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HARNACK ESTIMATES FOR SOME NON-LINEAR PARABOLIC EQUATION

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Abstract. We consider the following nonlinear parabolic equation

$$\begin{cases} \partial_t u - \Delta u + \dfrac{u}{\varepsilon}(|\nabla u|^2 - 1) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{\#}$$

which is derived by Goto-K. Ishii-Ogawa [6] to show the convergence of some numerical algorithms for the motion by mean curvature. They assumed that the solution of (#) is sufficiently regular. In this paper, we study the regularity of solutions of (#) from the Harnack estimate. We show the explicit dependence of a constant in the Harnack inequality using the De Giorgi-Nash-Moser method. We employ the Cole-Hopf transform to treat the nonlinear term.

1. Introduction and main result

We consider the following nonlinear parabolic equation:

$$\begin{cases} \partial_t u - \Delta u + \dfrac{u}{\varepsilon}(|\nabla u|^2 - 1) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u(t,x)$ is the unknown function, $u_0(x)$ is a given initial data and $\varepsilon > 0$ is a small parameter.

To compute the motion by mean curvature, Bence, Merriman, and Osher [3] proposed a numerical algorithm which is called the B-M-O algorithm, based on a simple procedure using solutions of heat equations. There are some mathematical justifications and extensions of the B-M-O algorithm given by Evans [4], Barles-Georgelin [2], H. Ishii [7] and H. Ishii-K. Ishii [8]. In considering the B-M-O algorithm, Goto-K. Ishii-Ogawa [6] introduced the

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singular limiting problem (1.1) of the non-linear parabolic equation. Moreover, Goto-K. Ishii-Ogawa gave another proof of the convergence of the B-M-O algorithm and a solution u of the limiting problem (1.1) satisfies the level set equation of the motion by mean curvature:

$$\partial_t u - \Delta u + \sum_{i,j=1}^{n} \frac{1}{|\nabla u|^2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0. \tag{1.2}$$

This problem (1.1) is similar to a singular limiting problem of the Allen-Cahn equation and the behavior of a solution of the limiting problem (1.1) might be singular as $\varepsilon \to 0$. In general, it is difficult to obtain the regularity of the solution of the limiting problem (1.2). Besides, the regularity of the limiting problem (1.1) is related to convergence of the B-M-O algorithm. Hence, it is important to study the regularity of a solution of (1.1) depending on the parameter $\varepsilon > 0$.

We note the existence of a solution of (1.1). Let $A_\varepsilon = \Delta + \frac{1}{\varepsilon}$ with a domain $D(A_\varepsilon) = H^2(\mathbb{R}^n)$ and e^{tA_ε} be the semigroup generated by A_ε on $\mathbb{R}^n$.

Definition 1.1. *We call $u = u(t,x)$ a mild solution of* (1.1) *if there exists $T > 0$ such that u satisfies the integral equation:*

$$u(t,x) = e^{tA_\varepsilon} u_0(x) - \frac{1}{\varepsilon} \int_0^t e^{(t-\tau)A_\varepsilon} u(\tau,x) |\nabla u(\tau,x)|^2 \, d\tau, \tag{1.3}$$

for all $0 < t < T$.

The existence of a mild solution of (1.1) is as follows.

Proposition 1.2. *Let $1 < p, r \leq \infty$ satisfy*

$$\frac{1}{p} + \frac{1}{r} < \frac{1}{n}, \quad \frac{1}{p} + \frac{2}{r} \leq 1.$$

For any initial data $u_0 \in L^p(\mathbb{R}^n)$ with $\nabla u_0 \in L^r(\mathbb{R}^n)$, we take $T > 0$ small enough so that

$$0 < T^{1-\gamma}(\|u_0\|_{L^p(\mathbb{R}^n)} + \|\nabla u_0\|^2_{L^r(\mathbb{R}^n)}) \ll 1\, ,\; e^{\frac{3T}{\varepsilon}} < \frac{3}{2}\, ,\; \gamma = \frac{n}{2}\Big(\frac{1}{p} + \frac{1}{r}\Big) + \frac{1}{2}.$$

Then there exists a unique mild solution of (1.3) *such that $u \in L^\infty(0,T; L^p(\mathbb{R}^n))$ and $\nabla u \in L^\infty(0,T; L^r(\mathbb{R}^n))$.*

We will show the proof of Proposition 1.2 in Appendix A.

In Proposition 1.2, we can obtain that the solution u is Hölder continuous in the spacial variable by the Sobolev embedding. Moreover, using the maximal regularity of heat equations, we find that the solution u is smooth in $(0, T)$. However, it is not clear how the regularity of the solution depends on the parameter $\varepsilon > 0$.

To study the regularity, we consider the Hölder estimate of a solution of (1.1). It is well known that the Harnack inequality gives the interior Hölder continuity for solutions of parabolic equations. The Harnack constant, the constant in the Harnack inequality, is related to the Hölder exponent of the solution, hence we can believe that the Harnack constant has some information about regularity of solutions of (1.1). Now, we study explicit dependence on the parameter $\varepsilon > 0$ of the Harnack constant for a non-negative solution of (1.1) and state our main theorem.

Theorem 1.3 (The Harnack inequality). *Let u_ε be a non-negative mild solution of* (1.1) *on $(0, 8T) \times B_{4R}$ and $0 < \varepsilon < 1$. Suppose that $0 \leq u_\varepsilon \leq M$ for some $M \geq 0$. Then we have*

$$\sup_{(T,2T)\times B_R} u_\varepsilon \leq CM \exp\left(\frac{\theta}{\varepsilon}\right) \inf_{(7T,8T)\times B_R} u_\varepsilon,$$

where the constant C depends on n, T, R and the constant θ depends on n, M.

The basic strategy to prove this theorem is to use the De Giorgi-Nash-Moser method. For a parabolic equation, Moser [11] showed the Harnack inequality and it is well known that his method can be extended to a non-linear case. However, we can not apply Moser's method directly since our equation has strong non-linearity and it is generally difficult to treat the equation by a perturbation method, whenever the parameter $\varepsilon > 0$ is small. To overcome this difficulty, we employ the Cole-Hopf transform. Formally, by using the Cole-Hopf transform, the non-linear equation (1.1) is transformed into a linear heat equation and hence Moser's method is applicable. Since we consider the mild solution, we need to justify the Cole-Hopf transform in the weak formulation. For this purpose, we modify Trudinger's argument [12] and we investigate the explicit dependence of the constant on ε.

Once we obtain the theorem, we obtain the Hölder continuity of solutions of (1.1) and an estimate of the Hölder exponent of solutions. Furthermore, our main theorem may be developed with a finer analysis of the singular limiting problem (1.1) as $\varepsilon \to 0$. For instance, our theorem is connected with the regularity of a derivative of a solution of the singular limiting problem (1.1). Moreover, by the regularity of a gradient of the solution, an interface of (1.1) makes sense and we study the mean curvature flow and B-M-O algorithm more clearly.

This paper is organized as follows. We first introduce the Ladyženskaja inequality, the weighted Poincaré inequality and the parabolic John-Nirenberg estimate in Section 2. In Section 3, we show the local maximum principle and the weak Harnack inequality and we prove Theorem 1.3. In appendix A, we give the existence theorem for the initial-value problem of (1.1).

At the end of this section, we introduce some notations. We denote a set of non-negative integer by $\mathbb{N}_0$. For $x \in \mathbb{R}^n$ and $R > 0$, we put $B_R(x) := \{y \in \mathbb{R}^n : |x - y| < R\}$ and $K_R(x) = \{y \in \mathbb{R}^n : \max_{1 \le i \le n} |x_i - y_i| < R\}$. We abbreviate $B_R(0)$ and $K_R(0)$ as B_R and K_R, respectively. For a function $f : \mathbb{R}^n \to \mathbb{R}$, we put $f_+(x) := \max\{f(x), 0\}$. We denote a set of infinitely differentiable functions with compact support in Ω by $C_0^\infty(\Omega)$ and by $H^1(\Omega)$ the Sobolev space with a weak derivative in $L^2(\Omega)$. We write a norm of $u \in H^1(\Omega)$ as $\|u\|_{H^1(\Omega)} := \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}$. The completion $C_0^\infty(\Omega)$ in $H^1(\Omega)$ is denoted by $H_0^1(\Omega)$. For any Banach spaces X and time intervals $I \subset \mathbb{R}$, we denote a set of X-valued p-th powered integrable functions in I by $L^p(I; X)$ and a set of X-valued essentially bounded maps in I by $L^\infty(I; X)$, endowed with a norm

$$\|u\|_{L^p(I;X)} := \Big(\int_I \|u(t)\|_X^p \, dt \Big)^{\frac{1}{p}}, \quad \|u\|_{L^\infty(I;X)} := \operatorname*{ess.sup}_{t \in I} \|u(t)\|_X.$$

2. Preliminary

First, we give the Ladyženskaja inequality.

