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REGULARITY CRITERIA FOR THE GENERALIZED NAVIER-STOKES AND RELATED EQUATIONS

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Abstract. We use the maximum principle type estimate and interpolation inequality on Besov spaces to show some regularity criteria for the generalized Navier-Stokes equations, the quasi-geostrophic equations, and the harmonic heat flow.

1. Introduction

In this paper, we study the regularity criteria for the generalized Navier-Stokes equations, quasi-geostrophic equations, and harmonic heat flow equations.

First, we consider the following generalized Navier-Stokes equations:

$$\begin{aligned} u_t + (u\cdot\nabla)u + \nabla\pi + \Lambda^\ell u = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0(x), \operatorname{div} u_0 = 0, x\in\mathbb{R}^3, \end{aligned} \tag{1.1}$$

where u is the velocity field, π is the scalar pressure, $\Lambda := (-\Delta)^{1/2}$ and ℓ is a positive constant. The case $\ell = 2$ corresponds to the usual Navier-Stokes equations.

The system (1.1) was first studied by J.L. Lions in [17], and the global regularity for $\ell \geq 5/2$ is shown there. When $0 < \ell < 2$, there were studies on the small data global well posedness for (1.1)-(1.3) by D. Chae [5], M. Cannone and G. Karch [2], D. Chae and J. Lee [3], and J. Wu [27]. D. Chae [5]

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showed the regularity condition:

$$\omega := \operatorname{curl} u \in L^r(0,T;L^p(\mathbb{R}^3)), \quad \text{with } \frac{3}{p} + \frac{\ell}{r} = \ell, \tag{1.2}$$

where $\frac{6}{\ell} < p \leq \infty$. We improved this condition (1.2) to the case $\frac{3}{\ell} < p$ and proved the following condition [13]:

$$u \in L^r(0,T;L^p(\mathbb{R}^3)) \text{ with } \frac{\ell}{r} + \frac{3}{p} = \ell - 1, \ \frac{3}{\ell-1} < p \leq \infty, \ \ell > 1. \tag{1.3}$$

When $\ell = 2$, (1.2) is due to H.B. da Veiga [1] and (1.3) corresponds to the well-known Serrin-Ohyama condition [24, 23, 12, 25]. The important case $r = \infty, p = 3$ in (1.3) was proved by Escauriaza-Sverak-Seregin [11]. Montgomery-Smith [21] proved that if

$$\int_0^T \frac{\|u(t)\|_{L^p}^r}{1 + \log^+ \|u(t)\|_{L^p}} dt < \infty, \frac{2}{r} + \frac{3}{p} = 1, \tag{1.4}$$

for $3 < p < \infty$, then u is regular. Kozono-Ogawa-Taniuchi [16] obtained a logarithmic Sobolev inequality to get the following regularity conditions:

$$u \in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)), \tag{1.5}$$

or

$$\omega \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)), \tag{1.6}$$

which refines (1.3) and (1.2) in the case $(r,p) = (1,\infty)$ or $(r,p) = (2,\infty)$. Here, $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov space.

We observe that the system (1.1) is invariant under the scaling transform $(u,\pi) \mapsto (u_\lambda, \pi_\lambda)$, where

$$u_\lambda(t,x) := \lambda^{\ell-1} u(\lambda^\ell t, \lambda x), \pi_\lambda(t,x) := \lambda^{2\ell-2}\pi(\lambda^\ell t, \lambda x), \lambda > 0,$$

which induces the scaling for the vorticity, $\omega \mapsto \omega_\lambda := \lambda^\ell \omega(\lambda^\ell t, \lambda x)$. Furthermore, we note that

$$\|u\|_{L^r(0,T;L^p)} = \|u_\lambda\|_{L^r(0,\lambda^\ell T;L^p)} \text{ if } \frac{\ell}{r} + \frac{3}{p} = \ell - 1;$$

$$\|\pi\|_{L^r(0,T;L^p)} = \|\pi_\lambda\|_{L^r(0,\lambda^\ell T;L^p)} \text{ if } \frac{\ell}{r} + \frac{3}{p} = 2\ell - 2;$$

$$\|\omega\|_{L^r(0,T;L^p)} = \|\omega_\lambda\|_{L^r(0,\lambda^\ell T;L^p)} \text{ if } \frac{\ell}{r} + \frac{3}{p} = \ell.$$

In this sense, the above conditions (1.2) and (1.3) are optimal.

