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LOCAL AND GLOBAL SOLUTIONS FOR THE NON-LINEAR SCHRÖDINGER-BOUSSINESQ SYSTEM

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Abstract. We study the local and global well posedness of the initial-value problem for the non-linear Schrödinger-Boussinesq System. Local existence results are proved for three initial data in Sobolev spaces of negative indices. Global results are proved using the arguments of Colliander Holmer and Tzirakis (2006 Arxiv preprint math.AP/0603595).

1. Introduction

In this paper we consider the initial-value problem (IVP) for the Schrödinger-Boussinesq system (hereafter referred to as the SB-system)

$$\begin{cases} iu_t + u_{xx} = vu, \\ v_{tt} - v_{xx} + v_{xxxx} = (|u|^2)_{xx}, \\ u(0,x) = u_0(x);\ v(0,x) = v_0(x);\ v_t(0,x) = (v_1)_x(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$ and $t > 0$.

Here, u and v are respectively a complex-valued and a real-valued function defined in space-time $\mathbb{R}^2$. The SB-system is considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction in a plasma [20] and diatomic lattice system [23]. The short-wave term $u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is described by a Schrödinger-type equation with a potential $v(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying some sort of Boussinesq equation and representing the intermediate long wave.

The non-linear Schrödinger (NLS) equation models a wide range of physical phenomena including self-focusing of optical beams in non-linear media, propagation of Langmuir waves in plasmas, etc. For an introduction to this topic, we refer the reader to [18]. The Boussinesq equation as a model of long waves was originally derived by Boussinesq [5] in his study of non-linear,

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dispersive wave propagation. We should remark that it was the first equation proposed in the literature to describe this kind of physical phenomena. This equation was also used by Zakharov [25] as a model of non-linear string and by Falk *et al* [8] in their study of shape-memory alloys.

Our principal aim here is to study the well posedness of the Cauchy problem for the SB-system (1.1). We refer to the expression "local well posedness" in the sense of Hadamard; that is, the solution uniquely exists in a certain time interval (unique existence), the solution has the same regularity as the initial data in a certain time interval (persistence), and the solution varies continuously depending upon the initial data (continuous dependence). Global well posedness requires that the same properties hold for all time $t > 0$. Natural spaces for the initial data are the classical Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$, which are defined as the completion of the Schwartz class $\mathcal{S}(\mathbb{R})$ with respect to the norm

$$\|f\|_{H^s(\mathbb{R})} = \|(1+\xi^2)^{s/2}\widehat{f}\|_{L^2(\mathbb{R})}.$$

Concerning the local well posedness question, some results have been obtained for the SB-system (1.1). Linares and Navas [17] proved that (1.1) is locally well posed for initial data $u_0 \in L^2(\mathbb{R})$, $v_0 \in L^2(\mathbb{R})$, $v_1 = h_x$ with $h \in H^{-1}(\mathbb{R})$ and $u_0 \in H^1(\mathbb{R})$, $v_0 \in H^1(\mathbb{R})$, $v_1 = h_x$ with $h \in L^2(\mathbb{R})$. Moreover, by using some conservations laws, in the latter case the solutions can be extended globally. Yongqian [24] established a similar result when $u_0 \in H^s(\mathbb{R})$, $v_0 \in H^s(\mathbb{R})$, $v_1 = h_{xx}$ with $h \in H^s(\mathbb{R})$ for $s \geq 0$ and assuming $s \geq 1$ these solutions are global.

Since scaling arguments cannot be applied to the Boussinesq-type equations to obtain a critical notion it is not clear what the lower Sobolev index s for which one has local (or maybe global) well posedness is. To obtain some idea on which spaces we should expect well posedness, we recall some results concerning the Schrödinger and Boussinesq equations.

For the single cubic nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + |u|^2u = 0,$$

Y. Tsutsumi [22] established local and global well posedness for data in $L^2(\mathbb{R})$. Moreover, by using the scaling and Galilean invariance with the special soliton solutions, it was proved by Kenig, Ponce and Vega [15] that the focusing cubic (NLS) equation is not locally well posed below $L^2(\mathbb{R})$. This ill-posed result is in the sense that the data-solution map is not uniformly continuous. Recently, Christ, Colliander and Tao [7] have obtained similar results for defocusing (NLS) equations. For the case of quadratics NLS

$$iu_t + u_{xx} + u^2 = 0, \tag{1.2}$$

$$iu_t + u_{xx} + \bar{u}^2 = 0, \tag{1.3}$$

$$iu_t + u_{xx} + u\bar{u} = 0, \tag{1.4}$$

where $\bar{u}$ denotes the complex conjugate of u, Kenig, Ponce and Vega [14] have proved local well posedness for data in $H^s(\mathbb{R})$ with $s > -3/4$ for (1.2)-(1.3) and $s > -1/4$ for (1.4). This result is sharp, in the sense that we cannot lower these Sobolev indices using the techniques of [14].

On the other hand, in the case of the Boussinesq equation

$$\begin{cases} v_{tt} - v_{xx} + v_{xxxx} + (f(v))_{xx} = 0, & x \in \mathbb{R}, t > 0, \\ v(0) = \phi;\ v_t(0) = \psi, \end{cases} \tag{1.5}$$

Bona and Sachs [3], using Kato's abstract theory for quasilinear evolution equations, showed local well posedness for $f \in C^\infty$ and initial data $\phi \in H^{s+2}(\mathbb{R})$, $\psi \in H^{s+1}(\mathbb{R})$ with $s > \frac{1}{2}$. Tsutsumi and Matahashi [21] established a similar result when $f(u) = |u|^{p-1}u$, $p > 1$ and $\phi \in H^1(\mathbb{R})$, $\psi = \chi_{xx}$ with $\chi \in H^1(\mathbb{R})$. These results were improved by Linares [16] who proved that (1.5) is locally well posed when $f(u) = |u|^{p-1}u$, $1 < p < 5$, $\phi \in L^2(\mathbb{R})$, $\psi = h_x$ with $h \in H^{-1}(\mathbb{R})$ and $f(u) = |u|^{p-1}u$, $1 < p < 5$, $\phi \in H^1(\mathbb{R})$, $\psi = h_x$ with $h \in L^2(\mathbb{R})$. Moreover, assuming smallness in the initial data, it was proved that these solutions can be extended globally in $H^1(\mathbb{R})$. The main tool used in [16] was the Strichartz estimates satisfied by solutions of the linear problem. Finally, using the techniques of [14], Farah [10] proved local well posedness for $f = u^2$, $\phi \in H^s(\mathbb{R})$, $\psi = h_x$ with $h \in H^{s-1}(\mathbb{R})$ and $s > -1/4$. Again, this last result is sharp in the same sense as above.

The local well posedness for single dispersive equations with quadratic non-linearities has been extensively studied in Sobolev spaces with negative indices. The proofs of these results are based on the Fourier restriction norm approach introduced by Bourgain [4] in his study of the non-linear Schrödinger equation $iu_t + u_{xx} + u|u|^{p-2} = 0$, with $p \geq 3$ (NLS) and the Korteweg-de Vries equation $u_t + u_{xxx} + u_x u = 0$ (KdV). This method was further developed by Kenig, Ponce and Vega in [13] for the KdV equation and [14] for the quadratics NLS $iu_t + u_{xx} + u^2 = 0$, $iu_t + u_{xx} + u\bar{u} = 0$, where $\bar{u}$ denotes the complex conjugate of u, in one spatial dimension and in the spatially continuous and periodic case.

The original Bourgain method makes extensive use of the Strichartz inequalities in order to derive the bilinear estimates corresponding to the nonlinearity. On the other hand, Kenig, Ponce and Vega simplified Bourgain's proof and improved the bilinear estimates using only elementary techniques, such as the Cauchy-Schwarz inequality and simple calculus inequalities.

Both arguments also use some arithmetic facts involving the symbol of the linearized equation. For example, the algebraic relation for the quadratic NLS $iu_t + u_{xx} + u^2 = 0$ is given by

$$2|\xi_1(\xi - \xi_1)| \leq |\tau - \xi^2| + |(\tau - \tau_1) - (\xi - \xi_1)^2| + |\tau_1 - \xi_1^2|. \tag{1.6}$$

Then, splitting the domain of integration into the sets where each term of the right side of (1.6) is the biggest, Kenig, Ponce and Vega made some cancellation in the symbol in order to use their calculus inequalities (see Lemma 3.1) and a clever change of variables to establish their crucial estimates.

This same kind of technique was used for the Boussinesq equation. However, we do not have good cancellations on the Boussinesq symbol. To overcome this difficulty, we observe that the dispersion in the Boussinesq case is given by the symbol $\sqrt{\xi^2 + \xi^4}$ and this is in some sense related to the Schrödinger symbol (see Lemma 3.2 below). Therefore, we can modify the symbols and work only with the algebraic relations for the Schrödinger equation already used in Kenig, Ponce and Vega [14] in order to derive our relevant bilinear estimates.

Taking into account the sharp local well posedness results obtained for the quadratic (NLS) and Boussinesq equations it is natural to ask whether the SB-system is, at least, locally well posed for initial data $(u_0, v_0, v_1) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > -1/4$. Here, we answer affirmatively this question. Indeed, we obtain local well posedness for weak initial data $(u_0, v_0, v_1) \in H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ for various values of k and s. The scheme of proof used to obtain our results is in the same spirit as the one implemented by Ginibre, Y. Tsutsumi and Velo [11] to establish their results for the Zakharov system

$$\begin{cases} iu_t + u_{xx} = vu, \\ \sigma v_{tt} - v_{xx} = (|u|^2)_{xx}, \\ u(0,x) = u_0(x);\ v(0,x) = v_0(x);\ v_t(0,x) = v_1, \end{cases} \tag{1.7}$$

where $x \in \mathbb{R}$ and $t > 0$.

In [1], it was shown that by a limiting procedure, as $\sigma \to 0$, the solution u_σ to (1.7) converges in a certain sense to the unique solution for the cubic (NLS). Hence, it is natural to expect that the system (1.7) is well posed for $u_0 \in L^2(\mathbb{R})$. In fact, for the case $\sigma = 1$, in [11] it is shown that (1.7) is locally well posed for $(u_0, v_0, v_1) \in L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R}) \times H^{-3/2}(\mathbb{R})$. Moreover, Holmer [12] shows that the one-dimensional local theory of [11] is effectively sharp, in the sense that for (k, s) outside the range given in [11], there exist ill-posedness results for the Zakharov system (1.7). In particular, we cannot have local well posedness for the initial data in Sobolev spaces of negative index.

Note that the system (1.7) is quite similar to the SB-system. In fact, taking $\sigma = 1$ and adding v_{xxxx} on the left-hand side of the second equation of (1.7) we obtain (1.1). In other words, the intermediate long wave in (1.7) is described by a wave equation instead of a Boussinesq equation.

Despite such similarity, there are strong differences in the local theory. According to Theorem 1.1 stated below, it is possible to prove that the system (1.1) is locally well posed for initial data $(u_0, v_0, v_1) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > -1/4$, which is not the case for the system (1.7). Therefore, in the sense of the local theory, we can say that the SB-system (1.1) is better behaved than the Zakharov system (1.7). This is due basically to the fact that (1.1) has more dispersion then (1.7).

To describe our results we define next the $X^S_{s,b}$ and $X^B_{s,b}$ spaces related respectively to the Schrödinger and Boussinesq equations. For the first equation, these spaces were introduced in [4]. In the case of the Boussinesq equation, the $X^B_{s,b}$ with $b = \frac{1}{2}$, were first defined by Fang and Grillakis [9] for the Boussinesq-type equations in the periodic case. Using these spaces and following Bourgain's argument introduced in [4] they proved local well posedness for (1.1) with the spatial variable in the unit circle assuming $u_0 \in H^s$, $u_1 \in H^{-2+s}$, with $0 \le s \le 1$ and $|f(u)| \le c|u|^p$, with $1 < p < \frac{3-2s}{1-2s}$ if $0 \le s < \frac{1}{2}$ and $1 < p < \infty$ if $\frac{1}{2} \le s \le 1$. Moreover, if $u_0 \in H^1$, $u_1 \in H^{-1}$ and $f(u) = \lambda |u|^{q-1}u - |u|^{p-1}u$, with $1 < q < p$ and $\lambda \in \mathbb{R}$, then the solution is global.

