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EXISTENCE OF GLOBAL SOLUTIONS FOR A SEMILINEAR PARABOLIC CAUCHY PROBLEM

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Abstract. In this paper, we consider the parabolic equation $w_t = \Delta w + |x|^l w^p$, $x \in \mathbf{R}^n$, $t > 0$ with $w(x,0) = f(x)$ and show the existence of global solution if $1+(2+l)/n < p < (n+2+2l)/(n-2)$ for each $n \geq 3$ and $l \in (-2, l^*]$, where $l^* = 0$ if $n \geq 4$ and $l^* = \sqrt{3}-1$ if $n = 3$. In order to prove this result, we need an upper solution for this Cauchy problem. If $f(x)$ satisfies some condition, then we can show the existence of upper solution by investigating the structure of positive radial solutions for related elliptic equation which has a gradient term.

1. Introduction

In this paper, we consider the following Cauchy problem

$$\begin{cases} w_t = \Delta w + |x|^l w^p, & x \in \mathbf{R}^n,\ t > 0, \\ w(x,0) = f(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 3$, $l > -2$ and $p > 1$ are parameters, and f is a non-negative bounded continuous function in $\mathbf{R}^n$. Our aim is to show that (1.1) has a global solution when p satisfies $1 + (2+l)/n < p < (n+2+2l)/(n-2)$ for each $n \geq 3$ and $l > -2$.

In 1966, Fujita [3] has proved that if $p < 1 + 2/n$, then the solution of

$$\begin{cases} w_t = \Delta w + w^p, & x \in \mathbf{R}^n,\ t > 0, \\ w(x,0) = f(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.2}$$

blows up in finite time for all $f \geq 0$ and $f \not\equiv 0$, and if $p > 1 + 2/n$, then (1.2) has a global classical solution when f satisfies $0 < f(x) < \delta \exp(-|x|^2)$ where δ is sufficiently small positive number. Moreover, Lee and Ni [6] have shown that if $p > 1+2/n$ and f satisfies $f(x) \sim (1+|x|^2)^{-1/(p-1)}$ as $x \to \infty$, then (1.2) has a global classical solution and the solution $w(x,t)$ satisfies

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$||w(\cdot,t)||_{L^\infty(\mathbf{R}^n)} \sim t^{-1/(p-1)}$ as $t \to \infty$. Furthermore, for the problem (1.1), Wang [8] has proved the following result.

Theorem A. ([8]) *Suppose $n \geq 3$, $l > -2$ and $p \geq (n+2+2l)/(n-2)$, then there exists a small $\mu > 0$ such that if $0 \leq f(x) \leq \mu(1+|x|)^{-(2+l)/(p-1)}$ in $\mathbf{R}^n$, then* (1.1) *has global solution $w(x,t)$ with*

$$||w(\cdot,t)||_{L^\infty(\mathbf{R}^n)} \leq Mt^{-(2+l)/\{2(p-1)\}}.$$

In order to prove Theorem A, the following semilinear elliptic equation with a gradient term is needed:

$$\Delta u + \frac{1}{2}x\cdot\nabla u + \frac{2+l}{2(p-1)}u + |x|^l u^p = 0, \quad x \in \mathbf{R}^n, \tag{1.3}$$

where n, l and p are parameters. It is easily seen that $w(x,t)$ given by

$$w(x,t) = t^{-(2+l)/\{2(p-1)\}}u\big(x/\sqrt{t}\big) \tag{1.4}$$

satisfies the equation of (1.1) if and only if $u(x) : \mathbf{R}^n \to \mathbf{R}$ in (1.4) satisfies (1.3). In the following, we set

$$\lambda := \frac{2+l}{2(p-1)}.$$

Here, we note that

$$1 + \frac{2+l}{n} < p < \frac{n+2+2l}{n-2} \Longleftrightarrow \frac{n-2}{4} < \lambda < \frac{n}{2} \tag{1.5}$$

holds. Since we will study the radial solutions ($u = u(r)$ with $r = |x|$) of (1.3), we need the following initial value problem

$$\begin{cases} u'' + \Big(\dfrac{n-1}{r} + \dfrac{r}{2}\Big)u' + \lambda u + r^l u^p = 0, \quad r > 0, \\ u(0) = \alpha > 0. \end{cases} \tag{1.6}$$

Note that

$$\begin{cases} u'' + \Big(\dfrac{n-1}{r} + \dfrac{r}{2}\Big)u' + \lambda u_+ + r^l (u_+)^p = 0, \quad r > 0, \\ u(0) = \alpha > 0, \end{cases} \tag{1.7}$$

where $u_+ := \max\{u, 0\}$, has a unique solution $u(r) \in C([0,\infty)) \cap C^2((0,\infty))$ with $\lim_{r\to 0} ru'(r) = 0$, whose proof will be given in Section 2. In this paper, we denote the unique solution of (1.7) by $u(r;\alpha)$, and we call $u(r;\alpha)$ an entirely positive solution when $u(r;\alpha)$ stays positive for all $r > 0$. (Note that if $u(r;\alpha)$ is an entirely positive solution, then (1.7) is equivalent to (1.6).) If $u(r;\alpha)$ is an entirely positive solution, then we can show the following result on the asymptotic behavior of $u(r;\alpha)$ as $r \to \infty$ by applying some arguments which have been used in [1], [4] and [7].

Theorem 1.1. *Let $u(r;\alpha)$ be an entirely positive solution of* (1.6)*. If $n \geq 3$, $-2 < l < 1$ and $p > 1$, then*

(i) $\lim_{r\to\infty} r^{2\lambda}u(r;\alpha)$ *exists in* $[0,\infty)$.

(ii) *Set $S := \lim_{r\to\infty} r^{2\lambda}u(r;\alpha)$. If $S = 0$, then there exists some positive constant C such that*

$$u(r;\alpha) = Cr^{2\lambda-n}\exp\Big(-\frac{r^2}{4}\Big)\big\{1+O(r^{-2})\big\} \quad as \quad r\to\infty. \tag{1.8}$$

Existence of the limit S has been stated in [8], but its proof was omitted and it seems that it is not trivial. So, in Section 2, we give the proof of not only (ii) but also (i) for Theorem 1.1.

On the initial value problem (1.6), Wang has shown the following result in [8].

Theorem B. ([8]) *If $n \geq 3$, $l > -2$ and $p \geq (n+2+2l)/(n-2)$, then there exists some positive number $\tilde{\alpha}$ such that $u(r;\tilde{\alpha}) > 0$ for all $r \geq 0$ and $u(r;\tilde{\alpha})$ satisfies $u(r;\tilde{\alpha}) \sim r^{-2\lambda}$ as $r \to \infty$.*

Wang has shown Theorem A by using Theorem B. We introduce the sketch of the proof in the following:

Sketch of the proof of Theorem A. When the parameters n, l and p satisfy the assumptions, it follows from Theorem B that there exists some positive solution $u(r)$ with $u(r) \sim r^{-2\lambda}$ as $r \to \infty$ for the problem (1.6). Now, we set

$$\hat{w}(x,t) = (t+1)^{-\lambda}u\big(|x|/\sqrt{t+1}\big),$$

then $\hat{w}(x,t)$ is an upper solution of (1.1) if μ is sufficiently small positive number. Moreover, since the trivial solution can be a lower solution of (1.1), there exists a global solution of (1.1). □

Namely, if we could show the existence of some entirely positive solution $u(r)$ with $u(r) \sim r^{-2\lambda}$ as $r \to \infty$ for the problem (1.6), then we could conclude the existence of a global solution to (1.1). Our main result is as follows.

Theorem 1.2. *Let $n \geq 3$. Suppose*

$$\begin{cases} -2 < l \leq 0 & for \quad n \geq 4, \\ -2 < l \leq \sqrt{3}-1 & for \quad n = 3, \end{cases} \tag{1.9}$$

and $1 + (2+l)/n < p < (n+2+2l)/(n-2)$, then there exists a positive number α_0 such that the structure of positive solutions to (1.6) *is as follows.*

(i) *For any* $\alpha \in (0, \alpha_0)$, $u(r; \alpha) > 0$ *for all* $r \geq 0$ *and* $u(r; \alpha)$ *satisfies* $u(r; \alpha) \sim r^{-2\lambda}$ *as* $r \to \infty$.

(ii) $u(r; \alpha_0) > 0$ *for all* $r \geq 0$ *and* $u(r; \alpha_0)$ *satisfies* $\lim_{r\to\infty} r^{2\lambda} u(r; \alpha_0) = 0$.

(iii) *For any* $\alpha \in (\alpha_0, \infty)$, $u(\cdot, \alpha)$ *has a zero in* $(0, \infty)$.

Remark 1.1. (i) If $n \geq 3$, $l > -2$ and $1 < p \leq 1 + (2 + l)/n$, then $\varphi(r)$, which is a solution of (1.10) below, has a zero in $(0, \infty)$. Therefore, $u(\cdot, \alpha)$ has a zero in $(0, \infty)$ for any $\alpha \in (0, \infty)$ independent of the behavior of $v(r; \alpha)$ which is a solution of (1.12) with (1.13) below.

