C^{|\beta|} \beta!^\sigma \|u\|_{\sigma,\rho;X}, \tag{2.5}$$

where $C = 2^\sigma \left(1 - (\tilde{\rho}/\rho)^{1/\sigma}\right)^{-\sigma} \rho^{-1}$.

In the proof of Lemma 2.1 we shall need a corollary of the following.

Lemma 2.2. *Let $x \geq 0$ and $y \geq 0$. Then for any $\varepsilon > 0$*

$$\frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} \leq \frac{(1+\varepsilon)^{x+y}}{\varepsilon^y}, \tag{2.6}$$

and equality holds if and only if $x = y = 0$.

Proof. Since $a\Gamma(a) = \Gamma(a+1)$ for $a > 0$ and

$$\Gamma(a)\Gamma(b)/\Gamma(a+b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt,$$

for $a > 0$ and $b > 0$ we get for $x \geq 0$, $y \geq 0$ and any $0 < \delta < 1$,

$$\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+1)} = (x+y+1) \int_0^1 t^x (1-t)^y dt$$

$$
\begin{aligned}
&\geq (x+y+1)\int_0^\delta t^x(1-\delta)^y dt + (x+y+1)\int_\delta^1 \delta^x(1-t)^y dt \\
&= \frac{x+y+1}{x+1}\delta^{x+1}(1-\delta)^y + \frac{x+y+1}{y+1}\delta^x(1-\delta)^{y+1} \\
&\geq \delta^{x+1}(1-\delta)^y + \delta^x(1-\delta)^{y+1}.
\end{aligned}
$$

Now, we put $\delta = (1+\varepsilon)^{-1}$ with $\varepsilon > 0$ to get

$$
\delta^{x+1}(1-\delta)^y + \delta^x(1-\delta)^{y+1} = \frac{\varepsilon^y}{(1+\varepsilon)^{x+y}},
$$

which proves (2.6). The last statement is clear. □

Corollary 2.3. *Let $\beta \in \mathbb{N}_0^n$. Then for any $\alpha \in \mathbb{N}_0^n$ and $\varepsilon > 0$ we have*

$$
\frac{(\alpha+\beta)!}{\alpha!\beta!} \leq \frac{(1+\varepsilon)^{|\alpha+\beta|}}{\varepsilon^{|\beta|}}. \tag{2.7}
$$

Proof of Lemma 2.1. Fix $\tilde{\rho} < \rho$ and choose $0 < \epsilon < 1$ such that $\tilde{\rho} \leq (1-\epsilon)^\sigma \rho$. Since $\epsilon < 1$ by Corollary 2.3 we get

$$
\begin{aligned}
\|\partial^\beta u\|_{\sigma,\tilde{\rho};X} &\leq \sup_{\alpha\in\mathbb{N}_0^n} \frac{\tilde{\rho}^{|\alpha|}(\alpha+\beta)!^\sigma}{\rho^{|\alpha+\beta|}\alpha!^\sigma} \cdot \sum_{\alpha\in\mathbb{N}_0^n} \frac{\rho^{|\alpha+\beta|}}{(\alpha+\beta)!^\sigma}\|\partial^{\alpha+\beta}u\|_X \\
&\leq \|u\|_{\sigma,\rho;X} \cdot \sup_{\alpha\in\mathbb{N}_0^n} \frac{\tilde{\rho}^{|\alpha|}}{\rho^{|\alpha+\beta|}}\Big(\frac{1+\epsilon}{\epsilon}\Big)^{|\beta|\sigma}\beta!^\sigma(1+\epsilon)^{|\alpha|\sigma} \\
&\leq \frac{2^{|\beta|\sigma}\beta!^\sigma}{\epsilon^{|\beta|\sigma}\rho^{|\beta|}}\|u\|_{\sigma,\rho;X}.
\end{aligned}
$$

Clearly this implies (2.5). □

We have the following.

Proposition 2.4. *Let $0 < \sigma < 1$ and Ω be an open domain of $\mathbb{R}^n$. Then*

(i) $G^\sigma_{un}(\Omega) = G^\sigma(\mathbb{R}^n) = \mathcal{O}^{exp}(\mathbb{C}^n; 1/(1-\sigma))$;

(ii) if $u \in G^\sigma(\rho, X)$ and $\tilde{u}$ denotes the extension of u to $\mathbb{C}^n$, then one can find $C < \infty$ and $m \in \mathbb{N}_0$ such that, for any $0 < \varepsilon < \rho$ and $z \in \mathbb{C}^n$,

$$
|\tilde{u}(z)| \leq C\frac{\rho^{\sigma m}(1+\rho^{-m})}{\varepsilon^{\sigma m}}\|u\|_{\sigma,\rho,;X}\exp\{(1-\sigma)\big(|z|/(\rho-\varepsilon)\big)^{1/(1-\sigma)}\}. \tag{2.8}
$$

Moreover, if $m = 0$, then (2.8) holds with $\varepsilon = 0$.

Proof. (i) Clearly $G^{\sigma}(\mathbb{R}^n) \subset G^{\sigma}_{un}(\Omega)$. We shall show that any $u \in G^{\sigma}_{un}(\Omega)$ extends to an entire function $\widetilde{u}$ of exponential order $\rho = 1/(1-\sigma)$. Indeed, assuming $0 \in \Omega$ we define

$$\widetilde{u}(z) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{\partial^{\alpha} u(0)}{\alpha!} z^{\alpha} \text{ for } z \in \mathbb{C}^n.$$

By (2.1) the series converges almost uniformly on $\mathbb{C}^n$ and

$$|\widetilde{u}(z)| \leq \sum_{\alpha \in \mathbb{N}_0^n} C \frac{C^{|\alpha|} \alpha!^{\sigma} |z^{\alpha}|}{\alpha!} \leq 2^n C \sup_{\alpha \in \mathbb{N}_0^n} (|\alpha|!)^{\sigma-1} (2Cn|z|)^{|\alpha|}$$
$$\leq 2^n C \exp\{(1-\sigma)(2Cn|z|)^{1/(1-\sigma)}\},$$

by the Stirling formula, since for any $r > 0$ and $0 < s < 1$,

$$\sup_{p \in \mathbb{N}_0} \left(\frac{e^s r}{p^s} \right)^p \leq \exp\{s r^{1/s}\}.$$

Conversely, let $u \in \mathcal{O}^{exp}(\mathbb{C}^n; 1/(1-\sigma))$ and fix $K = [-a, a]^n \Subset \mathbb{R}^n$, where $a > 0$. By the Cauchy integral formula in the polydisc $\{z \in \mathbb{C}^n : |z_j| \leq R, j = 1, \ldots, n\}$ for $R > a$ we obtain

$$\sup_{x \in K} |\partial_x^{\alpha} u(x)| \quad \leq \quad \frac{e^{LR^{1/(1-\sigma)}}}{(R-a)^{|\alpha|}}, \qquad R > 0. \tag{2.9}$$

Minimizing the right-hand side of (2.9) with respect to $R \in]a, +\infty[$, we get the desired G^{σ} estimates on K; i.e., $u \in G^{\sigma}(\mathbb{R}^n)$.

(ii) Let $m \in \mathbb{N}_0$ be such that if $\partial^{\beta} u \in X$ for $|\beta| \leq m$, then $u \in C(\Omega)$ and $\|u\|_{C(\Omega)} \leq C \sup_{|\beta| \leq m} \|\partial^{\beta} u\|_X$. Combining the proof of part (i) with that of Lemma 2.1 we derive, for $0 < \epsilon < 1$,

$$\begin{aligned}
|\tilde{u}(z)| &\leq C \sum_{\alpha \in \mathbb{N}_0^n} \sup_{|\beta| \leq m} \frac{\|\partial^{\alpha+\beta} u\|_X}{\alpha!} |z^{\alpha}| \\
&\leq C \sup_{|\beta| \leq m} \|u\|_{\sigma,\rho;X} \sup_{\alpha \in \mathbb{N}_0^n} \frac{(\alpha+\beta)!^{\sigma}}{\alpha!^{\sigma}} \frac{1}{\alpha!^{1-\sigma}} \frac{|z|^{|\alpha|}}{\rho^{|\alpha+\beta|}} \\
&\leq C \sup_{|\beta| \leq m} \frac{2^{|\beta|\sigma} \beta!^{\sigma}}{\epsilon^{|\beta|\sigma} \rho^{|\beta|}} \|u\|_{\sigma,\rho,X} \sup_{\alpha \in \mathbb{N}_0^n} \frac{\big((1+\epsilon)^{\sigma} |z|/\rho\big)^{|\alpha|}}{\alpha!^{1-\sigma}} \\
&\leq C C_m \frac{\rho^{\sigma m}(1+\rho^{-m})}{(\epsilon\rho)^{\sigma m}} \|u\|_{\sigma,\rho;X} \exp\{(1-\sigma)\big(\frac{(1+\epsilon)^{\sigma}|z|}{\rho}\big)^{\frac{1}{1-\sigma}}\},
\end{aligned}$$

which implies (2.8) since $(1+\epsilon)^{\sigma}/\rho \leq 1/(\rho(1-\epsilon))$.

The last statement is clear. ☐

Next, we investigate the Banach algebra properties of the spaces introduced above.

Proposition 2.5. *Let Ω be an open domain of $\mathbb{R}^n$ and let σ, ρ be positive numbers.*

(i) The vector spaces $G^\sigma(\Omega)$, $G^\sigma_{un}(\Omega)$ are differential rings with respect to the multiplication;

(ii) If X is $C^k(\overline{\Omega})$, $k \in \mathbb{N}_0$ or the Sobolev space $H^s_p(\Omega)$ for $1 \le p \le \infty$, $s > n/p$, then $G^\sigma(\rho, X)$ is a Banach algebra for $\sigma \ge 1$.

Moreover, for any $\sigma > 0$,

$$\|uv\|_{\sigma,\tilde{\rho};X} \le \omega \|u\|_{\sigma,\rho;X} \|v\|_{\sigma,\rho;X}, \tag{2.10}$$

where $\tilde{\rho} = min(1, 2^{\sigma-1})\rho$ and $\omega = \omega(X)$ is the Schauder constant for X.

Proof. (i) The statement is well known for $\sigma \ge 1$ (see [34]) while for $0 < \sigma < 1$ it follows from Proposition 2.4(i).