Our first result is:

Theorem 1.1. *Let* u_0 *be smooth and* $\operatorname{div} u_0 = 0$ *in* $\mathbb{R}^3$ *and* $\ell < 5/2$. *Assume that one of the following conditions is satisfied*

$$(1) \quad \ell \geq 2, \int_0^T \frac{\|u(t)\|_{L^p}^r}{1+\log^+ \|u(t)\|_{L^p}} dt < \infty,\ \frac{\ell}{r} + \frac{3}{p} = \ell - 1,\ \frac{3}{\ell-1} < p < \infty, \tag{1.7}$$

$$(2) \quad \ell > 0, \int_0^T \frac{\|\omega(t)\|_{L^p}^r}{1+\log^+ \|\omega(t)\|_{L^p}} dt < \infty,\ \frac{\ell}{r} + \frac{3}{p} = \ell,\ \frac{3}{\ell} < p < \infty. \tag{1.8}$$

Then there is no singularity up to T.

To prove this theorem, we will use the following maximum principle type estimates:

Lemma 1.1. *Let* $0 < \ell < 5/2, 2 \leq p < \infty$ *and* θ *be a smooth function on* $\mathbb{R}^n$. *We have:*

$$2 \int |\Lambda^{\ell/2}(|\theta|^{p/2})|^2 dx \leq p \int \theta |\theta|^{p-2} \Lambda^\ell \theta dx.$$

Next, we study the following 2-D quasi-geostrophic equations [9, 7, 14, 18]:

$$\begin{aligned} &\theta_t + u \cdot \nabla\theta + \Lambda^\ell \theta = 0, \\ &u := -R^\perp \theta = -(-R_2\theta, R_1\theta), \\ &\theta|_{t=0} = \theta_0(x), x \in \mathbb{R}^2, \end{aligned} \tag{1.9}$$

where θ is a scalar function representing temperature, u is the velocity field of the fluid. The Riesz transforms R_1 and R_2 are defined by

$$\widehat{R_k f}(\xi) = -\frac{i\xi_k}{|\xi|} \hat{f}(\xi),$$

where the Fourier transform $\hat{f} = \mathcal{F}(f)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-ix\cdot\xi} dx.$$

The case $1/2 < \ell \leq 1$, is called subcritical since smooth solutions are known to exist globally in time [8]. The case $\ell = \frac{1}{2}$ is called critical since there is a balance between the dissipation and the non-linear term, therefore is a good model for the 3D Navier-Stokes equations. Very recently, Kiselev-Nazarov-Volberg [15] showed the existence of global smooth solutions. The case $0 < \ell < \frac{1}{2}$ is called supercritical and is harder to deal with compared to the other cases. In order to make the similarities to the 3D Navier-Stokes equations more apparent we apply the operation $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$ to $(1.9)_1$ to obtain

$$\partial_t \nabla^\perp \theta + (u \cdot \nabla)\nabla^\perp \theta + \Lambda^\alpha \nabla^\perp \theta = (\nabla^\perp \theta \cdot \nabla) u. \tag{1.10}$$

Then, we observe that $\nabla^\perp \theta$ has the role of vorticity. Recently, D. Chae [4] gives the regularity condition:

$$\nabla^\perp \theta \in L^r(0, T; L^p(\mathbb{R}^2)) \text{ with } \frac{\ell}{r} + \frac{2}{p} = \ell, \ \ \frac{2}{\ell} < p \le \infty. \tag{1.11}$$

J. Yuan [26] improves it to the following condition:

$$\nabla^\perp \theta \in L^r(0, T; \dot{B}^0_{p,\infty}(\mathbb{R}^2)) \text{ with } \frac{\ell}{r} + \frac{2}{p} = \ell, \ \ \frac{2}{\ell} < p \le \infty. \tag{1.12}$$

In this paper, we will prove:

Theorem 1.2. *Let θ be a solution to the quasi-geostrophic equation* (1.9) *whose derivative $\nabla^\perp \theta$ satisfies*

$$\int_0^T \frac{\|\nabla^\perp \theta(t)\|^r_{L^p}}{1 + \log^+ \|\nabla^\perp \theta(t)\|_{L^p}} dt < \infty \quad \text{with} \ \ \frac{\ell}{r} + \frac{2}{p} = \ell, \ \frac{2}{\ell} < p < \infty. \tag{1.13}$$

Then there is no singularity up to T.

