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STATIONARY SOLUTIONS OF THE SCHRÖDINGER-NEWTON MODEL - AN ODE APPROACH

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Abstract. We prove the existence and uniqueness of stationary spherically symmetric positive solutions for the Schrödinger-Newton model in any space dimension d. Our result is based on an analysis of the corresponding system of second-order differential equations. It turns out that $d = 6$ is critical for the existence of finite energy solutions and the equations for positive spherically symmetric solutions reduce to a Lane-Emden equation for all $d \geq 6$. Our result implies, in particular, the existence of stationary solutions for two-dimensional self-gravitating particles and closes the gap between the variational proofs in $d = 1$ and $d = 3$.

1. Introduction

We consider the Schrödinger-Newton equations

$$i\psi_t + \Delta\ \psi - \gamma V \psi = 0, \quad \Delta\ V = |\psi|^2, \tag{1.1}$$

on $\mathbb{R}^d$ which are equivalent to the non-linear Schrödinger equation

$$i\psi_t + \Delta\psi + \gamma(G_d(|x|) * |\psi|^2)\psi = 0, \tag{1.2}$$

where $G_d(|x|)$ denotes the Green's function of the Laplacian on $\mathbb{R}^d$. Of physical interest are solutions having finite energy E and particle number (or charge) N given by

$$E(\psi) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi(x,t)|^2\, dx - \frac{\gamma}{4}\iint_{\mathbb{R}^{2d}} G_d(|x-y|)|\psi(x,t)|^2|\psi(y,t)|^2\, dxdy, \tag{1.3}$$

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and

$$N(\psi) = \int_{\mathbb{R}^d} |\psi(x,t)|^2 \, dx, \tag{1.4}$$

respectively.

In the present work, we study in the attractive case $\gamma > 0$ the existence and uniqueness of spherically symmetric quasi stationary solutions of the form

$$\psi(t,x) = u_\omega(|x|)e^{-i\omega t}, \quad u_\omega(|x|) > 0, \quad \lim_{|x|\to\infty} u_\omega(|x|) = 0, \tag{1.5}$$

which we call ground states. For solutions of the form (1.5), we have $V(t,x) = v_\omega(|x|)$ and $u_\omega(r), v_\omega(r)$ satisfy the following system of ordinary differential equations:

$$\begin{aligned} u_\omega'' + \frac{d-1}{r}\, u_\omega' &= (\gamma v_\omega - \omega)u_\omega, \\ v_\omega'' + \frac{d-1}{r}\, v_\omega' &= u_\omega^2, \quad r \geq 0. \end{aligned} \tag{1.6}$$

We suppose that $u_\omega(0), v_\omega(0)$ are finite and $u_\omega'(0) = v_\omega'(0) = 0$. The latter equation implies that $v_\omega' \geq 0$, and therefore for solutions u_ω vanishing at infinity we have $\omega - \gamma v_\omega(0) > 0$. By rescaling $u(r) = Au_\omega(r/\sigma)$, $V(r) = B(v_\omega(r/\sigma) - \omega/\gamma) + 1$ with

$$\sigma^2 = \omega - \gamma v_\omega(0), \quad A = \frac{\sqrt{\gamma}}{\sigma^2}, \quad B = \frac{\gamma}{\sigma^2},$$

we obtain the universal equations

$$\begin{aligned} u'' + \frac{d-1}{r}\, u' &= (V-1)u \\ V'' + \frac{d-1}{r}\, V' &= u^2, \end{aligned} \tag{1.7}$$

subject to the initial conditions

$$u(0) = u_0 \in \mathbb{R}^+, \ u'(0) = 0, \ V(0) = 0, V'(0) = 0. \tag{1.8}$$

Here, $d > 0$ may be regarded as a continuous parameter. By analyzing the solutions of the above initial-value problem, we shall prove the following result about the existence and uniqueness of ground states:

Theorem 1.1. *For any $d > 0$, the system* (1.7) *subject to the initial conditions* (1.8) *admits a unique solution (u, V) such that $u(r) > 0$, $u'(r) < 0$ on $(0, \infty)$ and*

$$\lim_{r\to\infty} u(r) = 0. \tag{1.9}$$

The critical dimension is $d = 6$ and the properties of ground states change at this value.