Lemma 2.1 (The Ladyženskaja inequality)**.** *Let $D \subset \mathbb{R}^n$ be an arbitrary domain and $T > 0$. Let numbers r, q satisfy*

$$\frac{2}{r} + \frac{n}{q} = \frac{n}{2},$$

with

$$\begin{aligned} 4 \le r \le \infty\,,\ 2 \le q \le \infty, &\quad \text{if } n = 1, \\ 2 < r \le \infty\,,\ 2 \le q < \infty, &\quad \text{if } n = 2, \\ 2 \le r \le \infty\,,\ 2 \le q \le \frac{2n}{n-2}, &\quad \text{if } n \ge 3. \end{aligned}$$

Then there exists a constant $C > 0$ depending on n, q only such that

$$\|w\|_{L^r(0,T;L^q(D))} \le C \|w\|_{L^2(0,T;H_0^1(D))}^{\frac{2}{r}} \|w\|_{L^\infty(0,T;L^2(D))}^{1-\frac{2}{r}},$$

for all $w \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;H_0^1(D))$*; i.e., we have the following embedding:*

$$L^\infty(0,T\,;L^2(D)) \cap L^2(0,T\,;H_0^1(D)) \subset L^r(0,T\,;L^q(D)).$$

We refer to Ladyženskaja-Solonnikov-Ural'ceva [9, page 74] for a proof. If $r = q$ in the Ladyženskaja inequality, we obtain

$$\|w\|_{L^{2(1+\frac{2}{n})}((0,T)\times D)} \le C(\|w\|_{L^2(0,T;H_0^1(D))} + \|w\|_{L^\infty(0,T;L^2(D))}). \tag{2.1}$$

Next, we give the weighted Poincaré inequality.

Lemma 2.2. *Let* $D \subset \mathbb{R}^n$ *be a bounded domain and* μ *be a non-negative continuous function in* D *with compact support. Furthermore,* $\{x \in D : \mu(x) \ge \lambda\}$ *is convex for all* $\lambda \ge 0$*. Then*

$$\int_D (g(x)-k)^2\mu(x)\,dx \le C\frac{(\operatorname{diam} D)^{n+2}}{A}\|\mu\|_{L^\infty(D)}\int_D |\nabla g(x)|^2\mu(x)\,dx,$$

for all $g \in H^1(D)$*, where*

$$A = \int_D \mu(x)\,dx\,,\ k = \frac{\int_D g(x)\mu(x)\,dx}{A}.$$

We refer to Lieberman [10, page 113, Lemma 6.12] for the proof.

Before giving the parabolic John-Nirenberg estimate, we introduce some notation. For $t_0 \in \mathbb{R},\ x_0 \in \mathbb{R}^n$ and $\rho > 0$, we denote $U_\rho^+(t_0,x_0) := (t_0, t_0 + \rho^2) \times B_\rho(x_0)$ and $U_\rho^-(t_0,x_0) := (t_0 - \rho^2, t_0) \times B_\rho(x_0)$.

Lemma 2.3 (Moser [11], Fabes-Garofalo [5])**.** *Let* $f : (0,T) \times K_R \to \mathbb{R}$*. Suppose that for* $0 < t_0 < T,\ x_0 \in K_R$ *and* $\rho > 0$ *with* $U^\pm = U_\rho^\pm(t_0,x_0) \subset (0,T) \times K_R$*, we have*

$$\begin{aligned} \frac{1}{|U^+|}\iint_{U^+}\sqrt{(f(t,x)-V_U)_+}\,dtdx \le C_0,\\ \frac{1}{|U^-|}\iint_{U^-}\sqrt{(V_U - f(t,x))_+}\,dtdx \le C_0, \end{aligned} \tag{2.2}$$

for some V_U *depending on* f, t_0, x_0, ρ *only and for some* $C_0 \ge 0$ *independent of* t_0, x_0, ρ*. Then there exist* $p_0, C_1 > 0$ *such that*

$$\Big(\iint_{(0,\frac{1}{8}T)\times K_{\frac{R}{2}}} e^{-p_0 f(t,x)}\,dtdx\Big)\Big(\iint_{(\frac{7}{8}T,T)\times K_{\frac{R}{2}}} e^{p_0 f(t,x)}\,dtdx\Big) \le C_1,$$

where the constant C_1 *depends on* n, T, R *and the constant* p_0 *depends on* n, C_0*.*

See Fabes-Garofalo [5] for a proof.

3. Proof of Theorem 1.3

In this section, we consider the Harnack estimate of a solution of the problem (1.1) and investigate the dependence on the parameter $\varepsilon > 0$ of the Harnack constant.

To prove Theorem 1.3, we show the local maximum principle, estimating the supremum of u by the L^p-norm of u, and show the weak Harnack inequality, estimating the L^p-norm of u by the infimum of u.

First, we give the local maximum principle.

Proposition 3.1 (the local maximum principle)**.** *Let u_ε be a non-negative mild solution of (1.1) on $(0,T) \times B_R$. Then, for all $p > 1$, $0 \le \tau < \tau' < T$, $0 < R' < R$ and $0 < \varepsilon < 1$, we have*

$$\sup_{(\tau',T)\times B_{R'}} u_\varepsilon \le C\varepsilon^{-\frac{n+2}{2p}} \|u_\varepsilon\|_{L^p((\tau,T)\times B_R)},$$

where the constant C depends on n, p, τ', τ, R, R'.

Remark 3.2. We consider the following problem:

$$\partial_t v - \Delta v - v = 0, \quad (t,x) \in (0,T) \times B_R. \tag{3.1}$$

For a non-negative subsolution v of (3.1) and for all $p > 1$, $0 \le \tau < \tau' < T$, $0 < R' < R$, we can obtain

$$\sup_{(\tau',T)\times B_{R'}} v \le C\|v\|_{L^p((\tau,T)\times B_R)},$$

where the constant C depends on n, p, τ, τ', R, R'. We put

$$v_\varepsilon(t,x) := v\Big(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}\Big),$$

then we have

$$\partial_t v_\varepsilon - \Delta v_\varepsilon - \frac{1}{\varepsilon} v_\varepsilon = 0\,, \quad (t,x) \in (0, \varepsilon T) \times B_{\sqrt{\varepsilon}R}.$$

By a change of variable, we find

$$\sup_{(\varepsilon\tau',\varepsilon T)\times B_{\sqrt{\varepsilon}R'}} v_\varepsilon \le C\varepsilon^{-\frac{n+2}{2p}} \|v_\varepsilon\|_{L^p((\varepsilon\tau,\epsilon T)\times B_{\sqrt{\varepsilon}R})}.$$

Therefore, the power of ε in Proposition 3.1 naturally arises.

Next, we give the weak Harnack inequality.

Proposition 3.3 (The weak Harnack inequality)**.** *Let u_ε be a non-negative mild solution of* (1.1) *on* $(0,T)\times B_R$. *Suppose that* $0\le u_\varepsilon\le M$ *for some* $M\ge 0$. *Then, for all* $p\ge 1$, $0<\tau\le \frac{T}{4}$ *and* $0<R'<R$, *we have*

$$\|u_\varepsilon\|_{L^p((0,\tau)\times B_{R'})}\le CM\exp\left(\frac{\theta}{\varepsilon}\right)\inf_{(3\tau,4\tau)\times B_{R'}}u_\varepsilon,$$

where the constant C depends on n,p,τ,R',R and the constant θ depends on n,M.

Using the local maximum principle and the weak Harnack inequality, we obtain the Harnack inequality.

3.1. **Proof of Proposition 3.1.** Hereafter, we abbreviate by u a solution u_ε of (1.1). Before proving Proposition 3.1, we show the reverse Hölder inequality.