Next, we give the precise definition of the $X^S_{s,b}$ and $X^B_{s,b}$ spaces in the continuous case.

Definition 1.1. *For $s, b \in \mathbb{R}$, $X^S_{s,b}$ denotes the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm*

$$\|F\|_{X^S_{s,b}} = \|\langle \tau + \xi^2 \rangle^b \langle \xi \rangle^s \widetilde{F}\|_{L^2_{\tau,\xi}},$$

where $\sim$ denotes the time-space Fourier transform and $\langle a \rangle \equiv 1 + |a|$.

Definition 1.2. *For $s, b \in \mathbb{R}$, $X^B_{s,b}$ denotes the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm*

$$\|F\|_{X^B_{s,b}} = \|\langle |\tau| - \gamma(\xi) \rangle^b \langle \xi \rangle^s \widetilde{F}\|_{L^2_{\tau,\xi}},$$

where $\gamma(\xi) \equiv \sqrt{\xi^2 + \xi^4}$.

We will also need the localized $X^S_{s,b}$ and $X^B_{s,b}$ spaces defined as follows:

Definition 1.3. *For $s, b \in \mathbb{R}$ and $T \geq 0$, $X^{S,T}_{s,b}$ (respectively $X^{B,T}_{s,b}$) denotes the space endowed with the norm*

$$\|u\|_{X^{S,T}_{s,b}} = \inf_{w \in X^S_{s,b}} \left\{ \|w\|_{X^S_{s,b}} : w(t) = u(t) \text{ on } [0,T] \right\}.$$

(respectively with $X^B_{s,b}$ instead of $X^S_{s,b}$).

Now we state the main results of this paper.

Theorem 1.1. *Let $1/4 < a < 1/2 < b$. Then, there exists $c > 0$, depending only on a, b, k, s, such that*

(i) $\|uv\|_{X^S_{k,-a}} \leq c \|u\|_{X^S_{k,b}} \|v\|_{X^B_{s,b}}$ *holds for $|k| - s \leq a$;*

(ii) $\|u_1 \bar{u}_2\|_{X^B_{s,-a}} \leq c \|u_1\|_{X^S_{k,b}} \|u_2\|_{X^S_{k,b}}$ *holds for*
- $s - k \leq a$, *if $s > 0$ and $k > 0$;*
- $s + 2|k| \leq a$, $2|k| \leq a$, *if $s > 0$ and $k \leq 0$;*
- $s + 2|k| \leq 1/2$, $2|k| \leq a$, *if $s \leq 0$ and $k \leq 0$.*

Theorem 1.2. *Let $k > -1/4$. Then, for any $(u_0, v_0, v_1) \in H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, provided,*

(i) $|k| - 1/2 < s < 1/2 + 2k$ *for $k \leq 0$,*

(ii) $k - 1/2 < s < 1/2 + k$ *for $k > 0$,*

there exist $T = T(\|u_0\|_{H^k}, \|v_0\|_{H^s}, \|v_1\|_{H^{s-1}})$, $b > 1/2$ and a unique solution (u, v) of the IVP (1.1), satisfying

$$u \in C([0,T] : H^k(\mathbb{R})) \cap X^{S,T}_{k,b} \text{ and } v \in C([0,T] : H^s(\mathbb{R})) \cap X^{B,T}_{s,b}.$$

Moreover, the map $(u_0, v_0, v_1) \mapsto (u(t), v(t))$ is locally Lipschitz from $H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ into $C([0,T] : H^k(\mathbb{R}) \times H^s(\mathbb{R}))$.

Next, we obtain bilinear estimates for the case $s = 0$ and $b, b_1 < 1/2$. These estimates will be useful to establish global solutions.

Theorem 1.3. *Let $a, a_1, b, b_1 > 1/4$, then there exists $c > 0$ depending only on a, a_1, b, b_1 such that*

(i) $\|uv\|_{X^S_{0,-a_1}} \leq c \|u\|_{X^S_{0,b_1}} \|v\|_{X^B_{0,b}}$ *and*

(ii) $\|u_1 \bar{u}_2\|_{X^B_{0,-a}} \leq c \|u_1\|_{X^S_{0,b_1}} \|u_2\|_{X^S_{0,b_1}}$.

These are the essential tools to prove the following global result.

Theorem 1.4. *The SB-system (1.1) is globally well posed for $(u_0, v_0, v_1) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$ and the solution (u, v) satisfies for all $t > 0$*

$$\|v(t)\|_{L^2} + \|(-\Delta)^{-1/2} v_t(t)\|_{H^{-1}} \lesssim e^{((\ln 2)\|u_0\|^2_{L^2} t)} \max\{\|v_0, v_1\|_{\mathfrak{B}}, \|u_0\|_{L^2}\}.$$

The argument used to prove this result follows the ideas introduced by Colliander, Holmer, Tzirakis [6] to deal with the Zakharov system. The intuition for this theorem comes from the fact that the non-linearity for the second equation of the SB-system (1.1) depends only on the first equation. Therefore, noting that the bilinear estimates given in Theorem 1.2 hold for $a, a_1, b, b_1 < 1/2$, it is possible to show that the time existence depends only on $\|u_0\|_{L^2}$. But since this norm is conserved by the flow, we obtain a global solution.

The plan of this paper is as follows: in Section 2, we prove some estimates for the integral equation in the $X^S_{s,b}$ and $X^B_{s,b}$ spaces introduced above. Bilinear estimates are proved in Section 3. Finally, the local and global well posedness results are treated in Sections 4 and 5, respectively.

2. Preliminary Results

First, we remark that for any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant θ such that $a \leq \theta b$. Also, we denote $a \sim b$ when, $a \lesssim b$ and $b \lesssim a$.

Consider the free Schrödinger equation

$$iu_t + u_{xx} = 0, \tag{2.1}$$

the solution for initial data $u(0) = u_0$

$$u(t) = U(t)u_0, \tag{2.2}$$

where $U(t)u_0 = (e^{-it\xi^2}\widehat{u}_0(\xi))^\vee$. On the other hand, for the linear Boussinesq equation

$$v_{tt} - v_{xx} + v_{xxxx} = 0, \tag{2.3}$$

it is well known that the solution for initial data $v(0) = v_0$ and $v_t(0) = (v_1)_x$, is given by

$$u(t) = V_c(t)v_0 + V_s(t)(v_1)_x, \tag{2.4}$$

where

$$V_c(t)v_0 = \Big(\frac{e^{it\sqrt{\xi^2+\xi^4}} + e^{-it\sqrt{\xi^2+\xi^4}}}{2}\widehat{v_0}(\xi)\Big)^\vee$$
$$V_s(t)(v_1)_x = \Big(\frac{e^{it\sqrt{\xi^2+\xi^4}} - e^{-it\sqrt{\xi^2+\xi^4}}}{2i\sqrt{\xi^2+\xi^4}}\widehat{(v_1)_x}(\xi)\Big)^\vee.$$

By Duhamel's Principle the solution of system (NLB) is equivalent to

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')(vu)(t')dt' \tag{2.5}$$

$$v(t) = V_c(t)v_0 + V_s(t)(v_1)_x + \int_0^t V_s(t-t')(|u|^2)_{xx}(t')dt'. \quad (2.6)$$

Let θ be a cutoff function satisfying $\theta \in C_0^\infty(\mathbb{R})$, $0 \le \theta \le 1$, $\theta \equiv 1$ in $[-1,1]$, $\operatorname{supp}(\theta) \subseteq [-2,2]$ and for $0 < T \le 1$ define $\theta_T(t) = \theta(t/T)$. In fact, to work in $X^S_{s,b}$ and $X^B_{s,b}$ we consider another version of (2.5); that is,

$$u(t) = \theta(t)U(t)u_0 - i\theta_T(t)\int_0^t U(t-t')(vu)(t')dt' \quad (2.7)$$

$$v(t) = \theta(t)\left(V_c(t)v_0 + V_s(t)(v_1)_x\right) + \theta_T(t)\int_0^t V_s(t-t')(|u|^2)_{xx}(t')dt' \quad (2.8)$$

and

$$u(t) = \theta_T(t)U(t)u_0 - i\theta_T(t)\int_0^t U(t-t')(vu)(t')dt' \quad (2.9)$$

$$v(t) = \theta_T(t)(V_c(t)v_0 + V_s(t)(v_1)_x) + \theta_T(t)\int_0^t V_s(t-t')(|u|^2)_{xx}(t')dt'. \quad (2.10)$$

We use equation (2.7) (respectively (2.9)) to study the local (respectively global) well posedness problem associated to the SB-system (1.1).

Note that the integral equations (2.7) and (2.9) are defined for all $(t,x) \in \mathbb{R}^2$. Moreover, if (u,v) is a solution of (2.7) or (2.9), then

$$(\tilde{u}, \tilde{v}) = (u|_{[0,T]}, v|_{[0,T]}),$$

will be a solution of (2.5) in $[0,T]$.

Before proceeding to the group and integral estimates for (2.7) we introduce the norm

$$\|v_0, v_1\|^2_{\mathfrak{B}^s} \equiv \|v_0\|^2_{H^s} + \|v_1\|^2_{H^{s-1}}.$$

For simplicity we denote $\mathfrak{B}^0$ by $\mathfrak{B}$ and, for functions of t, we use the shorthand

$$\|v(t)\|^2_{\mathfrak{B}^s} \equiv \|v(t)\|^2_{H^s} + \|(-\Delta)^{-1/2}v_t(t)\|^2_{H^{s-1}}.$$

The following lemmas are standard in this context. The difference here is on the exponent of T that appears in the group estimates. This exponent together with the growth control of the solution norm $\|v\|_{\mathfrak{B}}$ will be important for the proof of Theorem 1.4 in L^2.

Lemma 2.1 (Group estimates)**.** *Let* $0 < T \le 1$.

(a) *Linear Schrödinger equation*

(i) $\|U(t)u_0\|_{C(\mathbb{R}:H^s)} = \|u_0\|_{H^s}$.

(ii) *If* $0 \leq b_1 \leq 1$, *then*

$$\|\theta_T(t)U(t)u_0\|_{X^S_{s,b_1}} \lesssim T^{1/2-b_1}\|u_0\|_{H^s}.$$

(b) *Linear Boussinesq equation*
(i) $\|V_c(t)v_0 + V_s(t)(v_1)_x\|_{C(\mathbb{R}:H^s)} \leq \|v_0\|_{H^s} + \|v_1\|_{H^{s-1}}$.
(ii) $\|V_c(t)v_0 + V_s(t)(v_1)_x\|_{C(\mathbb{R}:\mathfrak{B})} = \|v_0, v_1\|_{\mathfrak{B}}$.
(iii) *If* $0 \leq b \leq 1$, *then*

$$\|\theta_T(t)\left(V_c(t)v_0 + V_s(t)(v_1)_x\right)\|_{X^B_{s,b}} \lesssim T^{1/2-b}\left(\|v_0\|_{H^s} + \|v_1\|_{H^{s-1}}\right).$$

Remark 2.1. We should notice that the first inequality of item (a) and the second one of item (b) do not have an implicit constant multiplying the right-hand side. This will be important in the proof of the global result in L^2 stated in Theorem 1.4, since we will make use of an iterated argument to control the growth of the solution norm.

Proof. (a) The first inequality comes from the fact that $S(\cdot)$ is a unitary group. The second one with $0 \leq b_1 \leq 1/2$ can be found, for instance, in [6] Lemma 2.1(a). The case $1/2 < b_1 \leq 1$ can be proved using the same arguments as the ones used in the previous case. Since in (b) we apply these same arguments in the context of the Boussinesq equation, we omit the proof of (ii).