(ii) If $l > 0$ for $n \geq 4$, $l > \sqrt{3} - 1$ for $n = 3$ and $1 + (2 + l)/n < p < (n + 2 + 2l)/(n - 2)$, then the structure of positive solutions to (1.6) remains open.

Now, we obtain the following result by Theorem 1.2 and the sketch of the proof of Theorem A.

Theorem 1.3. *Let* $n \geq 3$. *Suppose* (1.9) *and* $1 + (2 + l)/n < p < (n + 2 + 2l)/(n - 2)$, *then there exists a small* $\mu > 0$ *such that if* $0 \leq f(x) \leq \mu(1 + |x|)^{-(2+l)/(p-1)}$ *in* $\mathbf{R}^n$, *then* (1.1) *has global solution* $w(x, t)$.

In order to prove Theorem 1.2, we apply the classification theorem by Yanagida and Yotsutani [10]. Let $\varphi(r)$ be a solution of

$$\begin{cases} \varphi'' + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right)\varphi' + \lambda\varphi = 0, \quad r > 0, \\ \varphi(0) = 1. \end{cases} \tag{1.10}$$

For a solution $u(r)$ of (1.6), if we put

$$u(r) = \varphi(r)v(r), \tag{1.11}$$

then we see that $v(r)$ satisfies

$$\begin{cases} (g(r)v')' + g(r)K(r)v^p = 0, \quad r > 0, \\ v(0) = \alpha > 0, \end{cases} \tag{1.12}$$

where

$$g(r) := r^{n-1} \exp\left(r^2/4\right)\varphi(r)^2, \quad K(r) := r^l \varphi(r)^{p-1}. \tag{1.13}$$

We should note that $\varphi(r) > 0$ on $[0, \infty)$ if $\lambda < n/2$ by (i) of Proposition 3.1 in Section 3. So, in order to see whether $u(r)$ has a zero or not, we have only to check this property for $v(r)$. For this purpose, we employ the classification theorem by Yanagida and Yotsutani [10], which is stated as follows. Let $g(r)$ and $K(r)$ satisfy

$$\begin{cases} g(r) \in C^2([0, \infty)); \qquad g(r) > 0 \quad \text{on} \quad (0, \infty); \\ 1/g(r) \notin L^1(0, 1); \qquad 1/g(r) \in L^1(1, \infty), \end{cases} \tag{g}$$

and

$$\begin{cases} K(r) \in C(0,\infty); \quad K(r) \geq 0 \quad \text{and} \quad K(r) \not\equiv 0 \quad \text{on} \quad (0,\infty); \\ h(r)K(r) \in L^1(0,1); \quad g(r)\Big(h(r)/g(r)\Big)^p K(r) \in L^1(1,\infty), \end{cases} \tag{K}$$

where

$$h(r) := g(r)\int_r^\infty g(s)^{-1}ds.$$

Moreover, define

$$G(r) := \frac{2}{p+1}g(r)h(r)K(r) - \int_0^r g(s)K(s)ds, \tag{1.14}$$

$$H(r) := \frac{2}{p+1}h(r)^2\Big(\frac{h(r)}{g(r)}\Big)^p K(r) - \int_r^\infty h(s)\Big(\frac{h(s)}{g(s)}\Big)^p K(s)ds, \tag{1.15}$$

and

$$r_G := \inf\{r \in (0,\infty):\ G(r) < 0\}, \quad r_H := \sup\{r \in (0,\infty):\ H(r) < 0\}.$$

Theorem C. ([10]) *Suppose that $g(r)$ and $K(r)$ satisfy the conditions (g) and (K), respectively. Let $v(r;\alpha)$ be a solution of*

$$\begin{cases} (g(r)v')' + g(r)K(r)(v_+)^p = 0, \quad r > 0, \\ v(0) = \alpha > 0, \end{cases} \tag{1.16}$$

where $v_+ := \max\{v, 0\}$, and suppose that $G(r) \not\equiv 0$ on $(0,\infty)$. If

$$0 < r_H \leq r_G < \infty, \tag{1.17}$$

then there exists a unique positive number α_0 such that the structure of solutions to (1.16) *is as follows.*

(a) *For every $\alpha \in (\alpha_0,\infty)$, $v(r;\alpha)$ has a zero in $(0,\infty)$.*
(b) *If $\alpha = \alpha_0$, then $v(r;\alpha) > 0$ on $[0,\infty)$ and*

$$0 < \lim_{r\to\infty}\Big(\int_r^\infty g(s)^{-1}ds\Big)^{-1} v(r;\alpha) < \infty. \tag{1.18}$$

(c) *For every $\alpha \in (0,\alpha_0)$, $v(r;\alpha) > 0$ on $[0,\infty)$ and*

$$\lim_{r\to\infty}\Big(\int_r^\infty g(s)^{-1}ds\Big)^{-1} v(r;\alpha) = \infty. \tag{1.19}$$

Remark 1.2. Although we can not find the proof of Theorem C in [10], this result can be shown by using Theorem 1 in [9]. We will give the sketch of the proof of Theorem C in Appendix below.

In Sections 3 and 4, we will prove the following result.

Proposition 1.1. *Let* $n \geq 3$. *Suppose* (1.9) *and* $1 + (2 + l)/n < p < (n+2+2l)/(n-2)$. *Then the inequality* (1.17) *holds if* $g(r)$ *and* $K(r)$ *satisfy* (1.13).

Thus, we can get the structure of positive solutions to (1.12). Moreover, combining Theorem 1.1 with noting (1.11), we can conclude Theorem 1.2. (See Section 3 below.)

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. First of all, we show the following result.

Proposition 2.1. *Suppose that* $n \geq 3$, $l > -2$ *and* $p > 1$.

(i) *The following two conditions are equivalent:*
 (I) $u \in C([0,\infty)) \cap C^2((0,\infty))$ *satisfies* (1.7).
 (II) $u \in C([0,\infty))$ *satisfies*

$$u(r) = \alpha - \int_0^r t^{1-n} \exp\Big(-\frac{t^2}{4}\Big)\Big[\int_0^t s^{n-1}\exp\Big(\frac{s^2}{4}\Big) \times \Big\{\lambda u_+(s) + s^l(u_+(s))^p\Big\}\,ds\Big]dt. \tag{2.1}$$

Moreover, in both cases, the following properties hold.
 (a) $\lim_{r\to 0} r^{n-1}u'(r) = 0$.
 (b) u *is non-increasing in* $[0, r_+]$, *where* $r_+ := \inf\{r \in (0,\infty) : u(r) = 0\}$.
 (c) $\lim_{r\to 0} ru'(r) = 0$.

(ii) *There exists a unique solution of* (1.7) *which satisfies* (a), (b), (c) *of* (i).

Proof. (i) We will prove (I) implies (II). First, we show (a). Since the equation of (1.7) is equivalent to

$$\Big\{r^{n-1}\exp\Big(\frac{r^2}{4}\Big)u'\Big\}' + r^{n-1}\exp\Big(\frac{r^2}{4}\Big)\Big\{\lambda u_+ + r^l(u_+)^p\Big\} = 0, \tag{2.2}$$

and $u(0) = \alpha > 0$, there exists some sufficiently small positive number ζ such that $\{r^{n-1}\exp(r^2/4)u'\}' < 0$ for $r \in (0,\zeta)$, namely, $r^{n-1}\exp(r^2/4)u'$ is a decreasing function in $(0,\zeta)$. So noting $\lim_{r\to 0}\exp(r^2/4) = 1$, we obtain $\lim_{r\to 0} r^{n-1}u'(r) \in (-\infty, +\infty]$. Now, we will prove $\lim_{r\to 0} r^{n-1}u'(r) = 0$. Suppose that there exists some $\eta \in (0,\infty)$ such that $\lim_{r\to 0} r^{n-1}u'(r) = \eta$. (We can prove similarly in case $\eta \in (-\infty, 0) \cup \{+\infty\}$.) Then for any $\varepsilon \in$

$(0, \eta)$, there exists some positive number δ such that if $0 < r < \delta$, then $|r^{n-1}u'(r) - \eta| < \varepsilon$, namely,

$$r^{1-n}(\eta - \varepsilon) < u'(r) < r^{1-n}(\eta + \varepsilon) \tag{2.3}$$

holds. Integrating (2.3) in $[r, \delta]$, we get

$$\frac{\eta - \varepsilon}{n-2}\big(r^{2-n} - \delta^{2-n}\big) < u(\delta) - u(r) < \frac{\eta + \varepsilon}{n-2}\big(r^{2-n} - \delta^{2-n}\big).$$