Lemma 3.4. *Let u be a non-negative mild solution of* (1.1)*, Then for all* $\beta>0$, $0<s<s'<T$, $0<r'<r$ *and* $\varepsilon<1$, *we have the reverse Hölder inequality:*

$$\|u\|^{\beta+1}_{L^{(1+\frac{2}{n})(\beta+1)}((s',T)\times B_{r'})}\le C\Big(1+\frac{1}{\beta}\Big)^2\Big(\frac{1}{\varepsilon}(\beta+1)+\frac{1}{(r-r')^2}+\frac{1}{(s'-s)}\Big)\|u\|^{\beta+1}_{L^{\beta+1}((s,T)\times B_r)},\quad(3.2)$$

where the constant C depends on n only.

Proof. We consider that u is a classical solution. Set a cut-off function η satisfying

$$0\le\eta\le 1\,,\ \eta(t,x)=1\text{ on }(s',T)\times B_{r'}\,,\ |\partial_t\eta|\le\frac{4}{s'-s}\,,\ |\nabla\eta|\le\frac{4}{r-r'}.$$

Taking the test function η^2u^β in the equation of (1.1), integrating over $(s,t)\times B_r$ and neglecting the term $\frac{u}{\varepsilon}|\nabla u|^2$, we obtain

$$\frac{1}{\beta+1}\int_s^t\int_{B_r}\eta^2\partial_t(u^{\beta+1})\,d\tau dx+\beta\int_s^t\int_{B_r}\eta^2u^{\beta-1}|\nabla u|^2\,d\tau dx\le -2\int_s^t\int_{B_r}u^\beta\eta\nabla\eta\cdot\nabla u\,d\tau dx+\frac{1}{\varepsilon}\int_s^t\int_{B_r}\eta^2u^{\beta+1}\,d\tau dx.$$

Using the Young inequality with the first integral on the right-hand side and adding

$$\frac{1}{\beta+1}\int_s^t\int_{B_r}\partial_t(\eta^2)u^{\beta+1}\,d\tau dx,$$

we have

$$\frac{1}{\beta+1}\int_{B_r}\eta^2 u^{\beta+1}\,dx\Big|_t+\frac{2\beta}{(\beta+1)^2}\int_s^t\int_{B_r}\eta^2\big|\nabla\big(u^{\frac{\beta+1}{2}}\big)\big|^2\,d\tau dx$$
$$\le\frac{1}{\varepsilon}\int_s^T\int_{B_r}\eta^2u^{\beta+1}\,d\tau dx+\frac{2}{\beta+1}\int_s^T\int_{B_r}\eta|\partial_t\eta|u^{\beta+1}\,d\tau dx$$
$$+\frac{2}{\beta}\int_s^T\int_{B_r}|\nabla\eta|^2u^{\beta+1}\,d\tau dx.$$

Then we obtain

$$\big\|\eta u^{\frac{\beta+1}{2}}\big\|^2_{L^\infty(s,T\,;\,L^2(B_r))}$$
$$\le C\Big\{\frac{1}{\varepsilon}(\beta+1)+\Big(1+\frac{1}{\beta}\Big)\frac{1}{(r-r')^2}+\frac{1}{s'-s}\Big\}\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^2(s,T\,;\,L^2(B_r))},$$

and

$$\big\|\eta u^{\frac{\beta+1}{2}}\big\|^2_{L^2(s,T\,;\,H^1_0(B_r))}\le C\Big\{\frac{\beta+1}{\varepsilon}\Big(1+\frac{1}{\beta}\Big)$$
$$+\Big(1+\frac{1}{\beta}\Big)^2\frac{1}{(r-r')^2}+\Big(1+\frac{1}{\beta}\Big)\frac{1}{s'-s}\Big\}\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^2(s,T\,;\,L^2(B_r))},$$

where C is the universal constant. Using the Ladyženskaja inequality (2.1), we have

$$\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^{2(1+\frac{2}{n})}((s',T)\times B_{r'})}\le\big\|\eta u^{\frac{\beta+1}{2}}\big\|^2_{L^{2(1+\frac{2}{n})}((s',T)\times B_r)}$$
$$\le C(n)\Big(1+\frac{1}{\beta}\Big)^2\Big(\frac{1}{\varepsilon}(\beta+1)+\frac{1}{(r-r')^2}+\frac{1}{(s'-s)}\Big)\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^2((s,T)\times B_r)}.$$

This implies the inequality (3.2). □

Proof of Proposition 3.1. For $j\in\mathbb{N}_0$, we put

$$\tau_j:=\tau'-2^{-j}(\tau'-\tau)\,,\quad R_j:=R'+2^{-j}(R-R')\,,$$
$$\alpha_j:=\Big(1+\frac{2}{n}\Big)^j,\qquad D_j:=(\tau_j,T)\times B_{R_j}.$$

(cf. Figure 1) In the inequality (3.2), we set

$$\beta+1=p\alpha_j\,,\ s'=\tau_{j+1}\,,\ s=\tau_j\,,\ r'=R_{j+1}\,,\ r=R_j,$$

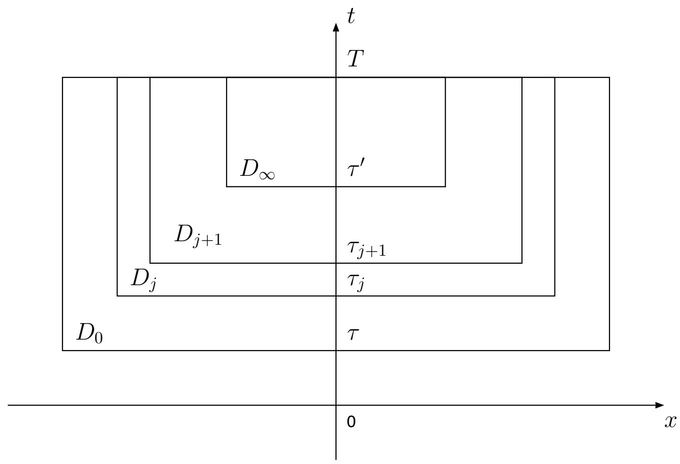

FIGURE 1. Figure D_j(We set $D_\infty = (\tau', T) \times B_{R'}$)

then we obtain

$$\|u\|_{L^{p\alpha_{j+1}}(D_{j+1})} \leq C(p,n)^{\frac{j}{\alpha_j}} \left(\frac{1}{\varepsilon} + \frac{1}{(\tau'-\tau)} + \frac{1}{(R-R')^2}\right)^{\frac{1}{p\alpha_j}} \|u\|_{L^{p\alpha_j}(D_j)}. \tag{3.3}$$

This inequality (3.3) asserts that if $\|u\|_{L^{p\alpha_j}(D_j)}$ is finite, then $\|u\|_{L^{p\alpha_{j+1}}(D_{j+1})}$ is also finite.

Iterating this inequality (3.3), we find

$$\begin{aligned}
&\|u\|_{L^{p\alpha_{j+1}}((\tau',T)\times B_{R'})} \leq \|u\|_{L^{p\alpha_{j+1}}(D_{j+1})} \\
&\leq \prod_{j=0}^{\infty} \left(C(p,n)^{\frac{j}{\alpha_j}} \left(\frac{1}{\varepsilon} + \frac{1}{(\tau'-\tau)} + \frac{1}{(R-R')^2}\right)^{\frac{1}{p\alpha_j}} \right) \|u\|_{L^p(D_0)} \\
&= C(p,n)^{\sum_{i=1}^{\infty} \frac{i}{\alpha_i}} \left(\frac{1}{\varepsilon} + \frac{1}{(\tau'-\tau)} + \frac{1}{(R-R')^2}\right)^{\frac{n+2}{2p}} \|u\|_{L^p((\tau,T)\times B_R)}.
\end{aligned}$$

We remark that $\sum_{i=1}^{\infty} \frac{i}{\alpha_i}$ is finite. Taking $j \to \infty$, we have

$$\sup_{(\tau',T)\times B_{R'}} u \leq C(n,p)\varepsilon^{-\frac{n+2}{2p}} \left(1 + \frac{1}{\tau'-\tau} + \frac{1}{(R-R')^2}\right)^{\frac{n+2}{2p}} \|u\|_{L^p((\tau,T)\times B_R)}.$$

□

Remark 3.5. In the proof of Proposition 3.1 and Lemma 3.4, we only consider the classical solution of (1.1). However, using the Steklov average, we can extend our results for weak solutions of (1.1).