(b) By the definitions of $V_c(\cdot)$ and $V_s(\cdot)$ it is easy to see that for all $t \in \mathbb{R}$

$$\|V_c(t)v_0\|_{H^s} \leq \|v_0\|_{H^s} \text{ and } \|V_s(t)(v_1)_x\|_{H^s} \leq \|v_1\|_{H^{s-1}}.$$

Let $f(t,x)$ be a solution of the linear Boussinesq equation

$$f_{tt} - f_{xx} + f_{xxxx} = 0, \quad f(0,x) = v_0, \quad f_t(0,x) = (v_1)_x. \tag{2.11}$$

Let $J^s = \mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}$, for $s \in \mathbb{R}$. Applying the operators $(-\Delta)^{-1}$ and J^{-1} to the equation (2.11), multiplying by $J^{-1}f_t$ and finally integrating with respect to x, we obtain (after an integration by parts) the following

$$\frac{d}{dt}\left\{\|f\|^2_{L^2} + \|(-\Delta)^{-1/2}f_t\|^2_{H^{-1}}\right\} = 0,$$

which implies for all $t \in \mathbb{R}$

$$\|V_c(t)v_0 + V_s(t)(v_1)_x\|_{\mathfrak{B}} = \|v_0, v_1\|_{\mathfrak{B}}.$$

Now, we turn to the proof of the third assertion in (b). A simple computation shows that

$$(\theta_T(t)\left(V_c(t)v_0 + V_s(t)(v_1)_x\right))^{\sim}(\tau,\xi)$$

$$= \frac{\widehat{\theta_T}(\tau-\gamma(\xi))}{2}\left(\widehat{v_0}(\xi)+\frac{i\xi\widehat{v_1}(\xi)}{\gamma(\xi)}\right)+\frac{\widehat{\theta_T}(\tau+\gamma(\xi))}{2}\left(\widehat{v_0}(\xi)-\frac{i\xi\widehat{v_1}(\xi)}{\gamma(\xi)}\right).$$

Thus, setting $h_1(\xi)=\widehat{v_0}(\xi)+\frac{i\xi\widehat{v_1}(\xi)}{\gamma(\xi)}$ and $h_2(\xi)=\widehat{v_0}(\xi)-\frac{i\xi\widehat{v_1}(\xi)}{\gamma(\xi)}$, we have

$$\begin{aligned}&\|\theta_T\left(V_c(t)v_0+V_s(t)(v_1)_x\right)\|^2_{X_{s,b}}\\ &\leq \int_{-\infty}^{+\infty}\langle\xi\rangle^{2s}|h_1(\xi)|^2\Big(\int_{-\infty}^{+\infty}\langle|\tau|-\gamma(\xi)\rangle^{2b}\Big|\frac{\widehat{\theta_T}(\tau-\gamma(\xi))}{2}\Big|^2 d\tau\Big)d\xi\\ &+\int_{-\infty}^{+\infty}\langle\xi\rangle^{2s}|h_2(\xi)|^2\Big(\int_{-\infty}^{+\infty}\langle|\tau|-\gamma(\xi)\rangle^{2b}\Big|\frac{\widehat{\theta_T}(\tau+\gamma(\xi))}{2}\Big|^2 d\tau\Big)d\xi.\end{aligned}$$

Since $||\tau|-\gamma(\xi)|\leq\min\{|\tau-\gamma(\xi)|,|\tau+\gamma(\xi)|\}$ we have

$$\begin{aligned}\|\theta_T\left(V_c(t)v_0+V_s(t)(v_1)_x\right)\|^2_{X_{s,b}} &\lesssim \left(\|h_1\|^2_{H^s}+\|h_2\|^2_{H^s}\right)\|\theta_T\|^2_{H^b_t}\\ &\lesssim \left(\|v_0\|_{H^s}+\|v_1\|_{H^{s-1}}\right)^2\|\theta_T\|^2_{H^b_t}.\end{aligned}$$

To complete the proof we note that (since $0<T\leq 1$)

$$\begin{aligned}\|\theta_T\|_{H^b_t} &\lesssim \|\theta_T\|_{L^2}+\|\theta_T\|_{\dot{H}^b_t}\\ &\lesssim T^{1/2}\|\theta_1\|_{L^2}+T^{1/2-b}\|\theta_1\|_{\dot{H}^b_t}\lesssim T^{1/2-b}\|\theta_1\|_{H^b_t}.\end{aligned}$$ □

Next, we estimate the integral parts of (2.7).

Lemma 2.2 (Integral estimates)**.** *Let* $0<T\leq 1$.

(a) *Non-homogeneous linear Schrödinger equation*

(i) *If* $0\leq a_1<1/2$, *then*

$$\Big\|\int_0^t U(t-t')z(t')dt'\Big\|_{C([0,T]:H^s)}\lesssim T^{1/2-a_1}\|z\|_{X^S_{s,-a_1}}.$$

(ii) *If* $0\leq a_1<1/2$, $0\leq b_1$ *and* $a_1+b_1\leq 1$, *then*

$$\Big\|\theta_T(t)\int_0^t U(t-t')z(t')dt'\Big\|_{X^S_{s,b_1}}\lesssim T^{1-a_1-b_1}\|z\|_{X^S_{s,-a_1}}.$$

(b) *Non-homogeneous linear Boussinesq equation*

(i) *If* $0\leq a<1/2$, *then*

$$\Big\|\int_0^t V_s(t-t')z_{xx}(t')dt'\Big\|_{C([0,T]:\mathfrak{B}^s)}\lesssim T^{1/2-a}\|z\|_{X^B_{s,-a}}.$$

(ii) *If* $0\leq a<1/2$, $0\leq b$ *and* $a+b\leq 1$, *then*

$$\Big\|\theta_T(t)\int_0^t V_s(t-t')z_{xx}(t')dt'\Big\|_{X^B_{s,b}}\lesssim T^{1-a-b}\|z\|_{X^B_{s,-a}}.$$

Proof. (a) Again we refer the reader to [6] Lemma 2.2(a). Since in (b) we apply these same arguments in the context of the Boussinesq equation, we omit the proof of this item.

(b) We know that (see [6] inequality (2.13))

$$\left\|\theta_T(t)\int_0^t f(t')dt'\right\|_{L_t^\infty} \lesssim T^{1/2-a}\left\|f\right\|_{H_t^{-a}}. \tag{2.12}$$

First, we will prove that

$$(I) \quad \left\|\theta_T(t)\int_0^t V_s(t-t')z_{xx}(t')dt'\right\|_{L_t^\infty H^s} \lesssim T^{1/2-a}\|z\|_{X^B_{s,-a}},$$

$$(II) \quad \left\|\theta_T(t)(-\Delta)^{-1/2}\partial_t\int_0^t V_s(t-t')z_{xx}(t')dt'\right\|_{L_t^\infty H^s} \lesssim T^{1/2-a}\|z\|_{X^B_{s,-a}}.$$

To prove (I), we observe that $\sup_{\xi\in\mathbb{R}}\frac{|\xi|^2}{\gamma(\xi)} < \infty$. Therefore, using the Minkowski inequality and (2.12) we obtain

$$\begin{aligned}
&\left\|\theta_T(t)\int_0^t V_s(t-t')z_{xx}(t')dt'\right\|_{L_t^\infty H^s}\\
&\lesssim \left\|\left\|\theta_T(t)\int_0^t e^{it'\gamma(\xi)}(1+|\xi|^2)^{s/2}z^{\wedge(x)}(t',\xi)dt'\right\|_{L_\xi^2}\right\|_{L_t^\infty}\\
&+\left\|\left\|\theta_T(t)\int_0^t e^{-it'\gamma(\xi)}(1+|\xi|^2)^{s/2}z^{\wedge(x)}(t',\xi)dt'\right\|_{L_\xi^2}\right\|_{L_t^\infty}\\
&\lesssim T^{1/2-a}\Big(\left\|\langle\tau+\gamma(\xi)\rangle^{-a}\langle\xi\rangle^s\widetilde{z}(\tau,\xi)\right\|_{L^2_{\xi,\tau}}+\left\|\langle\tau-\gamma(\xi)\rangle^{-a}\langle\xi\rangle^s\widetilde{z}(\tau,\xi)\right\|_{L^2_{\xi,\tau}}\Big).
\end{aligned}$$

Since $||\tau|-\gamma(\xi)| \leq \min\{|\tau-\gamma(\xi)|, |\tau+\gamma(\xi)|\}$ and $a\geq 0$, we obtain inequality (I).

To prove (II) we note that

$$\begin{aligned}
&\left\|\theta_T(t)(-\Delta)^{1/2}\partial_t\int_0^t V_s(t-t')z_{xx}(t')dt'\right\|_{L_t^\infty H^{s-1}}\\
&=\left\|\left\||\xi|^{-1}(1+|\xi|^2)^{(s-1)/2}\theta_T(t)\int_0^t\frac{\cos((t-t')\gamma(\xi))}{\gamma(\xi)}\gamma(\xi)|\xi|^2z^{\wedge(x)}(t',\xi)dt'\right\|_{L_\xi^2}\right\|_{L_t^\infty}.
\end{aligned}$$

Therefore, the same arguments used to prove inequality (I) yield (II).

Now, we need to prove the continuity statements. We will give the proof only for inequality (I), since for (II) it can be obtained applying analogous arguments.

By an $\varepsilon/3$ argument, it is sufficient to establish this statement for z belonging to the dense class $\mathcal{S}(\mathbb{R}^2) \subseteq X^B_{s,-a}$. A simple calculation shows

$$\partial_t \int_0^t V_s(t-t')z_{xx}(t')dt' = \int_0^t V_c(t-t')z_{xx}(t')dt'.$$

Moreover, with essentially the same proof given above, inequality (I) holds for $V_c(t-t')$ and $\|z_{xx}\|_{X^B_{s,-a}}$ instead of $V_s(t-t')$ and $\|z\|_{X^B_{s,-a}}$, respectively. Therefore, by the fundamental theorem of calculus we have for $t_1, t_2 \in [0,T]$

$$\begin{aligned}
&\Big\| \int_0^{t_1} V_s(t_1-t')z_{xx}(t')dt' - \int_0^{t_2} V_s(t_2-t')z_{xx}(t')dt' \Big\|_{H^s} \\
&= \Big\| \int_{t_1}^{t_2} \left(\int_0^t V_c(t-t')z_{xx}(t')dt' \right) dt \Big\|_{H^s} \\
&\lesssim (t_2-t_1) \Big\| \theta_T(t) \int_0^t V_c(t-t')z_{xx}(t')dt' \Big\|_{L^\infty_t H^s} \lesssim (t_2-t_1)\|z_{xx}\|_{X^B_{s,-a}},
\end{aligned}$$

which proves the continuity.

It remains to prove the second assertion. We will use an argument due to [11]. We have for $a, b \in \mathbb{R}$ such that $0 \leq a < 1/2$, $0 \leq b$ and $a + b \leq 1$ (see [11] inequality (3.11))

$$\Big\| \theta_T(t) \int_0^t g(t')dt' \Big\|_{H^t_b} \leq T^{1-a-b}\|g\|_{H^t_{-a}}. \tag{2.13}$$

A simple calculation shows that

$$\begin{aligned}
&\left(\theta_T(t) \int_0^t V_s(t-t')z_{xx}(t')dt' \right)^{\wedge_{(x)}}(t,\xi) \\
&= -e^{it\gamma(\xi)} \left(\theta_T(t) \int_0^t h_1(t',\xi)dt' \right) + e^{-it\gamma(\xi)} \left(\theta_T(t) \int_0^t h_2(t',\xi)dt' \right) \\
&\equiv e^{it\gamma(\xi)} w_1^{\wedge_{(x)}}(t,\xi) - e^{-it\gamma(\xi)} w_2^{\wedge_{(x)}}(t,\xi),
\end{aligned}$$

where $h_1(t',\xi) = \dfrac{e^{-it'\gamma(\xi)}|\xi|^2 z^{\wedge_{(x)}}(t',\xi)}{2i\gamma(\xi)}$ and $h_2(t',\xi) = \dfrac{e^{it'\gamma(\xi)}|\xi|^2 z^{\wedge_{(x)}}(t',\xi)}{2i\gamma(\xi)}$. Therefore,

$$\left(\theta_T(t) \int_0^t V_s(t-t')z_{xx}(t')dt' \right)^{\sim}(\tau,\xi) = \widetilde{w_1}(\tau-\gamma(\xi),\xi) - \widetilde{w_2}(\tau+\gamma(\xi),\xi).$$

Now, using the definition of $X^B_{s,b}$ we have

$$\Big\| \theta_T(t) \int_0^t V_s(t-t')z_{xx}(t')dt' \Big\|^2_{X^B_{s,b}}$$

$$\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle|\tau+\gamma(\xi)|-\gamma(\xi)\rangle^{2b}\langle\xi\rangle^{2s}|\widetilde{w_1}(\tau,\xi)|^2 d\tau d\xi$$
$$+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle|\tau-\gamma(\xi)|-\gamma(\xi)\rangle^{2b}\langle\xi\rangle^{2s}|\widetilde{w_2}(\tau,\xi)|^2 d\tau d\xi \equiv M.$$

Since $\gamma(\xi)\geq 0$ for all $\xi\in\mathbb{R}$, we have

$$\max\{||\tau+\gamma(\xi)|-\gamma(\xi)|, ||\tau-\gamma(\xi)|-\gamma(\xi)|\}\leq|\tau|.$$

Thus, applying (2.13) and the fact that $\sup_{\xi\in\mathbb{R}}\frac{|\xi|^2}{\gamma(\xi)}<\infty$ we obtain

$$M\lesssim\sum_{j=1}^{2}\int_{-\infty}^{+\infty}\langle\xi\rangle^{2s}\|w_j^{\wedge(x)}\|^2_{H_t^b}\lesssim T^{1-a-b}\sum_{j=1}^{2}\int_{-\infty}^{+\infty}\langle\xi\rangle^{2s}\|h_j\|^2_{H_t^{-a}}$$
$$=T^{1-a-b}\Big(\big\|\langle\tau-\gamma(\xi)\rangle^{-a}\langle\xi\rangle^{s}\widetilde{z}(\tau,\xi)\big\|_{L^2_{\xi,\tau}}+\big\|\langle\tau+\gamma(\xi)\rangle^{-a}\langle\xi\rangle^{s}\widetilde{z}(\tau,\xi)\big\|_{L^2_{\xi,\tau}}\Big).$$

Since $||\tau|-\gamma(\xi)|\leq\min\{|\tau-\gamma(\xi)|,|\tau+\gamma(\xi)|\}$ and $a\geq 0$ we obtain the desired inequality. □

The next lemma says that, for $b>1/2$, $X^S_{s,b}$ and $X^B_{s,b}$ are embedded in $C(\mathbb{R}:H^s)$. For the spaces associated to the Schrödinger equation this result is well known in the literature, so we will prove this inclusion only for the $X^B_{s,b}$ spaces.

