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GREEN'S FUNCTION OF NON-LINEAR DEGENERATE ELLIPTIC OPERATORS AND ITS APPLICATION TO REGULARITY

Zhaosheng Feng
Department of Mathematics, University of Texas-Pan American
Edinburg, TX 78539

Shenzhou Zheng and Hanfang Lu
Department of Mathematics, Beijing Jiaotong University
Beijing 100044, China

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Abstract. We are concerned with the Green's function associated with the non-linear degenerate elliptic operator of divergence form. We present several theorems on local estimates and comparisons with the p-Laplacian for the Green's function under certain conditions in the sense of distributions. As an application of these estimates of the Green's function, the regularity in Morrey spaces of the so-called inhomogeneous A-harmonic equation is derived.

1. Introduction

The main purpose of this paper is to investigate the Green's function associated with the non-linear degenerate elliptic operator $\mathcal{A}$ of divergence form:

$$\mathcal{A}u := -divA(x, \nabla u), \tag{1.1}$$

in an open subset Ω of $\mathbf{R}^n$ for $n \geq 3$. Assume that $A(x, \nabla u) : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ is a Carathédory's vector-valued function; that is, $A(x, \xi)$ is measurable in x for all $\xi \in \mathbf{R}^n$ and is continuous in ξ for almost every x. For some positive constants α, β and $1 < p < n$ it satisfies three conditions:

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(H1) the degenerate ellipticity

$$\langle A(x,\,\xi),\,\xi\rangle \geq \alpha|\xi|^p; \tag{1.2}$$

(H2) the growth estimate inequality

$$|A(x,\,\xi)| \leq \beta|\xi|^{p-1}; \tag{1.3}$$

(H3) the homogeneity condition

$$A(x,\,\lambda\xi) = \lambda|\lambda|^{p-2}A(x,\,\xi); \tag{1.4}$$

for almost every $x \in \Omega$, all $\xi \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$.

As we know that the operator $A(x,\,\xi)$ arises in many non-linear phenomena and a great number of theoretical issues concerning $A(x,\,\xi)$ have received considerable attention, for instance, in the theory of quasi-regular mappings and in mathematical modeling of non-Newtonian fluids for a discussion of the potential theory [1-6]. In this paper, we study the Green's function associated with non-linear degenerate elliptic operators of divergence form (1.1). In order to present our research results in a straightforward manner, here we introduce two definitions of the Green's function of the so-called A-harmonic operator $\mathcal{A}u = \delta(y)$ for any fixed point $y \in \Omega$ in the weak form:

Definition 1. A function $G(\cdot\,,\,\cdot) : \Omega \times \Omega \mapsto \mathbf{R} \cup \{\infty\}$ is called the Green's function of the A-harmonic operator (1), if, for each fixed point $y \in \Omega$ and any $R > 0$, $G(\cdot\,,\,y) \in W^{1,\,p}(\Omega \setminus B(y,\,R)) \cap W_0^{1,\,p-1}(\Omega)$ which satisfies

$$\mathcal{A}[G,\,\phi] := \int_\Omega \langle A(x,\,\nabla G(x,\,y)),\,\nabla\phi(x)\rangle\,dx = \phi(y), \tag{1.5}$$

for any test function $\phi(x) \in C_0^\infty(\Omega)$ [7-12].

We also consider an approximation of the Green's function for the degenerate elliptic operator $A(x,\,\xi)$. For the convenience of notation, in the following context we denote $B(y,\,R)$ by $B_R(y)$, a ball in $\mathbf{R}^n$ centered at y with radius R.

Definition 2. A function $G^h(\cdot\,,\,\cdot)$ (also denoted by $G_y^h(x)$): $\Omega \times \Omega \mapsto \mathbf{R} \cup \{\infty\}$ is called the modified Green's function of the A-harmonic operator (1.1), if, for each $y \in \Omega$ and any $h > 0$ small enough, $G^h(\cdot\,,\,y) \in W_0^{1,\,p}(\Omega)$ and

$$\mathcal{A}[G_y^h,\,\phi] := \int_\Omega \langle A(x,\,\nabla G_y^h(x)),\,\nabla\phi(x)\rangle\,dx = \int_\Omega \delta^h(y-x)\phi(x)\,dx, \tag{1.6}$$

for any $\phi(x) \in C_0^\infty(\Omega)$, where $\delta^h(x) = \frac{\chi_{B_h}}{|B_h|}$ and χ_{B_h} is a characteristic function on $B_h(y)$ and $|B_h|$ denotes the volume of a ball centered at $y \in \Omega$ with radius h in $\mathbf{R}^n$ [8, 9, 11].

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, and $\phi \mapsto \fint_{B_h} \phi(x)\, dx$ is a bounded linear functional on $W_0^{1,p}(\Omega)$ for any small $h > 0$ and the test function $\phi \in W_0^{1,p}(\Omega)$, it follows from (1.6) that

$$\mathcal{A}[G_y^h, \phi] := \int_\Omega \langle A(x, \nabla G_y^h), \nabla\phi(x)\rangle\, dx = \fint_{B_h(y)} \phi(x)\, dx. \qquad (1.7)$$

Before starting to present our results, let us make some bibliographical remarks. In the qualitative theory of classical partial differential equations, the Green's function is ultimately an important subject in many problems such as the existence of solutions, the representation formula and the Wiener criterion of regular boundary points for the classical Dirichlet problem of Laplacian operators [13]. The existence and uniqueness of weak solutions for A-harmonic equations with general measures such as (1.7) was discussed in [4]. If $\delta(y)$ is considered as a weak-star version of $\delta^r(y)$, the existence and uniqueness of the Green's function $G(\cdot\,, y)$ in the distribution sense is valid. Littman and his co-workers [7] presented the comparison of the Green's functions with Laplacian operators for uniformly elliptic operators of symmetrical, bounded and measurable coefficients defined in Ω, and established the same corresponding Wiener criterion of regular boundary points as the Laplacian. Furthermore, Grüter and Widman [8] studied the same problem of the Green's functions and regular points on the boundary for more general uniformly elliptic operators with bounded measurable coefficients. By way of non-linear potential theory and estimates of the Green's functions, Kilpeläinen et al. [1-4, 14-20] derived the regular test at a boundary point of solutions of quasilinear elliptic equations and degenerate elliptic equations.

Recently, Cancelier and Xu [12] studied the Green's function associated with the operator $Lu = \sum X_j^*(a_{ij} X_i u))$ under the hypotheses that the vector fields a_{ij} are C^∞ and satisfy the Hömander condition of rank $r \geq 2$. As a result, even in this general case the regular points on the boundary are the same as those for the Laplacian or subharmonic operators. It is shown that the multiple of the power for a cutting-off function and the Green's function can be presented as a test function in the sense of distributions of elliptic problems with divergence form [9, 21-22]. Labutin [21] derived a Serrin-type characterization [23-24] of isolated singularities for solutions of nonlinear elliptic equations $F(\nabla^2 u) = 0$ by means of special fundamental solutions. On the other hand, Chanillo and Wheeden [10] considered the existence and estimates of the Green's function for degenerate elliptic equations, and Mazzoni [11] obtained local estimates and a representation formula for X-elliptic operators with non-regular coefficients. In these pioneering works, the Green's function is closely related to the Wiener criterion for regular boundary points of domains [1-4, 9, 14-20], stochastic game problems [9], and the regularity of the weak solution under the weaker regular data [4, 9, 20, 25]. In fact, the Green's function and fundamental solutions of various differential operators have played a critical role in many scientific fields, especially in some convex problems and elliptic problems. More elegant results and applications can be seen in [9, 19, 22, 25-32] and references therein. In this paper, the concept of the Green's function for the nonlinear degenerate elliptic operator $\mathcal{A}$ is a straightforward generalization of the Laplacian operator in the nonlinear potential theory. We will show that the Green's function of the A-harmonic operator enjoys the fundamental comparison properties with the p-Laplacian operator in many aspects. Finally, as an application of our consequences, we will establish the regularity in Morrey spaces for the so-called inhomogeneous A-harmonic equations.

The rest of this paper is organized as follows. In the next section, we give some basic properties such as a maximum principle and Harnack inequality for the homogeneous A-harmonic equation. In Section 3, we present several theorems concerning local estimates and comparisons of the modified Green's function in the sense of distributions. In Section 4, we prove a theorem with regard to the estimate of the Green's function of the A-harmonic operator defined by (1.5) under structural conditions (H1)-(H3) (i.e., (1.2)-(1.4)). In Section 5, as an application, we show that the Green's function can be applied to the study of the regularity in Morrey spaces for non-linear elliptic problems. We conclude with Section 6 as a brief conclusion.

2. Maximum Principle and Harnack Inequality

To make the paper sufficiently self-contained, we introduce several preliminary properties for the homogeneous A-harmonic equation in this section.