3.2. **Proof of Proposition 3.3.** First, as Lemma 3.4, we show the reverse Hölder inequality.

Lemma 3.6. *Let u be a non-negative mild solution of* (1.1)*. Suppose that $0 \le u \le M$ for some $M \ge 0$. Then, for all $\beta < -1\,, 0 < s < s' < T$ and $0 < r' < r$, we have the reverse Hölder inequality:*

$$\|u^{\beta+1}\|_{L^{(1+\frac{2}{n})}((s',T)\times B_{r'})} \le Ce^{\frac{\theta}{\varepsilon}}\Big(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\Big)\|u^{\beta+1}\|_{L^1((s,T)\times B_r)}, \tag{3.4}$$

where the constant C depends on n and the constant θ depends on M, β only.

Lemma 3.7. *Let u be a non-negative mild solution of* (1.1)*. Suppose that $0 \le u \le M$ for some $M \ge 0$. Then, for all $-1 < \beta < 0\,, 0 < s' < s < T$ and $0 < r' < r$, we have the reverse Hölder inequality:*

$$\begin{aligned}\|u^{\beta+1}\|_{L^{(1+\frac{2}{n})}((0,s')\times B_{r'})} &\le Ce^{\frac{\theta}{\varepsilon}} \max\Big\{1, \big|1+\frac{1}{\beta}\big|, \big|1+\frac{1}{\beta}\big|^2\Big\} \\ &\quad \times \Big(\frac{1}{s-s'}+\frac{1}{(r-r')^2}\Big)\|u^{\beta+1}\|_{L^1((0,s)\times B_r)}, \end{aligned} \tag{3.5}$$

where the constant C depends on n and the constant θ depends on M, β only.

Since their proofs are similar, we show these lemmas at the same time.

Proof of Lemma 3.6 and Lemma 3.7. Set a cut-off function η satisfying $0 \le \eta \le 1$, and we require more conditions for η later. We put $b_0 = \frac{M}{\varepsilon}$ for our convenience. Taking a test function $\eta^2 e^{-b_0 u} u^\beta$ in the equation of (1.1), integrating over $(t_0, t) \times B_r$ and neglecting the term $\frac{u}{\varepsilon}$, we obtain

$$\begin{aligned} &-\int_{t_0}^t \int_{B_r} \eta^2 e^{-b_0 u} u^\beta \partial_t u \, d\tau dx - \int_{t_0}^t \int_{B_r} \eta^2 e^{-b_0 u}(\beta u^{\beta-1} - b_0 u^\beta)|\nabla u|^2 \, d\tau dx \\ &\quad \le 2\int_{t_0}^t \int_{B_r} \eta e^{-b_0 u} u^\beta \nabla\eta \cdot \nabla u \, d\tau dx + b_0 \int_{t_0}^t \int_{B_r} \eta^2 e^{-b_0 u} u^\beta |\nabla u|^2 \, d\tau dx. \end{aligned}$$

We can cancel out the integral of $\eta^2 e^{-b_0 u} u^\beta |\nabla u|^2$ in this inequality. Using the Young inequality, we have

$$-\int_{t_0}^t \int_{B_r} \eta^2 e^{-b_0 u} u^\beta \partial_t u \, d\tau dx - \frac{\beta}{2}\int_{t_0}^t \int_{B_r} \eta^2 e^{-b_0 u} u^{\beta-1} |\nabla u|^2 \, d\tau dx$$

$$\leq -\frac{2}{\beta}\int_{t_0}^{t}\int_{B_r} e^{-b_0 u}u^{\beta+1}|\nabla\eta|^2\,d\tau dx. \quad (3.6)$$

Next, for $\beta \neq -1$, we set

$$f(u) := \begin{cases} (\beta+1)\displaystyle\int_0^u e^{-b_0 s}s^\beta\,ds, & \text{if } \beta > -1,\\ -(\beta+1)\displaystyle\int_u^\infty e^{-b_0 s}s^\beta\,ds, & \text{if } \beta < -1.\end{cases}$$

Then $\partial_t f(u) = (\beta+1)e^{-b_0 u}u^\beta \partial_t u$.

If $\beta < -1$, by integration by parts we have

$$\begin{aligned} f(u) &= -(\beta+1)u^{\beta+1}\int_1^\infty e^{-b_0 ur}r^\beta\,dr \qquad (s = ur)\\ &= b_0 u^{\beta+2}\int_1^\infty e^{-b_0 ur}(1-r^{\beta+1})\,dr\\ &\geq b_0 u^{\beta+2}\int_{2^{-\frac{1}{\beta+1}}}^\infty e^{-b_0 ur}(1-r^{\beta+1})\,dr \geq \frac{1}{2}u^{\beta+1}e^{-b_0 M 2^{-\frac{1}{\beta+1}}}.\end{aligned}$$

If $-1 < \beta < 0$, we have

$$\begin{aligned} f(u) &= (\beta+1)u^{\beta+1}\int_0^1 e^{-b_0 ur}r^\beta\,dr \qquad (s = ur)\\ &\geq e^{-b_0 M}u^{\beta+1}(\beta+1)\int_0^1 r^\beta\,dr = e^{-b_0 M}u^{\beta+1}.\end{aligned}$$

On the other hand, since $f(u) \leq u^{\beta+1}$, there exists $0 < \theta = \theta(M,\beta) \leq 1$ such that

$$\frac{1}{2}e^{-b_0\theta(M,\beta)}u^{\beta+1} \leq f(u) \leq u^{\beta+1}. \quad (3.7)$$

We remark that

$$\begin{aligned} &\theta(M,\beta) \to \infty \quad \text{as } \beta \to -1,\\ &\theta(M,\beta) \to \theta(M,-\infty) < \infty \quad \text{as } \beta \to -\infty.\end{aligned}$$

From (3.6) we obtain

$$\begin{aligned} &-\frac{1}{\beta+1}\int_{t_0}^{t}\int_{B_r}\partial_t(\eta^2 f(u))\,d\tau dx - \frac{2\beta}{(\beta+1)^2}e^{-b_0 M}\int_{t_0}^{t}\int_{B_r}\eta^2|\nabla u^{\frac{\beta+1}{2}}|^2\,d\tau dx\\ &\quad\leq \frac{2}{|\beta|}\int_{t_0}^{t}\int_{B_r}u^{\beta+1}|\nabla\eta|^2\,d\tau dx + \frac{2}{|\beta+1|}\int_{t_0}^{t}\int_{B_r}\eta|\partial_t\eta|u^{\beta+1}\,d\tau dx. \quad (3.8)\end{aligned}$$

We now show the inequality (3.4) under the following additional conditions

$$\eta(t,x)=1 \text{ on } (s',T)\times B_{r'}\,,\ |\partial_t\eta|\le \frac{4}{s'-s}\,,\ |\nabla\eta|\le\frac{4}{r-r'}\,,\ t_0=s, \tag{3.9}$$

to the cut-off function η. Applying the estimates (3.8) and (3.9) to (3.7), and noting $-\frac{2\beta}{(\beta+1)^2}>0$, we have

$$\big\|\eta u^{\frac{\beta+1}{2}}\big\|^2_{L^\infty(s,T;L^2(B_r))}\le Ce^{b_0\theta}\Big(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\Big)\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^2((s,T)\times B_r)},$$

and

$$\big\|\eta u^{\frac{\beta+1}{2}}\big\|^2_{L^2(s,T;H^1_0(B_r))}\le Ce^{b_0\theta}\Big(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\Big)\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^2((s,T)\times B_r)}.$$

Using the Ladyženskaja inequality (2.1), we obtain

$$\begin{aligned}\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^{2(1+\frac{2}{n})}((s',T)\times B_{r'})}&\le\big\|\eta u^{\frac{\beta+1}{2}}\big\|^2_{L^{2(1+\frac{2}{n})}((s,T)\times B_r)}\\&\le C(n)e^{b_0\theta}\Big(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\Big)\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^2((s,T)\times B_r)}.\end{aligned}$$

This implies (3.4).