Lemma 2.3. *Let $b>\frac{1}{2}$. There exists $c>0$, depending only on b, such that*

$$\|u\|_{C(\mathbb{R}:H^s)}\leq c\|u\|_{X^B_{s,b}}.$$

Proof. First, we prove that $X^B_{s,b}\subseteq L^\infty(\mathbb{R}:H^s)$. Let $u=u_1+u_2$, where $\tilde{u}_1\equiv\tilde{u}\chi_{\{\tau\leq 0\}}$, $\tilde{u}_2\equiv\tilde{u}\chi_{\{\tau>0\}}$ and χ_A denotes the characteristic function of the set A. Then for all $t\in\mathbb{R}$

$$\begin{aligned}\|u_1(t)\|_{H^s} &= \Big\|\Big(e^{it\gamma(\xi)}(u_1)^{\wedge(x)}\Big)^{\vee(x)}(t,x)\Big\|_{H^s}\\ &= \Big\|\int_{-\infty}^{+\infty}\Big(\Big(e^{it\gamma(\xi)}(u_1)^{\wedge(x)}\Big)^{\vee(x)}\Big)^{\wedge(t)}(\tau,x)e^{it\tau}d\tau\Big\|_{H^s}\\ &\leq \int_{-\infty}^{+\infty}\Big\|\Big(\Big(e^{it\gamma(\xi)}(u_1)^{\wedge(x)}\Big)^{\vee(x)}\Big)^{\wedge(t)}(\tau,x)\Big\|_{H^s}d\tau.\end{aligned}$$

Using the Cauchy-Schwarz inequality we obtain

$$\|u_1(t)\|_{H^s}\leq\Big(\int_{-\infty}^{+\infty}\langle\tau\rangle^{-2b}\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\langle\tau+\gamma(\xi)\rangle^{2b}\langle\xi\rangle^{2s}|\tilde{u}(\tau,\xi)|^2d\tau d\xi\Big)^{\frac{1}{2}}.$$

On the other hand, by the same arguments

$$\|u_2(t)\|_{H^s} \leq \Big(\int_{-\infty}^{+\infty} \langle\tau\rangle^{-2b} \Big)^{\frac{1}{2}} \Big(\int_{-\infty}^{+\infty} \int_{0}^{+\infty} \langle\tau - \gamma(\xi)\rangle^{2b} \langle\xi\rangle^{2s} |\tilde{u}(\tau,\xi)|^2 d\tau d\xi \Big)^{\frac{1}{2}}.$$

Now, by the fact that $b > 1/2$, $|\tau + \gamma(\xi)| = ||\tau| - \gamma(\xi)|$ for $\tau \leq 0$ and $|\tau - \gamma(\xi)| = ||\tau| - \gamma(\xi)|$ for $\tau \geq 0$ we have $\|u\|_{L^\infty(\mathbb{R}:H^s)} \leq c\|u\|_{X^B_{s,b}}$. It remains to show continuity. Let $t, t' \in \mathbb{R}$ then

$$\begin{aligned} &\|u_1(t) - u_1(t')\|_{H^s} \\ &= \Big\| \int_{-\infty}^{+\infty} \Big(\Big(e^{it\gamma(\xi)} (u_1)^{\wedge(x)} \Big)^{\vee(x)} \Big)^{\wedge(t)} (\tau, x)(e^{it\tau} - e^{it'\tau}) d\tau \Big\|_{H^s}. \end{aligned} \tag{2.14}$$

Letting $t' \to t$, two applications of the dominated convergence theorem imply that the right-hand side of (2.14) goes to zero. Then $u_1 \in C(\mathbb{R} : H^s)$. It is clear that the same argument applies to u_2, which concludes the proof. □

We finish this section with the following standard Bourgain-Strichartz estimates. In the following, we denote by $a+$ a number slightly larger than a.

Lemma 2.4. *Let $\bar{X}^S_{s,b}$ denote the space with norm*

$$\|F\|_{\bar{X}^S_{s,b}} = \|\langle\tau - \xi^2\rangle^b \langle\xi\rangle^s \widetilde{F}\|_{L^2_{\tau,\xi}}.$$

Therefore, $\|u\|_{L^3_{x,t}} \leq c\min\{\|u\|_{X^S_{0,1/4+}}, \|u\|_{\bar{X}^S_{0,1/4+}}\}$, where $a+$ means that there exists $\varepsilon > 0$ such that $a+ = a + \varepsilon$.

Proof. This estimate is easily obtained by interpolating between

- $\|u\|_{L^6_{x,t}} \leq c\min\{\|u\|_{X^S_{0,1/2+}}, \|u\|_{\bar{X}^S_{0,1/2+}}\}$ (Strichartz inequality. See, for example, Lemma 2.4 in [11]) and
- $\|u\|_{L^2_{x,t}} = \|u\|_{X^S_{0,0}} = \|u\|_{\bar{X}^S_{0,0}}$ (by definition). □

3. Bilinear estimates

Before proceeding to the proof of Theorem 1.1, we state some elementary calculus inequalities that will be useful later.

Lemma 3.1. *For $p, q > 0$ and $r = \min\{p, q, p+q-1\}$ with $p + q > 1$, there exists $c > 0$ such that*

$$\int_{-\infty}^{+\infty} \frac{dx}{\langle x-\alpha\rangle^p \langle x-\beta\rangle^q} \leq \frac{c}{\langle\alpha-\beta\rangle^r}. \tag{3.1}$$

Moreover, let $C > 0$. For $a_0, a_1 \in \mathbb{R}$, $|a_2| \geq C > 0$ and $q > 1/2$, there exists $c > 0$ such that

$$\int_{-\infty}^{+\infty} \frac{dx}{\langle a_0 + a_1 x + a_2 x^2 \rangle^q} \leq c. \tag{3.2}$$

Proof. See Lemma 4.2 in [11] and Lemma 2.5 in [2]. □

Lemma 3.2. *There exists $c > 0$ such that*

$$\frac{1}{c} \leq \sup_{x,y \geq 0} \frac{1 + |x - y|}{1 + |x - \sqrt{y^2 + y}|} \leq c. \tag{3.3}$$

Proof. Since $y \leq \sqrt{y^2 + y} \leq y + 1/2$ for all $y \geq 0$ a simple computation shows the desired inequalities. □

Remark 3.1. In view of the previous lemma we have an equivalent way to estimate the $X^B_{s,b}$-norm: $\|u\|_{X^B_{s,b}} \sim \|\langle |\tau| - \xi^2 \rangle^b \langle \xi \rangle^s \widetilde{u}(\tau, \xi)\|_{L^2_{\tau,\xi}}$. This equivalence will be important in the proof of Theorem 1.1. As we said in the introduction, the Boussinesq symbol $\sqrt{\xi^2 + \xi^4}$ does not have good cancellations to make use of Lemma 3.1. Therefore, we modify the symbols as above and work only with the algebraic relations for the Schrödinger equation.

Now, we are in position to prove the bilinear estimates stated in Theorem 1.1

Proof of Theorem 1.1. (i) For $u \in X^S_{k,b}$ and $v \in X^B_{s,b}$ we define $f(\tau, \xi) \equiv \langle \tau + \xi^2 \rangle^b \langle \xi \rangle^k \widetilde{u}(\tau, \xi)$ and $g(\tau, \xi) \equiv \langle |\tau| - \gamma(\xi) \rangle^b \langle \xi \rangle^s \widetilde{v}(\tau, \xi)$. By duality the desired inequality is equivalent to

$$|W(f, g, \phi)| \leq c\|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|\phi\|_{L^2_{\xi,\tau}}, \tag{3.4}$$

where

$$W(f, g, \phi) = \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^k}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^k} \frac{g(\tau_1, \xi_1) f(\tau_2, \xi_2) \bar{\phi}(\tau, \xi)}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b} d\xi d\tau d\xi_1 d\tau_1,$$

and

$$\xi_2 = \xi - \xi_1, \quad \tau_2 = \tau - \tau_1, \tag{3.5}$$

$\sigma = \tau + \xi^2, \quad \sigma_1 = |\tau_1| - \gamma(\xi_1), \quad \sigma_2 = \tau_2 + \xi_2^2.$

In view of Remark 3.1, we know that $\langle |\tau_1| - \gamma(\xi_1) \rangle \sim \langle |\tau_1| - \xi_1^2 \rangle$. Therefore, splitting the domain of integration into the regions $\{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : \tau_1 < 0\}$ and $\{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : \tau_1 \geq 0\}$, it is sufficient to prove inequality (3.4) with $W_1(f, g, \phi)$ and $W_2(f, g, \phi)$ instead of $W(f, g, \phi)$, where

$$W_1(f, g, \phi) = \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^k}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^k} \frac{g(\tau_1, \xi_1) f(\tau_2, \xi_2) \bar{\phi}(\tau, \xi)}{\langle \sigma \rangle^a \langle \tau_1 + \xi_1^2 \rangle^b \langle \sigma_2 \rangle^b} d\xi d\tau d\xi_1 d\tau_1,$$

and

$$W_2(f,g,\phi)=\int_{\mathbb{R}^4}\frac{\langle\xi\rangle^k}{\langle\xi_1\rangle^s\langle\xi_2\rangle^k}\frac{g(\tau_1,\xi_1)f(\tau_2,\xi_2)\bar{\phi}(\tau,\xi)}{\langle\sigma\rangle^a\langle\tau_1-\xi_1^2\rangle^b\langle\sigma_2\rangle^b}d\xi d\tau d\xi_1 d\tau_1.$$

Let us first treat the inequality (3.4) with $W_1(f,g,\phi)$. In this case, we will make use of the following algebraic relation

$$-(\tau+\xi^2)+(\tau_1+\xi_1^2)+((\tau-\tau_1)+(\xi-\xi_1)^2)=2\xi_1(\xi_1-\xi). \tag{3.6}$$

By symmetry we can restrict ourselves to the set

$$A=\{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4:|(\tau-\tau_1)+(\xi-\xi_1)^2|\leq|\tau_1+\xi_1^2|\}.$$

First, we split A into three pieces

$$\begin{aligned}A_1&=\{(\xi,\tau,\xi_1,\tau_1)\in A:|\xi_1|\leq 10\},\\A_2&=\{(\xi,\tau,\xi_1,\tau_1)\in A:|\xi_1|\geq 10\text{ and }|2\xi_1-\xi|\geq|\xi_1|/2\},\\A_3&=\{(\xi,\tau,\xi_1,\tau_1)\in A:|\xi_1|\geq 10\text{ and }|\xi_1-\xi|\geq|\xi_1|/2\}.\end{aligned}$$