Consider the non-linear degenerate elliptic equation

$$-divA(x, \nabla u) = 0,$$

with the Dirichlet boundary values under conditions (H1)-(H3). The existence and uniqueness of the solution can be proved by using the theory of monotone operators or by the variational integral argument [2].

Proposition 1. *Suppose that Ω is a bounded domain and $v \in W^{1,p}(\Omega)$. Then there is a unique solution $u \in W^{1,p}(\Omega)$ satisfying the equation $\mathcal{A}u = 0$ under conditions* (H1)-(H3) *in Ω with $u - v \in W_0^{1,p}(\Omega)$.*

The proof of the maximum principle under structural conditions (H1)-(H3) can be seen in the literature [2-3, etc.]:

Proposition 2. *If u is any weak solution to the equation $\mathcal{A}u = 0$ in Ω with the boundary value v under conditions* (H1)-(H3), *then*

$$ess \inf_{x\in\Omega} v \le u \le ess \sup_{x\in\Omega} v, \quad \text{for almost every } x \in \Omega.$$

Under conditions (1.2)-(1.4), Proposition 1 guarantees that the equation $\mathcal{A}u = 0$ has a unique solution $u \in W^{1,p}(\Omega)$. Proposition 2 will yield some useful inequalities for the weak solution to the equation $\mathcal{A}u = 0$.

Using the same argument as the Moser-Nash iteration technique and extending the results in the space of bounded mean oscillatory functions introduced in [2, 13], one can prove a theorem of local upper bounds and Harnack's inequality to $\mathcal{A}u = 0$. We list those results below and will use them in proofs of our main results.

Proposition 3 (Local upper bound and weak Harnack's inequality)**.** *If $u \in W^{1,p}(\Omega)$ is a non-negative solution of equation $\mathcal{A}u = 0$ with conditions* (H1)-(H3), *then there is a positive constant $C = C(n, p, \alpha, \beta, q, s)$, such that for any $0 < s < \frac{n(p-1)}{n-p}$, we have*

$$ess \sup_{x\in\frac{1}{2}B_r} u \le C\Big(\fint_{B_r} u^q\, dx\Big)^{\frac{1}{q}}, \quad \forall q > 0, \tag{2.1}$$

and

$$\Big(\fint_{B_r} u^s\, dx\Big)^{\frac{1}{s}} \le C\, ess \inf_{B_r} u,$$

where $B_r = B(y, r)$ is a ball in $\mathbf{R}^n$ centered at $y \in \Omega$ with radius r and $2B_r \subset \Omega$.

Proposition 4 (Harnack's inequality)**.** *Let $u \in W^{1,p}(\Omega)$ be a weak solution of equation $\mathcal{A}[u, \phi] = 0$ with conditions* (H1)-(H3). *Then there exists a positive constant $C = C(n, p, \alpha, \beta)$ such that*

$$\sup_{x\in B_r(y)} u \le C \inf_{x\in B_r(y)} u,$$

for any fixed $y \in \Omega$ and small $r > 0$ such that $B_{2r}(y) \subset \Omega$.

From Harnack's inequality [2, 9, 13], it is easy to see the following.

Corollary 1 (Hölder continuity). *For any fixed $y \in \Omega$ and small $r > 0$ such that $B_{2r}(y) \subset \Omega$, suppose that $u \in W^{1,p}(\Omega)$ is a weak solution of equation $\mathcal{A}[u, \phi] = 0$ with conditions* (H1)-(H3). *Then there exists some index $\kappa = \kappa(n, p, \alpha, \beta)$ with $0 < \kappa < 1$ such that $u \in C^{0,\kappa}_{loc}(\Omega)$.*

3. Estimates of the Modified Green's Function

In this section, we will present some local estimates for the modified Green's function in weak-L^p space. First, let us recall the definition of the weak-L^p space [13, 33]: for $f \in L^p(\Omega)$ and $t > 0$, let $A_t(f) := \{x \in \Omega : |f(x)| > t\}$ and $\lambda_f(t) := |A_t(f)|$. The weak-L^p space is denoted by $L^p_w(\Omega)$, which satisfies

$$\|f\|_{L^p_w(\Omega)} := \inf\{A : \lambda_f(t) \le t^{-p}A^p,\ \forall t > 0\} < \infty,$$

for all $p \ge 1$. In particular, for any $1 \le q < p$, the following holds

$$\|f\|_{L^p_w(\Omega)} \le \|f\|_{L^p(\Omega)}, \tag{3.1}$$

$$\|f\|_{L^q(\Omega)} \le \left(\frac{p}{p-q}\right)^{\frac{1}{q}} |\Omega|^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p_w(\Omega)}. \tag{3.2}$$

Lemma 1. *For any fixed point $y \in \Omega$ and sufficiently small $h > 0$, suppose that the modified Green's function $G^h(x, y)$ is a weak solution of equation* (1.7) *with condition* (H1) *in the sense of distributions. Then $G^h(x, y) \ge 0$ for almost every x in Ω.*

Proof. Given any fixed point $y \in \Omega$ and small $h > 0$ such that $\bar{B}_h(y) \subset \Omega$, we choose $G^h_y \doteq G^h(\cdot, y)$ and $|G^h_y| \in W^{1,p}_0(\Omega)$ as test functions, respectively; then there holds

$$0 \le \mathcal{A}[G^h_y,\ G^h_y] = \fint_{B_h(y)} G^h(x, y)\, dx \le \fint_{B_h(y)} |G^h(x, y)|\, dx = \mathcal{A}[G^h_y,\ |G^h_y|],$$

which yields

$$\mathcal{A}\big[G^h_y,\ G^h_y - |G^h_y|\big] \le 0. \tag{3.3}$$

Setting $\Omega^+ = \{x \in \Omega : G^h_y \ge 0\}$ and $\Omega^- = \{x \in \Omega : G^h_y < 0\}$ and using inequality (3.3), we get

$$\begin{aligned} 0 &\ge \int_\Omega \langle A(x, \nabla G^h_y), \nabla(G^h_y - |G^h_y|)\rangle\, dx, \\ &= 2\int_{\Omega^-} \langle A(x, \nabla G^h_y), \nabla G^h_y\rangle\, dx \ge 2\alpha \int_{\Omega^-} |\nabla G^h_y|^p\, dx. \end{aligned}$$

Thus, we have $|\Omega^-| = 0$; that is, $G^h(x, y) \geq 0$, for almost every $x \in \Omega$. This completes the proof of Lemma 1. $\square$

Theorem 1. *For* $1 < p < n$, *suppose that* $G^h(x, y)$ *is a weak solution of equation* (1.7) *under assumption* (H1) *in the sense of distributions for small* $h > 0$ *and any fixed point* $y \in \Omega$. *Then the modified Green's function* $G^h_y(x)$ *has the following properties:*

(i) $G^h(\cdot\,, y) \in L^\gamma(\Omega)$ *for* $1 \leq \gamma < \frac{(p-1)n}{n-p}$, *with*

$$\|G^h(\cdot\,, y)\|_{L^\gamma(\Omega\setminus B_{h/2}(y))} < C(n,\, p,\, \alpha,\, \gamma);$$

(ii) $G^h(\cdot\,, y) \in L^\kappa_w(\Omega)$ *for* $\kappa = \frac{(p-1)n}{n-p}$, *with*

$$\|G^h(\cdot\,, y)\|_{L^\kappa_w(\Omega\setminus B_{h/2}(y))} < C(n,\, p,\, \alpha).$$

Proof. (i). Take $\phi(x) = \frac{G^h_y}{(1+(G^h_y)^s)^{\frac{1}{s}}} \in W^{1,\,p}_0(\Omega)$ as a test function to equation (1.7). Since $\frac{G^h_y}{[1+(G^h_y)^s]^{\frac{1}{s}}} \leq 1$, for any $0 < s < p-1$, we get

$$\begin{aligned} &\int_\Omega \Big\langle A(x,\, \nabla G^h_y),\ \nabla\Big(\frac{G^h_y}{[1+(G^h_y)^s]^{\frac{1}{s}}}\Big)\Big\rangle\, dx \\ = \ &\int_\Omega \delta(x)\cdot \frac{G^h_y}{[1+(G^h_y)^s]^{\frac{1}{s}}}\, dx \leq \int_\Omega \delta(x)\, dx = 1. \end{aligned}$$

That is,

$$\int_\Omega \Big\langle A(x,\, \nabla G^h_y),\ \frac{\nabla G^h_y}{[1+(G^h_y)^s]^{\frac{1+s}{s}}}\Big\rangle\, dx \leq 1.$$