Theorem 1.2. *If $d = 6$, then $u(r)$ is explicitly given by*

$$u(r) = \left(1 + \frac{r^2}{24}\right)^{-2}. \tag{1.10}$$

In addition, for all $d \geq 6$ the relation $u(r) = 1 - V(r)$ holds and the function $u(r)$ solves the Lane-Emden equation

$$u'' + \frac{d-1}{r}\, u' = -u^2, \tag{1.11}$$

subject to the initial conditions $u(0) = 1,\ u'(0) = 0$. If $d < 6$, then $u(0) > 1$.

The decay properties of these solutions will imply that they have finite energy and particle number if and only if $d \leq 6$.

In physical and mathematical literature, the Schrödinger-Newton system, in three space dimensions, has a long-standing history. With γ designating appropriate positive coupling constants it appeared first in 1954, then in 1976, and lastly in 1996 for describing the quantum mechanics of a Polaron at rest by S. J. Pekar [15], of an electron trapped in its own hole by the first author [11] and of self-gravitating matter by R. Penrose [18]. In 1977, E. Lieb [11] showed the existence of a unique ground state of the form (1.5) in three space dimensions by solving an appropriate minimization problem. This ground state solution $u_\omega(x), \omega > 0$ is a positive spherically symmetric strictly decreasing function. In [12], P.L. Lions proved the existence of infinitely many distinct spherically symmetric solutions and claimed a proof for the existence of anisotropic bound states in [13].

While Lieb's existence proof can be easily extended to dimensions $d = 4$ and $d = 5$, the situation is unclear for lower dimensions due to the lack of positivity of the Coulomb interaction energy term. For the one-dimensional problem, this difficulty has been overcome recently in [4] and the existence of a unique ground state of the form (1.5) has been shown by solving a minimization problem. The two-dimensional problem, however, remained open and so far only numerical studies are available indicating the existence of bound states, see e.g. [6]. Our main result proves the existence of such solutions.

From the variational point of view, the critical dimension $d = 6$ is related to a Hardy-Littlewood-Sobolev inequality of the form

$$\iint_{\mathbb{R}^{12}} |x-y|^{-4}|u(x)|^2|u(y)|^2 \, dxdy \leq C\Big(\int_{\mathbb{R}^6} |\nabla u(x)|^2 \, dx\Big)^2, \tag{1.12}$$

for a positive constant C. Our solution (1.10) is indeed an optimizer in this inequality. Instead of proving this inequality directly, we deduce it together with the optimal constant by simply combining two inequalities of [10] (see Appendix). From the ODE point of view the system of ordinary differential equations has a conformal invariance for $d = 6$ which leads to a one-dimensional autonomous system with a Yukawa-type interaction. In addition, the problem of finding a positive solution can be reduced to solving a (conformally invariant) Lane-Emden equation [3], [5], [9]. We do not make use of this property but we believe that this observation may be useful for further studies of these equations and we give the corresponding autonomous system in the Appendix.

To prove the main result we use a shooting method which various authors have successfully applied to existence and uniqueness of solutions in boundary-value problems for non-linear differential equations [1, 2, 7, 8, 16, 17]. Our paper is organized as follows: In Section 2, we employ a shooting method to prove the existence of ground states (Theorem 2.3). In Section 3, we study their decay properties to prove uniqueness by analyzing the Wronskian of solutions (Theorem 3.6). Finally, in Section 4, we prove Theorem 1.2 including the explicit solution for $d = 6$.

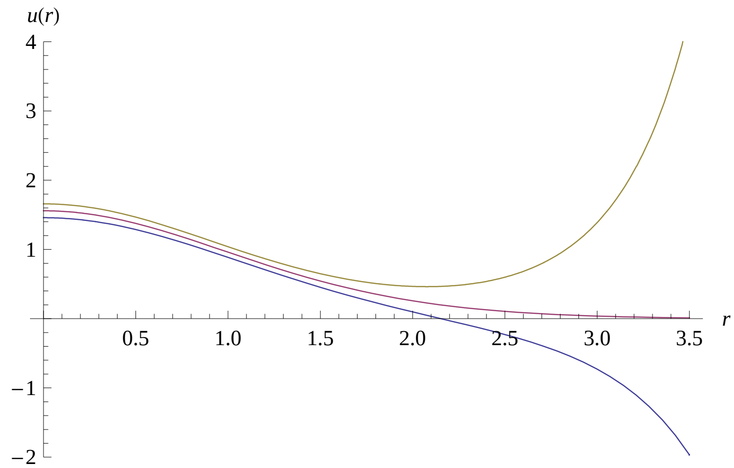
2. Existence of ground states

We begin our study with the discussion of some general properties of solutions of (1.7) with initial values (1.8). Standard results will guarantee local existence and uniqueness of solutions, their continuous dependence on the initial values as well as on the parameter d, and their regularity. As a consequence of local existence and uniqueness solutions cannot have double zeros. We shall frequently apply these properties in the sequel as well as in the following integral equations for u' and V':