Finally, we consider the regularity problem for smooth solutions to the time-dependent harmonic heat flow from $\mathbb{R}^n$ into a unit sphere $\mathbb{S}^m$:

$$\begin{aligned} &u_t - \Delta u = u|\nabla u|^2, \\ &u|_{t=0} = u_0(x), |u_0| = 1, x \in \mathbb{R}^n. \end{aligned} \tag{1.14}$$

The regularity of the weak solution fails in general because of the existence of a blow-up solution for large initial data. The example for the map from $B_1(0) \subset \mathbb{R}^n$ to a sphere was shown by Coron-Ghidaglia [10] for $n \ge 3$ and Chang-Ding-Ye [6] for $n = 2$. However, some smallness assumption on the initial data or integrability condition on the solution itself are sufficient to give the regularity. Ogawa [22] showed the following regularity conditions:

$$\nabla u \in L^r(0, T; L^p(\mathbb{R}^n)), \frac{2}{r} + \frac{n}{p} = 1, n < p \le \infty, \tag{1.15}$$

or

$$\nabla u \in L^2(0, T; \dot{F}^0_{\infty,2}(\mathbb{R}^n)). \tag{1.16}$$

Here, $\dot{F}^0_{\infty,2}$ is the homogeneous Triebel-Lizorkin space.

We will improve (1.15) and (1.16) to the following results.

Theorem 1.3. *Let u be a smooth solution to the harmonic heat flow* (1.14). *Assume that one of the following conditions is satisfied:*

$$(1) \quad \int_0^T \frac{\|\nabla u(t)\|_{L^p}^r}{1+\log^+\|\nabla u(u)\|_{L^p}} dt < \infty \quad \text{with} \quad \frac{2}{r}+\frac{n}{p}=1 \quad \text{and} \quad n<p<\infty; \tag{1.17}$$

$$(2) \quad \int_0^T \frac{\|\Delta u(t)\|_{L^p}^r}{1+\log^+\|\Delta u(t)\|_{L^p}} dt < \infty \quad \text{with} \quad \frac{2}{r}+\frac{n}{p}=2 \quad \text{and} \quad \frac{n}{2}<p<\infty; \tag{1.18}$$

$$(3) \quad 2\le n\le 4, \nabla u \in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)); \tag{1.19}$$

$$(4) \quad \Delta u \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)). \tag{1.20}$$

Then there is no singularity up to T.

Remark 1.1. Since $\dot{F}^0_{\infty,2}(\mathbb{R}^n) \subset \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$, our result improves that of Ogawa [22] when $2\le n\le 4$. Moreover, our proof of (1.19) below is simple.

Theorem 1.1 is proved in Section 2, Theorem 1.2 is proved in Section 3, and the final Section 4 is devoted to the proof of Theorem 1.3.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In the following proofs, we will use the following inequalities:

$$\|\pi\|_{L^s} \le C\|u\|_{L^{2s}}^2, 1<s<\infty; \tag{2.1}$$

$$\|\nabla f\|_{L^s} \le C\|\Lambda^\alpha f\|_{L^p}, 1-\frac{n}{s}=\alpha-\frac{n}{p}; \tag{2.2}$$

$$\|f\|_{L^q} \le C\|f\|_{L^2}^{1-\beta}\|\Lambda^\alpha f\|_{L^2}^\beta \quad \text{with} \quad -\frac{n}{q}=(1-\beta)\left(-\frac{n}{2}\right)+\beta\left(\alpha-\frac{n}{2}\right). \tag{2.3}$$

(1) First, we assume that the condition (1.7) holds.