Due to the fact that $\frac{1}{1+G^h_y} \leq \frac{2^{\frac{(1-s)}{s}}}{[1+(G^h_y)^s]^{\frac{1}{s}}}$ for any $0 < s < p-1$, by virtue of assumption (H1) we have

$$\int_\Omega \frac{|\nabla G^h_y|^p}{(1+G^h_y)^{1+s}}\, dx \leq \frac{2^{\frac{1-s^2}{s}}}{\alpha} \int_\Omega \frac{\langle A(x,\ \nabla G^h_y),\ \nabla G^h_y\rangle}{[1+(G^h_y)^s]^{\frac{1+s}{s}}}\, dx \leq \frac{1}{\alpha} 2^{\frac{1-s^2}{s}}.$$

Hence, we get

$$\int_\Omega \Big|\nabla(1+G^h_y)^{\frac{p-1-s}{p}}\Big|^p\, dx \leq \frac{1}{\alpha} 2^{\frac{1-s^2}{s}} \Big|\frac{p-1-s}{p}\Big|^p.$$

It follows that

$$\|(1+G^h_y)^{\frac{p-1-s}{p}} - 1\|_{W^{1,\,p}_0(\Omega)} \leq C,$$

where $C = C(p, s, \alpha)$. Using the Sobolev embedding theorem yields

$$\|(1+G_y^h)^{\frac{p-1-s}{p}}\|_{L^{\frac{np}{n-p}}(\Omega)} \le C(n, p, s, \alpha).$$

This implies that $(1+G_y^h) \in L^{\frac{n(p-1-s)}{n-p}}(\Omega)$ for any $0 < s < p-1$. Thus, we complete the proof of part (i).

(ii). For any $t > 0$, we set $A_t := \{x \in \Omega : G_y^h > t\}$ and consider the function

$$f(s) = [\frac{1}{t^{p-1}} - \frac{1}{s^{p-1}}]^+ := \max\left\{0, \frac{1}{t^{p-1}} - \frac{1}{s^{p-1}}\right\} \in Lip(\mathbf{R}).$$

Since $G_y^h(x) \in W_0^{1,p}(\Omega)$, using the property of the Sobolev functions [13], we have that $f[G_y^h(x)] \in W_0^{1,p}(\Omega)$ and $f[G_y^h(x)]^+ \in W_0^{1,p}(\Omega)$. Choosing

$$\phi(x) := f[G_y^h(x)]^+ = [\frac{1}{t^{p-1}} - \frac{1}{(G_y^h)^{p-1}}]^+,$$

as a test function and utilizing the fact that $f(s) < \frac{1}{t^{p-1}}$ for any $s \in \mathbf{R}$, we get

$$\mathcal{A}[G_y^h, \phi] = \int_\Omega \langle A(x, \nabla G_y^h), \nabla\phi\rangle\, dx = ⨍_\Omega \phi(x)\, dx \le ⨍_\Omega \frac{1}{t^{p-1}}\, dx = \frac{1}{t^{p-1}}.$$

On the other hand, using condition (H1) again we obtain

$$\begin{aligned} \mathcal{A}[G_y^h, \phi] &= \int_\Omega \langle A(x, \nabla G_y^h), \nabla\phi\rangle\, dx \ge \int_{A_t} \langle A(x, \nabla G_y^h), (p-1)\frac{\nabla G_y^h}{(G_y^h)^p}\rangle\, dx, \\ &= (p-1)\int_{A_t} \frac{1}{(G_y^h)^p}\langle A(x, \nabla G_y^h), \nabla G_y^h\rangle\, dx \qquad (3.4) \\ &\ge \alpha(p-1)\int_{A_t} |\nabla G_y^h|^p (G_y^h)^{-p}\, dx = \alpha(p-1)\int_{A_t} |\nabla \log G_y^h|^p\, dx. \end{aligned}$$

From the Sobolev imbedding inequality it follows that

$$\begin{aligned} \Big(\int_{A_t} \Big|\log\frac{G_y^h}{t}\Big|^{\frac{np}{n-p}} dx\Big)^{\frac{n-p}{n}} &= \Big(\int_{A_t} \Big|\log G_y^h - \log t\Big|^{\frac{np}{n-p}} dx\Big)^{\frac{n-p}{n}} \qquad (3.5) \\ &\le C\int_{A_t} \Big|\nabla \log G_y^h\Big|^p dx \le \frac{C}{\alpha(p-1)t^{p-1}}. \end{aligned}$$

Now, let us consider the domain A_{2t}. Clearly we have $A_{2t} \subset A_t$. Moreover, from (3.4) and (3.5), if $\frac{G_y^h}{t} > 2$ on A_{2t}, then we have

$$(\log 2)^{\frac{np}{n-p}} < (\log G_y^h/t)^{\frac{np}{n-p}},$$

on $x \in A_{2t}$, and

$$(\log 2)^p |A_{2t}|^{\frac{n-p}{n}} = \Big(\int_{A_{2t}} (\log 2)^{\frac{np}{n-p}} \, dx \Big)^{\frac{n-p}{n}} \le \Big(\int_{A_{2t}} (\log \frac{G_y^h}{t})^{\frac{np}{n-p}} \, dx \Big)^{\frac{n-p}{n}}$$
$$\le \Big(\int_{A_t} (\log \frac{G_y^h}{t})^{\frac{np}{n-p}} \, dx \Big)^{\frac{n-p}{n}} \le \frac{C}{\alpha(p-1)t^{p-1}}. \tag{3.6}$$

Re-writing (3.6) gives

$$2t^{p-1} |A_{2t}|^{\frac{n-p}{n}} \le C(n,\, p,\, \alpha).$$

That is,

$$G_y^h(x) \in L_w^{\kappa}(\Omega), \quad \kappa = \frac{(p-1)n}{n-p}.$$

Thus, the proof of part (ii) is completed. □

Theorem 2. *For any fixed point $y \in \Omega$ and small $h > 0$, suppose that $G_y^h \in W_0^{1,p}(\Omega)$ is a solution of equation* (1.7) *under assumptions* (H1), *then we have*

$$G^h(x,\, y) \le C(n,\, p,\, \alpha) |x - y|^{-\frac{n-p}{p-1}}, \tag{3.7}$$

for any $x,\, y \in \Omega$.

Proof. According to inequality (2.1) in Proposition 3, for any $x \in \Omega \backslash B_h(y)$ and $0 < s < \frac{n(p-1)}{n-p}$, we have

$$G_y^h(x) \le \sup_{x \in \Omega \backslash B_h(y)} G_y^h(x) \le C \Big(\fint_{\Omega \backslash B_{h/2}(y)} G^h(\zeta, y)^s \, d\zeta \Big)^{\frac{1}{s}}.$$

In view of formula (3.1) and (3.2), for any $0 < s < \frac{n(p-1)}{n-p}$, we have

$$\Big(\fint_{\Omega \backslash B_{h/2}(y)} G^h(\zeta, y)^s \, d\zeta \Big)^{\frac{1}{s}}$$
$$\le \quad \Big(\frac{\frac{n(p-1)}{n-p}}{\frac{n(p-1)}{n-p} - s} \Big) |\Omega \backslash B_{h/2}(y)|^{-\frac{n-p}{n(p-1)}} \cdot \|G^h(x,y)\|_{L_w^{\frac{n(p-1)}{n-p}}(\Omega \backslash B_{h/2}(y))}.$$

For any $x \in \Omega \backslash B_h(y)$ it follows that

$$G^h(x,y) \le C(n,p) |\Omega \backslash B_{h/2}(y)|^{-\frac{n-p}{n(p-1)}} \|G^h(x,y)\|_{L_w^{\frac{n(p-1)}{n-p}}(\Omega \backslash B_{h/2}(y))}.$$

Making use of part (ii) in Theorem 1, we obtain

$$G^h(x,\, y) \le C(n,\, p,\, \alpha) |x - y|^{-\frac{n-p}{p-1}}.$$

Thus, the proof of Theorem 2 is completed. □

Theorem 3. *Suppose that* $G_y^h \in W_0^{1,p}(\Omega)$ *is a weak solution of equation* (1.7) *under assumptions* (H1)-(H3) *for any fixed* $y \in \Omega$ *and small* $h > 0$. *Then there holds*

$$G^h(x, y) \geq C_1(n, p, \alpha, \beta)|x-y|^{-\frac{n-p}{p-1}}, \tag{3.8}$$

for any $x, y \in \Omega$ *such that* $|x-y| \leq \frac{1}{2}dist(y, \partial\Omega)$.