If $r \to +0$, then we have

$$+\infty \le \liminf_{r \to +0} \big(u(\delta) - u(r)\big) = u(\delta) - \alpha;$$

but this is a contradiction. Thus, we obtain (a). Moreover, integrating (2.2) on $[\varepsilon, r]$ and letting $\varepsilon \to +0$, we get

$$u'(r) = -r^{1-n} \exp\Big(-\frac{r^2}{4}\Big) \int_0^r s^{n-1} \exp\Big(\frac{s^2}{4}\Big) \Big\{\lambda u_+(s) + s^l (u_+(s))^p\Big\}\, ds \tag{2.4}$$

by (a) and $s^{n-1+l} \in L^1((0,1))$ for $l > -2$. It is readily seen that (b) holds from (2.4). Moreover, integrating (2.4) on $[0, r]$ and using $u(0) = \alpha$, we have (I). Furthermore, let r_0 be any sufficiently small positive number. Then noting the continuity of u, we have that there exists some positive number m such that $|u(r)| \le m$ for any $r \in [0, r_0]$. Therefore, we have

$$\begin{aligned}
|ru'(r)| &\le r^{2-n} \exp\big(-\frac{r^2}{4}\big) \\
&\quad \times \int_0^r \Big\{\lambda s^{n-1} \exp\Big(\frac{s^2}{4}\Big)|u_+(s)| + s^{n-1+l} \exp\Big(\frac{s^2}{4}\Big)|u_+(s)|^p\Big\} ds \\
&\le r^{2-n} \exp\Big(-\frac{r^2}{4}\Big)\Big\{\exp\Big(\frac{r^2}{4}\Big)\lambda m \int_0^r s^{n-1} ds + \exp\Big(\frac{r^2}{4}\Big) m^p \int_0^r s^{n-1+l} ds\Big\} \\
&= \frac{\lambda m}{n} r^2 + \frac{m^p}{n+l} r^{l+2}
\end{aligned}$$

for any $r \in [0, r_0]$. So we conclude (c) by noting $l > -2$. Finally, it is easily seen that (II) implies (I).

(ii) The uniqueness of the solution for (2.1) is easily proved by using Gronwall's inequality. Moreover, its existence is obtained by standard argument. Thus, by the equivalence of (2.1) and (1.7), we can finish the proof of Proposition 2.1. □

Now, we will prove Theorem 1.1. Let u be an entirely positive solution of (1.7) in the remaining part of this section. Following proofs of Propositions

2.2, 2.3, 2.4, Lemma 2.1 and (i) of Theorem 1.1 are essentially depended on the arguments used in [4].

Proposition 2.2. *Suppose $n \geq 3$, $l > -2$ and $p > 1$. If u satisfies*

$$0 < u(r) \leq M(1+r)^{\sigma}, \quad r \geq 1$$

for some $M > 0$ and $\sigma \leq 0$, then there is a positive constant N, which depends on M and σ, such that

$$|u'(r)| \leq N(1+r)^{\sigma-1+\max\{0,l\}}, \quad r \geq 1.$$

Proof. From (2.4) with noting $u_+ = u$ now, we obtain

$$\begin{aligned}
&|u'(r)| \leq r^{1-n} \exp\big(-\frac{r^2}{4}\big)\int_0^r s^{n-1} \exp\big(\frac{s^2}{4}\big)\{\lambda M(1+s)^{\sigma} + s^l M^p (1+s)^{\sigma p}\}ds \\
&\leq r^{1-n} \exp\Big(-\frac{r^2}{4}\Big)\int_0^r s^{n-1} \exp\Big(\frac{s^2}{4}\Big)\Big(\lambda M + s^l M^p\Big)\Big(1+s\Big)^{\sigma} ds \\
&= r^{1-n} \exp\Big(-\frac{r^2}{4}\Big)\Big[\int_0^{r/2} s^{n-1} \exp\Big(\frac{s^2}{4}\Big)\Big(\lambda M + s^l M^p\Big)\Big(1+s\Big)^{\sigma} ds \\
&\quad + \int_{r/2}^r s^{n-1} \exp\big(\frac{s^2}{4}\big)\Big(\lambda M + s^l M^p\Big)(1+s)^{\sigma} ds\Big] \\
&\leq r^{1-n} \exp\big(-\frac{r^2}{4}\big)\Big[\Big\{\big(\frac{r}{2}\big)^{n-1}\lambda M + \big(\frac{r}{2}\big)^{n-1+l} M^p\Big\} \exp\big(\frac{r^2}{16}\big)\int_0^{r/2}(1+s)^{\sigma} ds \\
&\quad + \Big(r^{n-1}\lambda M + r^{n-1+l} M^p\Big)\int_{r/2}^r \exp\Big(\frac{s^2}{4}\Big)\Big(1+s\Big)^{\sigma} ds\Big] \\
&\leq \exp\Big(-\frac{r^2}{4}\Big)\Big[\Big(\frac{\lambda M}{2^{n-1}} + \frac{M^p}{2^{n-1+l}} r^l\Big) \exp\Big(\frac{r^2}{16}\Big)(1+0)^{\sigma} \cdot \frac{r}{2} \\
&\quad + \Big(\lambda M + M^p r^l\Big)\Big(1+\frac{r}{2}\Big)^{\sigma-1}\int_{r/2}^r \exp\Big(\frac{s^2}{4}\Big)\Big(1+s\Big) ds\Big] \\
&= \Big(\frac{\lambda M}{2^n} + \frac{M^p}{2^{n+l}} r^l\Big) r \exp\Big(-\frac{3}{16} r^2\Big) \\
&\quad + \Big(\lambda M + M^p r^l\Big) \exp\Big(-\frac{r^2}{4}\Big)\Big(1+\frac{r}{2}\Big)^{\sigma-1}\int_{r/2}^r \exp\Big(\frac{s^2}{4}\Big)\Big(1+s\Big) ds.
\end{aligned}$$

Since it follows from $r/2 \leq s \leq r$ and $r \geq 1$ that $s \geq 1/2$, namely, $1 \leq 2s$ holds, we have

$$\int_{r/2}^r \exp\Big(\frac{s^2}{4}\Big)\Big(1+s\Big) ds \leq 3\int_{r/2}^r s \exp\Big(\frac{s^2}{4}\Big) ds$$

$$= 6\left\{\exp\left(\frac{r^2}{4}\right) - \exp\left(\frac{r^2}{16}\right)\right\} \le 6\exp\left(\frac{r^2}{4}\right).$$

Therefore, we get

$$\begin{aligned}\exp\left(-\frac{r^2}{4}\right)\left(1+\frac{r}{2}\right)^{\sigma-1}\int_{r/2}^{r}\exp\left(\frac{s^2}{4}\right)\left(1+s\right)ds &\le 6\cdot 2^{1-\sigma}\left(2+r\right)^{\sigma-1}\\ &\le 3\cdot 2^{2-\sigma}\left(1+r\right)^{\sigma-1}.\end{aligned}$$

Moreover, it is easily seen that there exists some positive number $C = C(\sigma)$ such that

$$r\exp\left(-\frac{3}{16}r^2\right) \le C(1+r)^{\sigma-1} \tag{2.5}$$

for all $r \ge 0$. So we obtain

$$|u'(r)| \le \left\{C\left(\frac{\lambda M}{2^n} + \frac{M^p}{2^{n+l}}r^l\right) + 3\cdot 2^{2-\sigma}\left(\lambda M + M^p r^l\right)\right\}\left(1+r\right)^{\sigma-1}$$

for all $r \ge 1$. If $-2 < l \le 0$, then it follows from $r^l \le 1$ that

$$|u'(r)| \le \left\{C\left(\frac{\lambda M}{2^n} + \frac{M^p}{2^{n+l}}\right) + 3\cdot 2^{2-\sigma}\left(\lambda M + M^p\right)\right\}\left(1+r\right)^{\sigma-1}.$$

Moreover, if $l > 0$, then it follows from $r^l \ge 1$ and $r^l \le (1+r)^l$ that

$$\begin{aligned}|u'(r)| &\le \left\{C\left(\frac{\lambda M}{2^n}r^l + \frac{M^p}{2^{n+l}}r^l\right) + 3\cdot 2^{2-\sigma}\left(\lambda M r^l + M^p r^l\right)\right\}\left(1+r\right)^{\sigma-1}\\ &\le \left\{C\left(\frac{\lambda M}{2^n} + \frac{M^p}{2^{n+l}}\right) + 3\cdot 2^{2-\sigma}\left(\lambda M + M^p\right)\right\}\left(1+r\right)^{\sigma-1+l}.\end{aligned}$$

Thus, we can conclude Proposition 2.2. □

Now set

$$E(r) := \frac{1}{2}u'(r)^2 + \frac{1}{2}\lambda u(r)^2 + \frac{1}{p+1}r^l u(r)^{p+1}. \tag{2.6}$$

Then we obtain the following lemma on $E(r)$.