Multiplying $(1.1)_1$ by $|u|^{p-2}u$ with $p\ge 2$, integrating by parts, using Lemma 1.1, $\operatorname{div} u=0$, Hölder's inequality, and (2.1)-(2.2), we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int |u|^p dx + \frac{2}{p}\int |\Lambda^{\ell/2}(|u|^{p/2})|^2 dx \le C\int |\pi||u|^{(p-2)/2}|\nabla |u|^{p/2}|dx \\
&\le C\|\pi\|_{L^{3(p+2)/(2(\ell+1))}}\||u|^{(p-2)/2}\|_{L^{6(p+2)/((p-2)(\ell+1))}}\|\nabla|u|^{p/2}\|_{L^{6/(5-\ell)}} \\
&\le C\|u\|^2_{L^{3(p+2)/(\ell+1)}}\|u\|^{(p-2)/2}_{L^{3(p+2)/(\ell+1)}}\|\Lambda^{\ell/2}(|u|^{p/2})\|_2 \qquad (2.4)\\
&\le C\|u\|^{(p+2)/2}_{L^{3(p+2)/(\ell+1)}}\|\Lambda^{\ell/2}(|u|^{p/2})\|_{L^2} \le \epsilon\|\Lambda^{\ell/2}(|u|^{p/2})\|^2_{L^2} + C\|u\|^{p+2}_{L^{3(p+2)/(\ell+1)}},
\end{aligned}$$

for any $\epsilon > 0$ by Young's inequality. We use (2.3) with $\beta = \frac{2p+6-p\ell}{(p+2)\ell}$ to infer that

$$\begin{aligned}
\|u\|_{L^{3(p+2)/(\ell+1)}}^{p+2} &= \||u|^{p/2}\|_{L^{6(p+2)/(p(\ell+1))}}^{2(p+2)/p} \\
&\leq C\||u|^{p/2}\|_{L^2}^{\frac{2(p+2)}{p}(1-\beta)}\|\Lambda^{\ell/2}(|u|^{p/2})\|_{L^2}^{\frac{2(p+2)}{p}\beta} \\
&\leq C\|u\|_{L^p}^{(p+2)(1-\beta)}\|\Lambda^{\ell/2}(|u|^{p/2})\|_{L^2}^{\frac{2(p+2)}{p}\beta} \\
&\leq \epsilon\|\Lambda^{\ell/2}(|u|^{p/2})\|_{L^2}^2 + C\|u\|_{L^p}^{(p+2)(1-\beta)\cdot\frac{p}{(1-\beta)p-2\beta}} \\
&\leq \epsilon\|\Lambda^{\ell/2}(|u|^{p/2})\|_{L^2}^2 + C\|u\|_{L^p}^{p+r},
\end{aligned} \tag{2.5}$$

with $r = \dfrac{\ell p}{\ell p - p - 3}$ so that $\frac{\ell}{r} + \frac{3}{p} = \ell - 1$. Inserting (2.5) into (2.4) and taking ϵ small enough, we see that

$$\begin{aligned}
\frac{d}{dt}\|u\|_{L^p} &\leq C\|u\|_{L^p}\|u\|_{L^p}^r \\
&\leq C\|u\|_{L^p}(1+\log^+\|u\|_{L^p})\cdot\frac{\|u\|_{L^p}^r}{1+\log^+\|u\|_{L^p}},
\end{aligned}$$

which leads to $\|u(t)\|_{L^\infty(0,T;L^p)} \leq C$, whence $\|u(t)\|_{L^r(0,T;L^p)} \leq C$. This proves Theorem 1.1 by (1.3).

(2) We assume that condition (1.8) holds true.

Applying curl to $(1.1)_1$, we see that

$$\omega_t + (u\cdot\nabla)\omega + \Lambda^\ell\omega = (\omega\cdot\nabla)u.$$

Multiplying the above equation by $|\omega|^{p-2}\omega$, integrating by parts, using Lemma 1.1, $\operatorname{div} u = 0$, Hölder's inequality, and (2.3), we get

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int|\omega|^p dx + \frac{2}{p}\int|\Lambda^{\ell/2}(|\omega|^{p/2})|^2 dx \\
\leq\ & \int|\omega||\nabla u||\omega|^{p-1}dx \leq \|\omega\|_{L^{p+1}}^p\|\nabla u\|_{L^{p+1}} \\
\leq\ & C\|\omega\|_{L^{p+1}}^{p+1} = C\||\omega|^{p/2}\|_{L^{\frac{2}{p}(p+1)}}^{\frac{2}{p}(p+1)} \\
\leq\ & C\||\omega|^{p/2}\|_{L^2}^{\frac{2}{p}(p+1)(1-\beta)}\|\Lambda^{\ell/2}(|\omega|^{p/2})\|_{L^2}^{\frac{2}{p}(p+1)\beta} \\
\leq\ & \epsilon\|\Lambda^{\ell/2}(|\omega|^{p/2})\|_{L^2}^2 + C\|\omega\|_{L^p}^{p+r},
\end{aligned}$$