(ii) By the Schauder lemma $\|uv\|_X \le \omega \|u\|_X \|v\|_X$ for $u, v \in X$. Next for $u, v \in G^\sigma(\rho, X)$ we estimate

$$\begin{aligned}
\|uv\|_{\sigma,\tilde{\rho};X} &= \sum_{\alpha \in \mathbb{N}_0^n} \frac{\tilde{\rho}^{|\alpha|}}{\alpha!^\sigma} \| \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta u \cdot \partial^{\alpha-\beta} v \|_X \\
&\le \omega \sum_{\alpha \in \mathbb{N}_0^n} \frac{\tilde{\rho}^{|\alpha|}}{\alpha!^\sigma} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \|\partial^\beta u\|_X \cdot \|\partial^{\alpha-\beta} v\|_X \\
&= \omega \sum_{\beta \in \mathbb{N}_0^n} \frac{\tilde{\rho}^{|\beta|} \|\partial^\beta u\|_X}{\beta!^\sigma} \sum_{\alpha \ge \beta} \frac{\tilde{\rho}^{|\alpha-\beta|} \alpha!^{1-\sigma}}{\beta!^{1-\sigma} (\alpha-\beta)!} \|\partial^{\alpha-\beta} v\|_X \\
&\le \omega \|u\|_{\sigma,\rho;X} \|v\|_{\sigma,\rho;X},
\end{aligned}$$

since

$$\Big(\frac{\alpha!}{\beta!(\alpha-\beta)!}\Big)^{1-\sigma} \le \begin{cases} 1 & \text{if } \sigma \ge 1, \\ 2^{(1-\sigma)|\alpha|} & \text{if } \sigma \le 1. \end{cases}$$

☐

Remark 2.6. If $0 < \sigma < 1$ and u, v are entire functions satisfying $|u(z)|$, $|v(z)| \le C \exp\{(1-\sigma)(|z|/\rho)^{1/(1-\sigma)}\}$, then

$$|uv(z)| \le C^2 \exp\{(1-\sigma)(|z|/\tilde{\rho})^{1/(1-\sigma)}\},$$

with $\tilde{\rho} = 2^{\sigma-1}\rho$. So, one cannot obtain the estimation of $\|uv\|_{\sigma,\rho;X}$ by $C\|u\|_{\sigma,\rho;X} \cdot \|v\|_{\sigma,\rho;X}$.

In order to deal with functions which have different behavior in time and space directions we introduce scales of anisotropic Gevrey spaces. Let $T, \sigma, \tau, \rho, \theta$ be positive numbers and let X be a Banach space of functions on Ω and $Y = L^\infty([0,T]; X)$ equipped with the norm

$$\|u\|_Y = \sup_{t\in]0,T[} \|u(t,\cdot)\|_X.$$

$G^{\tau,\sigma}(\theta,\rho,Y) = \{u \in Y : \partial_t^l \partial_x^\alpha u \in Y$ for any $l \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n$ and $|u|_{\tau,\sigma,\theta\rho;Y} < \infty\}$, where

$$|u|_{\tau,\sigma,\theta,\rho;Y} := \sum_{l=0}^{\infty} \sum_{\alpha\in\mathbb{N}_0^n} \frac{\theta^l \rho^{|\alpha|}}{(l!)^\tau (\alpha!)^\sigma} \sup_{t\in]0,T[} \|\partial_t^l \partial_x^\alpha u(t,\cdot)\|_X. \tag{2.11}$$

We have a counterpart of Proposition 2.5 for the spaces $G^{\sigma,\tau}$.

Proposition 2.7. *Let $T, \sigma, \tau, \rho, \theta$ be positive numbers and let X stand for $C^k(\overline{\Omega})$ or for the Sobolev spaces $H_p^s(\Omega)$ for $1 \le p \le \infty$, $s > n/p$. Then we have*

i) the vector spaces $G^{\tau,\sigma}([0,T[\times\Omega)$, $G_{un}^{\tau,\sigma}([0,T[\times\Omega)$ are differential rings with respect to the multiplication;

ii) the Banach space $G^{\tau,\sigma}(\rho,\theta,Y))$ is a Banach algebra for $\tau \ge 1$, $\sigma \ge 1$. Moreover, for any $\tau > 0, \sigma > 0$,

$$|uv|_{\tau,\sigma;\tilde\theta,\tilde\rho;Y} \le \omega |u|_{\tau,\sigma,\theta,\rho;Y} |v|_{\tau,\sigma,\theta,\rho;Y}, \tag{2.12}$$

where $\tilde\theta = min(1, 2^{\tau-1})\theta$, $\tilde\rho = min(1, 2^{\sigma-1})\rho$ and ω is the Schauder constant for X.

3. SHARP GEVREY ESTIMATES DOWN TO $t = 0$ FOR LINEAR HEAT EQUATIONS

We investigate the Cauchy problem for the well-known heat equation

$$\partial_t u = \Delta u, \quad u|_{t=0} = u^0, \quad t > 0,\ x \in \mathbb{R}^n. \tag{3.1}$$

Our main goal is to obtain sharp estimates of the Gevrey regularity of $u(t,x)$ down to $t = 0$. More precisely, we aim at finding all positive numbers $\tau, \sigma, \rho, \theta$ such that, for some $T > 0$,

$$|u|_{\tau,\sigma,\theta,\rho;Y} := \sum_{l=0}^{\infty} \sum_{\alpha\in\mathbb{N}_0^n} \frac{\theta^l \rho^{|\alpha|}}{(l!)^\tau (\alpha!)^\sigma} \sup_{t\in]0,T[} \|\partial_t^l \partial_x^\alpha u(t,\cdot)\|_X < +\infty. \tag{3.2}$$

As one motivation for the study of (3.1) in the framework of the norms (3.2) we can cite recent results on the formal Gevrey regularity via Borel transformations of the solutions of (3.1) cf. [2], [3], [19]. Although the IVP (3.1) is well known it seems that the estimates proposed in Propositions 4.1 and 4.2 below are a novelty, at least in the framework of the general theory of PDEs.

We denote by $E_n(t,x)$ the heat kernel

$$E_n(t,x) = (4\pi t)^{-n/2} \exp\{-x^2/4t\}. \tag{3.3}$$

Proposition 3.1. *Let* $X = H_p^k(\mathbb{R}^n)$, $1 \le p \le \infty$, $k \in \mathbb{N}_0$, $Y = L^\infty([0,T]; X),)$, $0 < T \le \infty$ *and let* $u_0 \in G^\sigma(\rho_0, X)$, $\sigma, \rho_0 > 0$. *Then the solution* $u = e^{t\Delta}u_0$ *of* (3.1) *belongs to* $G^{\tau,\sigma}(\theta,\rho;Y)$ *for any* $\rho < \rho_0$, $\theta > 0$ *if* $\tau > 2\sigma$, *and for any* $\rho < \rho_0$, $0 < \theta < \theta_{max} := (4^\sigma n C^2)^{-1}$ *if* $\tau = 2\sigma$, *where* $C = (2/\epsilon)^\sigma \cdot 1/\rho_0$ *if* $\rho = (1-\epsilon)^\sigma \rho_0$. *Moreover,*

$$|u|_{\tau,\sigma,\theta,\rho;Y} \le K\|u_0\|_{\sigma,\rho_0;X}, \tag{3.4}$$

where

$$K = \begin{cases} c_1(n)\exp\{c_2(n,\tau,\sigma)(\theta C^2)^{1/(\tau-2\sigma)}\} & \text{if } \tau > 2\sigma, \\ (1-4^\sigma n\theta C^2)^{-n} & \text{if } \tau = 2\sigma, 0 < \theta < \theta_{max}, \end{cases} \tag{3.5}$$

with $c_1(n) = 2^n$, $c_2(n,\tau,\sigma) = n\delta(2\cdot 4^\sigma n)^{1/\delta}, \delta = \tau - 2\sigma$.

Proof. Recall that $u(t,x) = E_n(t,\cdot) * u_0(x)$ and $\|E_n(t,\cdot)\|_{L^1} = 1$. Hence, the Young inequality gives $\|\partial^\alpha u(t,\cdot)\|_{L^p} \le \|\partial^\alpha u_0\|_{L^p}$. So, we can reduce the proof to the case $X = L^p(\mathbb{R}^n)$. Then clearly $u \in Y = L^\infty([0,T];X)$ with $\|u\|_Y \le \|u_0\|_X$. Next, for any $l \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$,

$$\partial_t^l \partial_x^\alpha u(t,x) = \Delta_x^l \partial_x^\alpha u(t,x) = \sum_{\ell \in \mathbb{N}_0^n, |\ell| = l} \binom{l}{\ell} \partial_x^{\alpha+2\ell} u(t,x).$$

Hence,

$$\|\partial_t^l \partial_x^\alpha u(t,\cdot)\|_X \le \sum_{\ell \in \mathbb{N}_0^n, |\ell| = l} \binom{l}{\ell} \|\partial_x^{\alpha+2\ell} u_0\|_X$$

and

$$\|\partial_t^l \partial_x^\alpha u\|_Y \le \sum_{\ell \in \mathbb{N}_0^n, |\ell| = l} \binom{l}{\ell} \|\partial_x^{\alpha+2\ell} u_0\|_X.$$

Now, for $\rho < \rho_0$ we estimate by Lemma 2.1

$$|u(t,x)|_{\tau,\sigma,\theta,\rho;Y} \le \sum_{l=0}^\infty \sum_{\alpha \in \mathbb{N}_0^n} \sum_{\ell \in \mathbb{N}_0^n, |\ell| = l} \frac{\theta^l \rho^{|\alpha|}}{l!^\tau \alpha!^\sigma} \binom{l}{\ell} \|\partial^{\alpha+2\ell} u_0\|_X$$

$$= \sum_{\ell\in\mathbb{N}_0^n} \frac{\theta^l}{l!^\tau}\binom{l}{\ell}\|\partial^{2\ell}u_0\|_{\sigma,\rho;X} \le K\|u_0\|_{\sigma,\rho_0;X},$$

where, with $\tau = 2\sigma+\delta$, $l = |\ell|$,

$$K = \sum_{\ell\in\mathbb{N}_0^n}(C^2\theta)^l\binom{l}{\ell}\frac{(2\ell)!^\sigma}{l!^\delta} \le \sum_{\ell\in\mathbb{N}_0^n}\frac{(4^\sigma n\theta C^2)^l}{l!^\tau}$$
$$\le \begin{cases} 2^n\exp\{n\delta(2\cdot 4^\sigma n\theta C^2)^{1/\delta}\} & \text{if } \delta>0,\\ (1-4^\sigma n\theta C^2)^{-n} & \text{if } \delta=0, 0<\theta<\theta_{max},\end{cases}$$

since $l!/\ell! \le n^l$, $(2\ell)!/l!^2 \le 4^l$ and for $\delta>0$ and $x\ge 1$,

$$\sum_{\ell\in\mathbb{N}_0^n}\frac{x^l}{l!^\delta} \le 2^n \sup_{\ell\in\mathbb{N}_0^n}\frac{(2x)^l}{l!^\delta} \le 2^n\exp\{n\delta(2x)^{1/\delta}\}. \qquad \square$$

In the next proposition, we treat inhomogeneous equations.