$$\begin{aligned} u'(r) &= \frac{1}{r^{d-1}} \int_0^r (V(s)-1)u(s)s^{d-1} \, ds \\ V'(r) &= \frac{1}{r^{d-1}} \int_0^r u^2(s)s^{d-1} \, ds. \end{aligned} \tag{2.1}$$

Viewed as a mechanical system we can associate an energy to (1.7) given by

$$2\mathcal{E} = u'^2 + u^2 + \tfrac{1}{2}V'^2 - Vu^2, \tag{2.2}$$



Solutions with initial values in $\mathcal{N}, \mathcal{G}, \mathcal{P}$

which satisfies

$$\mathcal{E}' = -\frac{d-1}{r}\, u'^2 - \frac{d-1}{2r}\, V'^2.$$

Therefore, $\mathcal{E}$ is a constant of motion if $d = 1$. However, we shall not use this property in the present work.

For the initial condition $u_0 > 0$ of the solution (u, V) we consider the following mutually disjoint sets:

Definition 2.1.

$$\mathcal{N} = \{u_0 \in \mathbb{R}_+ : \exists r_0 > 0 \textit{ such that } u(r_0) < 0 \textit{ and } u'(r) < 0 \textit{ on } (0, r_0]\ \}, \tag{2.3}$$

$$\mathcal{G} = \{u_0 \in \mathbb{R}_+ : u \geq 0,\ \lim_{r \longrightarrow \infty} u(r) = 0\}, \tag{2.4}$$

$$\mathcal{P} = \{u_0 \in \mathbb{R}_+ : \exists r_1 > 0 \textit{ such that } u'(r_1) > 0 \textit{ and } u(r) > 0 \textit{ on } (0, r_1]\ \}. \tag{2.5}$$

In order to see that $\mathcal{G}$ and $\mathcal{P}$ are disjoint note that since $u''(0) = -u_0/d$ all solutions start strictly decreasing. Therefore, any solution with initial condition u_0 in $\mathcal{P}$ has a local minimum before r_1 where $V \geq 1$. From (2.1) we deduce that V is strictly increasing. Therefore, u' will remain positive and bounded away from zero after r_1 and u becomes unbounded. If $u_0 \notin \mathcal{N} \cup \mathcal{P}$, then $u \geq 0$ and $u' \leq 0$. Therefore, it has a limit as r tends to infinity which must be zero. This implies $\mathcal{N} \cup \mathcal{G} \cup \mathcal{P} = \mathbb{R}_+$. From the continuous dependence on initial values we deduce that $\mathcal{N}, \mathcal{P}$ are open sets.

Our main result Theorem 1.1 states that $\mathcal{G}$ consists of exactly one element. Obviously, $\mathcal{G}$ is nonempty if $\mathcal{N}$ and $\mathcal{P}$ are nonempty which we show in the following lemma.

Lemma 2.2. *For any $d > 0$ the sets $\mathcal{N}$ and $\mathcal{P}$ are non-empty. In particular, $(0,1) \subset \mathcal{N}$.*

Proof. We consider the function

$$\phi = u + V - 1. \tag{2.6}$$

It satisfies the differential equation

$$\phi'' + \frac{d-1}{r}\,\phi' = u\phi,$$

and admits the Taylor expansion

$$\phi(r) = (u_0 - 1)\big(1 + \frac{u_0 r^2}{2d}\big) + o(r^2).$$

Let $u_0 < 1$ and suppose $u_0 \notin \mathcal{N}$. Then, ϕ is negative and strictly decreasing for all $r > 0$ sufficiently small. By hypothesis u remains strictly positive. Consequently, ϕ cannot have a critical point since then $\phi'' = u\phi < 0$ which is impossible. We conclude that $\phi(r) < \phi(0) = u_0 - 1$ for all $r > 0$ or, equivalently, $u(r) + V(r) < u_0$. Hence, $u_0 \in \mathcal{G}$ and $V(r) < u_0$ for all $r > 0$. Since V is always strictly increasing $V_\infty := \lim_{r\longrightarrow\infty} V(r)$ exists and $V_\infty \le u_0 < 1$. We consider the function $z := -\frac{u'}{u}$. Since $u_0 \in \mathcal{G}$ it follows that z exists for all $r > 0$ and $z(r) > 0$ for all $r > 0$. It satisfies the differential equation