with r satisfying $\frac{\ell}{r}+\frac{3}{p}=\ell$. Taking ϵ small enough yields

$$\begin{aligned}\frac{d}{dt}\|\omega\|_{L^p} &\leq C\|\omega\|_{L^p}\|\omega\|_{L^p}^r \\ &\leq C\|\omega\|_{L^p}(1+\log^+\|\omega\|_{L^p})\cdot\frac{\|\omega\|_{L^p}^r}{1+\log^+\|\omega\|_{L^p}},\end{aligned}$$

which implies $\|\omega\|_{L^\infty(0,T;L^p)}\leq C$. The result follows from (1.2). This completes the proof. □

3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. We assume that the condition (1.13) holds true. Multiplying (1.10) by $|\nabla^\perp\theta|^{p-2}\nabla^\perp\theta$, integrating by parts, using Lemma 1.1, $\operatorname{div} u=0$, Hölder's inequality, and (2.3) for $n=2$, we have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int|\nabla^\perp\theta|^p dx+\frac{2}{p}\int|\Lambda^{\ell/2}(|\nabla^\perp\theta|^{p/2})|^2dx \\ \leq\ &\int|\nabla^\perp\theta|^p\cdot|\nabla u|dx\leq\|\nabla^\perp\theta\|_{L^{p+1}}^p\|\nabla u\|_{L^{p+1}} \\ \leq\ &C\|\nabla^\perp\theta\|_{L^{p+1}}^{p+1}=C\||\nabla^\perp\theta|^{p/2}\|_{L^{\frac{2}{p}(p+1)}}^{\frac{2}{p}(p+1)} \\ \leq\ &C\||\nabla^\perp\theta|^{p/2}\|_{L^2}^{\frac{2}{p}(p+1)(1-\beta)}\|\Lambda^{\ell/2}(|\nabla^\perp\theta|^{p/2})\|_{L^2}^{\frac{2}{p}(p+1)\beta} \\ \leq\ &\epsilon\|\Lambda^{\ell/2}(|\nabla^\perp\theta|^{p/2})\|_{L^2}^2+C\|\nabla^\perp\theta\|_{L^p}^{p+r},\end{aligned}$$

with r satisfying $\frac{\ell}{r}+\frac{2}{p}=\ell$. Taking ϵ small enough leads to

$$\begin{aligned}\frac{d}{dt}\|\nabla^\perp\theta\|_{L^p} &\leq C\|\nabla^\perp\theta\|_{L^p}\|\nabla^\perp\theta\|_{L^p}^r \\ &\leq C\|\nabla^\perp\theta\|_{L^p}(1+\log^+\|\nabla^\perp\theta\|_{L^p})\cdot\frac{\|\nabla^\perp\theta\|_{L^p}^r}{1+\log^+\|\nabla^\perp\theta\|_{L^p}},\end{aligned}$$

which implies $\|\nabla^\perp\theta\|_{L^\infty(0,T;L^p)}\leq C$. The result follows from (1.11) and thus the proof is complete. □

4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3.

(1) First, we assume that the condition (1.17) holds true.

Applying Λ to $(1.14)_1$, we see that

$$\partial_t\Lambda u-\Delta\Lambda u=\Lambda(u|\nabla u|^2).$$

Multiplying the above equation by $|\Lambda u|^{p-2}\Lambda u$, integrating by parts, and using Hölder's inequality, we deduce that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int|\Lambda u|^p dx+\frac{2}{p}\int|\nabla|\Lambda u|^{p/2}|^2dx\\ \le\ & C\int|\nabla u|^2|\Lambda u|^{p-2}|\nabla\Lambda u|dx\le C\int|\nabla u|^{\frac{p}{2}+1}|\nabla|\Lambda u|^{p/2}|dx\\ \le\ & \epsilon\int|\nabla|\Lambda u|^{p/2}|^2dx+C\int|\Lambda u|^{p+2}dx, \end{aligned}\tag{4.1}$$