We have $A=A_1\cup A_2\cup A_3$. Indeed

$$|2\xi_1-\xi|+|\xi_1-\xi|\geq|(2\xi_1-\xi)-(\xi_1-\xi)|=|\xi_1|.$$

Next, we divide A_3 into two parts

$$\begin{aligned}A_{3,1}&=\{(\xi,\tau,\xi_1,\tau_1)\in A_3:|\tau_1+\xi_1^2|\leq|\tau+\xi^2|\},\\A_{3,2}&=\{(\xi,\tau,\xi_1,\tau_1)\in A_3:|\tau+\xi^2|\leq|\tau_1+\xi_1^2|\}.\end{aligned}$$

We can now define the sets R_i, $i=1,2$, as follows: $R_1=A_1\cup A_2\cup A_{3,1}$ and $R_2=A_{3,2}$. In what follows χ_R denotes the characteristic function of the set R. In view of the Cauchy-Schwarz and Hölder inequalities, it is easy to see that

$$\begin{aligned}|W_1|^2\leq{}&\|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\left\|\frac{\langle\xi\rangle^{2k}}{\langle\sigma\rangle^{2a}}\iint\frac{\chi_{R_1}d\xi_1 d\tau_1}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2k}\langle\tau_1+\xi_1^2\rangle^{2b}\langle\sigma_2\rangle^{2b}}\right\|_{L^\infty_{\xi,\tau}}\\&+\|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\left\|\frac{1}{\langle\xi_1\rangle^{2s}\langle\tau_1+\xi_1^2\rangle^{2b}}\iint\frac{\chi_{R_2}\langle\xi\rangle^{2k}d\xi d\tau}{\langle\xi_2\rangle^{2k}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\right\|_{L^\infty_{\xi_1,\tau_1}}.\end{aligned}$$

Noting that $\langle\xi\rangle^{2k}\leq\langle\xi_1\rangle^{2|k|}\langle\xi_2\rangle^{2k}$ for $k\geq 0$ and $\langle\xi_2\rangle^{-2k}\leq\langle\xi_1\rangle^{2|k|}\langle\xi\rangle^{-2k}$ for $k<0$, we have

$$\frac{\langle\xi\rangle^{2k}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2k}}\leq\langle\xi_1\rangle^{2|k|-2s}. \tag{3.7}$$

Therefore, in view of Lemma 3.1 it suffices to get bounds for

$$J_1(\xi,\tau)\equiv\frac{1}{\langle\sigma\rangle^{2a}}\int\frac{\langle\xi_1\rangle^{2|k|-2s}d\xi_1}{\langle\tau+\xi^2+2\xi_1^2-2\xi\xi_1\rangle^{2b}}\quad\text{on}\quad R_1,$$

$$J_2(\xi_1,\tau_1) \equiv \frac{\langle \xi_1\rangle^{2|k|-2s}}{\langle \tau_1+\xi_1^2\rangle^{2b}}\int \frac{d\xi}{\langle \tau_1-\xi_1^2+2\xi\xi_1\rangle^{2a}} \quad \text{on} \quad R_2.$$

In region A_1, we have $\langle \xi_1\rangle^{2|k|-2s} \lesssim 1$ and since $a>0$, $b>1/2$ we obtain

$$J_1(\xi,\tau) \lesssim \int_{|\xi_1|\le 10} d\xi_1 \lesssim 1.$$

In region A_2, by the change of variables $\eta = \tau+\xi^2+2\xi_1^2-2\xi\xi_1$ and the condition $|2\xi_1-\xi| \ge |\xi_1|/2$, we have

$$J_1(\xi,\tau) \lesssim \frac{1}{\langle\sigma\rangle^{2a}}\int \frac{\langle \xi_1\rangle^{2|k|-2s}}{|2\xi_1-\xi|\langle \eta\rangle^{2b}}d\eta \lesssim \frac{1}{\langle\sigma\rangle^{2a}}\int \frac{\langle \xi_1\rangle^{2|k|-2s-1}}{\langle \eta\rangle^{2b}}d\eta \lesssim 1,$$

since $a>0$, $|k|-s\le 1/2$ and $b>1/2$.

Now, by the definition of region $A_{3,1}$ and the algebraic relation (3.6) we have $\langle \xi_1\rangle^2 \lesssim |\xi_1|^2 \lesssim |\xi_1(\xi_1-\xi)| \lesssim \langle\sigma\rangle$. Therefore, by Lemma 3.1

$$J_1(\xi,\tau) \lesssim \int \frac{\langle \xi_1\rangle^{2|k|-2s-4a}}{\langle \tau+\xi^2+2\xi_1^2-2\xi\xi_1\rangle^{2b}}d\xi_1 \lesssim \int \frac{1}{\langle \tau+\xi^2+2\xi_1^2-2\xi\xi_1\rangle^{2b}}d\xi_1 \lesssim 1,$$

since $a>0$, $|k|-s\le 2a$ and $b>1/2$.

Next, we estimate $J_2(\xi_1,\tau_1)$. Making the change of variables, $\eta = \tau_1 - \xi_1^2+2\xi\xi_1$, using the restriction in the region $A_{3,2}$, we have

$$|\eta| \lesssim |(\tau-\tau_1)+(\xi-\xi_1)^2| + |\tau+\xi^2| \lesssim \langle \tau_1+\xi_1^2\rangle.$$

Moreover, in $A_{3,2}$

$$|\xi_1|^2 \lesssim |\xi_1(\xi_1-\xi)| \lesssim \langle \tau_1+\xi_1^2\rangle.$$

Therefore, since $|\xi_1|\ge 10$ we have

$$J_2(\xi_1,\tau_1) \lesssim \frac{|\xi_1|^{2|k|-2s}}{\langle \tau_1+\xi_1^2\rangle^{2b}}\int_{|\eta|\lesssim\langle\tau_1+\xi_1^2\rangle} \frac{d\eta}{|\xi_1|\langle\eta\rangle^{2a}} \lesssim \frac{|\xi_1|^{2|k|-2s-1}}{\langle \tau_1+\xi_1^2\rangle^{2b+2a-1}} \lesssim 1,$$

in view of $a>0$, $|k|-s\le 1/2$ and $b>1/2$.

Now, we turn to the proof of inequality (3.4) with $W_2(f,g,\phi)$. In the following estimates, we will make use of the algebraic relation

$$-(\tau+\xi^2)+(\tau_1-\xi_1^2)+((\tau-\tau_1)+(\xi-\xi_1)^2) = -2\xi_1\xi. \qquad (3.8)$$

First, we split $\mathbb{R}^4$ into four sets

$$\begin{aligned}
B_1 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\le 10\},\\
B_2 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\ge 10 \text{ and } |\xi|\le 1\},\\
B_3 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\ge 10, |\xi|\ge 1 \text{ and } |\xi|\ge |\xi_1|/2\},\\
B_4 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\ge 10, |\xi|\ge 1 \text{ and } |\xi|\le |\xi_1|/2\}.
\end{aligned}$$

Next, we separate B_4 into three parts

$$\begin{aligned} B_{4,1} &= \{(\xi,\tau,\xi_1,\tau_1)\in B_4 : |\tau_1-\xi_1^2|, |(\tau-\tau_1)+(\xi-\xi_1)^2| \le |\tau+\xi^2|\},\\ B_{4,2} &= \{(\xi,\tau,\xi_1,\tau_1)\in B_4 : |\tau+\xi^2|, |(\tau-\tau_1)+(\xi-\xi_1)^2| \le |\tau_1-\xi_1^2|\},\\ B_{4,3} &= \{(\xi,\tau,\xi_1,\tau_1)\in B_4 : |\tau_1-\xi_1^2|, |\tau+\xi^2| \le |(\tau-\tau_1)+(\xi-\xi_1)^2|\}. \end{aligned}$$

We can now define the sets S_i, $i=1,2,3$, as follows: $S_1 = B_1\cup B_3\cup B_{4,1}$, $S_2 = B_2\cup B_{4,2}$ and $S_3 = B_{4,3}$. Using the Cauchy-Schwarz and Hölder inequalities and duality it is easy to see that

$$\begin{aligned} |W_2|^2 \le{}& \|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\left\|\frac{\langle\xi\rangle^{2k}}{\langle\sigma\rangle^{2a}}\iint\frac{\chi_{S_1}d\xi_1d\tau_1}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2k}\langle\tau_1-\xi_1^2\rangle^{2b}\langle\sigma_2\rangle^{2b}}\right\|_{L^\infty_{\xi,\tau}}\\ &+\|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\left\|\frac{1}{\langle\xi_1\rangle^{2s}\langle\tau_1-\xi_1^2\rangle^{2b}}\iint\frac{\chi_{S_2}\langle\xi\rangle^{2k}d\xi d\tau}{\langle\xi_2\rangle^{2k}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\right\|_{L^\infty_{\xi_1,\tau_1}}\\ &+\|f\|^2_{L^2_{\xi,\tau}}\|g\|^2_{L^2_{\xi,\tau}}\|\phi\|^2_{L^2_{\xi,\tau}}\left\|\frac{1}{\langle\xi_2\rangle^{2k}\langle\sigma_2\rangle^{2b}}\iint\frac{\chi_{\widetilde{S}_3}\langle\xi_1+\xi_2\rangle^{2k}d\xi_1d\tau_1}{\langle\xi_1\rangle^{2s}\langle\tau_1-\xi_1^2\rangle^{2b}\langle\sigma\rangle^{2a}}\right\|_{L^\infty_{\xi_2,\tau_2}}, \end{aligned}$$

where σ, σ_2, ξ_2, τ_2 are given in (3.5) and

$$\widetilde{S}_3 \subseteq \left\{\begin{array}{r}(\xi_2,\tau_2,\xi_1,\tau_1)\in\mathbb{R}^4 : |\xi_1|\ge 10, |\xi_1+\xi_2|\ge 1, |\xi_1+\xi_2|\le|\xi_1|/2\\ \text{and } |\tau_1-\xi_1^2|, |(\tau_1+\tau_2)+(\xi_1+\xi_2)^2|\le|\tau_2+\xi_2^2|\end{array}\right\}.$$

Noting that $\langle\xi_1+\xi_2\rangle^{2k}\le\langle\xi_1\rangle^{2|k|}\langle\xi_2\rangle^{2k}$ for $k\ge 0$ and $\langle\xi_2\rangle^{-2k}\le\langle\xi_1\rangle^{2|k|}\langle\xi_1+\xi_2\rangle^{-2k}$ for $k<0$, we have

$$\frac{\langle\xi_1+\xi_2\rangle^{2k}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2k}}\le\langle\xi_1\rangle^{2|k|-2s}.$$

Therefore, in view of Lemma 3.1 and (3.7) it suffices to get bounds for

$$\begin{aligned} K_1(\xi,\tau) &\equiv \frac{1}{\langle\sigma\rangle^{2a}}\int\frac{\langle\xi_1\rangle^{2|k|-2s}d\xi_1}{\langle\tau+\xi^2-2\xi\xi_1\rangle^{2b}} \quad\text{on}\quad S_1,\\ K_2(\xi_1,\tau_1) &\equiv \frac{\langle\xi_1\rangle^{2|k|-2s}}{\langle\tau_1-\xi_1^2\rangle^{2b}}\int\frac{d\xi}{\langle\tau_1-\xi_1^2+2\xi\xi_1\rangle^{2a}} \quad\text{on}\quad S_2,\\ K_3(\xi_2,\tau_2) &\equiv \frac{1}{\langle\sigma_2\rangle^{2b}}\int\frac{\langle\xi_1\rangle^{2|k|-2s}d\xi_1}{\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_2\rangle^{2a}} \quad\text{on}\quad \widetilde{S}_3. \end{aligned}$$

In the region B_1 we have $\langle\xi_1\rangle^{2|k|-2s}\lesssim 1$ and since $a>0$, $b>1/2$ we obtain

$$K_1(\xi,\tau)\lesssim\int_{|\xi_1|\le 10}d\xi_1\lesssim 1.$$

In the region B_3, the change of variables $\eta = \tau + \xi^2 - 2\xi\xi_1$ and the condition $|\xi| \geq |\xi_1|/2$ imply

$$K_1(\xi,\tau) \lesssim \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{2|k|-2s}}{|\xi|\langle\eta\rangle^{2b}} d\eta \lesssim \frac{\langle\xi\rangle^{2|k|-2s-1}}{\langle\sigma\rangle^{2a}} \int \frac{1}{\langle\eta\rangle^{2b}} d\eta \lesssim 1,$$

since $a > 0$, $|k| - s \leq 1/2$ and $b > 1/2$.

