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CONSTRUCTION OF SINGULAR SOLUTIONS FOR ELLIPTIC PROBLEM OF FOURTH ORDER DERIVATIVE WITH A SUBCRITICAL NON-LINEARITY

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1. Introduction and statement of the results

Many authors have studied the existence of weak solutions for the semilinear elliptic equation

$$\Delta u + u^p = 0, \tag{1.1}$$

which are positive in a domain $\Omega \subset \mathbb{R}^n$, and prescribed isolated singularities or singular along arbitrary smooth submanifolds for special values of the exponent p.

The asymptotic behavior near an isolated singularity has been studied by Aviles in [2] when $p = \frac{n}{n-2}$, by Gidas and Spruck in [8], when $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$, and finally by Caffarelli, Gidas and Spruck in their paper [4], for the case $p = \frac{n+2}{n-2}$, which is the so-called critical exponent. They give some results about the asymptotic behavior of the singular solutions of (1.1).

Concerning the existence of weak solutions to the equation (1.1), which are positive in $\Omega \subset \mathbb{R}^n$, vanish at the boundary, it is known that if the exponent p is less than $\frac{n}{n-2}$, then any weak solution must be smooth on all of Ω. The existence of solutions of this equation with prescribed isolated singularities when p lies in the interval $\frac{n}{n-2} \leq p < p_0$, where p_0 is some value close to $\frac{n}{n-2}$ and in particular, less than $\frac{n+2}{n-2}$, has already been solved by Pacard in [15] and [16]. When $p = \frac{n+2}{n-2}$, the problem becomes conformally invariant. There is then a loss of compactness, and the problem consequently

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becomes much more difficult. Solutions now correspond to metrics of constant non-negative scalar curvature which are complete in a neighborhood of the singular points. It is geometrically more natural, in this case, to replace the domain by S^n (or in fact, any other compact manifold of non-negative scalar curvature); the operator Δ then needs to be replaced by the conformal Laplacian $\Delta - \frac{n-2}{4(n-1)} R_0$, where R_0 is the scalar curvature of the background manifold. An additional source of difficulties, in this case, is that the position of the singularities is no longer necessarily arbitrary. The general existence result for this geometric problem, when the background manifold is the sphere, was obtained by Schoen [19] and Mazzeo, Pollack and Uhlenbeck [12] examine the moduli space of solutions for this problem.

Rébaï, in her paper [21], has proved the existence of positive weak solutions to the equation (1.1), with $p > \frac{m}{m-2}$ and very close to this value such that its singular set is a given $(n-m)-$dimensional compact submanifold of Ω without boundary. Later, Mazzeo and Pacard in [11], construct solutions of the Yamabe problem on the sphere (S^N, g_0) with its standard metric, that are singular at a specified close set $\Lambda \subset S^N$. They seek a new metric g that is conformal to g_0, is complete on $\Lambda \setminus S^N$, and has constant positive scalar curvature R. They also, in [10], prove the existence of solutions to the equation (1.1), which are positive in a domain $\Omega \subset \mathbb{R}^n$, and which are singular along arbitrary smooth k-dimensional submanifolds in the interior of these domains provided p lies in the interval $(\frac{n-k}{n-k-2}, \frac{n-k+2}{n-k-2})$.

Baraket and Rebhi, in [3], have constructed positive weak solutions of a fourth order conformally invariant equation on S^N. These solutions give rise to complete metrics on the complement of finitely many points of S^N whose $Q-$curvature is constant.

In this paper, we are interested to the existence of positive weak solutions to the equation

$$\Delta^2 u = u^p, \tag{1.2}$$

with $p \in (\frac{m}{m-4}, \frac{m+4}{m-4}]$, where $4 < m < n$, such that its singular set is given by some $(n-m)-$dimensional submanifold of $\mathbb{R}^n$ without boundary.