for any $\epsilon>0$ by Young's inequality. We use (2.3) for $\alpha=1$ to bound

$$\begin{aligned} \int|\Lambda u|^{p+2}dx&=\||\Lambda u|^{p/2}\|_{L^{\frac{2}{p}(p+2)}}^{\frac{2}{p}(p+2)}\le C\||\Lambda u|^{p/2}\|_{L^2}^{\frac{2}{p}(p+2)(1-\beta)}\|\nabla|\Lambda u|^{p/2}\|_{L^2}^{\frac{2}{p}(p+2)\cdot\beta}\\ &\le\epsilon\|\nabla|\Lambda u|^{p/2}\|_{L^2}^2+C\|\Lambda u\|_{L^p}^{p+r}, \end{aligned}\tag{4.2}$$

with r satisfying $\frac{2}{r}+\frac{n}{p}=1$. Inserting (4.2) into (4.1) and taking ϵ small enough, we get

$$\begin{aligned} \frac{d}{dt}\|\Lambda u\|_{L^p}&\le C\|\Lambda u\|_{L^p}\|\Lambda u\|_{L^p}^r\\ &\le C\|\Lambda u\|_{L^p}(1+\log^+\|\Lambda u\|_{L^p})\cdot\frac{\|\Lambda u\|_{L^p}^r}{1+\log^+\|\Lambda u\|_{L^p}}, \end{aligned}$$

which implies $\|\Lambda u\|_{L^\infty(0,T;L^p)}\le C$. The result follows from (1.15).

(2) We assume that the condition (1.18) holds true.

Applying Δ to equation $(1.14)_1$, we find that

$$\Delta u_t-\Delta^2u=\Delta(u|\nabla u|^2).\tag{4.3}$$

Multiplying the above equation by $|\Delta u|^{p-2}\Delta u$, integrating by parts, and using Hölder's inequality, we see that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int|\Delta u|^p dx+\frac{2}{p}\int|\nabla|\Delta u|^{p/2}|^2dx\\ &\le C\int|\nabla(u|\nabla u|^2)|\cdot|\Delta u|^{p-2}|\nabla\Delta u|dx\\ &\le C\int(|\nabla u|^3+|\nabla u|\cdot|\Delta u|)|\Delta u|^{\frac{p}{2}-1}\cdot|\nabla|\Delta u|^{p/2}|dx\\ &\le C(\|\nabla u\|_{L^{2(p+1)}}^3+\|\nabla u\|_{L^{2(p+1)}}\|\Delta u\|_{L^{p+1}})\||\Delta u|^{\frac{p}{2}-1}\|_{L^{\frac{1}{p/2-1}(p+1)}}\|\nabla|\Delta u|^{p/2}\|_{L^2}\\ &\le C\|\Delta u\|_{L^{p+1}}^{\frac{p+1}{2}}\|\nabla|\Delta u|^{p/2}\|_{L^2} \end{aligned}$$

$$\leq \epsilon\|\nabla|\Delta u|^{p/2}\|_{L^2}^2 + C\|\Delta u\|_{L^{p+1}}^{p+1}, \tag{4.4}$$

for any $\epsilon > 0$ by Young's inequality.

Here we have used the Gagliardo-Nirenberg inequality:

$$\|\nabla u\|_{L^{2s}} \leq C\|u\|_{L^\infty}^{1/2}\|\Delta u\|_{L^s}^{1/2}. \tag{4.5}$$

We use (2.3) for $\alpha = 1$ to estimate

$$\begin{aligned}\|\Delta u\|_{L^{p+1}}^{p+1} &= \||\Delta u|^{p/2}\|_{L^{\frac{2}{p}(p+1)}}^{\frac{2}{p}(p+1)} \leq C\||\Delta u|^{p/2}\|_{L^2}^{\frac{2}{p}(p+1)(1-\beta)}\|\nabla|\Delta u|^{p/2}\|_{L^2}^{\frac{2}{p}(p+1)\beta} \\ &\leq \epsilon\|\nabla|\Delta u|^{p/2}\|_{L^2}^2 + C\|\Delta u\|_{L^p}^{p+r},\end{aligned} \tag{4.6}$$

with r satisfying $\frac{2}{r} + \frac{n}{p} = 2$.