Lemma 2.1. *Suppose $n \ge 3$, $-2 < l < 1$ and $p > 1$. Then*

(i) *Function $r^{-1}E(r)$ is decreasing in $(0,\infty)$.*
(ii) *The following inequality holds:*

$$\int_r^\infty \frac{E(x)}{x}dx \le \frac{E(r)}{4\lambda} + \frac{|u(r)u'(r)|}{2r} + \frac{n}{2}\int_r^\infty \frac{|u(x)u'(x)|}{x^2}dx + \int_r^\infty \frac{u'(x)^2}{x}dx. \tag{2.7}$$

(iii) *Suppose*

$$E(x) \le Cr^{\rho} \tag{2.8}$$

for all $r \ge 1$ and some fixed $\rho < 0$. Then there exist some positive constants A and B such that

$$\begin{cases} E(r) \le A\big(r^{-4\lambda} + r^{\rho-2+2\max\{0,l\}}\big) & \text{if} \quad 4\lambda+\rho-3+2\max\{0,l\} \ne -1, \\ E(r) \le Br^{\rho-1+\max\{0,l\}} & \text{if} \quad 4\lambda+\rho-3+2\max\{0,l\} = -1 \end{cases} \tag{2.9}$$

for all $r \ge 1$.

Proof. (i) Differentiating $r^{-1}E(r)$ and using (1.6), we have

$$\begin{aligned} \frac{d}{dr}\Big(r^{-1}E(r)\Big) &= -\Big(\frac{n-1}{r} + \frac{r}{2}\Big)r^{-1}u'(r)^2 \\ &\quad - \frac{1}{2}r^{-2}u'(r)^2 - \frac{1}{2}\lambda r^{-2}u(r)^2 + \frac{1}{p+1}(l-1)r^{l-2}u(r)^{p+1} \\ &< 0 \end{aligned}$$

for all $r > 0$ from $-2 < l < 1$. Therefore, $r^{-1}E(r)$ is decreasing in $(0,\infty)$.

(ii) Differentiating $u^2/4 + uu'/r$ and using (1.6), we get

$$\begin{aligned} &\frac{d}{dr}\Big(\frac{u(r)^2}{4} + \frac{u(r)u'(r)}{r}\Big) \\ &= -\frac{1}{r}\Big(\lambda u(r)^2 + r^l u(r)^{p+1}\Big) - \frac{n}{r^2}u(r)u'(r) + \frac{1}{r}u'(r)^2. \end{aligned}$$

So, noting $p > 1$, we obtain

$$\begin{aligned} \frac{E(r)}{r} &\le \frac{1}{2r}\Big(\lambda u(r)^2 + r^l u(r)^{p+1}\Big) + \frac{1}{2r}u'(r)^2 \\ &= -\frac{1}{2}\frac{d}{dr}\Big(\frac{u(r)^2}{4} + \frac{u(r)u'(r)}{r}\Big) - \frac{n}{2r^2}u(r)u'(r) + \frac{1}{r}u'(r)^2, \end{aligned}$$

and integrating this inequality from $r(\ge 1)$ to R, we have

$$\begin{aligned} \int_r^R \frac{E(x)}{x}dx &\le -\frac{1}{2}\Big(\frac{u(R)^2}{4} + \frac{u(R)u'(R)}{R} - \frac{u(r)^2}{4} - \frac{u(r)u'(r)}{r}\Big) \\ &\quad - \frac{n}{2}\int_r^R \frac{u(x)u'(x)}{x^2}dx + \int_r^R \frac{u'(x)^2}{x}dx \\ &\le \frac{u(r)^2}{8} + \frac{u(r)u'(r)}{2r} - \frac{u(R)^2}{8} - \frac{u(R)u'(R)}{2R} \\ &\quad + \frac{n}{2}\int_r^R \frac{|u(x)u'(x)|}{x^2}dx + \int_r^R \frac{u'(x)^2}{x}dx \end{aligned}$$

$$\leq \frac{E(r)}{4\lambda} + \frac{|u(r)u'(r)|}{2r} - \frac{u(R)u'(R)}{2R}$$
$$+ \frac{n}{2}\int_r^R \frac{|u(x)u'(x)|}{x^2}dx + \int_r^R \frac{u'(x)^2}{x}dx.$$

Noting $0 < u(r) \leq \alpha$ from (b) of Proposition 2.1, by Proposition 2.2 with $\rho = 0$ we get

$$|u'(r)| \leq Cr^{-1+\max\{0,l\}} \quad \text{for} \quad r \geq 1, \tag{2.10}$$

for some positive constant C. So, letting $R \to \infty$, we get (2.7). (Note that the right-hand side of (2.7) is finite.)

(iii) From (2.8) and (2.6), there exists some positive constant C_1 such that

$$u(r) \leq C_1 r^{\rho/2}, \quad r \geq 1.$$

Moreover, it follows from Proposition 2.2 that there exists some positive constant C_2 such that

$$|u'(r)| \leq C_2 r^{\rho/2-1+\max\{0,l\}}, \quad r \geq 1.$$

Therefore, it follows from (2.7) that

$$4\lambda \int_r^\infty \frac{E(x)}{x}dx \leq E(r) + C_3 r^{\rho-2+2\max\{0,l\}}, \quad r \geq 1,$$

for some positive constant C_3. Now setting

$$F(r) := \int_r^\infty \frac{E(x)}{x}dx,$$

we get

$$rF'(r) + 4\lambda F(r) \leq C_3 r^{\rho-2+2\max\{0,l\}},$$

namely,

$$\frac{d}{dr}\Big(r^{4\lambda}F(r)\Big) \leq C_3 r^{4\lambda+\rho-3+2\max\{0,l\}}, \quad r \geq 1. \tag{2.11}$$

Moreover, assuming $4\lambda + \rho - 3 + 2\max\{0,l\} \neq -1$ and integrating (2.11) from 1 to r, we obtain

$$\begin{aligned} F(r) &\leq C_4 r^{-4\lambda} + C_5 r^{\rho-2+2\max\{0,l\}} \\ &\leq \max\{C_4, C_5\}\Big(r^{-4\lambda} + r^{\rho-2+2\max\{0,l\}}\Big), \quad r \geq 1, \end{aligned} \tag{2.12}$$

where C_4 and C_5 are some positive constants. Furthermore, assume $4\lambda + \rho - 3 + 2\max\{0,l\} = -1$. It follows from (2.11) and $1 - \max\{0,l\} > 0$ that

$$\frac{d}{dr}\Big(r^{4\lambda}F(r)\Big) \leq C_3 r^{4\lambda+\rho-2+\max\{0,l\}}, \quad r \geq 1. \tag{2.13}$$

Noting $4\lambda + \rho - 2 + \max\{0, l\} > -1$ and integrating (2.13) from 1 to r, we obtain

$$\begin{aligned} F(r) &\leq C_6 r^{-4\lambda} + C_3 r^{\rho-1+\max\{0,l\}} \\ &\leq \max\{C_6, C_3\} r^{\rho-1+\max\{0,l\}}, \quad r \geq 1, \end{aligned} \tag{2.14}$$

for some positive constant C_6. Here, noting (i), we obtain

$$F(r) \geq \int_r^{2r} x^{-1} E(x) dx \geq (2r)^{-1} E(2r) \cdot r = \frac{E(2r)}{2} \quad \text{for} \quad r \geq 1,$$

namely $E(r) \leq 2F(r/2)$ for $r \geq 2$. Therefore, there exists some positive constants A' and B' such that

$$\begin{cases} E(r) \leq A'\big(r^{-4\lambda} + r^{\rho-2+2\max\{0,l\}}\big) & \text{if} \quad 4\lambda + \rho - 3 + 2\max\{0,l\} \neq -1, \\ E(r) \leq B' r^{\rho-1+\max\{0,l\}} & \text{if} \quad 4\lambda + \rho - 3 + 2\max\{0,l\} = -1, \end{cases} \tag{2.15}$$

for $r \geq 2$ from (2.12) and (2.14). Finally, since we have $E(r) \leq E(1)r$ for all $r \geq 1$ by using (i), $E(r) \leq 2E(1)$ for $1 \leq r \leq 2$ holds. Therefore, taking another constants A and B instead of A' and B' in (2.15), we get (2.9) for all $r \geq 1$. □

Proposition 2.3. *There exist some positive constants M and N such that*

$$0 < u(r) \leq M(1+r)^{-2\lambda} \quad \textit{and} \quad |u'(r)| \leq N(1+r)^{-2\lambda-1+\max\{0,l\}}$$

for all $r \geq 1$.