Proof. For any fixed point $y \in \Omega$, let

$$R := |x-y| < \frac{1}{5}dist(y, \partial\Omega),$$

B_R represent $B(y, R)$ and ω_n be the volume of a unit ball in $\mathbf{R}^n$, then we take a test function $\phi(x) \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ on B_{2R}, $\phi \equiv 0$ on $\Omega \setminus B_{3R}$ as well as $|\nabla\phi| \leq \frac{K}{R}$, where K is a positive constant. From (1.7) and Hölder's inequality it follows that

$$\begin{aligned} 1 &= \int_{B_{3R}\setminus B_{2R}} \langle A(x, \nabla G_y^h), \nabla\phi\rangle\, dx \leq \int_{B_{3R}\setminus B_{2R}} |A(x, \nabla G_y^h)| \cdot |\nabla\phi|\, dx \\ &\leq \frac{K\beta}{R}\int_{B_{3R}\setminus B_{2R}} |\nabla G_y^h|^{p-1}\, dx \\ &\leq [(3^n - 2^n)w_n]^{\frac{1}{p}}\, \frac{K\beta}{R} R^{\frac{n}{p}}\Big(\int_{B_{3R}\setminus B_{2R}} |\nabla G_y^h|^p\, dx\Big)^{1-\frac{1}{p}} \\ &= C(n,p,\beta)R^{\frac{n-p}{p}}\Big(\int_{B_{3R}\setminus B_{2R}} |\nabla G_y^h|^p\, dx\Big)^{1-\frac{1}{p}}. \end{aligned} \tag{3.9}$$

On the other hand, let us choose a cut-off function $\eta \in C_0^\infty(B_{4R} \setminus B_R)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{3R} \setminus B_{2R}$ and $|\nabla\eta| \leq \frac{K_1}{R}$. Putting a test function $\phi(x) = G_y^h(x)\eta^p(x)$ into equation (1.7), we obtain

$$\begin{aligned} &\alpha\int_{B_{4R}\setminus B_R} \eta^p|\nabla G_y^h|^p\, dx \leq \int_{B_{4R}\setminus B_R} \langle A(x, \nabla G_y^h), \eta^p\nabla G_y^h\rangle\, dx \\ &\leq \int_{B_{4R}\setminus B_R} |\langle A(x, \nabla G_y^h), p\eta^{p-1}\nabla\eta G_y^h|dx \leq p\beta\int_{B_{4R}\setminus B_R} |\eta\nabla G_y^h|^{p-1} \cdot |G_y^h\nabla\eta|dx \\ &\leq p\beta\Big(\int_{B_{4R}\setminus B_R} |\eta\nabla G_y^h|^p\, dx\Big)^{1-\frac{1}{p}} \cdot \Big(\int_{B_{4R}\setminus B_R} |\nabla\eta G_y^h|^p\, dx\Big)^{\frac{1}{p}}. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10) we have

$$\int_{B_{3R}\setminus B_{2R}} |\nabla G_y^h|^p\,dx \le \int_{B_{4R}\setminus B_R} \eta^p |\nabla G_y^h|^p\,dx \le \Big(\frac{Kp\beta}{\alpha R}\Big)^p \int_{B_{4R}\setminus B_R} [G_y^h]^p\,dx$$
$$\le C(n,p,\alpha,\beta) R^{n-p}\Big[\sup_{R\le|x-y|\le 4R}\Big(G_y^h(x)\Big)^p\Big]. \qquad (3.11)$$

By substituting (3.11) into inequality (3.9) and using Harnack's inequality, we get

$$\begin{aligned} 1 &\le CR^{\frac{n}{p}-1}\Big[R^{n-p}\sup_{R\le|x-y|\le 4R}(G_y^h(x))^p\Big]^{\frac{p-1}{p}} \\ &\le CR^{\frac{n-p}{p}+\frac{(n-p)(p-1)}{p}}\Big(\sup_{R\le|x-y|\le 4R} G_y^h(x)\Big)^{p-1} \\ &\le CR^{n-p}\Big(\inf_{R\le|x-y|\le 4R} G_y^h(x)\Big)^{p-1} \le C|x-y|^{n-p}(G_y^h(x))^{p-1}. \end{aligned}$$

This implies that formula (3.8) is true. Therefore, we have completed the proof of Theorem 3. □

Theorem 4. *For any given fixed point $y \in \Omega$ and small $h > 0$, suppose that $G_y^h \in W_0^{1,p}(\Omega)$ is a weak solution of equation* (1.7) *under assumptions* (H1)-(H3). *Then we have*

$$\int_{\Omega\setminus B_R} |\nabla G^h(x,\, y)|^p\,dx \le C(n,\, p,\, \alpha,\, \beta) R^{\frac{p-n}{p-1}}, \qquad (3.12)$$

for any $B_R \subset \Omega$.

Proof. We divide our arguments into two cases based on the various ranges of radius R.

Case 1. If $R < h$, we use G_y^h as a test function in

$$\mathcal{A}[G^h, \phi] := \fint_{B_h} \phi(x)\,dx,$$

and have

$$\alpha\int_\Omega |\nabla G^h(x,y)|^p\,dx \le \int_\Omega \Big\langle A\Big(x, \nabla G^h(x,y)\Big),\, \nabla G^h(x,y)\Big\rangle\,dx$$
$$= \fint_{B_h(y)} G^h(x,y)\,dx. \qquad (3.13)$$

By Hölder's inequality and Sobolev's inequality it follows that

$$\fint_{B_h(y)} G^h(x,\, y)\, dx \le \frac{1}{h^n}\Big(\int_{B_h(y)} |G^h_y|^{\frac{np}{n-p}}\, dx\Big)^{\frac{n-p}{np}} \cdot h^{n(1-\frac{n-p}{np})}$$
$$\le C(n,\, p) h^{\frac{p-n}{p}}\Big(\int_\Omega |\nabla G^h_y|^p\, dx\Big)^{\frac{1}{p}}. \tag{3.14}$$

By (3.13) and (3.14) we have

$$\int_\Omega |\nabla G^h(x,\, y)|^p\, dx \le C(n,\, p,\, \alpha) h^{\frac{p-n}{p-1}}.$$

Since $R < h$, we have

$$\int_{\Omega\setminus B_R} |\nabla G^h(x,\, y)|^p\, dx \le C(n,\, p,\, \alpha) R^{\frac{p-n}{p-1}}.$$

Case 2. If $R \ge h$, we choose a cut-off function $\eta = 1$ in $\Omega \setminus B_R$, $\eta = 0$ in $B_{\frac{R}{2}}$, as well as $|\nabla\eta| \le \frac{K}{R}$ for some positive constant K. Taking $\phi(x) = G^h\eta^p$ as a test function in formula (1.7), we have

$$\alpha\int_{\Omega\setminus B_R} \eta^p |\nabla G^h(x,\, y)|^p\, dx \le \int_{\Omega\setminus B_R} \Big\langle A(x,\, \nabla G^h_y(x)),\; \eta^p G^h_y(x)\Big\rangle\, dx$$
$$\le \int_{\Omega\setminus B_R} \Big|\Big\langle A\Big(x,\, \nabla G^h_y(x)\Big),\; p\eta^{p-1}\nabla\eta G^h_y(x)\Big\rangle\Big|\, dx$$
$$\le p\beta\int_{\Omega\setminus B_R} |\eta\nabla G^h_y(x)|^{p-1}\cdot|G^h_y(x)\nabla\eta|\, dx$$
$$\le p\beta\Big(\int_{\Omega\setminus B_R} |\eta\nabla G^h_y(x)|^p\, dx\Big)^{1-\frac{1}{p}}\cdot\Big(\int_{\Omega\setminus B_R} |\nabla\eta G^h_y(x)|^p\, dx\Big)^{\frac{1}{p}}.$$

So we have

$$\int_{\Omega\setminus B_R} |\nabla G^h_y(x)|^p\, dx \le \int_{\Omega\setminus B_R} \eta^p |\nabla G^h_y(x)|^p\, dx \le \Big(\frac{Kp\beta}{\alpha R}\Big)^p \int_{\Omega\setminus B_R} [G^h_y(x)]^p\, dx.$$

Utilizing inequality (3.7) in Theorem 2, we have

$$\int_{\Omega\setminus B_R} |\nabla G^h(x,y)|^p\, dx \le C(n,\, p,\, \alpha,\, \beta) R^{n-p}(R^{\frac{p-n}{p-1}})^p = C(n,\, p,\, \alpha,\, \beta) R^{\frac{p-n}{p-1}}.$$

This is exactly our formula (3.12). Hence, combining cases 1 with 2, we finish the proof of Theorem 4. □

Theorem 5. *For $1 < p < n$, suppose that $G^h(x\,,y)$ is any weak solution of equation* (1.7) *under assumptions* (H1)-(H3). *Then*