Next, we show the inequality (3.5). We assume further conditions on the test function η as

$$\eta(t,x)=1 \text{ on } (0,s')\times B_{r'}\,,\ |\partial_t\eta|\le\frac{4}{s-s'}\,,\ |\nabla\eta|\le\frac{4}{r-r'}\,,\ t_0=0.$$

Then it follows from (3.8) that

$$\begin{aligned}&\big\|\eta u^{\frac{\beta+1}{2}}\big\|^2_{L^\infty(0,s;L^2(B_r))}\\&\le Ce^{b_0\theta}\max\Big\{1,\Big|1+\frac{1}{\beta}\Big|\Big\}\Big(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\Big)\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^2((0,s)\times B_r)},\end{aligned}$$

and

$$\begin{aligned}&\big\|\eta u^{\frac{\beta+1}{2}}\big\|^2_{L^2(0,s;H^1_0(B_r))}\\&\le Ce^{b_0\theta}\max\Big\{1,\Big|1+\frac{1}{\beta}\Big|,\Big|1+\frac{1}{\beta}\Big|^2\Big\}\Big(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\Big)\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^2((0,s)\times B_r)}.\end{aligned}$$

Using the Ladyženskaja inequality (2.1), we obtain

$$\begin{aligned}&\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^{2(1+\frac{2}{n})}((0,s')\times B_{r'})}\le\big\|\eta u^{\frac{\beta+1}{2}}\big\|^2_{L^{2(1+\frac{2}{n})}((0,s)\times B_r)}\\&\le Ce^{b_0\theta}\max\Big\{1,\Big|1+\frac{1}{\beta}\Big|,\Big|1+\frac{1}{\beta}\Big|^2\Big\}\Big(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\Big)\big\|u^{\frac{\beta+1}{2}}\big\|^2_{L^2((0,s)\times B_r)}.\end{aligned}$$

This implies (3.5). □

Remark 3.8. Introducing the Cole-Hopf transform $v = e^{-\frac{M}{\varepsilon}u}$, if u is a classical solution of (1.1), then v is a subsolution of the linear heat equation. We can regard the test function $\phi = \eta^2 e^{-b_0 u} u^\beta$ as the justification of the Cole-Hopf transform for weak formulations. The original idea to cancel out the non-linear term may go back to Trudinger [12]. (Aronson and Serrin [1] also use this idea.)

Lemma 3.9. *Let u be a non-negative mild solution in $(0,T)\times B_R$ with $0 \le u \le M$. Then, for all $q>0$, $0\le\tau<\tau'<T$ and $0<R'<R$, we have*

$$\inf_{(\tau',T)\times B_{R'}} u \ge C\exp\Big(\frac{-M\theta(n+2)}{2q\varepsilon}\Big)\Big(\int_\tau^T\int_{B_R} u^{-q}\,dtdx\Big)^{-\frac{1}{q}}, \qquad (3.10)$$

where the constant C depends on $n, q, \tau'-\tau, R-R'$ and the constant θ depends on M, q.

Proof. For $j\in\mathbb{N}_0$, we put

$$\tau_j = (1-2^{-j})(\tau'-\tau)\,,\qquad r_j = R' + 2^{-j}(R-R')\,,$$
$$\alpha_j = \Big(1+\frac{2}{n}\Big)^j\,,\qquad D_j = (0,\tau_j)\times B_{R_j}.$$

In the inequality (3.4), we set

$$\beta+1 = p\alpha_j\,,\ s=\tau_j\,,\ s'=\tau_{j+1}\,,\ r'=R_{j+1}\,,\ r=R_j.$$

Then we obtain

$$\|u^{-q}\|_{L^{\alpha_{j+1}}(D_{j+1})} \le \Big\{C(n,q)e^{\frac{M}{\varepsilon}\theta}\Big(\frac{1}{\tau'-\tau}+\frac{1}{(R-R')^2}\Big)\Big\}^{\frac{1}{\alpha_j}} 2^{\frac{2j+2}{\alpha_j}}\|u^{-q}\|_{L^{\alpha_j}(D_j)}.$$

Iterating this inequality, we find

$$\sup_{(\tau',T)\times B_{R'}} u^{-q} \le C(n,q,\tau-\tau',R-R')e^{\frac{M\theta(n+2)}{2\varepsilon}}\|u^{-q}\|_{L^1(D_0)}.$$

Taking the $-\frac{1}{q}$-th power, we obtain (3.10). □

With almost the same argument, we obtain the following lemma:

Lemma 3.10. *Let u be a non-negative mild solution in $(0,T)\times B_R$ with $0\le u\le M$. Then, for all $0<q<1\le p$, $0<\tau'<\tau\le T$ and $0<R'<R$, we have*

$$\|u\|_{L^p((0,\tau')\times B_{R'})} \le C\exp\Big(\frac{M\theta(n+2)}{2q\varepsilon}\Big)\Big(\int_0^\tau\int_{B_R} u^q\,dtdx\Big)^{\frac{1}{q}},$$

where the constant C depends on $n, q, \tau - \tau', R - R'$ and the constant θ depends on M, q.

Next, we consider the case $\beta = -1$ in the proof of Lemma 3.6 and Lemma 3.7.

Lemma 3.11. *Let u be a non-negative mild solution of (1.1) in $(0, T) \times K_r$. Suppose that $0 \le u \le M$ for some $M \ge 0$. Then there exist $C, p_0 > 0$ such that*

$$\Big(\iint_{(0,\frac{1}{8}T)\times K_{\frac{r}{2}}} u^{p_0}\, dtdx \Big)^{\frac{1}{p_0}} \le CM \exp\Big(\int_0^M \frac{1 - e^{-\frac{M}{\varepsilon}s}}{s}\, ds \Big) \Big(\iint_{(\frac{7}{8}T,T)\times K_{\frac{r}{2}}} u^{-p_0}\, dtdx \Big)^{-\frac{1}{p_0}},$$

where the constant C depends on n, T, r only and the constant p_0 depends on n only.

Proof. We put $t > 0$ and $h \in \mathbb{R}$. We set $\beta = -1$, $t_0 = t$ and $t = t + h$ in the inequality (3.6) and we replace B_r with $K_r := \{x \in \mathbb{R}^n \ : \max_{1\le i\le n} |x_i| < r\}$, then

$$- \int_t^{t+h} \int_{K_r} \eta^2 e^{-b_0 u} u^{-1} \partial_t u \, d\tau dx + \frac{1}{2} \int_t^{t+h} \int_{K_r} \eta^2 e^{-b_0 u} u^{-2} |\nabla u|^2 \, d\tau dx \le 2 \int_t^{t+h} \int_{K_r} e^{-b_0 u} |\nabla \eta|^2 \, d\tau dx. \quad (3.11)$$

Letting

$$f(u) := - \int_1^u e^{-b_0 s} s^{-1} \, ds,$$

then since $\partial_t f(u) = -e^{-b_0 u} u^{-1} \partial_t u$ and $\nabla f(u) = -e^{-b_0 u} u^{-1} \nabla u$, we see from (3.11) that

$$\int_t^{t+h} \int_{K_r} \eta^2 \partial_t f(u) \, d\tau dx + \frac{1}{2} \int_t^{t+h} \int_{K_r} \eta^2 e^{b_0 u} |\nabla f(u)|^2 \, d\tau dx \le 2 \int_t^{t+h} \int_{K_r} e^{-b_0 u} |\nabla \eta|^2 \, d\tau dx.$$

We fix $\rho > 0$ and $x_0 \in K_r$ so that $K_\rho(x_0) \subset K_r$. We select a cut-off function η such that

$$\eta = \eta(x) = 1, \quad (x \in K_{\frac{\rho}{2}}(x_0)), \quad \operatorname{supp} \eta \subset\subset K_\rho(x_0),$$
$$0 \le \eta \le 1, \; |\nabla \eta| \le \frac{4}{\rho}, \quad \{x \in \mathbb{R}^n : \eta(x) \ge \lambda\} \text{ is convex for all } \lambda \ge 0.$$

Then we obtain

$$\int_{K_\rho(x_0)} \eta^2 f(u)\, dx \Big|_t^{t+h} + \frac{1}{2} \int_t^{t+h} \int_{K_\rho(x_0)} \eta^2 |\nabla f(u)|^2 \, d\tau dx \le C h \rho^{n-2},$$

where the constant C depends on n only.