Proposition 3.2. *Let* $X = H_p^k(\mathbb{R}^n)$, $1\le p\le\infty$, $k\in\mathbb{N}_0$, $Y = L^\infty([0,T];X),)$, $0<T\le\infty$. *Let* $g\in G^{\tau,\sigma}(\theta_0,\rho_0;Y)$ *and let* u *satisfy*

$$\partial_t u = \Delta u + g, \quad u|_{t=0} = 0, \quad t>0,\ x\in\mathbb{R}^n. \tag{3.6}$$

Then $u\in G^{\tau,\sigma}(\theta,\rho;Y)$ *for any* $\rho<\rho_0$, $\theta>0$ *if* $\tau>2\sigma$, *and for any* $\rho<\rho_0$, $0<\theta<\theta_{\max} = (4^\sigma nC^2)^{-1}$ *if* $\tau=2\sigma$, *where* $C = 2^\sigma\big(1-(\rho/\rho_0)^{1/\sigma}\big)^{-\sigma}\rho^{-1}$. *Moreover,*

$$|u|_{\tau,\sigma,\theta,\rho;Y} \le K(T+\theta)|g|_{\tau,\sigma,\theta_0,\rho_0;Y}, \tag{3.7}$$

where with $\tau = 2\sigma+\delta$,

$$K = \begin{cases} 2^n\exp\{n\delta(2\cdot 4^\sigma n\theta C^2)^{1/\delta}\} & \text{if } \delta>0,\\ (1-4^\sigma n\theta C^2)^{-n} & \text{if } \delta=0, 0<\theta<\theta_{max}.\end{cases} \tag{3.8}$$

Proof. As in the proof of Proposition 3.1 we can assume that $X = L^p(\mathbb{R}^n)$. We have

$$u(t,x) = \int_0^t E_n(t-s,\cdot)*g(s,\cdot)(x)ds.$$

Since $\|E_n(t-s,\cdot)\|_{L^1} = 1$ the Young inequality gives

$$\|u(t,\cdot)\|_X \le \int_0^t \|g(s,\cdot)\|_X ds.$$

Hence,

$$\|u\|_Y \le \|g\|_{L^1([0,T];X)} \le T\|g\|_Y.$$

Next note that, for any $l \in \mathbb{N}_0$,

$$\partial_t^l u = \Delta_x^l u + \sum_{j=0}^{l-1} \Delta_x^j \partial_t^{l-1-j} g \text{ and } \Delta_x^l u = \sum_{\ell \in \mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \partial_x^{2\ell} u.$$

So by the Young inequality, for any $l \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n$,

$$\|\partial_t^l \partial_x^\alpha u(t,\cdot)\|_X \le \sum_{\ell \in \mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \int_0^t \|\partial_x^{\alpha+2\ell} g(s,\cdot)\|_X ds + \sum_{j=0}^{l-1} \|\Delta_x^j \partial_x^\alpha \partial_t^{l-1-j} g(t,\cdot)\|_X.$$

Hence, for any $l \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n$, $\partial_t^l \partial_x^\alpha u \in Y$ and

$$\|\partial_t^l \partial_x^\alpha u\|_Y \le T \sum_{\ell \in \mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \|\partial_x^{\alpha+2\ell} g\|_Y + \sum_{j=0}^{l-1} \|\Delta_x^j \partial_x^\alpha \partial_t^{l-1-j} g\|_Y.$$

Now we decompose $|u|_{\tau,\sigma,\theta,\rho;Y}$ according to the above inequality.

$$\begin{aligned} |u|_{\tau,\sigma,\theta,\rho;Y} \le T \sum_{l=0}^{\infty} \sum_{\alpha \in \mathbb{N}_0^n} \frac{\theta^l \rho^{|\alpha|}}{l!^\tau \alpha!^\sigma} \sum_{\ell \in \mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \|\partial_x^{\alpha+2\ell} g\|_Y \\ + \sum_{l=0}^{\infty} \sum_{\alpha \in \mathbb{N}_0^n} \frac{\theta^l \rho^{|\alpha|}}{l!^\tau \alpha!^\sigma} \sum_{j=0}^{l-1} \|\Delta_x^j \partial_x^\alpha \partial_t^{l-1-j} g\|_Y =: T I_1 + I_2. \end{aligned}$$

In order to estimate the first term in the decomposition, we need the following counterpart of Lemma 2.1

Lemma 3.3. *Let $v \in G^{\tau,\sigma}(\theta_0, \rho_0, Y)$ and $m \in \mathbb{N}_0, \beta \in \mathbb{N}_0^n$. Then $\partial_t^m \partial_x^\beta v \in G^{\tau,\sigma}(\theta, \rho, Y)$ for any $\theta < \theta_0, \rho < \rho_0$ and*

$$|\partial_t^m \partial_x^\beta v|_{\tau,\sigma,\theta,\rho;Y} \le C_1^m C_2^{|\beta|} m!^\tau \beta!^\sigma |v|_{\tau,\sigma,\theta_0,\rho_0;Y}, \tag{3.9}$$

where $C_1 = 2^\tau \big(1-(\theta/\theta_0)^{1/\tau}\big)^{-\tau} \theta_0^{-1}$, $C_2 = 2^\sigma \big(1-(\rho/\rho_0)^{1/\sigma}\big)^{-\sigma} \rho_0^{-1}$.

Applying the above lemma with $v = g$, $m = 0, \beta = 2\ell$, we get with $C = C_2$

$$|\partial_x^{2\ell} g|_{\tau,\sigma,\theta,\rho;Y} \le C^{2|\ell|} (2\ell)!^\sigma |\, g \,|_{\tau,\sigma,\theta_0,\rho_0;Y},$$

which, in particular, implies

$$\sum_{\alpha \in \mathbb{N}_0^n} \frac{\rho^{|\alpha|}}{\alpha!^\sigma} \|\partial_x^{\alpha+2\ell} g\|_Y \le C^{2|\ell|} (2\ell)!^\sigma |g|_{\tau,\sigma,\theta_0,\rho_0;Y}.$$

Hence, the term I_1 is bounded by

$$\sum_{l=0}^{\infty}\frac{\theta^l}{l!^\tau}\sum_{|\ell|=l}\binom{l}{\ell}C^{2|\ell|}(2\ell)!^\sigma |g|_{\tau,\sigma,\theta_0,\rho_0;Y}\le K|g|_{\tau,\sigma,\theta_0,\rho_0;Y},$$

where, with $\tau=2\sigma+\delta$,

$$K=\begin{cases}2^n\exp\{n\delta(2\cdot 4^\sigma n\theta C^2)^{1/\delta}\} & \text{if } \delta>0,\\ (1-4^\sigma n\theta C^2)^{-n} & \text{if } \delta=0, 0<\theta<\theta_{max}.\end{cases}$$

Next we estimate I_2.

$$\begin{aligned}I_2&=\sum_{j=0}^{\infty}\sum_{\alpha\in\mathbb{N}_0^n}\sum_{l=j+1}^{\infty}\frac{\theta^l\rho^{|\alpha|}}{l!^\tau\alpha!^\sigma}\|\partial_t^{l-1-j}\Delta_x^j\partial_x^\alpha g\|_Y\\&=\sum_{l=0}^{\infty}\sum_{\alpha\in\mathbb{N}_0^n}\sum_{m=0}^{\infty}\frac{\theta^{l+1+m}\rho^{|\alpha|}}{(l+1+m)!^\tau\alpha!^\sigma}\|\partial_t^m\Delta_x^l\partial_x^\alpha g\|_Y\\&\le\sum_{l=0}^{\infty}\sum_{|\ell|=l}\frac{\theta^{l+1}}{(l+1)!^\tau}\binom{l}{\ell}\sum_{m=0}^{\infty}\sum_{\alpha\in\mathbb{N}_0^n}\frac{\theta^m\rho^{|\alpha|}}{m!^\tau\alpha!^\sigma}\|\partial_t^m\partial_x^{\alpha+2\ell}g\|_Y\\&\le\sum_{l=0}^{\infty}\sum_{|\ell|=l}\frac{\theta^{l+1}}{l!^\tau}\binom{l}{\ell}C^{2l}(2\ell)!^\sigma|g|_{\tau,\sigma,\theta_0,\rho_0;Y}\le K\theta|g|_{\tau,\sigma,\theta_0,\rho_0;Y}.\end{aligned}$$

Finally, $|u|_{\tau,\sigma,\theta,\rho;Y}\le TI_1+I_2\le K(T+\theta)|g|_{\tau,\sigma,\theta_0,\rho_0;Y}$. □

Remark 3.4. Note that Propositions 3.1 and 3.2 are not true for $\rho_0=\rho$. It resembles the so-called tamed estimates in Nash–Moser or KAM type methods for spaces of analytic functions.

4. Global Gevrey spaces $\widetilde{G}$

The purpose of this section is to introduce a new type of norm suitable for the simultaneous study of the uniform anisotropic critical $G^{2\sigma,\sigma}$ Gevrey regularity in $x\in\mathbb{R}^n$ and near $t=0$. The main advantage of the use of the new norms is that, in contrast to Propositions 3.1 and 3.2, no loss "a la Nash-Moser" type tamed estimates occurs.