Inserting (4.6) into (4.4) and taking ϵ small enough, we get

$$\begin{aligned}\frac{d}{dt}\|\Delta u\|_{L^p} &\leq C\|\Delta u\|_{L^p}\|\Delta u\|_{L^p}^r \\ &\leq C\|\Delta u\|_{L^p}(1 + \log^+\|\Delta u\|_{L^p}) \cdot \frac{\|\Delta u\|_{L^p}^r}{1 + \log^+\|\Delta u\|_{L^p}},\end{aligned}$$

which implies $\|\Delta u\|_{L^\infty(0,T;L^p)} \leq C$ and thus the solution u is regular.

(3) We assume that (1.19) holds true.

Multiplying (4.3) by Δu, integrating by parts, and using (4.5) for $s = 4$ and $s = 2$, we see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\Delta u|^2dx + \int|\nabla\Delta u|^2dx \\ \leq\ &\int|\nabla(u|\nabla u|^2)| \cdot |\nabla\Delta u|dx \\ \leq\ &C\int(|\nabla u|^3 + |\nabla u| \cdot |\Delta u|)|\nabla\Delta u|dx \\ \leq\ &C(\|\nabla u\|_{L^4}\|\nabla u\|_{L^8}^2 + \|\nabla u\|_{L^4}\|\Delta u\|_{L^4})\|\nabla\Delta u\|_{L^2} \\ \leq\ &C\|\nabla u\|_{L^4}\|\Delta u\|_{L^4}\|\nabla\Delta u\|_{L^2} \leq C\|\Delta u\|_{L^2}^{1/2} \cdot \|\nabla u\|_{\dot{B}^0_{\infty,\infty}}^{1/2}\|\nabla\Delta u\|_{L^2}^{3/2} \\ \leq\ &\frac{1}{2}\|\nabla\Delta u\|_{L^2}^2 + C\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}^2\|\Delta u\|_{L^2}^2.\end{aligned} \tag{4.7}$$

Here we have used the interpolation inequality [19, 20]:

$$\|\Delta u\|_{L^4} \leq C\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}^{1/2}\|\nabla\Delta u\|_{L^2}^{1/2}.$$

Now, Gronwall's inequality gives

$$\|\Delta u\|_{L^\infty(0,T;L^2)} + \|\nabla\Delta u\|_{L^2(0,T;L^2)} \le C. \tag{4.8}$$

(4) Finally, we assume that (1.20) holds true.

Testing (4.3) by $|\Delta u|^{p-2}\Delta u$ $(p > n)$, using the same calculations as that in Ogawa [22], we obtain

$$\frac{d}{dt}\int |\Delta u|^p dx + \frac{4(p-2)}{p}\int |\nabla|\Delta u|^{p/2}|^2 dx \le C\|\nabla u\|_{L^\infty}^2 \int |\Delta u|^p dx. \tag{4.9}$$

Using $\dot{B}^0_{\infty,1} \subset L^\infty \subset \dot{B}^0_{\infty,\infty}$ and another inequality in [19] gives:

$$\|\nabla u\|^2_{\dot{B}^0_{\infty,1}} \le C\|u\|_{\dot{B}^0_{\infty,\infty}} \|\Delta u\|_{\dot{B}^0_{\infty,\infty}}. \tag{4.10}$$

We see that

$$\begin{aligned} \|\nabla u\|^2_{L^\infty} &\le C\|\nabla u\|^2_{\dot{B}^0_{\infty,1}} \le C\|u\|_{\dot{B}^0_{\infty,\infty}}\|\Delta u\|_{\dot{B}^0_{\infty,\infty}} \\ &\le C\|u\|_{L^\infty}\|\Delta u\|_{\dot{B}^0_{\infty,\infty}} \le C\|\Delta u\|_{\dot{B}^0_{\infty,\infty}}. \end{aligned}$$

Plugging the above inequality into (4.9) and using Gronwall's inequality give

$$\|\Delta u\|_{L^\infty(0,T;L^p)} \le C.$$

This completes the proof. □

REFERENCES

[1] H.B. da Veiga, *A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$,* Chinese Ann. Math., 16 (1995), 407–412.

[2] M. Cannone and G. Karch, *Incompressible Navier-Stokes equations in abstract Banach spaces,* in "Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics," (2001), 27–41.