$$z' = z^2 - \frac{d-1}{r}\,z + 1 - V.$$

Choose $\tilde{r}$ such that $\frac{d-1}{r} \le \sqrt{2(1 - V_\infty)}$ for all $r \ge \tilde{r}$. Then, for $r \ge \tilde{r}$ we have

$$z' \ge \frac{1}{2}z^2 + \Big(\frac{1}{2}z^2 - \sqrt{2(1 - V_\infty)}z + 1 - V_\infty\Big) \ge \frac{1}{2}z^2.$$

This implies that z blows up in finite time, which is impossible. Hence, $u_0 \in \mathcal{N}$.

Next, we want to show that $u_0 \in \mathcal{P}$ for u_0 sufficiently large. Suppose on the contrary that $\mathcal{P}$ is empty and let $u_0 > 1$. Denote by $(0, R_0)$ the maximal interval where $u > 0$ and $u' < 0$. Therefore, from equation (2.1) for V' we obtain the bounds

$$\frac{u(r)^2 r}{d} \le V'(r) \le \frac{u_0^2 r}{d} \quad \text{on } (0, R_0).$$

Integrating these inequalities and using again the fact that u is decreasing yields the following estimates for V:

$$\frac{u(r)^2 r^2}{2d} \le V(r) \le \frac{u_0^2 r^2}{2d} \quad \text{on } (0, R_0).$$

By a similar reasoning as before we see that the function ϕ defined in (2.6) is strictly increasing on $(0, R_0)$. Hence,

$$u(r) > u_0 - V(r) \quad \text{on } (0, R_0 0.$$

Let $r_0 = \sqrt{2d/u_0}$. Inserting the upper bound for V we get

$$u(r) > u_0 \big(1 - \frac{r^2}{r_0^2}\big) \quad \text{on } (0, r_0),$$

and $r_0 \le R_0$. We want to show that $u'(r_0) > 0$ provided u_0 is sufficiently large, which yields the desired contradiction. Using our bounds on u and V in (2.1) we obtain

$$\begin{aligned} u'(r_0) &= \frac{1}{r_0^{d-1}} \int_0^{r_0} (V(r) - 1) u(r) r^{d-1}\, dr \\ &\ge \frac{1}{2d\, r_0^{d-1}} \int_0^{r_0} u^3(r) r^{d+1}\, dr - \frac{u_0 r_0}{d} \\ &\ge \frac{u_0^3}{2d\, r_0^{d-1}} \int_0^{r_0} \big(1 - \frac{r^2}{r_0^2}\big)^3 r^{d+1}\, dr - \frac{u_0 r_0}{d} \\ &\ge \frac{u_0 r_0}{d} \Big(d u_0 \int_0^1 (1 - s^2)^3 s^{d+1}\, ds - 1 \Big). \end{aligned}$$

We conclude that $u'(r_0) > 0$ for u_0 sufficiently large which contradicts the assumption that $\mathcal{P}$ is empty. □

Hence, we have proved by the preceding lemma the existence of ground states:

Theorem 2.3. *For any $d \ge 1$, the set $\mathcal{G}$ is non-empty; that is, there is a solution (u, V) (1.7) subject to the initial conditions (1.8) such that $u(r) > 0$, $u'(r) < 0$ on $(0, \infty)$ and $\lim_{r\to\infty} u(r) = 0$.*

Proof. It remains to prove $u(r) > 0$, $u'(r) < 0$. The first property follows from the fact that there are no double zeros. If u has a first critical point $r_1 > 0$, then $V(r_1) \ge 1$ and since V is strictly increasing (see (2.1)) it follows again from (2.1) that $u(r) > 0$ for all $r > r_1$ which is impossible. Hence, $u' < 0$. □

3. Uniqueness of ground states

In this section, we prove that $\mathcal{G}$ has exactly one element. First of all, we show that if $\mathcal{G}$ had more than one element the corresponding solutions cannot cross. This is an immediate consequence of the following lemma which states that any two solutions of the initial-value problem (1.7), (1.8) cannot cross as long as they stay positive.