Let X be a Banach space of functions on $\Omega\subset\mathbb{R}^n$. We define $\widetilde{G}^\sigma(\rho;X)=\{u\in X:\partial_x^\alpha u\in X$ for any $\alpha\in\mathbb{N}_0^n$ and $E^{\sigma,\rho;X}[u]<\infty\}$, where

$$E^{\sigma,\rho;X}[u]:=\sum_{\alpha\in\mathbb{N}_0^n}\frac{\rho^{|\alpha|}}{\Gamma(\sigma|\alpha|+1)}\|\partial_x^\alpha u\|_X. \tag{4.1}$$

Analogously for $0 < T < \infty$ we define
$\widetilde{G}^{\tau,\sigma}(\theta, \rho; Y) = \{u \in Y = L^\infty([0,T]; X) : \partial_t^l \partial_x^\alpha u \in Y$ for any $l \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n$ and $E^{\tau,\sigma,\theta\rho;Y}[u] < \infty\}$, where

$$E^{\tau,\sigma,\theta,\rho;Y}[u] := \sum_{l=0}^{\infty} \sum_{\alpha\in\mathbb{N}_0^n} \frac{\theta^l \rho^{|\alpha|}}{\Gamma(\tau l + \sigma|\alpha| + 1)} \|\partial_t^l \partial_x^\alpha u\|_Y. \tag{4.2}$$

We also introduce the anisotropic N-th partial anisotropic sum

$$E_N^{\tau,\sigma,\theta,\rho;Y}[u] := \sum_{2l+|\alpha|\le N} \frac{\theta^l \rho^{|\alpha|}}{\Gamma(\tau l + \sigma|\alpha| + 1)} \|\partial_t^l \partial_x^\alpha u\|_Y. \tag{4.3}$$

The relation between the spaces G and $\widetilde{G}$ is given in the following:

Proposition 4.1. *Let $\sigma, \rho > 0$ and $u \in G^\sigma(\rho; X)$. Then $u \in \widetilde{G}^\sigma(\widetilde{\rho}; X)$ for any $\widetilde{\rho} < \sigma^\sigma \rho$ and*

$$E^{\sigma,\,\widetilde{\rho};\,X}[u] \le C_{\sigma,\,\widetilde{\rho}/\rho} \|u\|_{\sigma,\,\rho;\,X}, \tag{4.4}$$

with some $C_{\sigma,\,\widetilde{\rho}/\rho} < \infty$.

Proof. Let $u \in G^\sigma(\rho; X)$ and $\widetilde{\rho} < \sigma^\sigma \rho$. Then by Lemma 4.2 (stated below) we get

$$\begin{aligned} E^{\sigma,\,\widetilde{\rho};\,X}[u] &= \sum_{\alpha\in\mathbb{N}_0^n} \frac{\widetilde{\rho}^{|\alpha|} \|\partial^\alpha u\|_X}{\Gamma(\sigma|\alpha| + 1)} \\ &= \sum_{\alpha\in\mathbb{N}_0^n} \frac{\rho^{|\alpha|} \|\partial^\alpha u\|_X}{\alpha!^\sigma} \cdot \frac{\alpha!^\sigma}{\Gamma(\sigma|\alpha| + 1)} \cdot \left(\frac{\widetilde{\rho}}{\rho}\right)^{|\alpha|} \le C_{\sigma,\,\widetilde{\rho}/\rho} \cdot \|u\|_{\sigma,\,\rho;\,X}. \end{aligned}$$

□

Lemma 4.2. *For $\sigma, \rho > 0$ and $\alpha \in \mathbb{N}_0^n$ put*

$$K_{\sigma,\rho}(\alpha) = \frac{\alpha!^\sigma}{\Gamma(\sigma|\alpha| + 1)} \rho^{|\alpha|}. \tag{4.5}$$

Then for $\rho < \sigma^\sigma$ one can find a constant $C_{\sigma,\rho} < \infty$ such that

$$K_{\sigma,\rho}(\alpha) \le C_{\sigma,\rho} \quad \textit{for any} \quad \alpha \in \mathbb{N}_0^n. \tag{4.6}$$

Proof. First of all note that without loss of generality we can assume that $\alpha \in \mathbb{N}^n$. Next note that by the Stirling formula we have

$$C_\sigma^{-1} \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \le \Gamma(x+1) \le C_\sigma \left(\frac{x}{e}\right)^x \sqrt{2\pi x}, \tag{4.7}$$

for $x \geq \min(1, \sigma)$ with some $1 < C_\sigma < \infty$. Hence,

$$
\begin{aligned}
K_{\sigma,\rho}(\alpha) &\leq \frac{C_\sigma^{n\sigma}\Big(\frac{\alpha_1}{e}\Big)^{\sigma\alpha_1}\dots\Big(\frac{\alpha_n}{e}\Big)^{\sigma\alpha_n}\Big(\sqrt{2\pi\alpha_1}\dots\sqrt{2\pi\alpha_n}\Big)^{\sigma}}{C_\sigma^{-1}\Big(\frac{\sigma|\alpha|}{e}\Big)^{\sigma|\alpha|}\sqrt{2\pi\sigma|\alpha|}}\rho^{|\alpha|} \\
&= C_\sigma^{n\sigma+1}\cdot\frac{\alpha_1^{\sigma\alpha_1}\dots\alpha_n^{\sigma\alpha_n}}{\big(\sigma|\alpha|\big)^{\sigma|\alpha|}}\cdot\frac{\big(\sqrt{2\pi\alpha_1}\dots\sqrt{2\pi\alpha_n}\big)^{\sigma}}{\sqrt{2\pi\sigma|\alpha|}}\ \rho^{|\alpha|} \\
&= C_\sigma^{n\sigma+1}\cdot\frac{(2\pi)^{n\sigma/2-1/2}}{\sqrt{\sigma}}\cdot\Big(\frac{\alpha_1^{\alpha_1}\dots\alpha_n^{\alpha_n}}{|\alpha|^{|\alpha|}}\Big)^{\sigma}\cdot\frac{\big(\sqrt{\alpha_1\dots\alpha_n}\big)^{\sigma}}{\sqrt{|\alpha|}}\cdot\Big(\frac{\rho}{\sigma^\sigma}\Big)^{|\alpha|} \\
&\leq C_\sigma^{n\sigma+1}\cdot\frac{(2\pi)^{n\sigma/2-1/2}}{\sqrt{\sigma}}\cdot|\alpha|^{n\sigma/2}\cdot\Big(\frac{\rho}{\sigma^\sigma}\Big)^{|\alpha|} \\
&\leq C_{\sigma,\rho}\quad\text{with}\quad C_{\sigma,\rho} = C_\sigma^{n\sigma+1}\cdot\frac{(2\pi)^{n\sigma/2-1/2}}{\sqrt{\sigma}}\cdot\Big(\frac{s}{-e\ln r}\Big)^{s},
\end{aligned}
$$

where $s = n\sigma/2, r = \rho/\sigma^\sigma$ since $\sup_{x>0} x^s \cdot r^x = \big(s/(-e\ln r)\big)^s$ for $r < 1$ and $s > 0$. $\square$

Remark 4.3. Let $u \in \widetilde{G}^\sigma(\rho; X)$. If $n > 1$, then in general one can not find any $\widetilde{\rho} > 0$ such that $u \in G^\sigma(\widetilde{\rho}; X)$. In fact following the proof of Lemma 4.2 we obtain the estimation

$$
\frac{\Gamma(\sigma|\alpha|+1)}{\alpha!^\sigma} \leq C_\sigma^{n\sigma+1}\cdot\frac{\sqrt{\sigma}}{(2\pi)^{n\sigma/2-1}}\cdot\Big(\frac{|\alpha|^{|\alpha|}}{\alpha_1^{\alpha_1}\dots\alpha_n^{\alpha_n}}\Big)^{\sigma}\cdot\frac{\sqrt{|\alpha|}}{\big(\sqrt{\alpha_1\dots\alpha_n}\big)^{\sigma}}\cdot\big(\sigma^\sigma\rho\big)^{|\alpha|}, \tag{4.8}
$$

in which the factor $\frac{|\alpha|^{|\alpha|}}{\alpha_1^{\alpha_1}\dots\alpha_n^{\alpha_n}}$ cannot be bounded by $C^{|\alpha|}$ except in the case $n = 1$.

Proposition 4.4. *Let $\sigma \geq 1, \rho > 0$ and $\beta \in \mathbb{N}_0^n$. If $u \in \widetilde{G}^\sigma(\rho, X)$, then $\partial^\beta u \in \widetilde{G}^\sigma(\widetilde{\rho}, X)$ for any $\widetilde{\rho} < \rho$ and*

$$
E^{\sigma,\tilde{\rho};X}[\partial^\beta u] \leq C^{|\beta|}\Gamma(\sigma|\beta|+1)\cdot E^{\sigma,\rho;X}[u], \tag{4.9}
$$

where $C = C_1^\sigma/\rho$ with $C_1 = 1 + \big((\rho/\tilde{\rho})^{1/\sigma} - 1\big)^{-1}$.

Proof. Fix $\widetilde{\rho} < \rho$. Then applying Lemma 2.2 with $\varepsilon = (\rho/\widetilde{\rho})^{1/\sigma} - 1$ we get

$$
E^{\sigma,\widetilde{\rho};X}[\partial^\beta u] = \sum_{\alpha\in\mathbb{N}_0^n}\frac{\widetilde{\rho}^{|\alpha|}\|\partial^{\alpha+\beta}u\|_X}{\Gamma(\sigma|\alpha|+1)}
$$

$$\leq \sup_{\alpha\in\mathbb{N}_0^n} \frac{\widetilde{\rho}^{|\alpha|}}{\rho^{|\alpha+\beta|}} \cdot \frac{\Gamma(\sigma|\alpha+\beta|+1)}{\Gamma(\sigma|\alpha|+1)} \cdot \sum_{\alpha\in\mathbb{N}_0^n} \frac{\rho^{|\alpha+\beta|}\|\partial^{\alpha+\beta}u\|_X}{\Gamma(\sigma|\alpha+\beta|+1)}$$

$$\leq E^{\sigma,\rho;X}[u] \cdot \sup_{\alpha\in\mathbb{N}_0^n} \frac{\widetilde{\rho}^{|\alpha|}}{\rho^{|\alpha+\beta|}} \cdot \frac{\Gamma(\sigma|\alpha+\beta|+1)}{\Gamma(\sigma|\alpha|+1)}$$

$$\leq E^{\sigma,\rho;X}[u] \cdot \sup_{\alpha\in\mathbb{N}_0^n} \frac{\widetilde{\rho}^{|\alpha|}}{\rho^{|\alpha+\beta|}} \cdot \frac{(1+\varepsilon)^{\sigma|\alpha+\beta|}}{\varepsilon^{\sigma|\beta|}} \cdot \Gamma(\sigma|\beta|+1)$$

$$= \frac{1}{\rho^{|\beta|}} \cdot \Big(\frac{1+\varepsilon}{\varepsilon}\Big)^{\sigma|\beta|} \cdot \Gamma(\sigma|\beta|+1) \cdot E^{\sigma,\rho;X}[u],$$

which implies (4.9). □

5. Gevrey estimates for a linear heat equation in spaces $\widetilde{G}$.

Now we shall prove two propositions about the solutions of a linear heat equation in the spaces $\widetilde{G}$.