(i) $\nabla G^h(\cdot\,,y) \in L_w^{\frac{p\kappa}{\kappa+1}}(\Omega)$ *for* $\kappa = \frac{(p-1)n}{n-p}$, *with a constant* $C_1 > 0$ *independent of* h *such that*

$$\|\nabla G_y^h\|_{L_w^{\frac{p\kappa}{\kappa+1}}(\Omega)} \leq C_1(n,\, p,\, \alpha,\, \beta);$$

and

(ii) $G^h(\cdot\,,y) \in W_0^{1,\,\nu}(\Omega)$ *for* $1 \leq \nu < \frac{(p-1)n}{n-1}$, *with a constant* $C_2 > 0$ *independent of* h *such that*

$$\|\nabla G_y^h\|_{W^{1,\,\nu}(\Omega)} \leq C_2(n,\, p,\, \alpha,\, \beta).$$

Proof. (i). Let $t > 0,\ R = t^{-\frac{p-1}{n-1}}$ and $A_t = \{x \in \Omega : |\nabla G_y^h| > t\}$. It is easy to see that $A_t \subset \Omega$, so we can write

$$|A_t| = |A_t \cap (\Omega \setminus B_R) \cup (A_t \cap B_R)| = |A_t \cap (\Omega \setminus B_R)| + |(A_t \cap B_R)|. \quad (3.15)$$

Consider the first term on the right-hand side. Using Theorem 4, we have

$$\begin{aligned} t^p\,|A_t \cap (\Omega \setminus B_R)| &= \int_{A_t \cap (\Omega\setminus B_R)} t^p\,dx < \int_{A_t \cap (\Omega\setminus B_R)} |\nabla G_y^h|^p\,dx \\ &\leq \int_{(\Omega\setminus B_R)} |\nabla G_y^h|^p\,dx \leq C(n,p,\alpha,\beta) R^{\frac{p-n}{p-1}} = C t^{\frac{n-p}{n-1}}, \end{aligned}$$

which implies

$$|A_t \cap (\Omega \setminus B_R)| \leq C t^{\frac{n-p}{n-1}-p} = C t^{-\frac{(p-1)n}{n-1}}. \quad (3.16)$$

Similarly, by considering $|B_R| = \omega_n R^n$ for all $R > 0$, the same form of estimate is also valid for the second term in (3.15); i.e.,

$$|(A_t \cap B_R)| \leq |B_R| = \omega_n R^n \leq C_0 t^{-\frac{(p-1)n}{n-1}}. \quad (3.17)$$

Combining the above two inequalities (3.16) and (3.17), we get $|A_t| \leq C t^{-\frac{(p-1)n}{n-1}}$, from which we have

$$\|\nabla G_y^h\|_{L_w^{\frac{p\kappa}{\kappa+1}}(\Omega)} = \inf_{t>0} |A_t| t^{\frac{(p-1)n}{n-1}} \leq C_1.$$

(ii). Let $y \in \Omega$ and $h > 0$. According to part (i) of Theorem 1 there exists $C_3 > 0$ independent of h such that$\||G_y^h|\|_{L_w^\kappa(\Omega)} \leq C_3$. Then, using formula (3.2) we have

$$\|G_y^h\|_{L^\nu(\Omega)} \leq \left(\frac{\kappa}{\kappa-\nu}\right)^{\frac{1}{\nu}} |\Omega|^{\frac{1}{\nu}-\frac{1}{\kappa}} \|G_y^h\|_{L_w^\kappa(\Omega)} \leq \left(\frac{\kappa}{\kappa-\nu}\right)^{\frac{1}{\nu}} |\Omega|^{\frac{1}{\nu}-\frac{1}{\kappa}} C_3 = C_4,$$

where $\nu \in [1,\, \kappa)$. Analogously, provided that h is sufficiently small, on the basis of part (i) of this theorem there exists a constant $C_5 > 0$ independent of h, such that $\|\nabla G_y^h\|_{L_w^{\frac{p\kappa}{\kappa+1}}(\Omega)} \leq C_5$. Using (3.2) again and assuming that $q = \frac{p\kappa}{\kappa+1}$ gives

$$\||\nabla G_y^h|\|_{L^\nu(\Omega)} \leq \left(\frac{q}{q-\nu}\right)^{\frac{1}{\nu}} |\Omega|^{\frac{1}{\nu}-\frac{1}{q}} \|\nabla G_y^h\|_{L_w^q(\Omega)} \leq \left(\frac{q}{q-\nu}\right)^{\frac{1}{\nu}} |\Omega|^{\frac{1}{\nu}-\frac{1}{q}} C_5 = C_2,$$

where $\nu \in [1, \frac{p\kappa}{\kappa+1})$. Thus, we can conclude that, when h is sufficiently small, we have $G^h(\cdot\,, y) \in W^{1,\nu}(\Omega)$ with $\nu \in [1, \frac{(p-1)n}{n-1})$. Due to the fact that $G_y^h \in W_0^{1,p}(\Omega)$ and $C_0^1(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, and using Hölder's inequality, we derive that $G_y^h \in W_0^{1,\nu}(\Omega)$ with $\nu \in [1, \frac{(p-1)n}{n-1})$. This implies that part (ii) is true too. □

4. Comparison of the Green's functions

As we know from [22] that if $E_p(x)$ is a special fundamental singular solution of the p-Laplace equation

$$\triangle_p = div(|\nabla u|^{p-2}\nabla u) = 0,$$

then $E_p(x) = \frac{1}{|x|^{(n-p)/(p-1)}}$ for $1 < p < n$. To the Green's function of the A-harmonic operator defined by (1.5), based on the theorems established in the preceding section we can derive the following result:

Theorem 6. *Let $n \geq 3$ and $1 < p < n$. For any fixed point $y \in \Omega$ and any $R > 0$, suppose that $G(\cdot\,, y) \in W^{1,p}(\Omega \setminus B(y, R)) \cap W_0^{p-1,1}(\Omega)$ is the Green's function of the A-harmonic operator defined by* (1.5) *in the sense of distributions. Then we have the following.*

(i) *For any $B_R(y) \subset \Omega$, there exists a positive constant $C_1 = C_1(n, p, \alpha, \beta)$ such that*

$$\int_{\Omega\setminus B_R} |\nabla G(x, y)|^p \, dx \leq C_1 \, R^{\frac{p-n}{p-1}}.$$

(ii) *For arbitrary $x, y \in \Omega$ and $x \neq y$, if $|x - y| \leq \frac{1}{2}dist(y, \partial\Omega)$ there exists a positive constant $C_2 = C_2(n, p, \alpha, \beta)$ such that*

$$G(x, y) \geq C_2 \, |x - y|^{\frac{p-n}{p-1}}.$$

(iii) *For arbitrary $x, y \in \Omega$, there exists a positive constant $C_3 = C_3(n, p, \alpha, \beta)$ such that*

$$G(x, y) \leq C_3 \, |x - y|^{\frac{p-n}{p-1}}. \tag{4.1}$$

Proof. Firstly, let us consider a sequence of weak solutions $\{G_y^{h_i}\}$ of

$$\mathcal{A}u = \fint_{B_{h_i}(y)} \phi(x)\, dx,$$

as $h_i \to 0$ (i.e., $i \to \infty$). Since $h_i \to 0$, we have

$$\fint_{B_{h_i}(y)} \phi(x)\, dx \to \phi(y).$$

In view of Theorem 4, we obtain that $\|G_y^{h_i}\|_{W_0^{1,p}(\Omega\setminus B_R)}$ is uniformly bounded with respect to h_i for any $R > 0$, so we have $G_y^{h_i} \rightharpoonup G$ in $W_0^{1,p}(\Omega \setminus y)$. Furthermore, we know that the L^p-norms are weakly lower semi-continuous, thus

$$\begin{aligned}\int_{\Omega\setminus B_R(y)} |\nabla G(x,\, y)|^p\, dx &\le \liminf_{i\to\infty} \int_{\Omega\setminus B_R(y)} |\nabla G^{h_i}(x,\, y)|^p\, dx \\ &\le \liminf_{i\to\infty} C_1(n,p,\alpha,\beta) R^{-\frac{n-p}{p-1}} = C_1(n,p,\alpha,\beta) R^{\frac{p-n}{p-1}}.\end{aligned}$$

Secondly, with the help of Rellich's compact embedding theorem, we know that $G^{h_i} \to G$ strongly in L^p. By Lebesgue's convergence theorem, it follows that $G^{h_i}(\cdot, y) \to G(\cdot\,, y)$ for almost all $x \in \Omega$. Corollary 1 in Section 2 tells us that $G(\cdot\,, y)$ is Hölder continuous in $\Omega \setminus \{y\}$. Using Theorems 2 and 3, for any $x, y \in \Omega$, we obtain

$$G(x,\, y) \le C_3(n,\, p,\, \alpha,\, \beta)|x - y|^{\frac{p-n}{p-1}},$$

and, for any $x,\, y \in \Omega$ such that $|x - y| \le \frac{1}{2} dist(y,\, \partial\Omega)$, we obtain

$$G(x, y) \ge C_2(n,\, p,\, \alpha,\, \beta)|x - y|^{\frac{p-n}{p-1}}.$$

Therefore, the proof of Theorem 6 is completed. □

From Theorem 6, we can derive a corollary immediately as follows:

Corollary 2. *Let $G(x,y)$ and $G'(x,y)$ be two Green's functions corresponding to operators $\mathcal{A}$ and $\mathcal{A}'$ which both satisfy conditions* (1.2) *and* (1.3) *with constants α, β and α', β', respectively. Then there exist constants $C_1(n,\alpha,\beta,\alpha',\beta')$ and $C_2(n,\alpha,\beta,\alpha',\beta')$, for any $x, y \in \Omega$ when $|x-y| \le \frac{1}{2} dist(y,\partial\Omega)$ such that*

$$C_1(n,\, \alpha,\, \beta,\, \alpha',\, \beta')G'(x,\, y) \le G(x,\, y) \le C_2(n,\, \alpha,\, \beta,\, \alpha',\, \beta')G'(x,\, y).$$

5. Application to nonlinear problems

The Green's function has many significant applications in quite a few scientific areas. In this section, we show that the Green's function can be applied to the study of regularity in Morrey spaces for non-linear elliptic problems acting as a kernel function.