Lemma 3.1. *Let $u_2(0) > u_1(0) > 0$ and suppose that $u_2(r), u_1(r)$ exist on $[0, R]$ such that $u_1(r) \geq 0$ on $[0, R]$. Then $u_2(r) > u_1(r)$ for all $r \in [0, R]$.*

Proof. We consider the Wronskian of u_1, u_2 defined by

$$w(r) = u_2'(r)u_1(r) - u_1'(r)u_2(r). \tag{3.1}$$

Then w satisfies the differential equation

$$w' + \frac{d-1}{r}\, w = (V_2 - V_1)u_1u_2. \tag{3.2}$$

Suppose there is $\bar{r} \in [0, R]$ such that $u_2(r) > u_1(r)$ on $[0, \bar{r})$ and $u_1(\bar{r}) = u_2(\bar{r}) \geq 0$. Then

$$w(\bar{r}) = (u_2'(\bar{r}) - u_1'(\bar{r}))u_1(\bar{r}) \leq 0.$$

On the other hand we have

$$V_2'(r) - V_1'(r) = \frac{1}{r^{d-1}} \int_0^r (u_2^2(s) - u_1^2(s))s^{d-1}\, ds > 0$$

on $(0, \bar{r}]$ and therefore $V_2(r) > V_1(r)$ on $(0, \bar{r}]$. We conclude then, from the differential equation (3.2), that wr^{d-1} is strictly increasing on $(0, \bar{r}]$ since $w(0) = 0$ we must have $w(\bar{r}) > 0$ which is the desired contradiction. □

Remark 3.2. From the no-crossing property stated in Lemma 3.1, it follows immediately that $\mathcal{N}, \mathcal{P}$ are intervals. More precisely, $\mathcal{N} = (0, a), \mathcal{P} = (b, \infty)$ with $0 \leq a \leq b \leq \infty$. Uniqueness of ground states is then equivalent to $a = b$.

The important conclusion from Lemma 3.1 is that two different ground state solutions can not intersect. From the differential equation (3.2) for their Wronskian $w(r)$ we see that $w(r)r^{d-1}$ is a non-negative strictly increasing function. However, we shall prove in the sequel that $w(r)r^{d-1}$ vanishes at infinity which yields the desired contradiction. Therefore, we have to analyze the decay properties of ground states at infinity.

Since V is always strictly increasing $V_\infty := \lim_{r \longrightarrow \infty} V(r)$ exists (including the case $V_\infty = +\infty$) and $V(r) < V_\infty$ for all $r > 0$.

Lemma 3.3. *Let $u_0 \in \mathcal{G}$. Then $1 \leq u_0 \leq V_\infty$. In particular, $u_0 = 1$ if and only if $V_\infty = 1$ and in this case $u = 1 - V$.*

Proof. Since $\mathcal{N} \subset (0,1)$ by Lemma 2.2 the inequality $1 \leq u_0$ is obvious. To prove the second inequality it is sufficient to consider the case $V_\infty < +\infty$. We consider the function ξ defined by

$$\xi = u + V - V_\infty.$$

Obviously, $\xi(0) = u_0 - V_\infty$, $\xi'(0) = 0$ and $\lim_{r\to\infty}\xi(r) = 0$. The function ξ satisfies the differential equation

$$\xi'' + \frac{d-1}{r}\,\xi' = u\xi + (V_\infty - 1)u,$$

and admits the Taylor expansion

$$\xi(r) = u_0 - V_\infty + \frac{u_0(u_0-1)r^2}{2d} + o(r^2).$$

Suppose $u_0 > V_\infty$. Then $\xi(r)$ is positive and strictly increasing for $r > 0$ sufficiently small. Therefore, ξ must have a critical point r_1 with $\xi(r_1) > u_0 - V_\infty$ and $\xi''(r_1) \leq 0$ which is impossible since $u > 0$.

For the initial condition $u(0) = 1$ we have thanks to the uniqueness of solutions for the initial-value problem (1.7), (1.8) that $u = 1 - V$. □

In the following lemma we determine the asymptotic behavior of ground states.