We apply Lemma 2.2 by $g = f(u)$, $\mu = \eta^2$ and $D = K_\rho(x_0)$, then we find

$$\int_{K_\rho(x_0)} \eta^2 f(u)\, dx \Big|_t^{t+h} + C_1 \frac{\int_{K_\rho(x_0)} \eta^2 dx}{\rho^{n+2}} \int_t^{t+h} \int_{K_\rho(x_0)} (f(u) - V(\tau))^2 \eta^2 \, d\tau dx$$
$$\le C h \rho^{n-2},$$

where C_1 is the constant depending on n and

$$V(\tau) := \frac{\int_{K_\rho(x_0)} f(u(\tau, x)) \eta^2 \, dx}{\int_{K_\rho(x_0)} \eta^2 \, dx}.$$

Dividing by $h \int_{K_\rho(x_0)} \eta^2 \, dx$ and letting $h \to 0$, we obtain

$$\frac{dV}{dt} + \frac{C_1}{\rho^{n+2}} \int_{K_{\frac{\rho}{2}}(x_0)} (f(u) - V(t))^2 \, dx$$
$$\le \frac{C \rho^{n-2}}{\int_{K_\rho(x_0)} \eta^2 \, dx} \le C_2 \rho^{-2}, \quad \text{a.e. } 0 < t < T,$$

where the constant C_2 depends on n only.

We put $0 < t_0 < T$ such that $0 < t_0 - \frac{\rho^2}{4} < t_0 + \frac{\rho^2}{4} < T$ and set

$$w_1(t, x) = f(u) - V(t_0) - C_2 \rho^{-2}(t - t_0),$$
$$W_1(t) = V(t) - V(t_0) - C_2 \rho^{-2}(t - t_0).$$

Then

$$\frac{dW_1}{dt} + \frac{C_1}{\rho^{n+2}} \int_{K_{\frac{\rho}{2}}(x_0)} (w_1 - W_1)^2 \, dx \le 0, \; W_1(t_0) = 0. \tag{3.12}$$

For $s>0$, we put $Q_{\rho,s}(t) := \{x \in K_\rho(x_0) \,:\, w_1(t,x) > s\}$. Since $W_1(t) \le 0$ for $t_0 \le t \le t_0 + \frac{\rho^2}{4}$ by (3.12), we have

$$w_1 - W_1 \ge s - W_1 > 0, \quad t \ge t_0\,,\, x \in Q_{\frac{\rho}{2},s}(t),$$

hence,

$$\frac{dW_1}{dt} + \frac{C_1}{\rho^{n+2}}(s - W_1(t))^2 |Q_{\frac{\rho}{2},s}(t)| \le 0.$$

Therefore,

$$\frac{|Q_{\frac{\rho}{2},s}(t)|}{\rho^{n+2}} \le C_1^{-1}(s - W_1(t))^{-2}\frac{d(s-W_1)}{dt} = C_1^{-1}\frac{d}{dt}\{-(s-W_1(t))^{-1}\}.$$

Integrating over $(t_0, t_0 + \frac{\rho^2}{4})$, we find

$$\frac{1}{\rho^{n+2}}\int_{t_0}^{t_0+\frac{\rho^2}{4}} |Q_{\frac{\rho}{2},s}(t)|\,dt \le C_1^{-1}\left\{\frac{1}{s - W_1(t_0)} - \frac{1}{s - W_1(t_0 + \frac{\rho^2}{4})}\right\} \le \frac{1}{C_1 s}.$$

We set $U_+ := (t_0, t_0 + \frac{\rho^2}{4}) \times K_{\frac{\rho}{2}}(x_0)$, then

$$\begin{aligned}
&\frac{1}{|U_+|}\iint_{U_+} \sqrt{(f(u) - V(t_0))_+}\,dtdx \\
&= \frac{1}{|U_+|}\iint_{U_+} \sqrt{(w_1(t,x) + C_2\rho^{-2}(t-t_0))_+}\,dtdx \\
&\le \frac{1}{|U_+|}\Big(\iint_{U_+} \sqrt{w_1(t,x)_+}\,dtdx + \iint_{U_+} \sqrt{C_2\rho^{-2}(t-t_0)}\,dtdx\Big) \\
&\le \frac{1}{|U_+|}\Big(\frac{1}{2}\int_{t_0}^{t_0+\frac{\rho^2}{4}} \Big(\int_0^\infty s^{-\frac{1}{2}}|Q_{\frac{\rho}{2},s}(t)|\,ds\Big)\,dt + \sqrt{\frac{C_2}{4}}|U_+|\Big).
\end{aligned} \tag{3.13}$$

Here, we write

$$\begin{aligned}
&\int_{t_0}^{t_0+\frac{\rho^2}{4}} \Big(\int_0^\infty s^{-\frac{1}{2}}|Q_{\frac{\rho}{2},s}(t)|\,ds\Big)\,dt \\
&\quad = \int_{t_0}^{t_0+\frac{\rho^2}{4}} \Big(\int_0^1 s^{-\frac{1}{2}}|Q_{\frac{\rho}{2},s}(t)|\,ds + \int_1^\infty s^{-\frac{1}{2}}|Q_{\frac{\rho}{2},s}(t)|\,ds\Big)dt =: I_1 + I_2,
\end{aligned}$$

and estimate

$$I_1 \le \int_{t_0}^{t_0+\frac{\rho^2}{4}} \Big(\int_0^1 s^{-\frac{1}{2}} |K_{\frac{\rho}{2}}|\, ds\Big)\, dt = 2|U_+|\,,$$

$$I_2 \le \int_1^\infty s^{-\frac{1}{2}} \Big(\int_{t_0}^{t_0+\frac{\rho^2}{4}} |Q_{\frac{\rho}{2},s}(t)|\, dt\Big) ds \le \int_1^\infty s^{-\frac{1}{2}} \frac{\rho^{n+2}}{C_1 s}\, ds = \frac{8}{C_1}|U_+|.$$

Substituting these estimates into (3.13), we obtain

$$\frac{1}{|U_+|} \iint_{U_+} \sqrt{(f(u) - V(t_0))_+}\, dtdx \le C,$$

where C is the constant depending on n only.

We set $U_- = (t_0 - \frac{\rho^2}{4}, \tau) \times K_{\frac{\rho}{2}}(x_0)$ and, by the same argument, we have

$$\frac{1}{|U_-|} \iint_{U_-} \sqrt{(V(t_0) - f(u))_+}\, dtdx \le C.$$

Consequently, for $0 < t_0 < T$, $x_0 \in K_r$ and $\rho > 0$ with $(t_0 - \frac{\rho^2}{4}, t_0 + \frac{\rho^2}{4}) \times K_{\frac{\rho}{2}}(x_0) \subset (0, T) \times K_r$ we have

$$\frac{1}{|U_+|} \iint_{U_+} \sqrt{(f(u) - V(t_0))_+}\, dtdx \le C,$$
$$\frac{1}{|U_-|} \iint_{U_-} \sqrt{(V(t_0) - f(u))_+}\, dtdx \le C.$$

By Lemma 2.3, we have

$$\Big(\iint_{(0,\frac{1}{8}T)\times K_{\frac{r}{2}}} e^{-p_0 f(u)}\, dtdx\Big)\Big(\iint_{(\frac{7}{8}T,T)\times K_{\frac{r}{2}}} e^{-p_0 f(u)}\, dtdx\Big) \le C. \tag{3.14}$$

Now, we give the following lemma, that gives an estimate of $f(u)$.

Lemma 3.12. *Let*

$$A = \exp\Big(- \int_1^M \frac{1 - e^{-b_0 s}}{s}\, ds\Big)\,,\ B = \exp\Big(\int_0^1 \frac{1 - e^{-b_0 s}}{s}\, ds\Big).$$

Then we have

$$-\log B\xi \le f(\xi) \le -\log A\xi, \tag{3.15}$$

for all $0 < \xi \le M$.

Proof of Lemma 3.12. We show that

$$F_1(\xi) := -\log A\xi - f(\xi) \geq 0,$$

for all $0 < \xi \leq M$. By differentiating F, we have

$$F_1'(\xi) := -\frac{1}{\xi} + \frac{e^{-b_0\xi}}{\xi} \leq 0.$$

Therefore, $F_1(\xi) \geq F_1(M)$ for $0 < \xi \leq M$. Since

$$F_1(M) = -\log A - \int_1^M \frac{1 - e^{-b_0 s}}{s}\, ds,$$

we have $F_1(M) = 0$ if and only if $A = \exp\Big(-\int_1^M \frac{1-e^{-b_0 s}}{s}\, ds\Big)$ and hence $F_1(\xi) \geq 0$ for all $0 < \xi \leq M$.