Now, by definition of the region $B_{4,1}$ and the algebraic relation (3.8) we have $\langle\xi_1\rangle \lesssim |\xi_1| \lesssim |\xi_1\xi| \lesssim \langle\sigma\rangle$. Therefore, by the change of variables $\eta = \tau + \xi^2 - 2\xi\xi_1$ and the condition $|\xi| \geq 1$ we have

$$K_1(\xi,\tau) \lesssim \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{2|k|-2s}}{|\xi|\langle\eta\rangle^{2b}} d\eta \lesssim \frac{\langle\sigma\rangle^{2|k|-2s-2a}}{|\xi|} \int \frac{1}{\langle\eta\rangle^{2b}} d\eta \lesssim 1,$$

since $a > 0$, $|k| - s \leq a$ and $b > 1/2$.

Next, we estimate $K_2(\xi_1, \tau_1)$. Making the change of variables $\eta = \tau_1 - \xi_1^2 + 2\xi\xi_1$ and using the restriction in the region B_2, we have

$$|\eta| \lesssim |\tau_1 - \xi_1^2| + |\xi\xi_1| \lesssim |\tau_1 - \xi_1^2| + |\xi_1|.$$

Therefore,

$$\begin{aligned} K_2(\xi_1,\tau_1) &\lesssim \frac{|\xi_1|^{2|k|-2s}}{\langle\tau_1 - \xi_1^2\rangle^{2b}} \int_{|\eta| \lesssim \langle\tau_1 - \xi_1^2\rangle + |\xi_1|} \frac{d\eta}{|\xi_1|\langle\eta\rangle^{2a}} \\ &\lesssim \frac{|\xi_1|^{2|k|-2s-2a}}{\langle\tau_1 - \xi_1^2\rangle^{2b}} + \frac{|\xi_1|^{2|k|-2s-1}}{\langle\tau_1 - \xi_1^2\rangle^{2b+2a-1}} \lesssim 1, \end{aligned}$$

since $a > 0$, $|k| - s \leq \min\{1/2, a\}$ and $b > 1/2$.

In the region $B_{4,2}$, from the algebraic relation (3.8) we obtain

$$\langle\xi_1\rangle \lesssim |\xi_1| \lesssim |\xi_1\xi| \lesssim \langle\tau_1 - \xi_1^2\rangle.$$

Moreover, making the change of variables $\eta = \tau_1 - \xi_1^2 + 2\xi\xi_1$, using the restriction in the region $B_{4,2}$ and (3.8), we obtain $|\eta| \lesssim \langle\tau_1 - \xi_1^2\rangle$. Therefore,

$$K_2(\xi_1,\tau_1) \lesssim \frac{\langle\xi_1\rangle^{2|k|-2s}}{\langle\tau_1 - \xi_1^2\rangle^{2b}} \int_{|\eta| \lesssim \langle\tau_1 - \xi_1^2\rangle} \frac{d\eta}{|\xi_1|\langle\eta\rangle^{2a}} \lesssim \frac{|\xi_1|^{2|k|-2s-1}}{\langle\tau_1 - \xi_1^2\rangle^{2b+2a-1}} \lesssim 1,$$

since $a > 0$, $|k| - s \leq 1/2$ and $b > 1/2$.

Finally, we estimate $K_3(\xi_2, \tau_2)$. In the region $B_{4,3}$ we have by the algebraic relation (3.8) that $\langle\xi_1\rangle \lesssim |\xi_1| \lesssim |\xi_1(\xi_1 + \xi_2)| \lesssim \langle\sigma_2\rangle$. Therefore, in view of Lemma 3.1 we have

$$K_3(\xi_2,\tau_2) \lesssim \langle\sigma_2\rangle^{2|k|-2s-2b} \int \frac{1}{\langle\tau_2 + \xi_2^2 + 2\xi_1^2 + 2\xi_1\xi_2\rangle^{2a}} d\xi_1 \lesssim 1,$$

since $a > 1/4$, $|k| - s \leq b$ and $b > 1/2$.

(ii) For $u_1 \in X^S_{k,b}$ and $u_2 \in X^S_{k,b}$ we define $f(\tau,\xi) \equiv \langle \tau+\xi^2\rangle^b \langle\xi\rangle^k \widetilde{u}_1(\tau,\xi)$ and $g(\tau,\xi) \equiv \langle \tau+\xi^2\rangle^b \langle\xi\rangle^k \widetilde{u}_2(\tau,\xi)$. By duality the desired inequality is equivalent to

$$|Z(f,g,\phi)| \le c\|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|\phi\|_{L^2_{\xi,\tau}}, \tag{3.9}$$

where

$$Z(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^k\langle\xi_2\rangle^k} \frac{h(\tau_1,\xi_1)f(\tau_2,\xi_2)\bar{\phi}(\tau,\xi)}{\langle\sigma\rangle^a\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b} d\xi d\tau d\xi_1 d\tau_1,$$

and $h(\tau_1,\xi_1) = \bar{g}(-\tau_1,-\xi_1)$, $\xi_2 = \xi-\xi_1$, $\tau_2 = \tau-\tau_1$, $\sigma = |\tau| - \gamma(\xi)$, $\sigma_1 = \tau_1 - \xi_1^2$, $\sigma_2 = \tau_2 + \xi_2^2$. Therefore, applying Lemma 3.2 and splitting the domain of integration according to the sign of τ it is sufficient to prove inequality (3.9) with $Z_1(f,g,\phi)$ and $Z_2(f,g,\phi)$ instead of $Z(f,g,\phi)$, where

$$Z_1(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^k\langle\xi_2\rangle^k} \frac{h(\tau_1,\xi_1)f(\tau_2,\xi_2)\bar{\phi}(\tau,\xi)}{\langle\tau+\xi^2\rangle^a\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b} d\xi d\tau d\xi_1 d\tau_1,$$

and

$$Z_2(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^k\langle\xi_2\rangle^k} \frac{h(\tau_1,\xi_1)f(\tau_2,\xi_2)\bar{\phi}(\tau,\xi)}{\langle\tau-\xi^2\rangle^a\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b} d\xi d\tau d\xi_1 d\tau_1.$$

Remark 3.2. Note that $Z_1(f,g,\phi)$ is not equal to $W_2(f,g,\phi)$ since the powers of the terms $\langle\xi\rangle$ and $\langle\xi_1\rangle$ are different.

First, we treat the inequality (3.9) with $Z_1(f,g,\phi)$. In this case, we will make use of the following algebraic relation

$$-(\tau+\xi^2) + (\tau_1-\xi_1^2) + ((\tau-\tau_1)+(\xi-\xi_1)^2) = -2\xi_1\xi. \tag{3.10}$$

We split $\mathbb{R}^4$ into five pieces

$$\begin{aligned}
A_1 &= \{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : |\xi| \le 10 \text{ and } |\xi_1| \le 100\},\\
A_2 &= \{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : |\xi| \le 10 \text{ and } |\xi_1| \ge 100\},\\
A_3 &= \{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : |\xi| \ge 10 \text{ and } [|\xi_1| \le 1 \text{ or } |\xi_2| \le 1]\},\\
A_4 &= \left\{ \begin{array}{r} (\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : |\xi| \ge 10, |\xi_1| \ge 1, |\xi_2| \ge 1 \\ \text{and } [|\xi_1| \ge 2|\xi_2| \text{ or } |\xi_2| \ge 2|\xi_1|] \end{array} \right\},\\
A_5 &= \left\{ \begin{array}{r} (\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4 : |\xi| \ge 10, |\xi_1| \ge 1, |\xi_2| \ge 1 \\ \text{and } |\xi_1|/2 \le |\xi_2| \le 2|\xi_1| \end{array} \right\}.
\end{aligned}$$

Next, we separate A_5 into three parts

$$\begin{aligned}
A_{5,1} &= \{(\xi,\tau,\xi_1,\tau_1) \in A_5 : |\tau_1-\xi_1^2|, |(\tau-\tau_1)+(\xi-\xi_1)^2| \le |\tau+\xi^2|\},\\
A_{5,2} &= \{(\xi,\tau,\xi_1,\tau_1) \in A_5 : |\tau+\xi^2|, |(\tau-\tau_1)+(\xi-\xi_1)^2| \le |\tau_1-\xi_1^2|\},\\
A_{5,3} &= \{(\xi,\tau,\xi_1,\tau_1) \in A_5 : |\tau_1-\xi_1^2|, |\tau+\xi^2| \le |(\tau-\tau_1)+(\xi-\xi_1)^2|\}.
\end{aligned}$$

Therefore, by the same argument as the one used in the proof of (i), it suffices to get bounds for

$$\begin{aligned} L_1(\xi,\tau) &\equiv \frac{1}{\langle \tau+\xi^2\rangle^{2a}} \int \frac{\langle \xi_1\rangle^{-2k}\langle \xi_2\rangle^{-2k}\langle \xi\rangle^{2s} d\xi_1}{\langle \tau+\xi^2-2\xi\xi_1\rangle^{2b}} \ \text{ on } \ V_1, \\ L_2(\xi_1,\tau_1) &\equiv \frac{1}{\langle \sigma_1\rangle^{2b}} \int \frac{\langle \xi_1\rangle^{-2k}\langle \xi_2\rangle^{-2k}\langle \xi\rangle^{2s} d\xi}{\langle \tau_1-\xi_1^2+2\xi\xi_1\rangle^{2a}} \ \text{ on } \ V_2, \\ L_3(\xi_2,\tau_2) &\equiv \frac{1}{\langle \sigma_2\rangle^{2b}} \int \frac{\langle \xi_1\rangle^{-2k}\langle \xi_2\rangle^{-2k}\langle \xi\rangle^{2s} d\xi_1}{\langle \tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_2\rangle^{2a}} \ \text{ on } \ \widetilde{V}_3, \end{aligned}$$

where $V_1 = A_3 \cup A_4 \cup A_{5,1}$, $V_2 = A_1 \cup A_2 \cup A_{5,2}$ and

$$\widetilde{V}_3 \subseteq \left\{ \begin{array}{r} (\xi_2,\tau_2,\xi_1,\tau_1) \in \mathbb{R}^4 : |\xi_1+\xi_2| \ge 10, |\xi_1| \ge 1, \\ |\xi_2| \ge 1, |\xi_1|/2 \le |\xi_2| \le 2|\xi_1| \\ \text{and } |\tau_1-\xi_1^2|, |(\tau_1+\tau_2)+(\xi_1+\xi_2)^2| \le |\tau_2+\xi_2^2| \end{array} \right\}.$$

First, we estimate $L_1(\xi,\tau)$. In the regions A_3 or A_4, it is easy to see that $\max\{|\xi_1|,|\xi_2|\} \sim |\xi|$, therefore, $\langle \xi_1\rangle^{-k}\langle \xi_2\rangle^{-k}\langle \xi\rangle^{s} \lesssim \langle \xi\rangle^{\gamma(k)}$, where

$$\gamma(k) = \begin{cases} s+2|k|, & \text{if } k \le 0 \\ s-k, & \text{if } k > 0. \end{cases}$$

Remark 3.3. Note that $\xi = N+1$ and $\xi_1 = N$ belong to A_3, for all $N \ge 100$. In all of these cases $|\xi_2| = 1$. Therefore, we cannot expect, in general, that both $|\xi_1|$ and $|\xi_2|$ are equivalent to $|\xi|$. Because of this fact, we define $\gamma(k) = s-k$, for $k > 0$.