Proof. It follows from $0 < u \leq \alpha$ and Proposition 2.2 with $\rho = 0$ that there exists some positive constant D_0 such that

$$|u'(r)| \leq D_0 r^{-1+\max\{0,l\}} \quad \text{for} \quad r \geq 1.$$

Using same argument as the proof of (iii) of Lemma 2.1, we obtain

$$\begin{cases} E(r) \leq D_1\Big(r^{-4\lambda} + r^{-2+2\max\{0,l\}}\Big) & \text{if} \quad 4\lambda - 3 + 2\max\{0,l\} \neq -1, \\ E(r) \leq D_2 r^{-1+\max\{0,l\}} & \text{if} \quad 4\lambda - 3 + 2\max\{0,l\} = -1, \end{cases} \tag{2.16}$$

for $r \geq 1$. If $-4\lambda < -2 + 2\max\{0,l\}$, then $E(r) \leq 2D_1 r^{-2+2\max\{0,l\}}$. So, from (iii) of Lemma 2.1, we get

$$\begin{cases} E(r) \leq D_3\Big(r^{-4\lambda} + r^{-4+4\max\{0,l\}}\Big) & \text{if } 4\lambda - 5 + 4\max\{0,l\} \neq -1, \\ E(r) \leq D_4 r^{-3+3\max\{0,l\}} & \text{if } 4\lambda - 5 + 4\max\{0,l\} = -1, \end{cases} \tag{2.17}$$

for $r \geq 1$. Moreover, by repeating to use (iii) of Lemma 2.1, if $4\lambda-(2k+1)+2k\max\{0,l\} \neq -1$ for any $k=1,2,\cdots,j-1$, then $-4\lambda < -2k+2k\max\{0,l\}$ for any $k=1,2,\cdots,j-1$ and we have

$$\begin{cases} E(r) \leq D_5\big(r^{-4\lambda}+r^{-2j+2j\max\{0,l\}}\big) & \text{if } 4\lambda-(2j+1)+2j\max\{0,l\} \neq -1, \\ E(r) \leq D_6 r^{-(2j-1)+(2j-1)\max\{0,l\}} & \text{if } 4\lambda-(2j+1)+2j\max\{0,l\} = -1, \end{cases} \tag{2.18}$$

for $r \geq 1$. It follows from the Archimedes' principle that for each $\lambda \in \big((n-2)/4, n/2\big)$ and $l \in (-2,1)$, there exists some natural number J such that $J[1-\max\{0,l\}] \geq 2\lambda$, namely,

$$-4\lambda \geq -2J+2J\max\{0,l\}. \tag{2.19}$$

In the following, let J be the minimum number which satisfies (2.19), namely, $-4\lambda < -2k+2k\max\{0,l\}$ for any $k=1,2,\cdots,J-1$. Then $4\lambda-(2k+1)+2k\max\{0,l\} \neq -1$ holds for any $k=1,2,\cdots,J-1$. If $4\lambda-(2J+1)+2J\max\{0,l\} \neq -1$, then there exists some positive constant D_7 such that

$$E(r) \leq D_7\Big(r^{-4\lambda}+r^{-2J+2J\max\{0,l\}}\Big) \leq 2D_7 r^{-4\lambda} \quad \text{for} \quad r \geq 1,$$

from (2.18). Moreover, if $4\lambda-(2J+1)+2J\max\{0,l\} = -1$, then there exists some positive constant D_8 such that

$$E(r) \leq D_8 r^{-(2J-1)+(2J-1)\max\{0,l\}} \quad \text{for} \quad r \geq 1,$$

from (2.18). So, by using the same argument as the proof of (2.9), we get

$$\begin{aligned} E(r) &\leq D_9\Big(r^{-4\lambda}+r^{-(2J+1)+(2J+1)\max\{0,l\}}\Big) \\ &= D_9\Big(r^{-4\lambda}+r^{-2J+2J\max\{0,l\}+\max\{0,l\}-1}\Big) \\ &= D_9\Big(r^{-4\lambda}+r^{-4\lambda+\max\{0,l\}-1}\Big) \leq 2D_9 r^{-4\lambda}, \end{aligned}$$

for $r \geq 1$. Thus, in any cases, we obtain $E(r) \leq Dr^{-4\lambda}$, namely, $E(r) \leq D'(1+r)^{-4\lambda}$ for all $r \geq 1$. Therefore, noting Proposition 2.2, we have that there exist some positive constant M and N such that $u(r) \leq M(1+r)^{-2\lambda}$ and $|u'(r)| \leq N(1+r)^{-2\lambda-1+\max\{0,l\}}$ for all $r \geq 1$. $\square$

From Proposition 2.3, we can show (i) of Theorem 1.1.

Proof of (i) of Theorem 1.1. By using the equation of (1.6), we have

$$\frac{d}{dr}\Big\{r^{2\lambda}u(r)+2r^{2\lambda-1}u'(r)\Big\} = 2r^{2\lambda-2}\Big\{(2\lambda-n)u'(r)-r^{l+1}u(r)^p\Big\}. \tag{2.20}$$

Moreover, integrating (2.20) from 1 to r, we obtain

$$\begin{aligned} &r^{2\lambda}u(r) + 2r^{2\lambda-1}u'(r) - u(1) - 2u'(1) \\ &= 2(2\lambda - n)\int_1^r x^{2\lambda-2}u'(x)dx - 2\int_1^r x^{2\lambda-1+l}u(x)^p dx. \end{aligned} \tag{2.21}$$

It follows from $\lambda = (2+l)/\{2(p-1)\}$ and the asymptotic estimates in Proposition 2.3 that $\lim_{r\to\infty} r^{2\lambda-1}u'(r) = 0$ and integrals in (2.21) are bounded as $r \to \infty$. Thus, $\lim_{r\to\infty} r^{2\lambda}u(r)$ exists in $[0,\infty)$ by noting $u(r)$ is an entirely positive solution now. □

Next, we will prove (ii) of Theorem 1.1.

Proposition 2.4. *If $S = 0$, then for all $m > 0$,*

$$r^m u(r) \to 0 \quad \textit{and} \quad r^m u'(r) \to 0 \quad \textit{as} \quad r \to \infty.$$

Proof. It follows from Proposition 2.3 that $\lim_{r\to\infty} r^{2\lambda}u'(r) = 0$. Therefore, by the assumption $S = 0$, namely, $\lim_{r\to\infty} r^{2\lambda}u(r) = 0$, we obtain $\lim_{r\to\infty} r^m u(r) = 0$ and $\lim_{r\to\infty} r^m u'(r) = 0$ for any $m \in (0, 2\lambda]$. Moreover, we integrate (2.20) from r to ∞ and get

$$r^{2\lambda}u(r) + 2r^{2\lambda-1}u'(r) = 2(n-2\lambda)\int_r^\infty x^{2\lambda-2}u'(x)dx + 2\int_r^\infty x^{2\lambda-1+l}u(x)^p dx. \tag{2.22}$$

Now, suppose $m \geq 2\lambda$ and there exists some positive constant M such that $u(r) \leq Mr^{-m}$ for $r \geq 1$. Then we have

$$|u'(r)| \leq Nr^{-m-1+\max\{0,l\}} \quad \text{for} \quad r \geq 1,$$

for some positive constant N by Proposition 2.2 and

$$u(r) \leq Cr^{-m-2+\max\{0,l\}} + Dr^{-pm+l} \quad \text{for} \quad r \geq 1,$$

from (2.22). Here, note that $-m-2+\max\{0,l\} < -m$ and $-pm+l < -m$ from $m \geq 2\lambda$. So, by induction with

$$m_1 = 2\lambda \quad \text{and} \quad m_{k+1} = \min\{m_k + 2 - \max\{0,l\},\ pm_k - l\}\ (k = 1, 2, 3, \cdots),$$

we can conclude this proposition. □

For each $\mu > 0$, we define the following domain in the phase plane

$$D_\mu := \{\ (u, U)\ :\ u > 0,\ U < 0,\ U \geq -\mu u\ \},$$

where $U(r) := u'(r)$. Moreover, noting Proposition 2.3, we have

$$\lim_{r\to\infty} r^l u(r)^{p-1} = 0,$$

namely, for any $\mu > 0$ there exists some $R_1 = R_1(\mu) > 0$ such that $r^l u(r)^{p-1} < \mu$ for all $r \in (R_1, \infty)$. Furthermore, we set $R_2 := \frac{2\lambda}{\mu} + 2\mu + 2$. Then we obtain the following result.

Proposition 2.5. *For each $\mu > 0$, set $R_0 := \max\{R_1, R_2\}$. If $R \in [R_0, \infty)$ satisfies $(u(R), U(R)) \in D_\mu$, then $(u(r), U(r)) \in D_\mu$ for all $r \geq R$.*

Proof. Note that the equation of (1.6) can be rewritten as

$$u' = U, \qquad U' = -\Big(\frac{n-1}{r} + \frac{r}{2}\Big)U - \lambda u - r^l u^p. \tag{2.23}$$

We investigate the vector field determined by (2.23) on the boundary of D_μ except for $(u, U) = (0, 0)$, which is defined by

$$L_1 \cup L_2 := \{\ (u, U)\ :\ u > 0,\ U = 0\ \} \cup \{\ (u, U)\ :\ u > 0,\ U = -\mu u\ \}.$$

First, it is impossible that (u, U) exists on L_1 for any $r > 0$ since $U < 0$ if $u > 0$ by (b) of Proposition 2.1. Moreover, if $(u, U) \in L_2$, then

$$\frac{U'}{u'} = -\Big(\frac{n-1}{r} + \frac{r}{2}\Big) - \frac{\lambda u + r^l u^p}{U} = -\Big(\frac{n-1}{r} + \frac{r}{2}\Big) + \frac{\lambda}{\mu} + \frac{r^l u^{p-1}}{\mu}.$$

Thus, if $r \geq R_0$, we obtain

$$\frac{U'}{u'} < -\frac{r}{2} + \frac{\lambda}{\mu} + \frac{\mu}{\mu} < -\mu, \quad \text{namely,} \quad U' > -\mu u'.$$

Therefore, when (u, U) arrives at some point on L_2, (u, U) will go back to the domain D_μ. Thus, we conclude Proposition 2.5. □

Following result can be proved similarly as Propositions 2.5, 2.6 and 2.7 in [7] with noting Proposition 2.5.