Consider the non-linear degenerate elliptic equation of divergence form

$$-divA(x,\nabla u) = g(x) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x). \tag{5.1}$$

Assume that $u \in L^p(\Omega)$ with $p \geq 1$ and $\lambda \geq 0$. Following tradition [9, 25], we use the definition of Morrey spaces:

$$L^{p,\lambda}(\Omega) = \Big\{ u \in L^p(\Omega) : \sup_{\substack{x_0 \in \Omega \\ 0<\rho\leq d}} \int_{\Omega(x_0,\rho)} \frac{|u|^p}{|x-x_0|^\lambda}\, dx < +\infty \Big\},$$

with a norm defined by

$$\|u\|_{L^{p,\lambda}(\Omega)} = \sup_{\substack{x_0 \in \Omega \\ 0<\rho\leq d}} \Big\{ \int_{\Omega(x_0,\rho)} \frac{|u|^p}{|x-x_0|^\lambda}\, dx \Big\}^{\frac{1}{p}},$$

where $\Omega(x_0,\, \rho) = B(x_0, \rho) \cap \Omega$ for $x_0 \in \Omega$, $B(x_0,\, \rho)$ denotes a ball in $\mathbf{R}^n$ of radius ρ centered at x_0, and d is the diameter of Ω.

Assume that $G = G(x,\, x_0)$ is the Green's function related to any fixed point $x_0 \in \Omega$, which satisfies

$$\int_\Omega \langle A(x,\, \nabla G),\, \nabla\phi(x)\rangle\, dx = \phi(x_0), \quad \forall \phi(x) \in W_0^{1,\,p}(\Omega).$$

Let $u \in W^{1,\,p}(\Omega)$. We rewrite equation (5.1) as a variational problem based on the sense of distributions:

$$\int_\Omega \langle A(x,\nabla u), \nabla\phi(x)\rangle\, dx = \int_\Omega g(x)\phi(x)dx + \int_\Omega \langle f(x), \nabla\phi(x)\rangle\, dx, \tag{5.2}$$

for any $\phi(x) \in W_0^{1,\,p}(\Omega)$, where the operator $A(x,\, \xi)$ satisfies structural assumptions (H1)-(H3). Assume that $g(x) \in L^q(\Omega)$ and $f(x) \in [L^s(\Omega, \mathbf{R})]^n$ with

$$q > \frac{n}{n - 2(\frac{n}{p} - 1)}, \quad s > \frac{p}{n - \frac{2n}{p} + 1}.$$

To prove our result, we need to use the hole-filling technique [27], the theorems presented in the last two sections and the following lemma:

Lemma 2. [13] *Let ω be a non-decreasing function on the interval* $(0, R_0]$ *for all $0 < R \le R_0$, and satisfy the inequality*

$$\omega(\tau R) \le \theta\omega(R) + KR^{\alpha},$$

where $0 < \theta,\ \tau < 1$. Then for $0 < \delta < \alpha$ we have

$$\omega(R) \le C\Big(\frac{R}{R_0}\Big)^{\delta}\Big(\omega(R_0) + KR_0^{\alpha}\Big),$$

where both $C = C(\tau, \theta)$ and $\delta = \delta(\tau, \theta, \alpha)$ are positive constants.

Theorem 7. *Let $u \in W^{1,p}_{loc}(\Omega)$ for $2 - \frac{2}{n+1} < p < n$ be a weak solution of a degenerate elliptic equation of divergence form* (5.2)*. Suppose that the A-harmonic operator $A(x, \nabla u)$ satisfies structural assumptions* (H1)-(H3), *and $g(x) \in L^q(\Omega)$, $f(x) \in [L^s(\Omega, \mathbf{R})]^n$ when q and s satisfy*

$$q > \frac{n}{n - 2(\frac{n}{p} - 1)}, \qquad s > \frac{p}{n - \frac{2n}{p} + 1}.$$

Then we have

$$\nabla u \in L^{p,\lambda}_{loc}(\Omega),$$

for every $0 < \lambda \le \frac{n-p}{p-1} + \vartheta$ with some positive constant ϑ which only depends on n, p, α, β, q and s.

Proof. Given a point $x_0 \in \Omega$ and a positive constant $0 < R_0 \le \frac{1}{4} dist(x_0, \partial\Omega)$, let $\eta(x)$ be a smooth cut-off function such that $\eta(x) = 1$ when $x \in B_R(x_0)$, $\eta = 0$, when $x \in \mathbf{R}^n \setminus B_{2R}(x_0)$, and $0 \le \eta \le 1$ when $|\nabla\eta| \le \frac{K}{R}$ for any $0 < R < R_0$ and some positive constant K. We define a constant u_R associated with u and $B_R(x_0)$ by

$$u_R = \frac{1}{|B_{2R} - B_{R/2}|}\int_{B_{2R}-B_{R/2}} u(x)\, dx, \quad \text{for } B_{2R} \subset \Omega.$$

Let $\psi(x) = G(x, x_0)\eta^p(x)$ and take $\phi(x) = (u - u_R)\psi(x)$ as a test function to equation (5.2). Since $\nabla\phi = \psi\nabla u + (u - u_R)\nabla\psi$, we have

$$\begin{aligned} &\int_{\Omega} \langle A(x, \nabla u),\ \psi\nabla u + (u - u_R)\nabla\psi\rangle\, dx, \\ =\ &\int_{\Omega} g(u - u_R)\psi dx + \int_{\Omega}\langle f(x),\ \psi\nabla u + (u - u_R)\nabla\psi\rangle\, dx. \end{aligned}$$

By virtue of assumptions (H1) and (H2), we have

$$\alpha\int_{\Omega}\psi|\nabla u|^p\, dx \le \int_{\Omega}\psi\langle A(x, \nabla u),\ \nabla u\rangle\, dx$$

$$\begin{aligned}
&\le \int_\Omega |u-u_R|\cdot|\langle A(x,\,\nabla u),\ \nabla\psi\rangle|\,dx + \int_\Omega |g||u-u_R|\psi\,dx \\
&+ \int_\Omega |f|\cdot|\psi\nabla u + (u-u_R)\nabla\psi|\,dx \qquad (5.3)\\
&\le \beta\int_\Omega |\nabla u|^{p-1}\cdot|u-u_R|\cdot|\nabla\psi|\,dx + \int_\Omega |g|\cdot|u-u_R|\psi\,dx \\
&+ \int_\Omega |f|\cdot|\nabla u|\psi\,dx + \int_\Omega |f|\cdot|u-u_R|\cdot|\nabla\psi|\,dx \equiv I_0 + II_0,
\end{aligned}$$

where

$$I_0 = \beta\int_\Omega |\nabla u|^{p-1}\cdot|u-u_R|\cdot|\nabla\psi|\,dx,$$

$$II_0 = \int_\Omega |g|\cdot|u-u_R|\psi\,dx + \int_\Omega |f|\cdot|\nabla u|\psi\,dx + \int_\Omega |f|\cdot|u-u_R|\cdot|\nabla\psi|\,dx.$$

Note that

$$\nabla\psi = p\eta^{p-1}G\nabla\eta + \eta^p\nabla G.$$

Substituting $\nabla\psi$ into I_0, and using Young's inequality and inequality (4.1) in Theorem 6, we can rewrite the first term of I_0 with arbitrary $\varepsilon_1 > 0$ as

$$\begin{aligned}
&\beta p\int_\Omega |\nabla u|^{p-1}\cdot|u-u_R|\cdot\eta^{p-1}G\cdot|\nabla\eta|\,dx \qquad (5.4)\\
&\le K\beta p\int_{B_{2R}-B_{R/2}} (\eta|\nabla u|)^{p-1}\cdot\frac{|u-u_R|}{R}G\,dx \\
&\le \varepsilon_1\int_{B_{2R}-B_{R/2}} (\eta|\nabla u|)^p G\,dx + C(p,\,\beta,\,\varepsilon_1)\int_{B_{2R}-B_{R/2}} \Big(\frac{|u-u_R|}{R}\Big)^p G\,dx \\
&\le \varepsilon_1 C_0\int_{B_{2R}-B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx + C\int_{B_{2R}-B_{R/2}} \frac{|u-u_R|^p}{R^p|x-x_0|^{\frac{n-p}{p-1}}}\,dx.
\end{aligned}$$