By a similar argument, we obtain $-\log B\xi \leq f(\xi)$ for all $0 < \xi \leq M$. □

By Lemma 3.12 and the estimate (3.14), we have

$$\Big(\iint_{(0,\frac{1}{8}T)\times K_{\frac{r}{2}}} e^{p_0 \log Au}\, dtdx\Big)\Big(\iint_{(\frac{7}{8}T,T)\times K_{\frac{r}{2}}} e^{-p_0 \log Bu}\, dtdx\Big) \leq C,$$

hence

$$\Big(\iint_{(0,\frac{1}{8}T)\times K_{\frac{r}{2}}} u^{p_0}\, dtdx\Big)^{\frac{1}{p_0}} \leq C\frac{B}{A}\Big(\iint_{(\frac{7}{8}T,T)\times K_{\frac{r}{2}}} u^{-p_0}\, dtdx\Big)^{-\frac{1}{p_0}}.$$

□

Using Lemma 3.9, 3.10 and 3.11, we obtain Proposition 3.3.

Appendix A. Existence of a mild solution

Now, we show Proposition 1.2, namely the existence of a mild solution of the following initial-value problem:

$$\begin{cases} \partial_t u - \Delta u + \dfrac{u}{\varepsilon}(|\nabla u|^2 - 1) = 0, \quad (t,x) \in (0,T)\times\mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{A.1}$$

To prove Proposition 1.2, we give key estimates.

Lemma A.1. *Let $1 \leq q \leq p \leq \infty$. Then for all $\phi \in L^q(\mathbb{R}^n)$ we have*

$$\|e^{tA_\varepsilon}\phi\|_p \leq C_1 e^{\frac{t}{\varepsilon}} t^{-\gamma}\|\phi\|_q,$$

$$\|\nabla e^{tA_\varepsilon}\phi\|_p \leq C_2 e^{\frac{t}{\varepsilon}} t^{-(\gamma+\frac{1}{2})}\|\phi\|_q,$$

where $\gamma = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)$. *and* C_1, C_2 *are constants depending on* p, q, n *only.*

Using the L^p-L^q estimate for $e^{t\Delta}$, we obtain Lemma A.1. In Lemma A.1, we can take

$$C_1 = (4\pi)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})},\ C_2 = C_0 4^{-\gamma}\Big(|\mathbb{S}^{n-1}|\Gamma\Big(\frac{n(n-2\gamma+1)}{2n(n-2\gamma)}\Big)\Big)^{1-\frac{2\gamma}{n}},$$

where the constant C_0 depends on n only, $|\mathbb{S}^{n-1}|$ is the area of the surface of the unit ball in $\mathbb{R}^n$, and Γ is the gamma function, namely

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,dt.$$

In this section, the constants C_1, C_2 are as in Lemma A.1. To construct the contraction mapping, we set the following function spaces.

Definition A.2. *Let* $1 \le p, r \le \infty$, $T, M > 0$. *We define*

$$X_M(T) = X_M, p, r(T) := \{u \in C([0,T]; L^p(\mathbb{R}^n)) : \nabla u \in C([0,T]; L^r(\mathbb{R}^n)),$$
$$\|u\|_{X_M} := \|u\|_{C([0,T];L^p(\mathbb{R}^n))} + \|\nabla u\|_{C([0,T];L^r(\mathbb{R}^n))} \le M\}.$$

We define distance in $X_M(T)$ *by*

$$d(u,v) := \|u-v\|_{C([0,T];L^p(\mathbb{R}^n))} + \|\nabla(u-v)\|_{C([0,T];L^r(\mathbb{R}^n))}.$$

We denote the homogeneous Sobolev space by $\dot{W}^{1,q}(\mathbb{R}^n)$. Since $X_M(T)$ is closed in $C([0,T]; L^p(\mathbb{R}^n)) \cap C([0,T]; \dot{W}^{1,q}(\mathbb{R}^n))$ and $C([0,T]; L^p(\mathbb{R}^n)) \cap C([0,T]; \dot{W}^{1,q}(\mathbb{R}^n))$ is complete, $X_M(T)$ is a complete metric space.

A.1. **Estimate of perturbation.**

Definition A.3. *Using* e^{tA_ε}, *we define*

$$\Phi(u) := e^{tA_\varepsilon}u_0 - \frac{1}{\varepsilon}\int_0^t e^{(t-\tau)A_\varepsilon}u(\tau)|\nabla u(\tau)|^2\,d\tau, \tag{A.2}$$

for $u \in X_M(T)$.

We show the existence of a fixed point for Φ. First, we take $T > 0$ such that we define Φ on $X_M(T)$.

Lemma A.4. *Let* $1 \le p, q \le \infty$ *satisfy*

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{n},\ \frac{1}{p} + \frac{2}{q} \le 1,$$

and let M, γ be

$$M := 2(\|u_0\|_p + \|\nabla u_0\|_q)\,,\ \gamma = \frac{n}{2}\Big(\frac{1}{p} + \frac{1}{q}\Big) + \frac{1}{2}.$$

We take $0 < T_0 < 1$ small enough such that

$$CT_0^{1-\gamma}M^2 \ll 1\,,\ e^{\frac{T_0}{\varepsilon}} < \frac{3}{2},$$

where C is the constant depending on n, p, q, ε only. Then $\Phi(u) \in X_M(T)$ for all $T < T_0$ and $u \in X_M(T)$.

Remark A.5. We can take T_0 explicitly so that

$$e^{\frac{T_0}{\varepsilon}} \leq \frac{3}{2}\,,\ \frac{1}{\varepsilon}\Big(\frac{C_1 r T_0^{1-\frac{n}{q}}}{r-n} + \frac{C_2 T_0^{1-\gamma}}{1-\gamma}\Big) \leq \frac{1}{4M^2}. \tag{A.3}$$

Proof of Lemma A.4. First, we estimate $\|\Phi(u)\|_{C([0,T];L^p(\mathbb{R}^n))}$. We put $r \geq 1$ as $\frac{1}{r} = \frac{1}{p} + \frac{2}{q}$. By Lemma A.1, we have

$$\begin{aligned} \|\Phi(u)\|_p &\leq \|e^{tA}u_0\|_p + \frac{1}{\varepsilon}\int_0^t \|e^{(t-\tau)A}u|\nabla u|^2\|_p\,d\tau \\ &\leq \|e^{tA}u_0\|_p + \frac{C_1}{\varepsilon}\int_0^t e^{\frac{t-\tau}{\varepsilon}}(t-\tau)^{-\frac{n}{q}}\|u|\nabla u|^2\|_r\,d\tau. \end{aligned} \tag{A.4}$$

Using the Hölder inequality for the integrand, we have

$$\|u|\nabla u|^2\|_r \leq \|u\|_p\|\nabla u\|_q^2,$$

hence,

$$\|\Phi(u)\|_p \leq \|e^{tA}u_0\|_p + \frac{C_1}{\varepsilon}\int_0^t e^{\frac{t-\tau}{\varepsilon}}(t-\tau)^{-\frac{n}{q}}\|u\|_p\|\nabla u\|_q^2\,d\tau.$$

We remark that $q > n$ since $\frac{1}{n} > \frac{1}{q} + \frac{1}{p}$. Therefore, taking a supremum for t in (A.4), we find

$$\begin{aligned} &\sup_{0\leq t\leq T}\|\Phi(u)\|_p \leq e^{\frac{T}{\varepsilon}}\|u_0\|_p + \frac{C_1}{\varepsilon}e^{\frac{T}{\varepsilon}}\sup_{0\leq t\leq T}\int_0^t (t-\tau)^{-\frac{n}{q}}\|u(\tau)\|_p\|\nabla u(\tau)\|_q^2\,d\tau \\ &\leq e^{\frac{T}{\varepsilon}}\|u_0\|_p + \frac{C_1}{\varepsilon}e^{\frac{T}{\varepsilon}}\sup_{0\leq t\leq T}\|u(t)\|_p\sup_{0\leq t\leq T}\|\nabla u(t)\|_q^2\sup_{0\leq t\leq T}\int_0^t (t-\tau)^{-\frac{n}{q}}\,d\tau \\ &\leq e^{\frac{T}{\varepsilon}}\|u_0\|_p + \frac{C_1}{\varepsilon}e^{\frac{T}{\varepsilon}}M^3\sup_{0\leq t\leq T}\int_0^t (t-\tau)^{-\frac{n}{q}}\,d\tau. \end{aligned}$$