Proposition 5.1. *Let $X = H_p^k(\mathbb{R}^n), 1 \leq p \leq \infty, k \in \mathbb{N}_0$, $Y = L^\infty([0,T]; X),)$, $0 < T \leq \infty$ and let $u_0 \in \widetilde{G}^\sigma(\rho; X)$, $\sigma, \rho > 0$. Then the solution $u = e^{t\Delta}u_0$ of (3.1) belongs to $\widetilde{G}^{\tau,\sigma}(\theta,\rho;Y)$ for any $\theta > 0$ if $\tau > 2\sigma$, and for $0 < \theta < \theta_{\max} = \rho^2/n$ if $\tau = 2\sigma$. Moreover,*

$$E^{\tau,\sigma,\theta,\rho;Y}[u] \leq K E^{\sigma,\rho;X}[u_0], \tag{5.1}$$

if $\tau > 2\sigma$, with some $K = K(\tau,\sigma,\theta/\rho^2,n)$, and for the critical case $\tau = 2\sigma$,

$$E^{2\sigma,\sigma,\theta,\rho;Y}[u] \leq K E^{\sigma,\rho;X}[u_0] \qquad \text{if } \theta < \rho^2/n, \tag{5.2}$$

with $K = \big(1 - n\theta/\rho^2\big)^{-n}$.

Proof. Recall that $u(t,x) = E_n(t,\cdot) * u_0(x)$ and $\|E_n(t,\cdot)\|_{L^1} = 1$. Hence, the Young inequality gives $\|\partial^\alpha u(t,\cdot)\|_{L^p} \leq \|\partial^\alpha u_0\|_{L^p}$. Thus we can reduce the proof to the case $X = L^p(\mathbb{R}^n)$. Then, clearly $u \in Y = L^\infty([0,T]; X)$, with $\|u\|_Y \leq \|u_0\|_X$. Next, for any $l \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$,

$$\partial_t^l \partial_x^\alpha u(t,x) = \Delta_x^l \partial_x^\alpha u(t,x) = \sum_{\ell\in\mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \partial_x^{\alpha+2\ell} u(t,x).$$

Hence,

$$\|\partial_t^l \partial_x^\alpha u(t,\cdot)\|_X \leq \sum_{\ell\in\mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \|\partial_x^{\alpha+2\ell} u_0\|_X,$$

and

$$\|\partial_t^l \partial_x^\alpha u\|_Y \le \sum_{\ell\in\mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \|\partial_x^{\alpha+2\ell} u_0\|_X.$$

Therefore, after the change of the summation index $\beta = \alpha + 2\ell$ we get

$$\begin{aligned} E^{\tau,\sigma,\theta,\rho;Y}[u] &\le \sum_{l=0}^\infty \sum_{\alpha\in\mathbb{N}_0^n} \sum_{\ell\in\mathbb{N}_0^n,|\ell|=l} \frac{\theta^l \rho^{|\alpha|}}{\Gamma(\tau l+\sigma|\alpha|+1)} \binom{l}{\ell} \|\partial_x^{\alpha+2\ell} u_0\|_X \\ &\le \sum_{\beta\in\mathbb{N}_0^n} \Big(\sum_{\ell\le\beta/2} \frac{1}{\Gamma(\sigma|\beta|+(\tau-2\sigma)l+1)} \binom{l}{\ell} \frac{\theta^l}{\rho^{2l}} \Big) \rho^{|\beta|} \|\partial_x^\beta u_0\|_X \\ &\le \sum_{\beta\in\mathbb{N}_0^n} Q_{\tau,\sigma}(\beta) \frac{\rho^{|\beta|}}{\Gamma(\sigma|\beta|+1)} \|\partial^\beta u_0\|_X, \end{aligned} \tag{5.3}$$

where

$$Q_{\tau,\sigma}(\beta) = \sum_{2\ell\le\beta} \frac{\Gamma(\sigma|\beta|+1)}{\Gamma(\sigma|\beta|+(\tau-2\sigma)l+1)} \binom{l}{\ell} \frac{\theta^l}{\rho^{2l}}, \qquad \beta\in\mathbb{N}_0^n. \tag{5.4}$$

We shall prove that one can find a constant $K = K(\tau,\sigma,\theta/\rho^2,n) < \infty$ such that for any $\beta\in\mathbb{N}_0^n$,

$$Q_{\tau,\sigma}(\beta) \le K. \tag{5.5}$$

In order to prove (5.5) assume first that $\tau = 2\sigma$. Then

$$Q_{2\sigma,\sigma}(\beta) = \sum_{2\ell\le\beta} \binom{l}{\ell} \Big(\frac{\theta}{\rho^2}\Big)^l \le \sum_{\ell\in\mathbb{N}_0^n} \Big(\frac{n\theta}{\rho^2}\Big)^l = \Big(\frac{1}{1-n\theta/\rho^2}\Big)^n.$$

So in that case

$$K(2\sigma,\sigma,\theta/\rho^2,n) = \Big(\frac{1}{1-n\theta/\rho^2}\Big)^n < \infty \quad \text{for} \quad 0<\theta<\rho^2/n. \tag{5.6}$$

Now let $\sigma > 2\tau$. By the Stirling formula there exists a constant $C(\sigma) < \infty$ such that

$$\frac{\Gamma(z+1)}{\Gamma(z+\zeta+1)} \le C(\sigma) \frac{e^\zeta}{(z+\zeta)^\zeta} \qquad \text{for} \qquad z\ge\sigma,\ \zeta\ge 0.$$

Hence,

$$\frac{\Gamma(\sigma|\beta|+1)}{\Gamma(\sigma|\beta|+(\tau-2\sigma)l+1)} \le C(\sigma) \frac{e^{(\tau-2\sigma)l}}{(\sigma|\beta|+(\tau-2\sigma)l)^{(\tau-2\sigma)l}}, \qquad |\beta|\ge 1.$$

Therefore, we get

$$\begin{aligned} Q_{\tau,\sigma}(\beta) &\leq C(\sigma)\sum_{2\ell\leq\beta}\frac{e^{(\tau-2\sigma)l}}{(\sigma|\beta|+(\tau-2\sigma)l)^{(\tau-2\sigma)l}}\binom{l}{\ell}\Big(\frac{\theta}{\rho^2}\Big)^l \\ &\leq C(\sigma)\sum_{2\ell\leq\beta}\Big(\frac{n\theta e^{(\tau-2\sigma)}}{\rho^2(\sigma|\beta|+(\tau-2\sigma)l)^{(\tau-2\sigma)}}\Big)^l \\ &\leq C(\sigma)\Big\{\sum_{l=0}^{|\beta|/2}\Big(\frac{n\theta e^{(\tau-2\sigma)}}{\rho^2(\sigma|\beta|+(\tau-2\sigma)l)^{(\tau-2\sigma)}}\Big)^l\Big\}^n. \end{aligned} \tag{5.7}$$

For $\delta=\tau-2\sigma>0$, $A=ne^{\delta}\theta/\rho^2$, put

$$R(s) = \sum_{l=0}^{s/2}\Big(\frac{A}{(\sigma s+\delta l)^{\delta}}\Big)^l, \qquad s\in\mathbb{N}_0. \tag{5.8}$$

Then $R(0)=R(1)=1$. Next if $2\leq s<\frac{1}{\sigma}(2A)^{1/\delta}$, then

$$R(s)\leq 1+\frac{A}{(s\sigma)^{\delta}}+\cdots+\Big(\frac{A}{(s\sigma)^{\delta}}\Big)^{\lfloor s/2\rfloor}\leq\Big(\frac{s}{2}+1\Big)\cdot\max\Big\{1,\Big(\frac{A}{(2\sigma)^{\delta}}\Big)^{\lfloor s/2\rfloor}\Big\}.$$

Finally if $s\geq\frac{1}{\sigma}(2A)^{1/\delta}$, then $A/((s\sigma)^{\delta})\leq 1/2$ and $R(s)\leq 2$. Hence, one can find a constant $C(A,\sigma,\delta)$ such that $R(s)\leq C(A,\sigma,\delta)$ for any $s\in\mathbb{N}_0$. Combining this with (5.4) and (5.7) we get (5.5) with $K(\tau,\sigma,\theta/\rho^2,n)=C(\sigma)\cdot C^n(A,\sigma,\delta)$, where $\delta=\tau-2\sigma, A=ne^{\delta}\theta/\rho^2$. Finally (5.3) and (5.5) give (5.1) and (5.2) which completes the proof. □

Proposition 5.2. *Let* $X=H_p^k(\mathbb{R}^n), 1\leq p\leq\infty, k\in\mathbb{N}_0$, $Y=L^{\infty}([0,T];X),)$, $0<T<\infty$. *Let* $g\in\widetilde{G}^{\tau,\sigma}(\theta,\rho;Y)$ *and let* u *satisfy*

$$\partial_t u=\Delta u+g,\quad u|_{t=0}=0,\quad t>0,\ x\in\mathbb{R}^n. \tag{5.9}$$

Then $u\in\widetilde{G}^{\tau,\sigma}(\theta,\rho;Y)$ *for any* $\theta>0$ *if* $\tau>2\sigma$, *and for* $0<\theta<\theta_{\max}=\rho^2/n$ *if* $\tau=2\sigma$. *Moreover,*

$$E^{\tau,\sigma,\theta,\rho;Y}[u]\leq K(T+\theta)E^{\tau,\sigma,\theta,\rho;Y}[g], \tag{5.10}$$

where $K=K(\tau,\sigma,\theta/\rho^2,n)<\infty$ *for any* $\theta>0$ *if* $\tau>2\sigma$, *and* $K(2\sigma,\sigma,\theta/\rho^2,n)=\big(1-n\theta/\rho^2\big)^{-n}<\infty$ *for* $0<\theta<\theta_{\max}$ *if* $\tau=2\sigma$.