Since the second term in (5.4) satisfies

$$C\int_{B_{2R}-B_{R/2}} \frac{|u-u_R|^p}{R^p|x-x_0|^{\frac{n-p}{p-1}}}\,dx \le \frac{C}{R^{p+\frac{n-p}{p-1}}}\int_{B_{2R}-B_{R/2}} |u-u_R|^p\,dx,$$

by using Poincaré's inequality we have

$$\begin{aligned}
C\int_{B_{2R}-B_{R/2}} \frac{|u-u_R|^p}{R^p|x-x_0|^{\frac{n-p}{p-1}}}\,dx &\le \frac{C}{R^{\frac{n-p}{p-1}}}\int_{B_{3R}-B_{R/2}} |\nabla u|^p\,dx \\
&\le C\int_{B_{3R}-B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx. \qquad (5.5)
\end{aligned}$$

Combining (5.4) and (5.5) we have

$$\beta p \int_\Omega |\nabla u|^{p-1} \cdot |u - u_R| \eta^{p-1} G |\nabla \eta| \, dx \le C \int_{B_{3R} - B_{R/2}} \frac{|\nabla u|^p}{|x - x_0|^{\frac{n-p}{p-1}}} \, dx. \tag{5.6}$$

Similarly, by Young's inequality for arbitrary $\varepsilon_2 > 0$, for the second term in I_0, we have

$$\begin{aligned} &\beta \int_\Omega |\nabla u|^{p-1} \cdot |u - u_R| \cdot \eta^p \cdot |\nabla G| \, dx \qquad (5.7)\\ &\le \beta \int_{B_{2R} - B_{R/2}} \Big(|\eta \nabla u|^{p-1} G^{\frac{p-1}{p}} \Big) \Big(\eta |u - u_R| \frac{|\nabla G|}{G^{\frac{p-1}{p}}} \Big) \, dx \\ &\le \varepsilon_2 C_0 \int_{B_{2R} - B_{R/2}} \frac{|\nabla u|^p}{|x - x_0|^{\frac{n-p}{p-1}}} \, dx + C \int_{B_{2R} - B_{R/2}} |\eta (u - u_R)|^p \frac{|\nabla G|^p}{G^{p-1}} \, dx. \end{aligned}$$

But it remains to estimate the second term on the right-hand side of (5.7); i.e.,

$$\int_{B_{2R} - B_{R/2}} |\eta (u - u_R)|^p \frac{|\nabla G|^p}{G^{p-1}} \, dx.$$

We separate our discussions into two cases:

Case 1. If $p \ne 2$, let us introduce a new smooth cut-off function satisfying:

$$\xi(x) = \begin{cases} 0, & \text{for} \quad |x| \le \frac{R}{2}, \\ \eta, & \text{for} \quad |x| \ge R. \end{cases}$$

For the Green's Function G defined by (1.5) we take the test function as

$$\phi(x) = [\xi(x)(u - u_R)]^p G^{2-p}.$$

Notice that $\phi(x_0) = 0$, and

$$\nabla \phi = (2 - p)[\xi(u - u_R)]^p G^{1-p} \nabla G + p G^{2-p} [\xi(u - u_R)]^{p-1} \nabla [\xi(u - u_R)].$$

Using assumptions (H1)-(H2) and Young's inequality, for arbitrary $\varepsilon_3 > 0$ we derive

$$\begin{aligned} &\alpha |2 - p| \int_{B_{2R} - B_{R/2}} |\xi(u - u_R)|^p \cdot \frac{|\nabla G|^p}{G^{p-1}} \, dx \\ &\le p\beta \int_{B_{2R} - B_{R/2}} |\nabla G|^{p-1} G^{2-p} |\xi(u - u_R)|^{p-1} |\nabla(\xi(u - u_R))| \, dx \\ &= p\beta \int_{B_{2R} - B_{R/2}} |\nabla G|^{p-1} |\xi(u - u_R)|^{p-1} \frac{1}{G^{\frac{(p-1)^2}{p}}} \cdot G^{\frac{1}{p}} |\nabla(\xi(u - u_R))| \, dx \end{aligned}$$

$$\leq \varepsilon_3 \int_{B_{2R}-B_{R/2}} |\xi(u-u_R)|^p \frac{|\nabla G|^p}{G^{p-1}} dx + C \int_{B_{2R}-B_{R/2}} G \cdot |\nabla(\xi(u-u_R))|^p \, dx.$$

Let $\varepsilon_3 < \alpha|2-p|$, then the above inequality becomes

$$\int_{B_{2R}-B_{R/2}} |\xi(u-u_R)|^p \frac{|\nabla G|^p}{G^{p-1}} \, dx \leq C \int_{B_{2R}-B_{R/2}} G|\nabla\xi(u-u_R) + \xi\nabla u|^p \, dx. \tag{5.8}$$

Using inequality (4.1) and Poincaré's inequality again, from (5.8) we have

$$\begin{aligned}
&\int_{B_{2R}-B_{R/2}} |\xi(u-u_R)|^p \frac{|\nabla G|^p}{G^{p-1}} \, dx \\
&\leq C \int_{B_{2R}-B_{R/2}} G \cdot \Big(\frac{|u-u_R|^p}{R^p} + |\nabla u|^p \Big) \, dx \\
&\leq C \int_{B_{2R}-B_{R/2}} \frac{1}{|x-x_0|^{\frac{n-p}{p-1}}} \Big(\frac{|u-u_R|^p}{R^p} + |\nabla u|^p \Big) \, dx \\
&\leq C \int_{B_{2R}-B_{R/2}} \frac{|u-u_R|^p}{R^p|x-x_0|^{\frac{n-p}{p-1}}} \, dx + C \int_{B_{3R}-B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}} \, dx \\
&\leq C \int_{B_{3R}-B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}} \, dx.
\end{aligned} \tag{5.9}$$

Case 2. If $p = 2$, we take the test function to equation (1.5) as

$$\phi(x) = G^{-\frac{1}{2}} (u-u_R)^2 \xi^2.$$

Due to the fact that $\phi(x_0) = 0$ and

$$\nabla\phi = -\frac{1}{2} G^{-\frac{3}{2}} (u-u_R)^2 \xi^2 \nabla G + 2G^{-\frac{1}{2}} \xi (u-u_R) \cdot \nabla(\xi(u-u_R)),$$

using Young's inequality again, for arbitrary $\varepsilon_4 > 0$, we get

$$\frac{1}{2}\alpha \int_{B_{2R}-B_{R/2}} |\nabla G|^2 \cdot G^{-\frac{3}{2}} \Big(\xi|u-u_R|\Big)^2 \, dx \tag{5.10}$$

$$\leq 2\beta \int_{B_{2R}-B_{R/2}} |\xi(u-u_R)||\nabla G| G^{-\frac{1}{2}} |\nabla(\xi|u-u_R|)| \, dx$$

$$\leq \varepsilon_4 \int_{B_{2R}-B_{R/2}} |\xi(u-u_R)|^2 |\nabla G|^2 G^{-\frac{3}{2}} \, dx + C \int_{B_{2R}-B_{R/2}} G^{\frac{1}{2}} |\nabla(\xi|u-u_R|)|^2 \, dx.$$

According to inequality (4.1), the first term on the right-hand side of (5.10) without ε_4 satisfies

$$\begin{aligned}
&\int_{B_{2R}-B_{R/2}} |\xi(u-u_R)|^2 \cdot |\nabla G|^2 G^{-\frac{3}{2}}\, dx \\
&\le C(n,\,\alpha,\,\beta)\int_{B_{2R}-B_{R/2}} G^{\frac{1}{2}}\cdot\Big(\frac{|u-u_R|^2}{R^2}+|\nabla u|^2\Big)\, dx \\
&\le C\int_{B_{2R}-B_{R/2}} |x-x_0|^{\frac{2-n}{2}}\frac{|u-u_R|^2}{R^2}\, dx + C\int_{B_{3R}-B_{R/2}} |\nabla u|^2|x-x_0|^{\frac{2-n}{2}}\, dx \\
&\le C\, R^{\frac{2-n}{2}}\int_{B_{3R}-B_{R/2}} |\nabla u|^2\, dx.
\end{aligned}$$