Recently, Gazzola and Grunau, in [7], proved the existence and uniqueness of positive radial entire solutions of the supercritical semilinear biharmonic equation

$$\Delta^2 u = |u|^{p-1} u, \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

where $n \geq 5$ and $p > \frac{n+4}{n-4}$.

The first result treat the case when the exponent p is chosen as $p > \frac{m}{m-4}$, and close to $\frac{m}{m-4}$. We consider Ω a bounded subset of $\mathbb{R}^n$, and Σ some compact submanifold of Ω without boundary of dimension $(n-m)$, we prove the following result:

Theorem 1. *Let Ω be a subset of $\mathbb{R}^n$ with a smooth boundary, Σ a compact submanifold of Ω without boundary of dimension $(n-m)$, $n \geq 5$ and $4 < m < n$. Then, if $p > \frac{m}{m-4}$ and close enough to this value, the problem*

$$\begin{cases} \Delta^2 u &= u^p \quad \text{in } \Omega \\ u = \Delta u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

has at least one solution which is singular on Σ.

Also, we give the following result which holds for every $p \in (\frac{m}{m-4}, p^*]$ and for almost every $p \in (p^*, \frac{m+4}{m-4}]$, where

$$p^* = \frac{m + 2 + \sqrt{m^2 + 4 - \sqrt{m^4 - 8m^3 + 30m^2 + 8m - 15}}}{m - 6 + \sqrt{m^2 + 4 - \sqrt{m^4 - 8m^3 + 30m^2 + 8m - 15}}}. \tag{1.5}$$

More precisely, denote by $\mathbb{T}^{n-m}$ the taurus defined as

$$\mathbb{T}^{n-m} := \mathbb{R}^{n-m}/\mathbb{Z}^{n-m}, \tag{1.6}$$

then,

Theorem 2. *Let p^* is given by (1.5), then for every $p \in (\frac{m}{m-4}, p^*]$, for almost every $p \in (p^*, \frac{m+4}{m-4}]$ and for every $\mathcal{C}^{5,\alpha}$ diffeomorphism τ of $(\mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\})$ close to the identity, the equation*

$$\Delta^2 u = u^p,$$

has a solution defined on $\mathbb{T}^{n-m} \times \mathbb{R}^m - \tau(\mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\})$ and which is singular at $\tau(\mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\})$.

1.1. **Radial positive weak solutions in $\mathbb{R}^m$.** It is easy to check that the function

$$u_0(z) = a_0\, |z|^{-\frac{4}{p-1}},$$

where the constant a_0 is given by

$$a_0^{p-1} = \frac{8(p+1)}{(p-1)^2}\Big(m - 2\frac{p+1}{p-1}\Big)\Big(m - \frac{4p}{p-1}\Big),$$

is a radial weak positive solution of

$$\Delta^2\, u = u^p \quad \text{in} \quad \mathbb{R}^m - \{0\}. \tag{1.7}$$

provided $p > \frac{m}{m-4}$.

1.2. **Function spaces.** Assume that Σ is a smooth $(n-m)-$dimensional submanifold of Ω. For all $\sigma > 0$ small enough, we define N_σ to be the geodesic tubular neighborhood of radius σ around Σ.

Assume that $r \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$ are given. For all function $u \in \mathcal{C}^{r,\alpha}_{loc}(\Omega - \Sigma)$, we define the family of semi-norms

$$| \, u \, |_{r,\alpha,s} = \sum_{j=0}^{r} s^j \sup_{N_s \setminus N_{s/2}} |\nabla^j u| + s^{r+\alpha} \sup_{x,x' \in N_s \setminus N_{s/2}} \frac{|\nabla^r u(x) - \nabla^r u(x')|}{|x - x'|^\alpha}, \quad (1.8)$$

where $s \in (0, \sigma)$. With this definition in mind, we define the weighted Hölder norm

$$\|u\|_{\mathcal{C}^{r,\alpha}_\nu} := |u|_{\mathcal{C}^{r,\alpha}(\Omega - N_\sigma)} + \sup_{0 \le s \le \sigma} s^{-\nu} \, | \, u \, |_{r,\alpha,s} \, . \quad (1.9)$$