Then, making the change of variables $\eta = \tau+\xi^2-2\xi\xi_1$, we have

$$L_1(\xi,\tau) \lesssim \frac{\langle \xi\rangle^{2\gamma(k)}}{\langle \tau+\xi^2\rangle^{2a}} \int \frac{d\eta}{|\xi|\langle \eta\rangle^{2b}} \lesssim 1,$$

since $a > 0$, $b > 1/2$, and $\gamma(k) \le 1/2$, that is, $s-k \le 1/2$, if $k > 0$ and $s+2|k| \le 1/2$, if $k \le 0$. In region A_5 we have

$$\langle \xi_1\rangle^{-k}\langle \xi_2\rangle^{-k}\langle \xi\rangle^{s} \lesssim \langle \xi_1\rangle^{\gamma(s,k)}, \tag{3.11}$$

where

$$\gamma(s,k) = \begin{cases} 0, & \text{if } s \le 0, k > 0 \\ 2|k|, & \text{if } s \le 0, k \le 0 \\ s-2k, & \text{if } s > 0, k > 0 \\ s+2|k|, & \text{if } s > 0, k \le 0. \end{cases}$$

Moreover, the restriction in the region $A_{5,1}$, the condition $|\xi| > 10$ and the algebraic relation (3.10) give us $\langle \xi_1 \rangle \lesssim |\xi_1| \lesssim |\xi_1 \xi| \lesssim \langle \tau + \xi^2 \rangle$. Therefore,

$$L_1(\xi, \tau) \lesssim \int \frac{\langle \xi_1 \rangle^{2\gamma(s,k)-2a} d\eta}{|\xi| \langle \eta \rangle^{2b}} \lesssim \frac{1}{|\xi|} \int \frac{d\eta}{\langle \eta \rangle^{2b}} \lesssim 1,$$

if $a > 0$, $b > 1/2$ and $\gamma(s,k) \leq a$; that is, $2|k| \leq a$, if $s \leq 0$, $k \leq 0$ and $s - 2k \leq a$, if $s > 0$.

Next, we estimate $L_2(\xi_1, \tau_1)$. In the region A_1, we have

$$\langle \xi_1 \rangle^{-2k} \langle \xi_2 \rangle^{-2k} \langle \xi \rangle^{2s} \lesssim 1$$

and since $a, b > 0$, we obtain

$$L_2(\xi_1, \tau_1) \lesssim \int_{|\xi| \leq 10} d\xi \lesssim 1.$$

In the region A_2, we have $|\xi_1| \sim |\xi_2|$, therefore, $\langle \xi_1 \rangle^{-k} \langle \xi_2 \rangle^{-k} \langle \xi \rangle^{2s} \lesssim \langle \xi_1 \rangle^{\theta(k)}$, where

$$\theta(k) = \begin{cases} 0, & \text{if } k > 0 \\ 2|k|, & \text{if } k \leq 0. \end{cases}$$

Making the change of variables, $\eta = \tau_1 - \xi_1^2 + 2\xi\xi_1$, using the restriction in the region A_2, we have $|\eta| \lesssim |\tau_1 - \xi_1^2| + |\xi\xi_1| \lesssim |\tau_1 - \xi_1^2| + |\xi_1|$. Therefore,

$$\begin{aligned} L_2(\xi_1, \tau_1) &\lesssim \frac{\langle \xi_1 \rangle^{2\theta(k)}}{\langle \tau_1 - \xi_1^2 \rangle^{2b}} \int_{|\eta| \lesssim \langle \tau_1 - \xi_1^2 \rangle + |\xi_1|} \frac{d\eta}{|\xi_1| \langle \eta \rangle^{2a}} \\ &\lesssim \frac{|\xi_1|^{2\theta(k)-2a}}{\langle \tau_1 - \xi_1^2 \rangle^{2b}} + \frac{|\xi_1|^{2\theta(k)-1}}{\langle \tau_1 - \xi_1^2 \rangle^{2b+2a-1}} \lesssim 1, \end{aligned}$$

since $a > 0$, $b > 1/2$ and $\theta(k) \leq \min\{1/2, a\}$; that is, $|k| \leq \min\{1/4, a/2\}$, if $k \leq 0$.

Now, we turn to the region $A_{5,2}$. From (3.10) and the condition $|\xi| > 10$ we have $\langle \xi_1 \rangle \lesssim |\xi_1| \lesssim |\xi_1 \xi| \lesssim \langle \tau_1 - \xi_1^2 \rangle$ and $|\eta| \lesssim |\tau_1 - \xi_1^2| + |\xi\xi_1| \lesssim \langle \tau_1 - \xi_1^2 \rangle$. Therefore, making the change of variables, $\eta = \tau_1 - \xi_1^2 + 2\xi\xi_1$, and using (3.11), we obtain

$$L_2(\xi_1, \tau_1) \lesssim \frac{\langle \xi_1 \rangle^{2\gamma(s,k)}}{\langle \tau_1 - \xi_1^2 \rangle^{2b}} \int_{|\eta| \lesssim \langle \tau_1 - \xi_1^2 \rangle} \frac{d\eta}{|\xi_1| \langle \eta \rangle^{2a}} \lesssim \frac{\langle \xi_1 \rangle^{2\gamma(s,k)-1}}{\langle \tau_1 - \xi_1^2 \rangle^{2b+2a-1}} \lesssim 1,$$

since $a > 0$, $b > 1/2$ and $\gamma(s,k) \leq 1/2$.

Finally, we bound $L_3(\xi_2, \tau_2)$. Again, we use (3.10), so in the region $A_{5,3}$ we have $\langle \xi_1 \rangle \lesssim \langle \sigma_2 \rangle$. From Lemma 3.1 it follows that

$$L_3(\xi_2, \tau_2) \lesssim \langle \sigma_2 \rangle^{2\gamma(s,k)-2b} \int \frac{1}{\langle \tau_2 + \xi_2^2 + 2\xi_1^2 + 2\xi_1 \xi_2 \rangle^{2a}} d\xi_1 \lesssim 1,$$

since $a > 1/4$, $b > 1/2$ and $\gamma(s,k) \le b$.

Now, we turn to the proof of inequality (3.9) with $Z_2(f,g,\phi)$. First, we make the change of variables $\tau_2 = \tau - \tau_1$, $\xi_2 = \xi - \xi_1$ to obtain

$$\begin{aligned} Z_2(f,g,\phi) &= \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^s}{\langle \xi - \xi_2 \rangle^k \langle \xi_2 \rangle^k} \\ &\quad \times \frac{h(\tau - \tau_2, \xi - \xi_2) f(\tau_2, \xi_2) \bar{\phi}(\tau,\xi)}{\langle \tau - \xi^2 \rangle^a \langle (\tau - \tau_2) - (\xi - \xi_2)^2 \rangle^b \langle \tau_2 + \xi_2^2 \rangle^b} d\xi d\tau d\xi_2 d\tau_2, \end{aligned}$$

then changing the variables $(\xi, \tau, \xi_2, \tau_2) \mapsto -(\xi, \tau, \xi_2, \tau_2)$ we can rewrite $Z_2(f,g,\phi)$ as

$$\begin{aligned} Z_2(f,g,\phi) &= \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^s}{\langle \xi - \xi_2 \rangle^k \langle \xi_2 \rangle^k} \\ &\quad \times \frac{k(\tau - \tau_2, \xi - \xi_2) l(\tau_2, \xi_2) \bar{\psi}(\tau,\xi)}{\langle \tau + \xi^2 \rangle^a \langle \tau - \tau_2 + (\xi - \xi_2)^2 \rangle^b \langle \tau_2 - \xi_2^2 \rangle^b} d\xi d\tau d\xi_2 d\tau_2, \end{aligned}$$

where $k(a,b) = h(-a,-b)$, $l(a,b) = f(-a,-b)$ and $\psi(a,b) = \phi(-a,-b)$. But this is exactly $Z_1(f,g,\phi)$ with ξ_1, h, f, ϕ replaced respectively by ξ_2, l, k, ψ. Since the L^2-norm is preserved under the reflection operation the result follows from the estimate for $Z_1(f,g,\phi)$. □

Now we turn to the proof of the bilinear estimates with $b < 1/2$ and $s = 0$.

Proof of Theorem 1.3. (i) For $u \in X^S_{0,b_1}$ and $v \in X^B_{0,b}$ we define $f(\tau,\xi) \equiv \langle \tau + \xi^2 \rangle^{b_1} \widetilde{u}(\tau,\xi)$ and $g(\tau,\xi) \equiv \langle |\tau| - \gamma(\xi) \rangle^b \widetilde{v}(\tau,\xi)$. By duality the desired inequality is equivalent to

$$|R(f,g,\phi)| \le c \|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|\phi\|_{L^2_{\xi,\tau}}, \tag{3.12}$$

where

$$R(f,g,\phi) = \int_{\mathbb{R}^4} \frac{g(\tau_1,\xi_1) f(\tau_2,\xi_2) \bar{\phi}(\tau,\xi)}{\langle \sigma \rangle^{a_1} \langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^{b_1}} d\xi d\tau d\xi_1 d\tau_1,$$

and

$$\xi_2 = \xi - \xi_1, \quad \tau_2 = \tau - \tau_1, \tag{3.13}$$

$\sigma = \tau + \xi^2$, $\sigma_1 = |\tau_1| - \gamma(\xi_1)$, $\sigma_2 = \tau_2 + \xi_2^2$.

Without loss of generality we can suppose that f, g, ϕ are real valued and non-negative. Therefore, by Lemma 3.2 we have

$$\begin{aligned} R(f,g,\phi) \le &\int_{\mathbb{R}^4} \frac{g(\tau_1,\xi_1) f(\tau_2,\xi_2) \bar{\phi}(\tau,\xi)}{\langle \sigma \rangle^{a_1} \langle \tau_1 + \xi_1^2 \rangle^b \langle \sigma_2 \rangle^{b_1}} d\xi d\tau d\xi_1 d\tau_1 \\ &+ \int_{\mathbb{R}^4} \frac{g(\tau_1,\xi_1) f(\tau_2,\xi_2) \bar{\phi}(\tau,\xi)}{\langle \sigma \rangle^{a_1} \langle \tau_1 - \xi_1^2 \rangle^b \langle \sigma_2 \rangle^{b_1}} d\xi d\tau d\xi_1 d\tau_1 \equiv R_+ + R_-. \end{aligned}$$

Applying Plancherel's identity and Hölder's inequality we obtain

$$\begin{aligned} R_{\pm} &= \int_{\mathbb{R}^2} \Big(\frac{g(\tau,\xi)}{\langle \tau \pm \xi^2\rangle^{b}}\Big)^{\sim^{-1}} \Big(\frac{f(\tau,\xi)}{\langle \tau + \xi^2\rangle^{b_1}}\Big)^{\sim^{-1}} \Big(\frac{\bar{\phi}(\tau,\xi)}{\langle \tau + \xi^2\rangle^{a_1}}\Big)^{\sim^{-1}} d\xi d\tau \\ &\leq \Big\| \Big(\frac{g(\tau,\xi)}{\langle \tau \pm \xi^2\rangle^{b}}\Big)^{\sim^{-1}} \Big\|_{L^3_{x,t}} \Big\| \Big(\frac{f(\tau,\xi)}{\langle \tau + \xi^2\rangle^{b_1}}\Big)^{\sim^{-1}} \Big\|_{L^3_{x,t}} \Big\| \Big(\frac{\bar{\phi}(\tau,\xi)}{\langle \tau + \xi^2\rangle^{a_1}}\Big)^{\sim^{-1}} \Big\|_{L^3_{x,t}}. \end{aligned}$$

Now, the fact that $a_1, b, b_1 > 1/4$ together with Lemma 2.4 yields the result.