Notice that, when $p=2$, we also have

$$\begin{aligned}
&\int_{B_{2R}-B_{R/2}} |\xi(u-u_R)|^p\cdot\frac{|\nabla G|^p}{G^{p-1}}\, dx \\
&\le CR^{\frac{2-n}{2}}\int_{B_{3R}-B_{R/2}} \xi^2|u-u_R|^2G^{-\frac{3}{2}}|\nabla G|^2\, dx \\
&\le CR^{2-n}\int_{B_{3R}-B_{R/2}} |\nabla u|^2\, dx \le C\int_{B_{3R}-B_{R/2}} |\nabla u|^2|x-x_0|^{2-n}\, dx.
\end{aligned}\tag{5.11}$$

Thus, from (5.6)-(5.11), we have

$$I_0=\beta\int_\Omega |\nabla u|^{p-1}|u-u_R||\nabla\phi|\, dx \le C\int_{B_{3R}-B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\, dx. \tag{5.12}$$

Now we are ready to prove that

$$\begin{aligned}
II_0 &= \int_\Omega |g|\cdot|u-u_R|\psi\, dx+\int_\Omega |f|\cdot|\nabla u|\psi\, dx+\int_\Omega |f|\cdot|u-u_R|\cdot|\nabla\psi|\, dx \\
&\le C\int_{B_{3R}-B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\, dx + CR^{\vartheta_0}, \qquad \vartheta_0>0.
\end{aligned}\tag{5.13}$$

Choose $\psi = G\eta^p$ for the first term of the right-hand side of II_0. Using Young's inequality and Poincaré's inequality we get

$$\begin{aligned}
&\int_\Omega |g|\cdot|u-u_R|G\eta^p\, dx \\
&\le C\int_{B_{2R}-B_{R/2}} |gR|^{\frac{p}{p-1}}G\, dx + C\int_{B_{2R}-B_{R/2}} \frac{|u-u_R|^p}{R^p}G\, dx
\end{aligned}$$

$$\leq CR^{\frac{p}{p-1}}\int_{B_{2R}-B_{R/2}}|g|^{\frac{p}{p-1}}G\,dx+C\int_{B_{3R}-B_{R/2}}\frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx. \tag{5.14}$$

Looking at the first term of the right-hand side of (5.14), from Theorem 1 we know that $\|G\|_{L^\kappa_w(\Omega)}\leq C(n,\,p,\,\alpha,\,\beta)$ with $\kappa=\frac{(p-1)n}{n-p}$. Using inequality (3.2) and Hölder's inequality gives

$$\begin{aligned}&\int_{B_{2R}-B_{R/2}}|gR|^{\frac{p}{p-1}}G\,dx\\&\leq CR^{\frac{p}{p-1}}\Big(\int_{B_{2R}-B_{R/2}}G^\gamma\,dx\Big)^{\frac{1}{\gamma}}\cdot\Big(\int_{B_{2R}-B_{R/2}}|g|^{\frac{p}{p-1}\gamma'}\Big)^{\frac{1}{\gamma'}}\\&\leq CR^{\frac{p}{p-1}}|B_{2R}|^{(\frac{1}{\gamma}-\frac{1}{\kappa})}\|G\|_{L^\kappa_w}\cdot|B_{2R}|^{(\frac{1}{\gamma'}-\frac{p}{(p-1)q})}\|g\|_{L^q}^{\frac{p}{p-1}}\\&\leq CR^{\frac{p}{p-1}}|B_{2R}|^{1-\frac{1}{\kappa}-\frac{p}{(p-1)q}}=CR^{\frac{p}{p-1}+n(1-\frac{1}{\kappa}-\frac{p}{(p-1)q})},\end{aligned} \tag{5.15}$$

where γ' is the Hölder conjugate number of γ with $1<\gamma<\frac{(p-1)n}{n-p}$. Since $2-\frac{2}{n+1}<p<n$ and $q>\frac{n}{n-2(\frac{n}{p}-1)}$, the exponent of R is positive.

Using the same arguments, we have

$$\begin{aligned}&\int_\Omega|f||\nabla u|G\eta^p\,dx\leq C\int_{B_{2R}-B_{R/2}}|f|^{\frac{p}{p-1}}G\,dx+C\int_{B_{2R}-B_{R/2}}|\nabla u|^pG\,dx\\&\leq C\int_{B_{2R}-B_{R/2}}|f|^{\frac{p}{p-1}}G\,dx+C\int_{B_{3R}-B_{R/2}}\frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx;\end{aligned} \tag{5.16}$$

$$\int_{B_{2R}-B_{R/2}}|f|^{\frac{p}{p-1}}G\,dx\leq C|B_{2R}|^{(\frac{1}{\gamma}-\frac{1}{\kappa})+(\frac{1}{\gamma'}-\frac{p}{(p-1)s})}\leq CR^{n\left(1-\frac{1}{\kappa}-\frac{p}{(p-1)s}\right)}; \tag{5.17}$$

and

$$\begin{aligned}&\int_\Omega|f|\cdot|u-u_R|\cdot|pG\eta^{p-1}\nabla\eta|\,dx\\&\leq C\int_{B_{2R}-B_{R/2}}|f|^{\frac{p}{p-1}}G\,dx+C\int_{B_{2R}-B_{R/2}}\frac{|u-u_R|^p}{R^p}G\,dx\\&\leq C\int_{B_{2R}-B_{R/2}}|f|^{\frac{p}{p-1}}G\,dx+C\int_{B_{3R}-B_{R/2}}\frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx.\end{aligned} \tag{5.18}$$

Using Young's inequality and the result in Case 1, we have

$$\begin{aligned}
&\int_{B_{2R}-B_{R/2}} |f||u-u_R||\nabla G|\eta^p\,dx\\
&\le C\int_{B_{2R}-B_{R/2}} |f|^{\frac{p}{p-1}}G\eta^p\,dx + C\int_{B_{2R}-B_{R/2}} |\eta(u-u_R)|^p\frac{|\nabla G|^p}{G^{p-1}}\,dx\\
&\le C\int_{B_{2R}-B_{R/2}} |f|^{\frac{p}{p-1}}G\,dx + C\int_{B_{3R}-B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx\\
&\le CR^{n\left(1-\frac{1}{\kappa}-\frac{p}{(p-1)s}\right)} + C\int_{B_{3R}-B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx. \qquad (5.19)
\end{aligned}$$

Since $2-\frac{2}{n+1}<p<n$ and $s\ge\frac{p}{n-\frac{2n}{p}+1}$, by a direct calculation we get

$$1-\frac{n-p}{(p-1)n}-\frac{p}{(p-1)s}>0.$$

Hence, we take

$$\vartheta_0=\min\left\{\frac{p}{p-1}+n(1-\frac{n-p}{(p-1)n}-\frac{p}{(p-1)q}),n(1-\frac{n-p}{(p-1)n}-\frac{p}{(p-1)s})\right\},$$

then from (5.14)-(5.19), we obtain the estimate of inequality (5.13).

Combining (5.3), (5.12), and (5.13), we have

$$\begin{aligned}
\int_{B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx &\le \alpha\int_\Omega \psi|\nabla u^p|\,dx\\
&\le C_7\int_{B_{3R}-B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx + C_8R^{\vartheta_0}.
\end{aligned}$$

Thus, there is a basic estimate for positive constants K_0 and C which only depends on n, p, α and β, such that

$$\int_{B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx \le K_0\int_{B_{3R}-B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx + CR^{\vartheta_0};$$

i.e.,

$$\int_{B_{R/2}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx \le \frac{K_0}{K_0+1}\int_{B_{3R}} \frac{|\nabla u|^p}{|x-x_0|^{\frac{n-p}{p-1}}}\,dx + CR^{\vartheta_0}.$$

Since $\frac{K_0}{K_0+1} < 1$, using Lemma 2, we can obtain

$$\int_{B_R} \frac{|\nabla u|^p}{|x - x_0|^{\frac{n-p}{p-1}}}\, dx \le CR^{\vartheta}.$$

Using the hole-filling technique [27] directly, we conclude that $\nabla u \in L^{p,\lambda}_{loc}(\Omega)$ for every $0 < \lambda \le \frac{n-p}{p-1} + \vartheta$ with some $\vartheta = \min\{\vartheta_0,\, p\log_2 \frac{1+K_0}{K_0}\} > 0$. Therefore, the proof of Theorem 7 is completed. □

Conclusion. In this paper, first we study the Green's function associated with non-linear degenerate elliptic operators of divergence form, then establish several theorems in regard to local estimates and comparison with the p-Laplacian for the Green's function in the sense of distributions. Under the structural conditions (H1)-(H3), we also prove an estimate of the Green's function of the A-harmonic operator defined by (1.5). Finally, as an application of these estimates of the Green's function, we show that the Green's function can be applied to the regularity in the Morrey space for some non-linear elliptic problems.

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