Then, we define the function space

$$\mathcal{C}^{r,\alpha}_\nu(\Omega - \Sigma) := \{u \in \mathcal{C}^{k,\alpha}_{loc}(\Omega - \Sigma) : \|u\|_{\mathcal{C}^{r,\alpha}_\nu} < \infty\}. \quad (1.10)$$

Reducing the value of σ if this is necessary, we can choose in N_σ Fermi coordinates defined as follows : For every $y_0 \in \Sigma$, there exists an orthonormal frame field $E_1, \ldots, E_m$, basis of the normal bundle of Σ. Then, we consider the coordinate system

$$(y, z) \in \Sigma \times \mathbb{R}^m \longrightarrow y + \sum z_i \, E_i(y),$$

where $(y_1, \ldots, y_{n-m})$ is a local coordinate system on Σ. These are well defined coordinates in a neighborhood of y_0, provided $|z|$ remains small enough, say $|z| \le \sigma$.

In the above defined coordinates, the Euclidean metric can be expanded as

$$g_{eucl} = g_{\mathbb{R}^m} + g_\Sigma + \mathcal{O}(|z|) \, dz \, dy + \mathcal{O}(|z|) \, dy^2.$$

It then follows from the formula,

$$\Delta_g = \frac{1}{\sqrt{\det g}} \, \partial_i (g^{ij} \sqrt{\det g} \, \partial_j), \quad (1.11)$$

that

$$\Delta = \Delta_{\mathbb{R}^m} + \Delta_\Sigma + \tilde{e}_1 \nabla + \tilde{e}_2 \, \nabla^2,$$

where $\tilde{e}_1$ and $\tilde{e}_2$ satisfy

$$\|\tilde{e}_1\|_{\mathcal{C}^{0,\alpha}_0} + \|\tilde{e}_2\|_{\mathcal{C}^{0,\alpha}_1} \le c_0,$$

for some constant $c_0 > 0$. Therefore, we also have

$$\Delta^2 = \Delta^2_{\mathbb{R}^m} + \Delta^2_\Sigma + 2\,\Delta_{\mathbb{R}^m}\,\Delta_\Sigma + \sum_{i=1}^{4} e_i \nabla^i, \tag{1.12}$$

where

$$\|e_1\|_{\mathcal{C}^{0,\alpha}_{-2}} + \|e_2\|_{\mathcal{C}^{0,\alpha}_{-1}} + \|e_3\|_{\mathcal{C}^{0,\alpha}_{0}} + \|e_4\|_{\mathcal{C}^{0,\alpha}_{1}} \leq c_0, \tag{1.13}$$

for some constant $c_0 > 0$.

2. PROOF OF THEOREM 1

We now proceed with the proof of Theorem 1. As usual in these constructions, we start by building an approximate solution which is then perturbed into a solution of our equation. The perturbation result requires the analysis of the linearized operator about the approximate solution and the application of a simple fixed point theorem.

2.1. **Construction of the approximate solution.** Let χ_σ be a cutoff function identically equal to 1 in $N_{\sigma/2}$, and identically equal to 0 in $\Omega - N_\sigma$. We define the function $\tilde{u}$ to be identically equal to 0 in $\Omega - N_\sigma$ and to be equal to

$$\tilde{u}(y,z) := u_0(z)\chi_\sigma(y,z) \qquad \text{in } \; N_\sigma. \tag{2.1}$$

We fix $\nu < 5 - m$. Observe that

$$\lim_{p \to \frac{m}{m-4}} \frac{p-5}{p-1} = 5 - m.$$

Using the expansion given in (1.12), it is easy to check that there exists a constant C_p which tends to 0 as p tends to $\frac{m}{m-4}$ such that

$$\|\Delta^2 \tilde{u} - \tilde{u}^p\|_{\mathcal{C}^{0,\alpha}_{\nu-4}} \leq C_p. \tag{2.2}$$

2.2. **A linear problem.** We consider the mapping properties of the bilaplace operator when defined between the weighted spaces defined in (1.10). In particular, we will prove that Δ^2 with Navier boundary conditions is an isomorphism when $\nu \in (4-m, 0)$.