[3] D. Chae and J. Lee, *On the Global well-posedness and stability of the Navier-Stokes and the related equations,* Adv. Math. Fluid Mech., "Contributions to Current Challenges in Mathematical Fluid Mechanics," Birkhäuser, Basel, (2004), 31–51.

[4] D. Chae, *On the regularity conditions for the dissipative quasi-geostrophic equations,* SIAM J. Math. Anal., 37 (2006), 1649–1656.

[5] D. Chae, *On the regularity conditions for the Navier-Stokes and related equations,* Revista Mat. Iberoamericana, 23 (2007), 373–386.

[6] K. C. Chang, W.Y. Ding, and R. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces,* J. Diff. Geom., 36 (1992), 507–515.

[7] P. Constantin, D. Córdoba, and J. Wu, *On the critical dissipative quasi-geostrophic equation,* Indiana Univ. Math. J., 50 (2001), 97–107.

[8] P. Constantin and J. Wu, *Behavior of solutions of 2D quasi-geostrophic equations,* SIAM J. Math. Anal., 30 (1999), 937–948.

[9] A. Córdoba and D. Córdoba, *A maximum principle applied to quasi-geostrophic equations,* Comm. Math. Phys., 249 (2004), 511–528.

[10] J.M. Coron and J.M. Ghidaglia, *Explision en temps fini pour le flot des applications harmoniques,* C.R. Acad. Sci. Paris, 308 (1989), 339–344.

[11] L. Escauriaza, G. Seregin, and V. Sverak, $L^{3,\infty}$*-solutions of Navier-Stokes equations and backward uniqueness,* Russian Math. Surveys, 58 (2003), 211–250.

[12] E. Fabes, B. Jones, and N. Riviere, *The initial value problem for the Navier-Stokes equations with data in* L^p*,* Arch. Rat. Mech. Anal., 45 (1972), 222–248.

[13] J. Fan and T. Ozawa, *On the regularity criteria for the generalized Navier-Stokes equations and Lagrangian averaged Euler equations,* Differential and Integral Equations, 21 (2008), 443–457.

[14] N. Ju, *The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations,* Commun. Math. Phys., 255 (2005), 161–181.

[15] A. Kiselev, F. Nazarov, and A. Volberg, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation,* arXiv, 2006.

[16] H. Kozono, T. Ogawa, and Y. Taniuchi, *The critical Sobolev inequalities in Besov spaces and regularity criterion to some semilinear evolution equations,* Math. Z., 242 (2002), 251–278.

[17] J.L. Lions, “Quelques Méthods de Résolution des Problémes aux Limites Non-linéaires,” Dunod, Paris, (1969).

[18] F. Marchand, *Existence and regularity of weak solutions to the quasi-geostrophic equations in the spaces* L^p *or* $\dot{H}^{-1/2}$*,* Commun. Math. Phys., online, 2007.

[19] S. Machihara and T. Ozawa, *Interpolation inequalities in Besov spaces,* Proc. Amer. Math. Soc., 131 (2002), 1553–1556.

[20] Y. Meyer, *Oscillating patterns in some nonlinear evolution equations,* pp. 101-187, Mathematical Foundation of Turbulent Viscous Flows, Lecture Notes in Math., 1871, Editors: M. Cannone, T. Miyakawa, Springer-Verlag, 2006.

[21] S. Montgomery-Smith, *Conditions implying regularity of the three dimensional Navier-Stokes equation,* Applications of Mathematics, 50 (2005), 451–464.

[22] T. Ogawa, *Sharp Sobolev inequality of logarithmic type and the limiting regularity condition to the harmonic heat flow,* SIAM J. Math. Anal., 34 (2003), 1318–1330.

[23] T. Ohyama, *Interior regularity of weak solutions to the Navier-Stokes equations,* Proc. Japan Acad., 36 (1960), 273–277.

[24] J. Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations,* Arch. Rat. Mech. Anal., 9 (1962), 187–191.

[25] M. Struwe, *On partial regularity results for the Navier-Stokes equations,* Comm. Pure Appl. Math., 41 (1988), 437–458.

[26] J. Yuan, *On regularity criterion for the dissipative quasi-geostrophic equations,* J. Math. Anal. Appl., to appear.

[27] J. Wu, *The generalized incompressible Navier-Stokes equations in Besov spaces,* Dynamics of PDE, 1 (2004), 381–400.