Proof. As in the proof of Proposition 3.1 we can assume that $X=L^p(\mathbb{R}^n)$. We have

$$u(t,x)=\int_0^t E_n(t-s,\cdot)*g(s,\cdot)(x)ds.$$

Since $\|E_n(t-s,\cdot)\|_{L^1} = 1$ the Young inequality gives

$$\|u(t,\cdot)\|_X \le \int_0^t \|g(s,\cdot)\|_X ds.$$

Hence,

$$\|u\|_Y \le \|g\|_{L^1([0,T];X)} \le T\|g\|_Y.$$

Next, note that for any $l \in \mathbb{N}_0$,

$$\partial_t^l u = \Delta_x^l u + \sum_{j=0}^{l-1} \Delta_x^j \partial_t^{l-1-j} g \quad \text{and} \quad \Delta_x^l u = \sum_{\ell\in\mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \partial_x^{2\ell} u.$$

So by the Young inequality for any $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned}\|\partial_t^l \partial_x^\alpha u(t,\cdot)\|_X \quad &\le \quad \sum_{\ell\in\mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \int_0^t \|\partial_x^{\alpha+2\ell} g(s,\cdot)\|_X ds \\ &+ \quad \sum_{j=0}^{l-1} \|\Delta_x^j \partial_x^\alpha \partial_t^{l-1-j} g(t,\cdot)\|_X.\end{aligned}$$

Hence, for any $l \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n$, $\partial_t^l \partial_x^\alpha u \in Y$ and

$$\|\partial_t^l \partial_x^\alpha u\|_Y \quad \le \quad T \sum_{\ell\in\mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \|\partial_x^{\alpha+2\ell} g\|_Y + \sum_{j=0}^{l-1} \|\Delta_x^j \partial_x^\alpha \partial_t^{l-1-j} g\|_Y.$$

Now we decompose $E^{\tau,\sigma,\theta,\rho;Y}[u]$ according to the above inequality.

$$\begin{aligned}E^{\tau,\sigma,\theta,\rho;Y}[u] \quad &\le \quad T\sum_{l=0}^\infty \sum_{\alpha\in\mathbb{N}_0^n} \frac{\theta^l \rho^{|\alpha|}}{\Gamma(\tau l + \sigma|\alpha| + 1)} \sum_{\ell\in\mathbb{N}_0^n, |\ell|=l} \binom{l}{\ell} \|\partial_x^{\alpha+2\ell} g\|_Y \\ &+ \quad \sum_{l=0}^\infty \sum_{\alpha\in\mathbb{N}_0^n} \frac{\theta^l \rho^{|\alpha|}}{\Gamma(\tau l + \sigma|\alpha| + 1)} \sum_{j=0}^{l-1} \|\Delta_x^j \partial_x^\alpha \partial_t^{l-1-j} g\|_Y \\ &=: \quad TI_1 + I_2.\end{aligned}$$

Following the proof of Proposition 5.1, we get $I_1 \le KE^{\tau,\sigma,\theta,\rho;Y}[g]$.

Next we estimate I_2.

$$I_2 \quad = \quad \sum_{j=0}^\infty \sum_{\alpha\in\mathbb{N}_0^n} \sum_{l=j+1}^\infty \frac{\theta^l \rho^{|\alpha|}}{\Gamma(\tau l + \sigma|\alpha| + 1)} \|\partial_t^{l-1-j} \Delta_x^j \partial_x^\alpha g\|_Y$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty} \sum_{\alpha\in\mathbb{N}_0^n} \sum_{m=0}^{\infty} \frac{\theta^{l+1+m}\rho^{|\alpha|}}{\Gamma(\tau(l+1+m)+\sigma|\alpha|+1)} \|\partial_t^m \Delta_x^l \partial_x^\alpha g\|_Y \\
&\leq \sum_{\ell\in\mathbb{N}_0^n} \sum_{\alpha\in\mathbb{N}_0^n} \sum_{m=0}^{\infty} \frac{\theta^{l+1+m}\rho^{|\alpha|}}{\Gamma(\tau(l+1+m)+\sigma|\alpha|+1)} \binom{l}{\ell} \|\partial_t^m \partial_x^{\alpha+2\ell} g\|_Y \\
&\leq \sum_{m=0}^{\infty} \sum_{\beta\in\mathbb{N}_0^n} \Big\{ \sum_{\ell\leq\beta/2} \frac{\theta^{l+1}/\rho^{2l}\cdot\Gamma(\tau m+\sigma|\beta|+1)}{\Gamma(\tau(l+1+m)+\sigma(|\beta|-2l)+1)} \binom{l}{\ell} \Big\} \\
&\quad \times \frac{\theta^m \rho^{|\beta|}}{\Gamma(\tau m+\sigma|\beta|+1)} \|\partial_t^m \partial_x^\beta g\|_Y \\
&\leq \theta \cdot \sum_{m=0}^{\infty} \sum_{\beta\in\mathbb{N}_0^n} Q(m,\beta) \frac{\theta^m \rho^{|\beta|}}{\Gamma(\tau m+\sigma|\beta|+1)} \|\partial_t^m \partial_x^\beta g\|_Y \\
&\leq K\theta E^{\tau,\sigma,\theta,\rho;Y}[g], \qquad (5.11)
\end{aligned}
$$

since $\Gamma(x+a)\Gamma(b+a) \leq \Gamma(x+b+a)\Gamma(a)$ for $a>0, b>0,\ x\geq 0$, and so

$$
Q(m,\beta) := \sum_{\ell\leq\beta/2} \frac{\Gamma(\tau m+\sigma|\beta|+1)}{\Gamma(\tau m+\sigma|\beta|+(\tau-2\sigma)l+1)} \binom{l}{\ell} \frac{\theta^l}{\rho^{2l}} \leq Q(\beta) \leq K.
$$

Finally, $E^{\tau,\sigma,\theta,\rho;Y}[u] \leq TI_1 + I_2 \leq K(T+\theta)E^{\tau,\sigma,\theta,\rho;Y}[g]$. □

6. Nonlinear Estimates in Gevrey Spaces

In order to deal with semilinear equations in spaces $\widetilde{G}$, we need precise estimations of non-linear superpositions.

Proposition 6.1. *Let σ,ρ be positive numbers. If X is $C^k(\overline{\Omega})$, $k\in\mathbb{N}_0$ or the Sobolev space $H_p^s(\Omega)$ for $1\leq p\leq\infty$, $s>n/p$, then $\widetilde{G}^\sigma(\rho,X)$ is a Banach algebra for $\sigma\geq 1$. Moreover,*

$$
E^{\sigma,\rho;X}[uv] \leq \omega E^{\sigma,\rho;X}[u]E^{\sigma,\rho;X}[v], \qquad (6.1)
$$

where $\omega=\omega(X)$ is the Schauder constant for X.

Proof. By the Schauder lemma $\|uv\|_X \leq \omega\|u\|_X\|v\|_X$ for $u,v\in X$. Next for $u,v\in\widetilde{G}^\sigma(\rho,X)$ we estimate

$$
E^{\sigma,\rho;X}[uv] \;=\; \sum_{\alpha\in\mathbb{N}_0^n} \frac{\rho^{|\alpha|}}{\Gamma(\sigma|\alpha|+1)} \|\sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta u\cdot\partial^{\alpha-\beta} v\|_X
$$

$$
\begin{aligned}
&\leq \omega \sum_{\alpha\in\mathbb{N}_0^n} \frac{\rho^{|\alpha|}}{\Gamma(\sigma|\alpha|+1)} \cdot \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \|\partial^\beta u\|_X \cdot \|\partial^{\alpha-\beta} v\|_X \\
&= \omega \sum_{\beta\in\mathbb{N}_0^n} \frac{\rho^{|\beta|}\|\partial^\beta u\|_X}{\Gamma(\sigma|\beta|+1)} \cdot \sum_{\alpha\geq\beta} \rho^{|\alpha-\beta|} \binom{\alpha}{\beta} \frac{\Gamma(\sigma|\beta|+1)}{\Gamma(\sigma|\alpha|+1)} \|\partial^{\alpha-\beta} v\|_X \\
&= \omega \sum_{\beta\in\mathbb{N}_0^n} \frac{\rho^{|\beta|}\|\partial^\beta u\|_X}{\Gamma(\sigma|\beta|+1)} \cdot \sum_{\gamma\in\mathbb{N}_0^n} \frac{\rho^{|\gamma|}\|\partial^\gamma v\|_X}{\Gamma(\sigma|\gamma|+1)} \cdot C_\sigma(\beta,\gamma),
\end{aligned}
$$

where

$$
C_\sigma(\beta,\gamma) = \frac{(\beta+\gamma)!}{\beta!\gamma!} \cdot \frac{\Gamma(\sigma|\beta|+1)\Gamma(\sigma|\gamma|+1)}{\Gamma(\sigma|\beta+\gamma|+1)}, \quad \beta,\gamma\in\mathbb{N}_0^n.
$$

We shall prove that $C_\sigma(\beta,\gamma) \leq 1$ for any $\beta,\gamma\in\mathbb{N}_0^n$ if $\sigma\geq 1$ which implies (6.1). Clearly for $\sigma = 1$ we have

$$
C_1(\beta,\gamma) = \binom{\beta+\gamma}{\beta} \Big/ \binom{|\beta+\gamma|}{|\beta|} \leq 1.
$$