Proposition 2.6.

(i) *Either $\lim_{r\to\infty} \frac{U(r)}{u(r)} = -\infty$ or $\lim_{r\to\infty} \frac{U(r)}{u(r)} = 0$ holds.*

(ii) *If $\lim_{r\to\infty} \frac{U(r)}{u(r)} = 0$, then $\lim_{r\to\infty} \frac{rU(r)}{u(r)} = -2\lambda$.*

(iii) *If $\lim_{r\to\infty} \frac{U(r)}{u(r)} = -\infty$, then $S = 0$.*

(iv) *If $\lim_{r\to\infty} \frac{U(r)}{u(r)} = 0$, then $S \neq 0$.*

From Proposition 2.6, if $S = 0$, then $\lim_{r\to\infty} U(r)/u(r) = -\infty$ holds. Moreover, in order to show the following proposition, it is sufficiently to change $u(x)^{p-1}$ to $-x^l u(x)^{p-1}$ in the proof of Lemmas 13, 14 and 15 in [1].

Proposition 2.7. *Suppose u and U satisfy $\lim_{r\to\infty} \frac{U(r)}{u(r)} = -\infty$. Then*

(i) $\lim_{r\to\infty} \frac{U(r)}{ru(r)} = -\frac{1}{2}$.

(ii) *Set* $\mathcal{E}_1(r) := rU(r) + \frac{1}{2}r^2u(r)$. *Then* $\lim_{r\to\infty} \frac{\mathcal{E}_1(r)}{u(r)} = 2\lambda - n$ *holds.*

(iii) *Set* $\mathcal{E}_2(r) := r^2\mathcal{E}_1(r) - (2\lambda - n)r^2u(r)$. *Then* $\lim_{r\to\infty} \frac{\mathcal{E}_2(r)}{u(r)} = 4(2\lambda - n)(\lambda - 1)$ *holds.*

Proof of (ii) of Theorem 1.1. By using Proposition 2.7 and same argument as in the proof of Theorem 2 in [1], we can readily show this result. □

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we prepare the following proposition which is shown in [2] and [5].

Proposition 3.1. *Let* φ *be the solution of* (1.10). *If* $n \geq 3$, $l > -2$ *and* $p > 1 + (2 + l)/n$ *(i.e.,* $0 < \lambda < n/2$*), then*

(i) $\varphi(r) > 0$ *and* $\varphi'(r) < 0$ *in* $(0, \infty)$.

(ii) $\lim_{r\to\infty} r^{2\lambda}\varphi(r)$ *exists, and it is positive.*

(iii) $\varphi(r) = 1 - \frac{\lambda}{2n}r^2 + o(r^2)$ *as* $r \to 0$.

(iv) $\exp(r^2/4)\varphi(r)$ *is an increasing function of* $r \in [0, \infty)$.

(v) *If* $(n + l)/(n - 2) \leq p < (n + 2 + 2l)/(n - 2)$, *then*

$$-2\lambda < \frac{r\varphi'(r)}{\varphi(r)} < 0, \tag{3.1}$$

and if $1 + (2 + l)/n < p < (n + l)/(n - 2)$, *then*

$$-\frac{4\lambda}{n - 2\lambda} < \frac{r\varphi'(r)}{\varphi(r)} < 0 \tag{3.2}$$

for all $r \in (0, \infty)$.

(vi) *Set*

$$m(\lambda) := \begin{cases} 2\lambda & \text{if} \quad \dfrac{n + l}{n - 2} \leq p < \dfrac{n + 2 + 2l}{n - 2}, \\ \dfrac{4\lambda}{n - 2\lambda} & \text{if} \quad 1 + \dfrac{2 + l}{n} < p < \dfrac{n + l}{n - 2}. \end{cases}$$

Then for each $p \in (1 + (2 + l)/n, (n + 2 + 2l)/(n - 2))$, $r^{m(\lambda)}\varphi(r)$ *is an increasing function of* $r \in [0, \infty)$.

By using Proposition 3.1 and noting the proof of Proposition 3.1 in [2], we can easily check the conditions imposed on the coefficients of equation of (1.12).

Lemma 3.1. *Functions* $g(r) := r^{n-1}\exp(r^2/4)\varphi(r)^2$ *and* $K(r) := r^l\varphi(r)^{p-1}$ *satisfy* (g) *and* (K), *respectively.*

Therefore, $g(r)$ and $K(r)$ are admissible. Substituting their definition (1.13) into $G(r)$ and $H(r)$, we obtain

$$G(r) = \frac{2}{p+1} r^{2n-2+l} \exp\Big(\frac{r^2}{2}\Big)\varphi(r)^{p+3} \Big\{\int_r^\infty s^{1-n} \exp\Big(-\frac{s^2}{4}\Big)\varphi(s)^{-2} ds\Big\} - \int_0^r s^{n-1+l} \exp\Big(\frac{s^2}{4}\Big)\varphi(s)^{p+1} ds,$$

$$H(r) = \frac{2}{p+1} r^{2n-2+l} \exp\Big(\frac{r^2}{2}\Big)\varphi(r)^{p+3} \Big\{\int_r^\infty s^{1-n} \exp\Big(-\frac{s^2}{4}\Big)\varphi(s)^{-2} ds\Big\}^{p+2} - \int_r^\infty s^{n-1+l} \exp\big(\frac{s^2}{4}\big)\varphi(s)^{p+1} \Big\{\int_s^\infty t^{1-n} \exp\Big(-\frac{t^2}{4}\Big)\varphi(t)^{-2} dt\Big\}^{p+1} ds.$$

Now, we will prove Proposition 1.1. To do so, we must investigate the profiles of $G(r)$ and $H(r)$. First, we study monotonically properties of $G(r)$ and $H(r)$. Differentiation yields

$$G'(r) = \Big(\int_r^\infty g(s)^{-1} ds\Big)^{-p-1} H'(r) = \frac{2}{p+1} g(r)K(r)\Big(\Phi(r) - \frac{p+3}{2}\Big), \quad (3.3)$$

where

$$\begin{aligned}\Phi(r) &:= \Big(2g'(r) + \frac{g(r)K'(r)}{K(r)}\Big)\int_r^\infty g(s)^{-1} ds \\ &= r^{n-2} \exp\Big(\frac{r^2}{4}\Big)\varphi(r)^2 \Big[r^2 + \{2(n-1)+l\} + (p+3)\Big(\frac{r\varphi'(r)}{\varphi(r)}\Big)\Big] \quad (3.4) \\ &\times \int_r^\infty s^{1-n} \exp\Big(-\frac{s^2}{4}\Big)\varphi(s)^{-2} ds.\end{aligned}$$

In view of (3.3), $G(r)$ and $H(r)$ have the same extremal points, namely those $r > 0$ which satisfy $\Phi(r) = (p+3)/2$. So, in order to know the sign of $G'(r)$ and $H'(r)$, we must study the relation between $\Phi(r)$ and $(p+3)/2$. We first consider the behavior of $\Phi(r)$ near $r = 0$ and $r = \infty$.

Lemma 3.2. $\lim_{r\to 0}\Phi(r) = \frac{2n-2+l}{n-2}$ *and* $\lim_{r\to\infty}\Phi(r) = 2$.

Proof. Using l'Hospital's theorem, in view of the definition of $\Phi(r)$ we derive

$$\begin{aligned}&\lim_{r\to\infty}\Phi(r) \\ &= \lim_{r\to\infty} \frac{\frac{d}{dr}\int_r^\infty s^{1-n}\exp(-s^2/4)\varphi(s)^{-2} ds}{\frac{d}{dr}\Big[r^{n-2}\exp(r^2/4)\varphi(r)^2\Big\{r^2 + 2(n-1) + l + (p+3)\Big(\frac{r\varphi'}{\varphi}\Big)\Big\}\Big]^{-1}} \\ &= 2,\end{aligned}$$

with noting (3.1) and (3.2). The case $r \to 0$ is done similarly. □

Note that $\Phi(r)$ is continuous in $[0,\infty)$ and $(p+3)/2$ satisfies

$$(2 <)\ 2 + \frac{2+l}{2n} < \frac{p+3}{2} < \frac{2n-2+l}{n-2}, \tag{3.5}$$

if and only if $1 + \frac{2+l}{n} < p < \frac{n+2+2l}{n-2}$. Moreover, we get the following lemma which will be proved in Section 4 below.