Proposition 1. *Assume that $\nu \in (4-m, 0)$ is fixed. Then, there exists a bounded operator $G_\nu : \mathcal{C}^{0,\alpha}_{\nu-4}(\Omega - \Sigma) \longrightarrow \mathcal{C}^{4,\alpha}_{\nu}(\Omega - \Sigma)$ such that, for all $f \in \mathcal{C}^{0,\alpha}_{\nu-4}(\Omega - \Sigma)$, the function $v = G_\nu\, f$ is a solution of*

$$\begin{cases} \Delta^2 v &= f \qquad \textit{in } \; \Omega - \Sigma \\ v = \Delta v &= 0 \qquad \textit{on } \; \partial\Omega. \end{cases} \tag{2.3}$$

Proof. This result follows from the general theory developed in [10]. However, using the maximum principle we can give a simple direct proof.

First, observe that since $\nu > 4 - m$, then $f \in L^1(\Omega)$ and one can solve

$$\begin{cases} -\Delta\, w &= f \qquad \text{in } \Omega - \Sigma \\ \quad w &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Moreover, in N_σ, we compute thanks to the expansion (1.12)

$$\Delta |z|^{\nu-2} = (\nu - 2)\,(m - 4 + \nu)\,|z|^{\nu-4} + \mathcal{O}(|z|^{\nu-3}).$$

Observe that, by our choice, $(\nu - 2)(m - 4 + \nu) < 0$ and, therefore, the function $(y, z) \to |z|^{\nu-2}$ can be used as a barrier function and the maximum principle together with standard elliptic estimates imply that, in N_σ, w is bounded by a constant times $|z|^{\nu-2}$ times the norm of f in $\mathcal{C}^{0,\alpha}_{\nu-4}(\Omega - \Sigma)$. Next, we solve

$$\begin{cases} -\Delta\, v &= w \qquad \text{in } \Omega - \Sigma \\ \quad v &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Again, $w \in L^1(\Omega)$, and this time we compute, thanks to the expansion, (1.12)

$$\Delta |z|^{\nu} = \nu\,(m - 2 + \nu)\,|z|^{\nu-2} + \mathcal{O}(|z|^{\nu-1}) \qquad \text{in } N_\sigma.$$

Again, $\nu(m-2+\nu) < 0$ and the maximum principle together with standard elliptic estimates imply that, in N_σ, v is bounded by a constant times $|z|^\nu$ times the norm of w. This provides both the existence of $G_\nu(f) := v$ and the estimate

$$\|v\|_{\mathcal{C}^{0,\alpha}_{\nu}} \leq c\,\|f\|_{\mathcal{C}^{0,\alpha}_{\nu-4}}.$$

The estimates for the derivatives now follow from Schauder estimates applied in $\Omega - \Sigma$. □

2.3. **A non-linear problem.** We now have all the ingredients necessary for the proof of Theorem 1. We perturb the approximate solution $\tilde{u}$ into $u := \tilde{u} + v$ so that

$$\begin{cases} \Delta^2(\tilde{u} + v) &= |\tilde{u} + v|^p \qquad \text{in } \Omega - \Sigma \\ \quad v = \Delta v &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

We fix $\nu \in (4 - m, 5 - m)$. Using the result of Proposition 1, we can write

$$v = G_\nu\,(|\tilde{u} + v|^p - \tilde{u}^p) - G_\nu\,(\Delta^2 \tilde{u} - \tilde{u}^p). \tag{2.4}$$

So, that it remains to find a function $v \in \mathcal{C}^{4,\alpha}_{\nu}(\Omega - \Sigma)$ as a solution of a fixed point problem.