Since

$$\int_0^t (t-\tau)^{-\frac{n}{q}}\,d\tau = \frac{q}{q-n}\Big[-(t-\tau)^{-\frac{n}{q}+1}\Big]_0^t = \frac{q}{q-n}t^{1-\frac{n}{q}},$$

we obtain

$$\sup_{0\le t\le T}\|\Phi(u)\|_p \le e^{\frac{T}{\varepsilon}}\|u_0\|_p + \frac{C_1}{\varepsilon}e^{\frac{T}{\varepsilon}}M^3\frac{qT^{1-\frac{n}{q}}}{q-n}.$$

Next, we consider $\|\nabla\Phi(u)\|_{C([0,T];L^q(\mathbb{R}^n))}$. Differentiating (A.2), we can write

$$\nabla\Phi(u) = \nabla e^{tA}u_0 - \frac{1}{\varepsilon}\int_0^t e^{(t-\tau)A}\nabla(u(\tau)|\nabla u(\tau)|^2)\,d\tau.$$

Considering the L^p-L^q estimate of the derivative in Lemma A.1, we find

$$\begin{aligned}\|\nabla\Phi(u)\|_q &\le \|e^{tA}\nabla u_0\|_q + \frac{1}{\varepsilon}\int_0^t \|\nabla e^{(t-\tau)A}u|\nabla u|^2\|_q\,d\tau\\ &\le \|e^{tA}\nabla u_0\|_q + \frac{C_2}{\varepsilon}\int_0^t e^{\frac{t-\tau}{\varepsilon}}(t-\tau)^{-\gamma}\|u|\nabla u|^2\|_r\,d\tau,\end{aligned}$$

where $\frac{1}{r}=\frac{1}{p}+\frac{2}{q}$. Using the Hölder inequality for the integrand, we have $\|u|\nabla u|^2\|_r \le \|u\|_p\|\nabla u\|_q^2$. Since $\frac{1}{n}>\frac{1}{p}+\frac{1}{q}$, we obtain

$$\gamma = \frac{n}{2}\Big(\frac{1}{p}+\frac{1}{q}\Big)+\frac{1}{2}<1,$$

therefore,

$$\|\nabla\Phi(u)\|_q \le \|e^{tA}\nabla u_0\|_q + \frac{C_2}{\varepsilon}e^{\frac{T}{\varepsilon}}\int_0^t (t-\tau)^{-\gamma}\|u\|_p\|\nabla u\|_q^2\,d\tau.$$

As in the previous estimate, taking the supremum for t, we obtain

$$\sup_{0\le t\le T}\|\nabla\Phi(u)\|_q \le e^{\frac{T}{\varepsilon}}\|\nabla u_0\|_q + \frac{C_2}{\varepsilon}e^{\frac{T}{\varepsilon}}M^3\frac{T^{1-\gamma}}{1-\gamma}.$$

By the above estimate, we have

$$\|\Phi(u)\|_{X_M} \le \frac{M}{2}e^{\frac{T}{\varepsilon}} + \frac{M^3e^{\frac{T}{\varepsilon}}}{\varepsilon}\Big(\frac{C_1qT^{1-\frac{n}{q}}}{q-n}+\frac{C_2T^{1-\gamma}}{1-\gamma}\Big).$$

We take T_0 as in (A.3). Then we obtain

$$\|\Phi(u)\|_{X_M} \le \frac{3M}{4}+\frac{M}{4}\le M,$$

for $T<T_0$, therefore if $u\in X_M(T)$, then $\Phi(u)\in X_M(T)$. □

A.2. **Contraction of Φ.**

Lemma A.6. *Let p, q be as in Lemma A.4. Then for small $T > 0$, Φ is a contraction mapping on $X_M(T)$.*

Proof of Lemma A.6. By Lemma A.1 we find

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_p &\le \frac{1}{\varepsilon}\int_0^t \|e^{(t-\tau)A}(u|\nabla u|^2 - v|\nabla v|^2)\|_p\,d\tau\\ &\le \frac{C_1}{\varepsilon}\int_0^t e^{\frac{t-\tau}{\varepsilon}}(t-\tau)^{-\frac{n}{q}}\|u|\nabla u|^2 - v|\nabla v|^2\|_r\,d\tau\end{aligned}$$

for $u, v \in X_M(T)$, where $\frac{1}{r} = \frac{1}{p} + \frac{2}{q}$. By the Hölder inequality, we have

$$\begin{aligned}\|u|\nabla u|^2 - v|\nabla v|^2\|_r &\le \|(u-v)|\nabla u|^2\|_r + \|(|\nabla u|^2 - |\nabla v|^2)v\|_r\\ &\le \|(u-v)\|_p\|\nabla u\|_q^2 + \|\nabla u + \nabla v\|_q\|\nabla(u-v)\|_q\|v\|_p\\ &\le M^2\|u-v\|_p + 2M^2\|\nabla(u-v)\|_q,\end{aligned}$$

therefore,

$$\begin{aligned}&\sup_{0\le t\le T}\|\Phi(u)-\Phi(v)\|_p\\ &\le \frac{C_1 q e^{\frac{T}{\varepsilon}}T^{1-\frac{n}{q}}}{\varepsilon(q-n)}\Big(M^2\sup_{0\le t\le T}\|u-v\|_p + 2M^2\sup_{0\le t\le T}\|\nabla(u-v)\|_q\Big)\\ &\le \frac{2M^2 C_1 q e^{\frac{T}{\varepsilon}}T^{1-\frac{n}{q}}}{\varepsilon(q-n)}\Big(\sup_{0\le t\le T}\|u-v\|_p + \sup_{0\le t\le T}\|\nabla(u-v)\|_q\Big).\end{aligned}$$

As in the similar estimate, putting $\gamma = \frac{n}{2}\big(\frac{1}{p}+\frac{1}{q}\big)+\frac{1}{2}$ we find

$$\begin{aligned}&\sup_{0\le t\le T}\|\nabla(\Phi(u)-\Phi(v))\|_q\\ &\le \frac{2M^2 C_2 e^{\frac{T}{\varepsilon}}T^{1-\gamma}}{\varepsilon(1-\gamma)}\Big(\sup_{0\le t\le T}\|u-v\|_p + \sup_{0\le t\le T}\|\nabla(u-v)\|_q\Big).\end{aligned}$$

As in the above estimate, we obtain

$$\|\Phi(u)-\Phi(v)\|_{X_M} \le \frac{2M^2 e^{\frac{T}{\varepsilon}}}{\varepsilon}\Big(\frac{C_1 q T^{1-\frac{n}{q}}}{q-n} + \frac{C_2 T^{1-\gamma}}{1-\gamma}\Big)\|u-v\|_{X_M}.$$

Therefore we take $T > 0$ small enough so that

$$\frac{2M^2 e^{\frac{T}{\varepsilon}}}{\varepsilon}\Big(\frac{C_1 q T^{1-\frac{n}{q}}}{q-n} + \frac{C_2 T^{1-\gamma}}{1-\gamma}\Big) \le \frac{3}{4}. \tag{A.5}$$

Then we have

$$\|\Phi(u) - \Phi(v)\|_{X_M} \leq \frac{3}{4}\|u - v\|_{X_M}.$$

Therefore, we find that Φ is a contraction mapping on $X_M(T)$. □

Remark A.7. We take $T_0 > 0$ satisfying (A.3). Then the inequality (A.5) is satisfied for all $T < T_0$.

Proof of Proposition 1.2. By Lemma A.4 and Lemma A.6, we find that Φ is a contraction mapping on $X_M(T)$. By Cauchy's fixed point theorem, Φ has a fixed point; that is, there uniquely exists $u \in X_M(T)$ such that $\Phi(u) = u$. This u satisfies (1.3) and is unique in $\{u \in C([0,T]; L^p(\mathbb{R}^n)) : \nabla u \in C([0,T]; L^q(\mathbb{R}^n))\}$. □

Remark A.8. We consider the following initial-boundary problem:

$$\begin{cases} \partial_t u - \Delta u + \dfrac{u}{\varepsilon}(|\nabla u|^2 - 1) = 0, \quad (t,x) \in (0,T) \times \Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{A.6}$$

If Lemma A.1 holds, then we can use our argument and show the existence of a solution of (A.6).

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