Lemma 3.3. *Under the assumptions on n, l and p in Proposition* 1.1*, there exists a unique number $r^* \in (0,\infty)$ satisfying $\Phi(r^*) = (p+3)/2$ and*

$$\begin{cases} \Phi(r) > \dfrac{p+3}{2} & in \quad [0, r^*), \\[2ex] \Phi(r) \le \dfrac{p+3}{2} & in \quad (r^*, \infty). \end{cases} \tag{3.6}$$

Therefore, the following result follows from (3.3) and Lemma 3.3.

Lemma 3.4. *Under the assumptions on n, l and p in Proposition* 1.1*, there exists a unique number $r^* \in (0,\infty)$ such that $G(r)$ and $H(r)$ are increasing in $[0, r^*)$ and decreasing in (r^*, ∞).*

Moreover, in order to locate r_G and r_H, we need to determine the behavior of $G(r)$ and $H(r)$ near $r = 0$ and $r = \infty$. Similarly, to the proof of Proposition 3.6 in [2], we can prove the following result by using Proposition 3.1.

Lemma 3.5. *Under the assumptions on n, l and p in Proposition* 1.1*,*

(i) $\lim_{r\to\infty} G(r) = -\infty$. (ii) $\lim_{r\to 0} G(r) = 0$.
(iii) $\liminf_{r\to\infty} H(r) \ge 0$. (iv) $\limsup_{r\to 0} H(r) \in [-\infty, 0)$.

Proposition 1.1 follows from Lemmas 3.4 and 3.5 as follows.

Proof of Proposition 1.1. As is already seen in Lemma 3.4, both $G(r)$ and $H(r)$ have exactly one local maximum at $r^* \in (0,\infty)$. Moreover, in view of Lemma 3.5 $H(r)$ is negative near $r = 0$ and positive for large r. Thus, $H(r^*) > 0$ and $0 < r_H < r^*$. Besides, we obtain $G(r^*) > 0$ from $G(0) = 0$, and the negativity of $G(r)$ for large r yields $0 < r^* < r_G < \infty$; so we conclude that condition (1.17) holds. □

Now, we can prove Theorem 1.2.

Proof of Theorem 1.2. From Proposition 1.1 and Theorem C there exists a unique positive number α_0 such that $v(\cdot;\alpha)$, i.e., $u(\cdot;\alpha)$ has a zero in $(0,\infty)$

for every $\alpha \in (\alpha_0, \infty)$. Moreover, for every $\alpha \in (0, \alpha_0]$, $v(\cdot;\alpha)$, i.e., $u(\cdot;\alpha)$ is positive in $[0,\infty)$ and

$$\lim_{r\to\infty}\left\{\int_r^\infty s^{1-n}\exp\Big(-\frac{s^2}{4}\Big)\varphi(s)^{-2}ds\right\}^{-1} v(r;\alpha)\begin{cases}\in(0,\infty) & \text{if} \quad \alpha=\alpha_0,\\ =\infty & \text{if} \quad \alpha\in(0,\alpha_0).\end{cases}$$

Integrating by parts, we obtain

$$\int_r^\infty s^{1-n}\exp\Big(-\frac{s^2}{4}\Big)\varphi(s)^{-2}ds = 2r^{-n}\exp\Big(-\frac{r^2}{4}\Big)\varphi(r)^{-2}(1+o(1))$$

as $r\to\infty$. (Here, we use the boundedness of $r\varphi'/\varphi$.) Taking the decay rate of $\varphi(r)$ and the definition of $v(r)$ into account (see (ii) of Proposition 3.1), this estimate immediately shows that (1.18) with $g(r) = r^{n-1}\exp(r^2/4)\varphi^2$ is equivalent to $\lim_{r\to\infty} r^{2\lambda}u(r) = 0$ in view of Theorem 1.1. At the same time, we obtain that (1.19) with $g(r) = r^{n-1}\exp(r^2/4)\varphi^2$ is equivalent to $\lim_{r\to\infty} r^{2\lambda}u(r) \in (0,\infty)$, i.e., $u(r;\alpha)$ satisfies $u(r;\alpha)\sim r^{-2\lambda}$ as $r\to\infty$ for every $\alpha\in(0,\alpha_0)$. □

4. Proof of Lemma 3.3

We will show there is exactly one crossing point of $q = \Phi(r)$ and $q = (p+3)/2$ in (r,q)-plane. Our strategy is to investigate the sign of $\Phi'(r_*)$ for r_* satisfying $\Phi(r_*) = (p+3)/2$. Here, the existence of r_* is guaranteed by Lemma 3.2, the continuity of $\Phi(r)$ and (3.5). Define

$$\begin{cases}\Omega_1 := \{r_*\in(0,\infty) \ : \ \Phi'(r_*)>0\},\\ \Omega_2 := \{r_*\in(0,\infty) \ : \ \Phi'(r_*)=0\},\\ \Omega_3 := \{r_*\in(0,\infty) \ : \ \Phi'(r_*)<0\}.\end{cases}$$

Then we obtain the following result.

Lemma 4.1. *Suppose the assumptions on n, l and p in Proposition* 1.1. *Then*

(i) Ω_1 *is empty.*
(ii) Ω_2 *consists of at most one element.*
(iii) Ω_3 *consists of at most one element.*

Proof. From (1.10), we can express $X'(r)$ in terms of $X(r)$ as

$$X'(r) = -\frac{1}{r}X(r)^2 - \frac{n-2}{r}X(r) - \frac{r}{2}X(r) - \lambda r. \tag{4.1}$$

So differentiating $\Phi(r)$, we get

$$\Phi'(r) = \Big[\frac{r^4}{2} + \{(2n-1)-\lambda(p+3)\}r^2 + 2(n-1)(n-2) \tag{4.2}$$

$$+2\{r^2+2(n-1)+l\}\Big(\frac{r\varphi'}{\varphi}\Big)+(p+3)\Big(\frac{r\varphi'}{\varphi}\Big)^2\Big]$$
$$\times\, r^{n-3}\exp\Big(\frac{r^2}{4}\Big)\varphi(r)^2\int_r^\infty s^{1-n}\exp\Big(-\frac{s^2}{4}\Big)\varphi(s)^{-2}ds$$
$$-\frac{1}{r}\left\{r^2+2(n-1)+l+(p+3)\Big(\frac{r\varphi'}{\varphi}\Big)\right\}.$$

For any r_* satisfying $\Phi(r_*)=(p+3)/2$ the equality

$$\begin{aligned} &r_*^{n-3}\exp\Big(\frac{r_*^2}{4}\Big)\varphi(r_*)^2\int_{r_*}^\infty s^{1-n}\exp\Big(-\frac{s^2}{4}\Big)\varphi(s)^{-2}ds\\ &\quad=\frac{p+3}{2r_*\left\{r_*^2+2(n-1)+l+(p+3)\Big(\frac{r_*\varphi'(r_*)}{\varphi(r_*)}\Big)\right\}}\end{aligned}\tag{4.3}$$

holds. Note that

$$2r_*\left\{r_*^2+2(n-1)+l+(p+3)\Big(\frac{r_*\varphi'(r_*)}{\varphi(r_*)}\Big)\right\}>0,\tag{4.4}$$

since the left-hand side of (4.3) is positive. Combining (4.2) with $r=r_*$ and (4.3), we obtain

$$\Phi'(r_*)=\frac{\Psi(r_*)}{2r_*\left\{r_*^2+2(n-1)+l+(p+3)\Big(\frac{r_*\varphi'(r_*)}{\varphi(r_*)}\Big)\right\}},$$

where

$$\begin{aligned}\Psi(r):=\frac{p-1}{2}r^4+&\left\{(2n-1)p-2n+5-\lambda(p+3)^2+\frac{p-5}{2}l\right\}r^2\\ &+\{2(n-1)+l\}\{(n-2)p-(n+2)-2l\}\\ -2(p+3)&\{r^2+2(n-1)+l\}\Big(\frac{r\varphi'}{\varphi}\Big)-(p+3)^2\Big(\frac{r\varphi'}{\varphi}\Big)^2.\end{aligned}\tag{4.5}$$

In view of (4.4), we will investigate the sign of $\Psi(r_*)$ instead of $\Phi'(r_*)$. Using (3.1) and (3.2), it is easily seen that

$$\begin{cases}\lim_{r\to\infty}\Psi(r)=+\infty,\\ \lim_{r\to0}\Psi(r)=\{2(n-1)+l\}\{(n-2)p-(n+2)-2l\}<0.\end{cases}\tag{4.6}$$

Moreover, we have the following lemma whose proof will be given at the end of this section.

Lemma 4.2. *Under the assumptions on n, l and p in Proposition* 1.1, *there exists a unique number $\hat r\in(0,\infty)$ satisfying $\Psi(\hat r)=0$ such that*

$$\Psi(r)<0\quad in\quad(0,\hat r)\quad and\quad\Psi(r)>0\quad in\quad(\hat r,\infty).$$

Recalling that the sign of $\Phi'(r_*)$ is equivalent to that of $\Psi(r_*)$, from Lemma 4.2 we get the sign of $\Phi'(r_*)$ as follows:

$$\begin{cases} \Phi'(r_*) < 0 & \text{if} \quad r_* \in [0, \hat{r}), \\ \Phi'(r_*) = 0 & \text{if} \quad r_* = \hat{r}, \\ \Phi'(r_*) > 0 & \text{if} \quad r_* \in (\hat{r}, \infty). \end{cases} \tag{4.7}$$

First, we will show (i). If $q = \Phi(r)$ and $q = (p+3)/2$ cross in $(\hat{r}, \infty)$, then there exists a unique number r'_* in $(\hat{r}, \infty)$ such that $\Phi(r'_*) = (p+3)/2$ with $\Phi'(r'_*) > 0$. But, it is impossible because of $\Phi(r) \to 2$ as $r \to \infty$. Therefore, $\Omega_1 = \emptyset$.