The key point is that, according to (2.2), the error term $\Delta^2 \tilde{u} - \tilde{u}^p$ tends to 0 , as p tends to $\frac{m}{m-4}$. Moreover, provided p is chosen close enough to $\frac{m}{m-4}$ the non-linear operator $v \longrightarrow |\tilde{u}+v|^p - \tilde{u}^p$ is a contraction. Indeed, we have the estimate

$$\| |\tilde{u}+v|^p - |\tilde{u}+v'|^p \|_{\mathcal{C}^{0,\alpha}_{\nu-4}} \leq c(\, a_0 + \|v\|_{\mathcal{C}^{4,\alpha}_{\nu}} + \|v\|_{\mathcal{C}^{4,\alpha}_{\nu}}) \|v - v'\|_{\mathcal{C}^{4,\alpha}_{\nu}},$$

provided $v, v' \in \mathcal{C}^{4,\alpha}_{\nu}(\Omega - \Sigma)$ satisfy $\|v\|_{\mathcal{C}^{4,\alpha}_{\nu}} + \|v'\|_{\mathcal{C}^{4,\alpha}_{\nu}} \leq \frac{C_p}{4}$.

It is now a simple exercise to prove the existence of a fixed point v, solution of (2.3), which belongs to the ball of radius $\frac{C_p}{4}$ in $\mathcal{C}^{4,\alpha}_{\nu}(\Omega - \Sigma)$, provided p is close enough to $\frac{m}{m-4}$.

Now, observe that $\tilde{u} + v$ and $\tilde{u}$ certainly have the same singular set if p is close enough to $\frac{m}{m-4}$. This follows from the choice of $\nu > 4 - m$ which implies that $\nu > -\frac{4}{p-1}$ for p close enough to $\frac{m}{m-4}$. In addition, this shows that $\tilde{u} + v > 0$ and $\Delta(\tilde{u} + v) < 0$ in the neighborhood of Σ. On the other hand, knowing that $\Delta^2(\tilde{u}+v) \geq 0$ in Ω, the maximum principle implies that $\Delta(\tilde{u}+v) \leq 0$ in Ω and next, applying the maximum principle once more, we prove that $\tilde{u} + v \geq 0$ in Ω. Therefore, we conclude that $\tilde{u} + v$ is a solution of the problem (1.4). This completes the proof of Theorem 1.

3. Proof of Theorem 2

3.1. **Linear analysis.** Let us recall that $\mathbb{T}^{n-m} = \mathbb{R}^{n-m}/\mathbb{Z}^{n-m}$. We consider the function $\tilde{u}(y,z) := u_0(z)$, where $x = (y,z) \in \mathbb{T}^{n-m} \times \mathbb{R}^m$, and we denote by $r = |z|$. Let $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$. Denote by $\mathcal{C}^{k,\alpha}_{\nu}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\}\big)$ the subspace of functions in $\mathcal{C}^{k,\alpha}_{loc}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\}\big)$ which are bounded by constants times r^{ν}. We would like to understand the mapping properties of the operator

$$L := \Delta^2 - p\, \tilde{u}^{p-1}, \tag{3.1}$$

when defined between the space $\mathcal{C}^{4,\alpha}_{\nu}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\}\big)$ and $\mathcal{C}^{0,\alpha}_{\nu-4}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\}\big)$.