Lemma 3.4. *Let $u_0 \in \mathcal{G}$. Then*

$$\lim_{r\longrightarrow\infty}\frac{u'}{u} = -\sqrt{V_\infty - 1}.$$

Moreover, if $1 < V_\infty \leq \infty$, then for any $\kappa \in (0, \sqrt{V_\infty - 1})$,

$$\limsup_{r\longrightarrow\infty} u(r)e^{\kappa r} < \infty.$$

Proof. First of all, let $V_\infty < \infty$. We consider the function $z := -\frac{u'}{u}$ which is well defined for all $r \geq 0$ and satisfies the differential equation

$$z' = z^2 - \frac{d-1}{r}\,z + 1 - V.$$

Now, choose $\tilde{r}$ such that $\frac{d-1}{r} \leq \frac{1}{2}\sqrt{V_\infty}$ for all $r \geq \tilde{r}$. Consider the direction field in the (r,z) plane for the preceding differential equation. In the set $r \geq \tilde{r}$, $z \geq 2\sqrt{V_\infty}$ we have

$$z' \geq \frac{1}{2}z^2 + \left(\frac{1}{2}z^2 - \frac{1}{2}\sqrt{V_\infty}z + 1 - V_\infty\right) \geq \frac{1}{2}z^2 + 1.$$

It follows that, should $z(r)$ ever enter this region, it would blow up at a finite time after $\tilde{r}$, which is impossible. Hence, z remains bounded. This also implies

$$\lim_{r\longrightarrow\infty} u'(r) = 0.$$

Therefore, we may apply l'Hôspital's rule. We obtain

$$\lim_{r\longrightarrow\infty} z^2 = \lim_{r\longrightarrow\infty} \frac{u''}{u} = \lim_{r\longrightarrow\infty} \left(\frac{d-1}{r}\, z + V - 1\right) = V_\infty - 1.$$

Finally, if V_∞ is infinite, then z is also unbounded since otherwise applying l'Hôspital's rule as above yields the desired contradiction. This proves the first part of the lemma.

Now, let $V_\infty > 1$. Then, for any $\kappa \in (0, \sqrt{V_\infty - 1})$ and r sufficiently large, $-\frac{u'}{u} \geq \kappa$ and the proof is completed by integrating this inequality and taking exponentials on both sides. □

Remark 3.5. If $V_\infty = +\infty$, the asymptotic behavior of ground states can be given more precisely. Indeed, by analyzing the differential equation for $Z := -\frac{u'}{u} V^{-1/2}$ and taking into account the fact that $\lim_{r\longrightarrow\infty} V'/V = 0$ it can be easily shown by mimicking the proof of the preceding lemma that $\lim_{r\longrightarrow\infty} Z = 1$.

Now we are in position to prove our uniqueness result:

Theorem 3.6. *The set $\mathcal{G}$ has exactly one element.*

Proof. Let $u_1(0), u_2(0) \in \mathcal{G}$ such that $u_2(0) > u_1(0)$. By Lemma 3.1 the corresponding solutions u_1, u_2 cannot intersect and we have $u_2(r) > u_1(r) > 0$ for all $r \geq 0$. From the differential equation (3.2) for their Wronskian $w(r)$ we see that $w(r)r^{d-1}$ is a non-negative strictly increasing function since

$$\left(wr^{d-1}\right)' = (V_2 - V_1)u_1u_2r^{d-1} > 0,$$

and $w(0) = 0$. On the other hand, we claim that

$$\lim_{r\longrightarrow\infty} wr^{d-1} = 0.$$

Indeed, by Lemma 3.3 we have $\lim_{r\to\infty} V_2(r) > 1$. Trivially, $V_2(r) \leq \frac{u_2(0)^2 r^2}{2d}$. From the integral equation (2.1) for u',

$$u_2'(r)r^{d-1} = \int_0^r (V_2(s) - 1)u_2(s)s^{d-1}\, ds,$$

and the decay properties of u_2 given in Lemma 3.4, it follows then that $u_2' r^{d-1}$ and $u_2 r^{d-1}$ are uniformly bounded. Therefore,

$$|w(r)r^{d-1}| \leq |u_1||u_2' r^{d-1}| + |u_1'||u_2 r^{d-1}| \leq c_1|u_1| + c_2|u_1'|,$$

for some positive constants c_1, c_2 which concludes the proof. □

4. Proof of Theorem 1.2

For the initial condition $u(0) = 1$ we have, thanks to the uniqueness of solutions for the initial-value problem (1.7), (1.8) that $u = 1 - V$ and therefore we have to solve the following initial-value problem

$$u'' + \frac{d-1}{r}\, u' = -u^2, \quad u(0) = 1,\ u'(0) = 0. \tag{4.1}$$

This is the d-dimensional Lane-Emden equation. The behavior of its solutions has been widely studied in the mathematical literature. However, in the following we will give alternative proofs of the results relevant for our work. Analyzing the behavior of the solutions of the initial-value problem (4.1) we prove the final part of our main result:

Theorem 4.1. *If $d \geq 6$, then $1 \in \mathcal{G}$ and if $d < 6$, then $1 \in \mathcal{N}$.*

Proof. By Lemma 2.2 we know that $(0,1) \subset \mathcal{N}$. On the other hand, $\mathcal{P}$ is open, and therefore $1 \notin \mathcal{P}$ for any $d > 0$. Now, let $d \geq 6$. We introduce a new Lyapunov function $L(r)$ defined by

$$L(r) = E(r)r^d + \frac{d}{3}u(r)u'(r)r^{d-1}. \tag{4.2}$$

Then $L(0) = 0$ and

$$L'(r) = -\frac{d-6}{6}u'^2(r)r^{d-1} \leq 0,$$

since $d \geq 6$. Hence, $L(r) \leq 0$ for all $r \geq 0$. We suppose that $1 \in \mathcal{N}$. Then there exists $r_0 > 0$ such that $u(r_0) = 0$. Computing L at this point we get

$$L(r_0) = \tfrac{1}{2}u'^2(r_0)r_0^d > 0,$$

which is impossible. The explicit solution (1.10) for $d = 6$ is readily verified (see also Appendix A).

Let $d < 6$ and assume $1 \in \mathcal{G}$. We may then use the Milne variables (see e.g. [14], [3])

$$y := \frac{-ru'}{u}, \quad z := -\frac{-ru^2}{u'},$$

which are well defined for all $r \geq 0$. Indeed, $y(0) = 0, z(0) = d$ and $y, z > 0$ for all $r > 0$. They satisfy the differential equations

$$y' = \frac{y}{r}(2 - d + y + z), \quad z' = \frac{z}{r}(d - 2y - z).$$

If $d \leq 2$, then the first differential equation implies that y blows up in finite time which is impossible. If $d > 2$, then y and z remain bounded, which implies that for all r sufficiently large

$$u(r) \leq Cr^{-2}, \quad -u' \leq Cr^{-3},$$

for an appropriate constant $C > 0$. We consider now the Lyapunov functional $L(r)$ defined in (4.2). Since $d < 6$, the bounds on u, u' imply that

$$\lim_{r\to\infty} L(r) = 0.$$

On the other hand, $L(0) = 0$ and

$$L'(r) = -\frac{d-6}{6}u'^2(r)r^{d-1} > 0,$$

which yields the desired contradiction. □

Corollary 4.2. *Let $d < 6$ and $u_0 \in \mathcal{G}$. Then $V_\infty > u_0 > 1$.*

Proof. The inequality $u_0 > 1$ is an immediate consequence of the previous theorem. The strict inequality $V_\infty > u_0$ follows from the proof of Lemma 3.3: If $V_\infty = u_0 > 1$, then $\xi = u + V - V_\infty$ is still positive increasing for $r > 0$, which is impossible. □

Appendix A. Transformation to an autonomous system

Putting $u(r) = e^{-2s}\phi(s)$ and $V(r) - 1 = e^{-2s}W(s)$ with $s = \ln r$ the system (1.7) transforms into the autonomous system

$$\begin{aligned} \ddot{\phi} + (d-6)\,\dot{\phi} - 2(d-4)\phi &= W\phi \\ \ddot{W} + (d-6)\,\dot{W} - 2(d-4)W &= \phi^2, \end{aligned} \tag{A.1}$$