So it is sufficient to show that for any $k,l\in\mathbb{N}_0$ the function

$$
F_{k,l}(\sigma) = \frac{\Gamma(\sigma k+1)\Gamma(\sigma l+1)}{\Gamma(\sigma(k+l)+1)}
$$

is non-increasing for $\sigma\geq 1$. Since $F_{k,l}(\sigma) = 1$ if $k=0$ or $l=0$ we can assume that $k\geq 1$ and $l\geq 1$. By the properties of the Euler Γ and B functions we have

$$
\begin{aligned}
F_{k,l}(\sigma) &= \big(\sigma(k+l)+1\big)\cdot B(\sigma k+1,\sigma l+1) \\
&= \big(\sigma(k+l)+1\big)\cdot \int_0^1 t^{\sigma k}(1-t)^{\sigma l}dt.
\end{aligned}
$$

Hence,

$$
\frac{dF_{k,l}(\sigma)}{d\sigma} = \int_0^1 \Big\{k+l+\big(\sigma(k+l)+1\big)\big(k\ln t + l\ln(1-t)\big)\Big\} t^{\sigma k}(1-t)^{\sigma l}dt.
$$

We shall show that the expression in parenthesis is non-positive for any $0<t<1$ which implies $\frac{dF_{k,l}(\sigma)}{d\sigma}\leq 0$, proving our claim. To this end, note that since $k\ln t + l\ln(1-t) < 0$ we can assume $\sigma = 1$ and so we need to show that

$$
\frac{k+l}{k+l+1} \leq -k\ln t + l\ln(1-t) \quad \text{for any} \quad 0<t<1. \tag{6.2}
$$

Next note that the function $f(t) = -k\ln t - l\ln(1-t)$, $0 < t < 1$ assumes a minimum at the point $t = k/(k+l)$ equal $(k+l)\ln(k+l) - k\ln k - l\ln l$. So (6.2) is reduced to

$$\frac{k+l}{k+l+1} \le (k+l)\ln(k+l) - k\ln k - l\ln l \quad \text{for} \quad k,l \in \mathbb{N}. \tag{6.3}$$

Finally, the left-hand side of (6.3) is < 1 while the right one is > 1. □

Remark 6.2. As in the case of Proposition 2.5 if $0 < \sigma < 1$ the E norm of uv cannot be estimated by that of u and v without the loss of ρ.

The next proposition generalizes Proposition 6.1 to the case of polynomial superpositions.

Proposition 6.3. *Let X be $C^k(\overline{\Omega})$, $k \in \mathbb{N}_0$, or the Sobolev space $H_p^s(\Omega)$ for $1 \le p \le \infty$, $s > n/p$ with $\omega \ge 1$ being its Schauder constant. Let $f(z) = \sum_{j=2}^M f_j z^j$ be a polynomial of order $M \ge 2$ vanishing at the origin together with its derivative and let*

$$F(|z|) = \sum_{j=2}^{M} |f_j||z|^j, \qquad z \in \mathbb{C}, \tag{6.4}$$

be the standard majorant function for f. Then we have

$$E^{\sigma,\rho;X}[f(u)] \le F\big(\omega \cdot E^{\sigma,\rho;X}[u]\big), \tag{6.5}$$

$$E^{\sigma,\rho;X}[f(u) - f(v)] \le E^{\sigma,\rho;X}[u-v] \cdot F'\big(\omega \max\{E^{\sigma,\rho;X}[u], E^{\sigma,\rho;X}[v]\}\big), \tag{6.6}$$

for all $u, v \in \widetilde{G}^{\sigma}(\rho; X)$ with $\sigma \ge 1$, $\rho > 0$. Analogously

$$E^{\tau,\sigma,\theta,\rho;Y}[f(u)] \le F\big(\omega \cdot E^{\tau,\sigma,\theta,\rho;Y}[u]\big), \tag{6.7}$$

$$E^{\tau,\sigma,\theta,\rho;Y}[f(u) - f(v)] \le E^{\tau,\sigma,\theta,\rho;Y}[u-v] \cdot F'\big(\omega \max\{E^{\tau,\sigma,\theta,\rho;Y}[u], E^{\tau,\sigma,\theta,\rho;Y}[v]\}\big), \tag{6.8}$$

for all $u, v \in \widetilde{G}^{\tau,\sigma}(\theta, \rho; Y)$ with $\tau, \sigma \ge 1$ and $\theta, \rho > 0$.

Proof. Clearly, since $\omega \ge 1$, (6.5) follows immediately from Proposition 6.1, while (6.6) is a consequence of that proposition and the equality

$$u^j - v^j = (u-v)\sum_{i=0}^{j-1} u^i v^{j-i-1} \text{ for } j \in \mathbb{N}.$$

Next (6.7) and (6.8) hold by an analogue of Proposition 6.1 for the spaces $\widetilde{G}^{\tau,\sigma}(\theta, \rho; Y)$. □

7. The main result

For the sake of simplicity we consider polynomial non-linearities of the type $F[u] = f(u)$. The case of $f(u, \nabla u)$ is treated by the same method, modulo minor technical complications.

Theorem 7.1. *Let $\sigma \geq 1, \tau \geq 2\sigma$ and $\rho, \theta > 0$. Let X be the Sobolev space $H_p^k(\mathbb{R}^n)$ for $1 \leq p \leq \infty$, $k \in \mathbb{N}$, $k > n/p$. Let $f(u)$ be a polynomial of order $M \geq 2$ vanishing at the origin together with its derivative. Assume that $u_0 \in \widetilde{G}^\sigma(\rho; X)$. Then the solution to the initial-value problem*

$$\partial_t u = \Delta u + f(u), \quad u|_{t=0} = u_0 \tag{7.1}$$

belongs to $\widetilde{G}^{\tau,\sigma}(\theta, \rho; Y)$ provided that $(T + \theta)\big(E^{\sigma,\rho;X}[u_0] + (E^{\sigma,\rho;X}[u_0])^{M-1}\big)$ is small enough and $0 < \theta < \rho^2/n$ if $\tau = 2\sigma$.

Proof. Define the approximation scheme by

$$\begin{aligned} \partial_t U_N &= \Delta U_N + f(U_{N-1}), \\ U_N(0, x) &= u_0(x), \qquad N \in \mathbb{N}, \end{aligned}$$

with $U_0 = e^{t\Delta} u_0$. Then by Propositions 5.1 and 5.2 we get

$$\begin{aligned} &E^{\tau,\sigma,\theta,\rho;Y}[U_0] \leq K E^{\sigma,\rho;X}[u_0], \qquad\qquad (7.2)\\ &E^{\tau,\sigma,\theta,\rho;Y}[U_{N+1}] \leq K E^{\sigma,\rho;X}[u_0] + K(T+\theta) E^{\tau,\sigma,\theta,\rho;Y}[f(U_N)], \quad N \in \mathbb{N}_0, \end{aligned}$$

for any $\theta > 0$ if $\tau > 2\sigma$, and for $0 < \theta < \rho^2/n$ if $\tau = 2\sigma$. Next if $\tau, \sigma \geq 1$, by Proposition 6.3 we can find a constant $C < \infty$ depending on ω and the polynomial f such that

$$E^{\tau,\sigma,\theta,\rho;Y}[f(U_N)] \leq C\Big(\big(E^{\tau,\sigma,\theta,\rho;Y}[U_N]\big)^2 + \big(E^{\tau,\sigma,\theta,\rho;Y}[U_N]\big)^M\Big). \tag{7.3}$$

Hence, combining (7.2) with (7.3), we get

$$\begin{aligned} &E^{\tau,\sigma,\theta,\rho;Y}[U_{N+1}] \qquad\qquad (7.4)\\ &\leq K E^{\sigma,\rho;X}[u_0] + CK(T+\theta)\Big(\big(E^{\tau,\sigma,\theta,\rho;Y}[U_N]\big)^2 + \big(E^{\tau,\sigma,\theta,\rho;Y}[U_N]\big)^M\Big). \end{aligned}$$

Now we apply the following:
Claim. *If*

$$4K^2 L C_0\big(1 + (2KC_0)^{M-2}\big) \leq 1 \tag{7.5}$$

and

$$C_{N+1} = KC_0 + KL\big(C_N^2 + C_N^M\big) \text{ for } N \in \mathbb{N}_0, \tag{7.6}$$

then

$$C_N \leq \big(2 - 2^{-N}\big)KC_0 < 2KC_0 \text{ for } N \in \mathbb{N}_0. \tag{7.7}$$

Proof. Clearly (7.7) holds for $N = 0$. Next, assuming (7.7) we get by (7.5)

$$\begin{aligned} C_{N+1} &\le KC_0 + C_N \cdot KL\big(1 + C_N^{M-2}\big)C_N \\ &\le KC_0 + C_N/2 \le (2 - 2^{-N-1})KC_0 < 2KC_0. \end{aligned}$$

□

Applying the claim with $L = C(T + \theta), C_0 = E^{\sigma,\rho;X}[u_0]$ and $C_N = E^{\tau,\sigma,\theta,\rho;Y}[U_N]$ we conclude that U_N belongs to a ball in $\widetilde{G}^{\tau,\sigma}(\theta, \rho; Y)$ of radius $2KC_0$ provided that $(T + \theta)\big(2KC_0 + (2KC_0)^{M-1}\big) \le 1/(2CK)$. Now, for $N \in \mathbb{N}$ put $V_N = U_N - U_{N-1}$. Then V_N satisfies (with $U_{-1} \equiv 0$)

$$\partial_t V_N = \Delta V_N + f(U_{N-1}) - f(U_{N-2}), \quad V_N(0, x) = 0, \qquad N \in \mathbb{N}.$$

By Propositions 5.2 and 6.3 we get with some $D < \infty$ depending on ω and f,

$$\begin{aligned} E^{\tau,\sigma,\theta,\rho;Y}[V_N] &\le K(T + \theta)E^{\tau,\sigma,\theta,\rho;Y}[f(U_{N-1}) - f(U_{N-2})] \\ &\le K(T + \theta)E^{\tau,\sigma,\theta,\rho;Y}[V_{N-1}] \\ &\quad \times\ F'\big(\omega \max\{E^{\tau,\sigma,\theta,\rho;Y}[U_{N-1}], E^{\tau,\sigma,\theta,\rho;Y}[U_{N-2}]\}\big) \\ &\le K(T + \theta)E^{\tau,\sigma,\theta,\rho;Y}[V_{N-1}] \cdot D\big(2KC_0 + (2KC_0)^{M-1}\big) \\ &\le \big\{(T + \theta)DK\big(2KC_0 + (2KC_0)^{M-1}\big)\big\} \cdot E^{\tau,\sigma,\theta,\rho;Y}[V_{N-1}]. \end{aligned}$$

Hence, if $(T + \theta)DK\big(2KC_0 + (2KC_0)^{M-1}\big) < 1$ the sequence

$$U_N = U_0 + \sum_{j=1}^{N} V_j,$$

converges to a function $u \in \widetilde{G}^{\tau,\sigma}(\theta, \rho; Y)$. Clearly, u satisfies (7.1). □

Remark 7.2. The case of Gevrey non-linearities leads to substantial technical difficulties from the point of view of the nonlinear estimates. We can deal with such non-linearities, demonstrating suitable non-linear estimates in the anisotropic spaces $\tilde{G}$ (for non-linear Gevrey estimates in different spaces and applications for problems in PDEs and Dynamical Systems we refer to [7], [14], [15], [25]).