We denote by (φ_j, λ_j) the eigendata of $-\Delta_{S^{m-1}}$, and assume that the eigenvalues are counted with multiplicity

$$\lambda_0 = 0, \lambda_1 = \ldots = \lambda_m = m-1,\ \lambda_{m+1} = 2m \ldots$$

The indicial roots of the operator L are defined to be real γ_j for which

$$L\,(|z|^{\gamma_j}\, \varphi_j) = \mathcal{O}(|z|^{\gamma_j - 4}).$$

We find with little work that for each j there are 4 distinct indicial roots given by

$$\gamma_j^{\eta_1\eta_2} := \frac{4-m}{2} + \eta_1 \sqrt{\frac{(m-4)^2}{4} + m - 2 + \lambda_j + \eta_2 \sqrt{4\lambda_j + (m-2)^2 + p\, a_0^{p-1}}},$$

where $\eta_1, \eta_2 \in \{\pm 1\}$. The exact expression of the indicial roots is not so important and all information we need are included in

$$\gamma_0^{-+} < 4 - m < -\frac{4}{p-1} < \Re(\gamma_0^{--}) \leq \frac{4-m}{2} \leq \Re(\gamma_0^{+-}) < 0 < \gamma_0^{++}.$$

Note that after a computation, the indicial roots for the other values of $j \geq 1$, satisfy

$$\gamma_j^{-\pm} < \frac{-4}{p-1} \qquad \text{and} \qquad \Re(\gamma_0^{+-}) < \gamma_j^{+\pm}.$$

Next, we will choose

$$\frac{-4}{p-1} < \nu < \Re(\gamma_0^{--}) \leq \frac{4-m}{2} \leq \Re(\gamma_0^{+-}) < \mu < \gamma_1^{+-} - 1. \tag{3.2}$$

We consider the operator

$$L : \mathcal{C}^{4,\alpha}_{\nu}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\}\big) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-4}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\}\big),$$

given by

$$L := (\Delta_{\mathbb{R}^m} + \Delta_{\mathbb{T}^{n-m}})^2 - \frac{p a_0^{p-1}}{r^4}.$$

We claim that L is surjective. The interval in which μ lies is chosen to guarantee injectivity of L on $\mathcal{C}^{4,\alpha}_{\mu}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^{n-m}}\}\big)$; ν is determined by duality, i.e., $\nu + \mu = 4 - m$.

Proposition 2. *Let p^* the scalar introduced in* (1.5). *Then, for every $p \in (\frac{m}{m-4}, p^*]$ and for almost every $p \in (p^*, \frac{m+4}{m-4}]$, the only solution $w \in \mathcal{C}^{4,\alpha}_{\mu}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\}\big)$, where μ is chosen as* (3.2), *which satisfies $Lw = 0$ is $w = 0$.*

Proof. Let $w \in \mathcal{C}^{4,\alpha}_{\mu}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\}\big)$, then w can be viewed as function of $(y,z) \in \mathbb{R}^{n-m} \times \mathbb{R}^m - \mathbb{R}^{n-m} \times \{0_{\mathbb{R}^m}\}$ 1-periodic with respect to each component of the variable y in $\mathbb{R}^{n-m}$. So, we can write

$$w(x) = \sum_{l \in \mathbb{N}^{n-m}} w_l(z) e^{2i\pi l \cdot y},$$

where $l = (l_1, ..., l_{n-m})$ with $l.y = \sum_{j=1}^{n-m} l_j y_j$. Solving $Lw = 0$ is equivalent to solve for every $l \in \mathbb{N}^{n-m}$, $\mathcal{L}_{|l|} w_l = 0$ in $\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^m - \{0\})$, where

$$\mathcal{L}_{|l|} = (\Delta_{\mathbb{R}^m} - |l|^2)^2 - \frac{pa_0^{p-1}}{r^4},$$

and $|l|^2 = l_1^2 + ... + l_{n-m}^2$.