(ii) For $u_1 \in X^S_{0,b_1}$ and $u_2 \in X^S_{0,b_1}$ we define $f(\tau,\xi) \equiv \langle \tau + \xi^2\rangle^{b_1} \widetilde{u}_1(\tau,\xi)$ and $g(\tau,\xi) \equiv \langle \tau + \xi^2\rangle^{b_1} \widetilde{u}_2(\tau,\xi)$. By duality the desired inequality is equivalent to

$$|S(f,g,\phi)| \leq c\|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|\phi\|_{L^2_{\xi,\tau}}, \tag{3.14}$$

where

$$S(f,g,\phi) = \int_{\mathbb{R}^4} \frac{\bar{g}(\tau_2,\xi_2) f(\tau_1,\xi_1) \bar{\phi}(\tau,\xi)}{\langle\sigma\rangle^{a} \langle\sigma_1\rangle^{b_1} \langle\sigma_2\rangle^{b_1}} d\xi d\tau d\xi_1 d\tau_1,$$

and $\xi_2 = \xi_1 - \xi$, $\tau_2 = \tau_1 - \tau$, $\sigma = |\tau| - \gamma(\xi)$, $\sigma_1 = \tau_1 + \xi_1^2$, $\sigma_2 = \tau_2 + \xi_2^2$. We note that the estimate above is similar to that in item (i). □

4. LOCAL WELL POSEDNESS

Proof of Theorem 1.2. The proof proceeds by a standard contraction principle method applied to the integral equations associated to the IVP (1.1). Given $(u_0, v_0, v_1) \in H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ and $0 < T \leq 1$ we define the integral operators

$$\Gamma^S_T(u,v)(t) = \theta(t)U(t)u_0 - i\theta_T(t)\int_0^t U(t-t')(vu)(t')dt' \tag{4.1}$$

$$\Gamma^B_T(u,v)(t) = \theta(t)\left(V_c(t)v_0 + V_s(t)(v_1)_x\right) + \theta_T(t)\int_0^t V_s(t-t')(|u|^2)_{xx}(t')dt'.$$

Our goal is to use the Picard fixed point theorem to find a solution of

$$\begin{aligned} \Gamma^S_T(u,v) &= u, \\ \Gamma^B_T(u,v) &= v. \end{aligned}$$

Let k, s satisfy the conditions $(i)-(ii)$ of Theorem 1.2. It is easy to see that we can find $\varepsilon > 0$ small enough such that for $b = 1/2 + \varepsilon$ and $a = 1/2 - 2\varepsilon$, Theorem 1.1 holds. Therefore using Lemmas 2.1-2.2, Theorem 1.1 and $0 < T \leq 1$, we have

$$\|\Gamma^S_T(u,v)\|_{X^S_{k,b}} \leq c\|u_0\|_{H^k} + cT^{\varepsilon}\|uv\|_{X^S_{k,-a}} \leq c\|u_0\|_{H^k} + cT^{\varepsilon}\|u\|_{X^S_{k,b}}\|v\|_{X^B_{s,b}},$$

$$\|\Gamma^B_T(u,v)\|_{X^B_{s,b}} \leq c\|v_0,v_1\|_{\mathfrak{B}^s} + cT^{\varepsilon}\|u\bar{u}\|_{X^B_{s,-a}} \leq c\|v_0,v_1\|_{\mathfrak{B}^s} + cT^{\varepsilon}\|u\|^2_{X^S_{k,b}}.$$

Similarly,

$$\|\Gamma_T^S(u,v)-\Gamma_T^S(z,w)\|_{X^S_{k,b}} \le cT^\varepsilon\left(\|u\|_{X^S_{k,b}}\|v-w\|_{X^B_{s,b}}+\|u-z\|_{X^S_{k,b}}\|w\|_{X^B_{s,b}}\right),$$

$$\|\Gamma_T^B(u,v)-\Gamma_T^B(z,w)\|_{X^B_{s,b}} \le c\,T^\varepsilon\left(\|u\|_{X^S_{k,b}}+\|z\|_{X^S_{k,b}}\right)\|u-z\|_{X^S_{k,b}}.$$

We define

$$X^S_{k,b}(d_S)=\left\{u\in X^S_{k,b}:\|u\|_{X^S_{k,b}}\le d_S\right\},\quad X^B_{s,b}(d_B)=\left\{v\in X^B_{s,b}:\|v\|_{X^B_{s,b}}\le d_B\right\},$$

where $d_S=2c\|u_0\|_{H^k}$ and $d_B=2c\|v_0,v_1\|_{\mathfrak{B}^s}$.

Consider $X^S_{k,b}(d_S)\times X^B_{s,b}(d_B)$ endowed with the sum norm. Then, choosing

$$0<T^\varepsilon\le\frac{1}{4}\min\left\{\frac{1}{cd_B},\frac{d_B}{cd_S^2},\frac{1}{c(d_S+d_B)},\frac{1}{2cd_S}\right\},\tag{4.2}$$

we have that $(\Gamma_T^S,\Gamma_T^B):X^S_{k,b}(d_S)\times X^B_{s,b}(d_B)\to X^S_{k,b}(d_S)\times X^B_{s,b}(d_B)$ is a contraction mapping and we obtain a unique fixed point which solves the integral equation (4.1) for any T that satisfies (4.2).

Remark 4.1. Note that the choice of suitable values of a,b is essential for our argument. In fact, since $1-(a+b)=\varepsilon>0$, the factor T^ε can be used directly to obtain a contraction factor for T sufficient small.

Moreover, by Lemma 2.3, we have that $\tilde{u}=u|_{[0,T]}\in C([0,T]:H^s)\cap X^{S,T}_{k,b}$ and $\tilde{v}=v|_{[0,T]}\in C([0,T]:H^s)\cap X^{B,T}_{s,b}$ is a solution of (2.5) in $[0,T]$.

Using an argument due to Bekiranov, Ogawa and Ponce [2] one can prove that the solution (u,v) of (2.5) obtained above is unique in the whole space $X^{S,T}_{k,b}\times X^{B,T}_{s,b}$. Finally, we remark that since we established the existence of a solution by a contraction argument, the proof that the map $(u_0,v_0,v_1)\mapsto(u(t),v(t))$ is locally Lipschitz follows easily. □

5. Global Well posedness

Proof of Theorem 1.4. Let $(u_0,v_0,v_1)\in L^2(\mathbb{R})\times L^2(\mathbb{R})\times H^{-1}(\mathbb{R})$ and $0<T\le 1$. Based on the integral formulation (2.9), we define the integral operators

$$G_T^S(u,v)(t)=\theta(t)U_T(t)u_0-i\theta_T(t)\int_0^t U(t-t')(vu)(t')dt'\tag{5.1}$$

$$G_T^B(u,v)(t)=\theta_T(t)\left(V_c(t)v_0+V_s(t)(v_1)_x\right)+\theta_T(t)\int_0^t V_s(t-t')(|u|^2)_{xx}(t')dt'.$$

Therefore, applying Lemmas 2.1-2.2 and Theorem 1.3, we obtain

$$\|G_T^S(u,v)\|_{X^S_{0,b_1}}\le cT^{1/2-b_1}\|u_0\|_{L^2}+cT^{1-(a_1+b_1)}\|uv\|_{X^S_{0,-a_1}}\tag{5.2}$$

$$\leq cT^{1/2-b_1}\|u_0\|_{L^2} + cT^{1-(a_1+b_1)}\|u\|_{X^S_{0,b_1}}\|v\|_{X^B_{0,b}},$$

$$\|G^B_T(u,v)\|_{X^B_{0,b}} \leq cT^{1/2-b}\|v_0,v_1\|_{\mathfrak{B}} + cT^{1-(a+b)}\|u\bar{u}\|_{X^B_{0,-a}}$$

$$\leq cT^{1/2-b}\|v_0,v_1\|_{\mathfrak{B}} + cT^{1-(a+b)}\|u\|^2_{X^S_{0,b_1}},$$

and also

$$\|G^S_T(u,v) - G^S_T(z,w)\|_{X^S_{0,b_1}} \tag{5.3}$$

$$\leq cT^{1-(a_1+b_1)}\left(\|u\|_{X^S_{0,b_1}}\|v-w\|_{X^B_{0,b}} + \|u-z\|_{X^S_{0,b_1}}\|w\|_{X^B_{0,b}}\right),$$

$$\|G^B_T(u,v) - G^B_T(z,w)\|_{X^B_{0,b}} \leq cT^{1-(a+b)}\left(\|u\|_{X^S_{0,b_1}} + \|z\|_{X^S_{0,b_1}}\right)\|u-z\|_{X^S_{0,b_1}}.$$

We define

$$X^S_{0,b_1}(d_1) = \left\{u \in X^S_{0,b_1} : \|u\|_{X^S_{0,b_1}} \leq d_1\right\}, \quad X^B_{0,b}(d) = \left\{v \in X^B_{0,b} : \|v\|_{X^B_{0,b}} \leq d\right\},$$

where $d_1 = 2cT^{1/2-b_1}\|u_0\|_{L^2}$ and $d = 2cT^{1/2-b}\|v_0,v_1\|_{\mathfrak{B}}$.

For (G^S_T, G^B_T) to be a contraction in $X^S_{0,b_1}(d_1) \times X^B_{0,b}(d)$ it needs to satisfy

$$d_1/2 + cT^{1-(a_1+b_1)}d_1 d \leq d_1 \Leftrightarrow T^{3/2-(a_1+b_1+b)}\|v_0,v_1\|_{\mathfrak{B}} \lesssim 1, \tag{5.4}$$

$$d/2 + cT^{1-(a+b)}d_1^2 \leq d \Leftrightarrow T^{3/2-(a+2b_1)}\|u_0\|^2_{L^2} \lesssim \|v_0,v_1\|_{\mathfrak{B}}, \tag{5.5}$$

$$2cT^{1-(a+b)}d_1 \leq 1/2 \Leftrightarrow T^{3/2-(a+b+b_1)}\|u_0\|_{L^2} \lesssim 1, \tag{5.6}$$

$$2cT^{1-(a_1+b_1)}d_1 \leq 1/2 \Leftrightarrow T^{3/2-(a_1+2b_1)}\|u_0\|_{L^2} \lesssim 1. \tag{5.7}$$

Therefore, we conclude that there exists a solution $(u,v) \in X^S_{0,b_1} \times X^B_{0,b}$ satisfying

$$\|u\|_{X^S_{0,b_1}} \leq 2cT^{1/2-b_1}\|u_0\|_{L^2} \quad \text{and} \quad \|v\|_{X^B_{0,b}} \leq 2cT^{1/2-b}\|v_0,v_1\|_{\mathfrak{B}}. \tag{5.8}$$

On the other hand, applying Lemmas 2.1-2.2 we have that, in fact, $(u,v) \in C([0,T] : L^2) \times C([0,T] : L^2)$. Moreover, since the L^2-norm of u is conserved by the flow we have $\|u(T)\|_{L^2} = \|u_0\|_{L^2}$.

Now, we need to control the growth of $\|v(t)\|_{\mathfrak{B}}$ in each time step. If, for all $t > 0$, $\|v(t)\|_{\mathfrak{B}} \lesssim \|u_0\|^2_{L^2}$ we can repeat the local well posedness argument and extend the solution globally in time. Thus, without loss of generality, we suppose that after some number of iterations we reach a time $t^* > 0$ where $\|v(t^*)\|_{\mathfrak{B}} \gg \|u_0\|^2_{L^2}$.

Hence, since $0 < T \leq 1$, condition (5.5) is automatically satisfied and conditions (5.4)-(5.7) imply that we can select a time increment of size

$$T \sim \|v(t^*)\|_{\mathfrak{B}}^{-1/(3/2-(a_1+b_1+b))}. \tag{5.9}$$

Therefore, applying Lemmas 2.1(*b*)-2.2(*b*) to $v = G_T^B(u,v)$ we have

$$\|v(t^* + T)\|_{\mathfrak{B}} \le \|v(t^*)\|_{\mathfrak{B}} + cT^{3/2-(a+2b_1)}\|u_0\|_{L^2}^2.$$

Thus, we can carry out m iterations on time intervals, each of length (5.9), before the quantity $\|v(t)\|_{\mathfrak{B}}$ doubles, where m is given by

$$mT^{3/2-(a+2b_1)}\|u_0\|_{L^2}^2 \sim \|v(t^*)\|_{\mathfrak{B}}.$$

The total time of existence we obtain after these m iterations is

$$\Delta T = mT \sim \frac{\|v(t^*)\|_{\mathfrak{B}}}{T^{1/2-(a+2b_1)}\|u_0\|_{L^2}^2} \sim \frac{\|v(t^*)\|_{\mathfrak{B}}}{\|v(t^*)\|_{\mathfrak{B}}^{-(1/2-(a+2b_1))/(3/2-(a_1+b_1+b))}\|u_0\|_{L^2}^2}.$$

Taking a, b, a_1, b_1 such that $\frac{a+2b_1-1/2}{(3/2-(a_1+b_1+b))} = 1$ (for instance, $a = b = a_1 = b_1 = 1/3$), we have that ΔT depends only on $\|u_0\|_{L^2}$, which is conserved by the flow. Hence, we can repeat this entire argument and extend the solution (u, v) globally in time.

Moreover, since in each step of time ΔT the size of $\|v(t)\|_{\mathfrak{B}}$ will at most double it is easy to see that, for all $\widetilde{T} > 0$

$$\|v(\widetilde{T})\|_{\mathfrak{B}} \lesssim \exp\left((\ln 2)\|u_0\|_{L^2}^2 \widetilde{T}\right) \max\{\|v_0, v_1\|_{\mathfrak{B}}, \|u_0\|_{L^2}\}. \tag{5.10}$$

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