8. EXAMPLES OF ANALYTICITY DOWN TO $t = 0$ AND STABILITY PROPERTIES

The aim of this section is to exhibit solutions for semilinear parabolic equations with conservative terms defined by rotation type vector fields which are analogues to the celebrated Oseen solution to the 2–D Navier–Stokes equation (see C. Oseen [29] in 1911).

We consider semilinear parabolic equations of the following type:

$$\partial_t u - \Delta u = F[u], \qquad t > 0,\ x \in \mathbb{R}^n, \tag{8.1}$$

where F is a non-linear map satisfying

$$F[u] \equiv 0 \text{ if } u \text{ is radially symmetric with respect to } x\ (u = u(t, |x|^2)). \tag{8.2}$$

As an example of such nonlinear terms for $n \geq 2$ we have

$$F[u] := \sum_{j=1}^{k} X_j(F_j(u)), \tag{8.3}$$

where $F_j(u)$, $j = 1, \dots, k$ are polynomials with real coefficients and

$$X_j,\ j = 1, \dots, k \text{ are smooth vector fields tangential to } S_r,\ r > 0, \tag{8.4}$$

where $S_r = \{x \in \mathbb{R}^n : |x| = r\}$, see [13]. One also recaptures by (8.2) the non-linear term for the 2–D Navier–Stokes equation.

Example 8.1. Vector fields of the type $\sum_{1 \leq j < k \leq n} Q_{jk}(x)(x_j \partial_k - x_k \partial_j)$, Q_{jk}, $1 \leq j < k \leq n$, being smooth functions, are tangential to S_r, $r > 0$.

Before the formulation of the next theorem recall that a function φ belongs to the Gelfand-Shilov space $S^{\mu}_{\nu}(\mathbb{R}^n)$ if and only if one can find $C < \infty, D < \infty$ and $a > 0$ such that for any $\alpha \in \mathbb{N}_0^n$

$$|\partial^{\alpha} \varphi(x)| \leq C D^{|\alpha|} \alpha!^{\mu} \exp\{-a|x|^{1/\nu}\} \text{ for } x \in \mathbb{R}^n, \tag{8.5}$$

(cf. Gelfand–Shilov [11], Chapter IV). We consider the initial data

$$u(0, x) = u_0(x) = \theta(|x|^2), \tag{8.6}$$

where θ is a real-valued continuous function on $[0, \infty)$. Then we have the following.

Theorem 8.2. *Suppose that $\theta(y)$ grows infraexponentially for $y \to +\infty$; i.e., for every $\varepsilon > 0$ one can find $C_\varepsilon > 0$ such that*

$$|\theta(y)| \leq C_\varepsilon e^{\varepsilon y}, \qquad y \geq 0. \tag{8.7}$$

Then the IVP (8.1), (8.6) *has a unique solution $u(t, x) = e^{t\Delta} u_0(x)$, $t \geq 0$, $x \in \mathbb{R}^n$, satisfying for every $\varepsilon > 0$,*

$$|u(t, x)| = O(e^{\varepsilon |x|^2}), \qquad |x| \to \infty. \tag{8.8}$$

Next, assume that $\theta \in G^{\sigma}([0, \infty) : \mathbb{R})$, $\sigma > 0$, and for some $\nu \in (0, 1]$, $\theta \in S^{\sigma}_{\nu}(\overline{\mathbb{R}}_+)$; i.e., there exist two positive constants A and B such that

$$|\theta^{(j)}(y)| \leq A^{j+1} j!^{\sigma} \exp\{-B y^{1/\nu}\}, \tag{8.9}$$

for all $y \geq 0$, $j \in \mathbb{N}_0$. *Then*

$$u_0(x) = \theta(x^2) \in S_{\nu/2}^{\sigma+1-\nu/2}(\mathbb{R}^n) \tag{8.10}$$

and

$$u(t,x) \in G^{2\sigma+2-\nu}([0,\infty) : S_{\nu/2}^{\sigma+1-\nu/2}(\mathbb{R}^n)). \tag{8.11}$$

In particular, if

$$\theta(y) = \theta_\varepsilon(y) = \frac{1}{(4\pi\varepsilon)^{n/2}} e^{-y/(4\varepsilon)}, \qquad \varepsilon > 0, \tag{8.12}$$

then

$$u^\varepsilon(t,x) = \frac{1}{(4\pi(t+\varepsilon)^{n/2}} e^{-x^2/(4(t+\varepsilon))}, \qquad t \geq 0, x \in \mathbb{R}^n \tag{8.13}$$

solves (8.1) *with initial data*

$$u^\varepsilon(0,x) = \frac{1}{(4\pi\varepsilon)^{n/2}} e^{-x^2/(4\varepsilon)}, \qquad x \in \mathbb{R}^n. \tag{8.14}$$

Proof. The arguments are simple (as for the 2-D NS, see also [13]). By (8.6) and (8.7)

$$|u_0(x)| = O(e^{\varepsilon|x|^2}), \qquad \text{as } |x| \to \infty, \tag{8.15}$$

for all $\varepsilon > 0$, therefore $u(t,x) = e^{t\Delta}u_0$ is well defined and is a solution to the heat equation $\partial_t u - \Delta u = 0$ with initial data u_0, unique in the class of functions satisfying (8.8). Next, we note that if $u_0(x)$ is radially symmetric, then $e^{t\Delta}u_0$ remains radially symmetric with respect to x for all $t > 0$ and by the hypothesis (8.4) we observe that $X_j\big(F_j(e^{t\Delta}u_0)\big) \equiv 0$.

We show the Gelfand–Shilov property for space dimension $n = 1$, since the general case brings only notational complications. We have the following identity:

$$\begin{aligned}
\partial_x^\alpha(\theta(x^2)) &= \sum_{j=1}^{\alpha} \frac{\theta^{(j)}(x^2)}{j!} \partial_z^\alpha \left((z^2 - x^2)^j\right)|_{z=x} \\
&= \sum_{j=1}^{\alpha} \theta^{(j)}(x^2) \frac{\alpha!}{(\alpha-j)!j!} \partial_z^{\alpha-j} \left((z+x)^j\right)|_{z=x} \\
&= \sum_{\alpha/2 \leq j \leq \alpha} \theta^{(j)}(x^2) \frac{\alpha!}{(\alpha-j)!j!} \frac{j!}{(2j-\alpha)!} (2x)^{2j-\alpha}.
\end{aligned} \tag{8.16}$$

Next, combining (8.16) with (8.9), we derive (see Biagioni–Gramchev [6] for similar arguments for $\sigma = 1/2$), by using Stirling type formulas, the following estimates:

$$
\begin{aligned}
|\partial_x^\alpha(\theta(x^2))| &\le \sum_{\alpha/2\le j\le\alpha} |\theta^{(j)}(x^2)| \frac{\alpha!}{(\alpha-j)!j!} 2^{2j-\alpha} \frac{j!}{(2j-\alpha)!} |x|^{2j-\alpha} \\
&\le \sum_{\alpha/2\le j\le\alpha} A^{j+1} j!^\sigma e^{-B|x|^{2/\nu}} \frac{\alpha!}{(\alpha-j)!j!} 2^{2j-\alpha} \frac{j!}{(2j-\alpha)!} |x|^{2j-\alpha} \\
&\le A \sum_{\alpha/2\le j\le\alpha} (4A)^j j!^\sigma e^{-B|x|^{2/\nu}} \frac{j!}{(2j-\alpha)!} (|x|^{2/\nu})^{\nu j-\nu\alpha/2} \\
&\quad (\text{using } e^{-az} z^\gamma \le a^{-\gamma}\Gamma(\gamma+1) \text{ with } a = B/2, z = |x|^{2/\nu}, \gamma = \nu j - \nu\alpha/2) \\
&\le A \sum_{\alpha/2\le j\le\alpha} (2/B)^{\nu j-\nu\alpha/2} (4A)^j j!^\sigma e^{-B/2|x|^{2/\nu}} \frac{j!}{(2j-\alpha)!} \Gamma(\nu j - \nu\alpha/2 + 1) \\
&\le A e^{-B/2|x|^{2/\nu}} \sum_{\alpha/2\le j\le\alpha} (2/B)^{\nu j-\nu\alpha/2} (8A)^j j!^\sigma (\alpha-j)! \Gamma(\nu j - \nu\alpha/2 + 1) \\
&\le A C^\alpha e^{-B/2|x|^{2/\nu}} \sum_{\alpha/2\le j\le\alpha} j!^\sigma (\alpha-j)! \Gamma(\nu j - \nu\alpha/2 + 1) \\
&\le A C^\alpha e^{-B/2|x|^{2/\nu}} \sum_{\alpha/2\le j\le\alpha} j!^\sigma \Gamma((1-\nu/2)\alpha - (1-\nu)j + 1)! \\
&\le A C^\alpha (1+\alpha/2) e^{-B/2|x|^{2/\nu}} \alpha!^\sigma \Gamma((1-\nu/2)\alpha + 1),
\end{aligned}
\tag{8.17}
$$

for all $\alpha \in \mathbb{N}_0$, where $C = 8A(2/B)^{\nu/2}$. By the Stirling formula we get that (8.17) yields (8.10). As it concerns (8.11), it follows from our linear estimates. □

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