Let $l \in \mathbb{N}^{n-m}$ and $v_l \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^m - \{0\})$ such that

$$\mathcal{L}_{|l|}\, v_l = 0.$$

By the invariance of $\mathcal{L}_{|l|}$ under the group of dilatations we can reduce to the case where $|l| = 1$ by setting $v_l(z) = v(\frac{z}{|l|})$ so that

$$\mathcal{L}_1 v = (\Delta_{\mathbb{R}^m} - 1)^2 v - \frac{pa_0^{p-1}}{r^4} v = 0.$$

We decompose v over the eigenfunctions of the Laplacian in S^{m-1}, we have $v = \sum_{j=0}^{+\infty} v_j(r)\varphi_j(\theta)$ and v_j satisfies $\mathcal{L}_{1,j} v_j = 0$ where

$$\mathcal{L}_{1,j} = \Delta_r^2 - 2(\frac{\lambda_j}{r^2} + 1)\, \Delta_r + \frac{4}{r^3}\, \lambda_j\, \partial_r + \Big(\frac{\lambda_j^2}{r^4} + \frac{2(m-4)}{r^4}\lambda_j + \frac{2\lambda_j}{r^2} + 1 - \frac{pa_0^{p-1}}{r^4}\Big). \tag{3.3}$$

Note that $\mathcal{L}_{1,j}$ can be written as $\mathcal{L}_{1,j} = \mathcal{P} + \mathcal{Q}_j$ with $\mathcal{P} = \Delta_r^2 - 2\Delta_r + 1$. Then, any solution of $\mathcal{P}w = 0$ defined in $\mathbb{R}^m$ is a linear combination of e^r, e^{-r}, re^r and re^{-r}. Clearly, for every $j \geq 1$ the equation $\mathcal{L}_{1,j} w = 0$ has only one solution bounded by a constant times r^μ as r tends to 0, one of which w_j^+ behaves like $r^{\gamma_j^{++}}$ as $r \longrightarrow 0$. At infinity, using a perturbation argument, then for any solution of $\mathcal{L}_{1,j}\, w = 0$ which is polynomially bounded, there exists $(\alpha, \beta) \neq (0, 0)$, such that

$$w(r) = (\alpha + \beta\, r)\, e^{-r} + \mathcal{O}(r^{-1}\, e^{-r}).$$

Multiplying $\mathcal{L}_{1,j} w_j = 0$ by $r^{m-1} w_j$ and integrating by parts, we conclude that after a painful computation

$$\begin{aligned}&\Big(m - 1 + 2\lambda_j + \frac{(m-1)(m-3)}{4}\Big) \int_0^\infty [\partial_r(r^{\frac{m-3}{2}} w_j)]^2 dr \\ &+ \int_0^\infty [\partial_r(r^{\frac{m-1}{2}} \partial_r w_j)]^2 dr + 2\int_0^\infty r^{m-1}(\partial_r w_j)^2 dr + 2\lambda_j \int_0^\infty r^{m-3} w_j^2 dr \\ &+ \int_0^\infty r^{m-1} w_j^2 dr + A_j \int_0^\infty r^{m-5} w_j^2 dr = 0,\end{aligned}$$

where we have defined

$$A_j = \lambda_j^2 + 2(m-4)\lambda_j + (m-1+2\lambda_j)\frac{(m-3)(m-5)}{4} + \frac{(m-1)(m-3)^2(m-5)}{16} - p\, a_0^{p-1}.$$

For $j \geq 1$ one checks that $A_j > 0$. Therefore, we already deduce that $w_j = 0$ for $j \geq 1$. It remains to prove that $w_0 = 0$. In this case

$$A_0 = \frac{(m^2-1)(m-3)(m-5)}{16} - pa_0^{p-1}.$$

Observe that when $p \in (\frac{m}{m-4}, p^*]$, then $A_0 \geq 0$, and we conclude that $w_0 = 0$.

Finally, we will deal with the case $p \in (p^*, \frac{m+4}{m-4}]$. We introduce the family of operators

$$L_c = (\Delta - 1)^2 - \frac{c}{r^4},$$

when $c = pa_0^{p-1}$, $\mathcal{L}_1 = L_c$.