subject to the boundary conditions

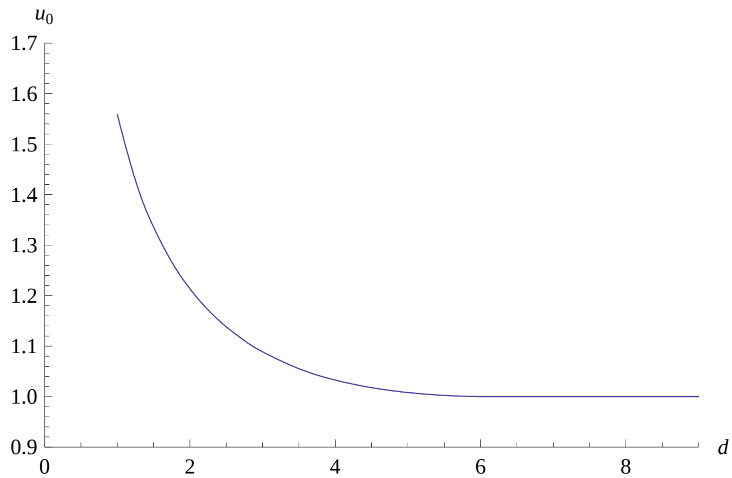
$$\lim_{s\to-\infty} e^{-2s}\phi(s) = u_0 \in \mathbb{R}^+, \quad \lim_{s\to-\infty} e^{-2s}W(s) = -1. \tag{A.2}$$

For $u_0 \in \mathcal{G}$ we have the asymptotic behavior

$$\lim_{s\to\infty} e^{-2s}\phi(s) = 0, \quad \lim_{s\to\infty} e^{-2s}W(s) = V_\infty - 1. \tag{A.3}$$

We can associate an energy of the system given by

$$2E = \dot{\phi}^2 - 2(d-4)\phi^2 + \tfrac{1}{2}\dot{W}^2 - (d-4)W^2 - W\phi^2, \tag{A.4}$$



The curve $d \mapsto \mathcal{G}$

which satisfies

$$\dot{E} = -(d-6)(\dot{\phi}^2 + \tfrac{1}{2}\,\dot{W}^2). \tag{A.5}$$

We should note that though the new system is invariant under translations in s, the boundary conditions (A.2) break this symmetry, and therefore the solutions are not translation invariant. However, $e^{-2s}\phi(s)$ and $e^{-2s}W(s)$ are translation invariant which corresponds to the dilation invariance of the original system (1.7).

If $d = 6$, then system (A.1) is Hamiltonian and solutions satisfying the boundary conditions (A.2) have zero energy. We look for a solution such that $\phi = -W$ (i.e. $u_0 = 1$). Then the zero energy condition reads

$$\dot{\phi}^2 - 4\phi^2 + \tfrac{2}{3}\phi^3 = 0. \tag{A.6}$$

Since ϕ vanishes at $\pm\infty$ there is $s_0 \in \mathbb{R}$ such that $\dot{\phi}(s_0) = 0$. Equation (A.6) yields then $\phi(s_0) = 6$. Integrating (A.6) we get

$$2(s - s_0) = \int_6^{\phi(s)} \frac{du}{u\sqrt{1 - u/6}}, \tag{A.7}$$

and we obtain the solution

$$\phi(s) = \frac{6}{\cosh^2(s - s_0)}. \tag{A.8}$$

The boundary condition (A.2) yields $e^{2s_0} = 24$ which after changing variables gives the solution (1.10).

Appendix B. A Sobolev inequality

In this part, we make the simple observation that the Sobolev inequality (1.12) and its optimizers can be obtained by combining two inequalities of [10]. We denote by $H^1(\mathbb{R}^d)$ the space of functions $f : \mathbb{R} \to \mathbb{C}$ for which f and ∇f are square integrable.

Proposition B.1. *Let $d > 4$. For all $f, g \in H^1(\mathbb{R}^d)$*

$$\iint_{\mathbb{R}^{2d}} |x-y|^{-4}|f(x)|^2|g(y)|^2\, dxdy \leq C_d \int_{\mathbb{R}^d} |\nabla f(x)|^2\, dx \int_{\mathbb{R}^d} |\nabla g(x)|^2\, dx, \tag{B.1}$$

with

$$C_d = \frac{4(d-1)}{d^2(d-2)^2(d-4)}, \tag{B.2}$$

and optimizers (up to translations and dilations) given by

$$f(x) = g(x) = (1+|x|^2)^{-\frac{d-2}{2}}. \tag{B.3}$$

Proof. We apply inequality (1.1) of [10] with $p = t = \frac{d}{d-2}$ to get

$$\iint_{\mathbb{R}^{2d}} |x-y|^{-4}|f(x)|^2|g(y)|^2\, dxdy \leq N_{p,4,d}||f||_{2p}^2||g||_{2p}^2,$$

with $N_{p,4,d}$ given by equation (3.2) of [10] and then the Sobolev inequality (1.2) with K_d given in equation (4.11) of [10]. Both inequalities have the same optimizers which concludes the proof. □

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