We define w_c to be the unique solution of $L_c w_c = 0$ which satisfies $w_c = r^{\gamma_0^{++}} + \mathcal{O}(r^{\gamma_0^{++}+1})$ at $r = 0$. At infinity, this solution can be expanded as

$$w_c(r) = ((a(c) + b(c)\, r)\, e^r + \mathcal{O}(r^{-1} e^r).$$

We define $\psi(c) := (a(c))^2 + (b(c))^2$. This is a real analytic function of c which is not identically equal to 0 since $\psi(c) \neq 0$ at least when $c < 0$. Therefore, the zero set of ψ is discreet. This implies that the set of powers p corresponding to the zeros of ψ is also discreet.

Finally, almost there exists a finite number of p when this case occurs. This complete the proof of the injectivity of $\mathcal{L}_{|l|}$ for every $p \in (\frac{m}{m-4}, p^*]$ and for almost every $p \in (p^*, \frac{m+4}{m-4}]$, which complete the proof of Proposition 2.

By duality, we have

Corollary 1. *For every $p \in (\frac{m}{m-4}, p^*]$ and for almost every $p \in (p^*, \frac{m+4}{m-4}]$, the operator*

$$L : \mathcal{C}^{4,\alpha}_{\nu}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\}\big) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-4}\big(\mathbb{T}^{n-m} \times \mathbb{R}^m - \mathbb{T}^{n-m} \times \{0_{\mathbb{R}^m}\}\big),$$

is surjective.

3.2. **A non-linear argument.** Let τ a $\mathcal{C}^{5,\alpha}(\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\})$ diffeomorphism close to the the identity of $(\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\})$ to it self and

$$w\in\mathcal{C}^{4,\alpha}_{\nu}\big(\mathbb{T}^{n-m}\times\mathbb{R}^m-\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\}\big).$$

Consider the non-linear operator $\mathcal{N}$ defined on

$$\mathcal{C}^{5,\alpha}(\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\})\times\mathcal{C}^{4,\alpha}_{\nu}\big(\mathbb{T}^{n-m}\times\mathbb{R}^m-\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\}\big),$$

by

$$\mathcal{N}(\tau,w)(x)=\Delta^2(u_0+w)\circ(\tau)^{-1}(y,z)-(u_0+w)^p\circ(\tau)^{-1}(y,z),$$

where $x=(y,z)$. We shall determine solutions of $\mathcal{N}(\tau,w)=0$ close to the known solution $(I,0)$ by using the implicit function Theorem. We have $\mathcal{N}(I,0)=0$, furthermore $D_w\mathcal{N}(I,0)=L$, which is an isomorphism of $\mathcal{C}^{4,\alpha}_{\nu}\big(\mathbb{T}^{n-m}\times\mathbb{R}^m-\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\}\big)$ to $\mathcal{C}^{0,\alpha}_{\nu-4}\big(\mathbb{T}^{n-m}\times\mathbb{R}^m-\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\}\big)$. By using the implicit function Theorem, there exist $\mathcal{U}\in\mathcal{C}^{5,\alpha}(\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\})$ neighborhood of I, $\mathcal{V}\in\mathcal{C}^{4,\alpha}_{\nu}\big(\mathbb{T}^{n-m}\times\mathbb{R}^m-\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\}\big)$ a neighborhood of 0 and a map $\varphi:\mathcal{U}\longrightarrow\mathcal{V}$ satisfying

$$\mathcal{N}(\tau,\varphi(\tau))=0\qquad\forall\,\tau\in\mathcal{U}.$$

Hence, we have constructed a solutions u of $\Delta^2u-u^p=0$ on $\mathbb{T}^{n-m}\times\mathbb{R}^m-\tau(\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\})$ which are singular on $\tau(\mathbb{T}^{n-m}\times\{0_{\mathbb{R}^m}\})$.

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