Moreover, if $q = \Phi(r)$ and $q = (p+3)/2$ cross in $[0, \hat{r})$, then there exists a unique number $r''_* \in [0, \hat{r})$ such that $\Phi(r''_*) = (p+3)/2$ with $\Phi'(r''_*) < 0$. Therefore, $\Omega_3 = \emptyset$ or $\Omega_3 = \{r''_*\}$; so we conclude (iii). Statement (ii) is trivial in view of (4.7). □

Thus, we conclude that the relation between $q = \Phi(r)$ and $q = (p+3)/2$ in (r, q)-plane is one of the following:

(a) $\Phi(r) > (p+3)/2$ in $[0, r''_*)$ and $\Phi(r) < (p+3)/2$ in (r''_*, ∞),
(b) $\Phi(r) > (p+3)/2$ in $[0, \hat{r})$ and $\Phi(r) < (p+3)/2$ in $(\hat{r}, \infty)$,
(c) $\Phi(r) > (p+3)/2$ in $[0, r''_*)$ and $\Phi(r) < (p+3)/2$ in $(r''_*, \infty) \backslash \hat{r}$.

(Here, we used the notation introduced in the proof of Lemma 4.1.) Therefore, by putting $r^* := r''_*$ in cases (a) and (c) or $r^* = \hat{r}$ in case (b), Lemma 3.3 holds.

Proof of Lemma 4.2. In order to prove Lemma 4.2, we will investigate the sign of $\Psi'(\tilde{r})$ for $\tilde{r}$ satisfying $\Psi(\tilde{r}) = 0$. If we set

$$\begin{cases} X(r) := r\varphi'/\varphi; \ A := -(p+3)^2; \ B(r) := -2(p+3)\{r^2 + 2(n-1) + l\}; \\ C(r) := \dfrac{p-1}{2} r^4 + \big\{(2n-1)p - 2n + 5 - \lambda(p+3)^2 + \dfrac{l}{2}(p-5)\big\} r^2 \\ \qquad\quad + \{2(n-1) + l\}\{(n-2)p - (n+2) - 2l\}, \end{cases}$$

then $\Psi(r)$ can be rewritten as

$$\Psi(r) \equiv AX(r)^2 + B(r)X(r) + C(r). \tag{4.8}$$

Differentiating (4.8) and using (4.1), we obtain

$$\begin{aligned} \Psi'(r) = &-\frac{2A}{r} X(r)^3 - \Big\{\frac{2A(n-2) + B(r)}{r} + Ar\Big\} X(r)^2 \\ &- \Big\{\frac{(n-2)B(r)}{r} - B'(r) + \Big(2\lambda A + \frac{B(r)}{2}\Big) r\Big\} X(r) - \lambda B(r) r + C'(r). \end{aligned} \tag{4.9}$$

From $\Psi(\tilde{r})=0$, $X(\tilde{r})^2$ and $X(\tilde{r})^3$ can be replaced by

$$\begin{cases} X(\tilde{r})^2 = -\dfrac{B(\tilde{r})}{A}X(\tilde{r}) - \dfrac{C(\tilde{r})}{A}, \\ X(\tilde{r})^3 = \Big(\dfrac{B(\tilde{r})^2}{A^2} - \dfrac{C(\tilde{r})}{A}\Big)X(\tilde{r}) + \dfrac{B(\tilde{r})C(\tilde{r})}{A^2}. \end{cases} \tag{4.10}$$

Substituting (4.10) into (4.9), all the terms containing $X(\tilde{r})$ vanish and $\tilde{r}\Psi'(\tilde{r})$ turns out to be a polynomial in $\tilde{r}^2$ of degree 3:

$$\begin{aligned}(p+3)\tilde{r}\Psi'(\tilde{r}) = {}& \frac{(p+1)(p-1)}{2}\tilde{r}^6 + \eta(l,p,n,\lambda)\tilde{r}^4 + \kappa(l,p,n,\lambda)\tilde{r}^2 \\ &+ 2\{2(n-1)+l\}\{(n-2)p+n-4-l\}\{(n-2)p-(n+2)-2l\},\end{aligned} \tag{4.11}$$

where

$$\eta(l,p,n,\lambda) := -\lambda p^3 + \Big(3n-5\lambda-1+\frac{l}{2}\Big)p^2 - 3(\lambda-2+l)p - 3\Big(n-3\lambda-1+\frac{l}{2}\Big)$$

and

$$\begin{aligned}\kappa(l,p,n,\lambda) := {}& -3(p-1)l^2 + \Big[(2n-3)p^2 - 6(2n-1)p - 3(2n+1) \\ &+ 4\lambda(p+3)^2\Big]l + 2(n-1)(p-1)\{-\lambda p^2 + 3(n-2\lambda-1)p + 3(n-3\lambda-1)\}.\end{aligned}$$

We want to determine the sign of $\Psi'(\tilde{r})$. Setting $x=\tilde{r}^2$, we have $(p+3)\tilde{r}\Psi'(\tilde{r})=\Theta(x)$, where $\Theta(x)$ is a polynomial of degree 3 given by

$$\begin{aligned}\Theta(x) := {}& \frac{(p+1)(p-1)}{2}x^3 + \eta(l,p,n,\lambda)x^2 + \kappa(l,p,n,\lambda)x \\ &+ 2\{2(n-1)+l\}\{(n-2)p+n-4-l\}\{(n-2)p-(n+2)-2l\},\end{aligned} \tag{4.12}$$

for $x\in[0,\infty)$. Concerning the profile of $\Theta(x)$, we readily see

$$\begin{cases} \Theta(0) = 2\{2(n-1)+l\}\{(n-2)p+n-4-l\}\{(n-2)p-(n+2)-2l\}, \\ \lim\limits_{x\to\infty}\Theta(x) = +\infty. \end{cases} \tag{4.13}$$

(Here, we must note that $(n-2)p+n-4-l>0$ holds if $n\geq 3$, $p>1+(2+l)/n$ and $-2<l<n^2-2n-2$.) We will investigate what may happen in $(0,\infty)$. It is readily seen that $l^2+2(n+4)l+4(2n+1)>0$ for any $n\geq 3$ and $l\in(-2,1)$. Moreover, we get the following result.

Lemma 4.3. *Let $n\geq 3$. For each $l\in(-2,1)$, set*

$$p^\natural(l) := 1 + \frac{2(l+2)^2\,\{l+2(n-1)\}}{(n-2)\{l^2+2(n+4)l+4(2n+1)\}}.$$

Then

(i) *For any* $n \geq 3$ *and* $l \in (-2,1)$, $p^{\natural}(l) < (n+2+2l)/(n-2)$ *holds.*
(ii) *For each* $l \in (-2,1)$, *if*

$$\max\left\{1+\frac{2+l}{n},\ p^{\natural}(l)\right\} < p < \frac{n+2+2l}{n-2},$$

then $\Theta(x)$ *has exactly one zero in* $(0,\infty)$.

Proof. (i) This follows from

$$\frac{n+2+2l}{n-2} - p^{\natural}(l) = \frac{8(l+2)(2l+n+2)}{(n-2)\{l^2+2(n+4)l+4(2n+1)\}}.$$

(ii) Note that it follows from $n^2-2n-2 \geq 1$ for any $n \geq 3$ that $\Theta(0) < 0$ under the assumptions on n, l and p. Differentiating $\Theta(x)$, we obtain

$$\Theta'(x) = \frac{3}{2}(p+1)(p-1)x^2 + 2\eta(l,p,n,\lambda)x + \kappa(l,p,n,\lambda).$$

Now we will show $\eta(l,p,n,\lambda)$ is positive under the assumptions on n, l, p and λ. In fact, define

$$f_1(p) := \eta(l,p,n,\lambda), \qquad p \in \Big(1, \frac{n+2+2l}{n-2}\Big),$$

then we observe

$$f_1(1) = 8-4l > 0 \tag{4.14}$$

and

$$\begin{aligned}&(n-2)^3 f_1\Big(\frac{n+2+2l}{n-2}\Big) \\ &= 2\{l+2(n-1)\}\big[(n-2)\big\{l^2+2(n+4)l+4(2n+1)\big\} \\ &-4(l+2)\{l+2(n-1)\}\lambda\big] > 0,\end{aligned} \tag{4.15}$$

since $p^{\natural}(l) < p$ is equivalent to

$$\lambda < \frac{(n-2)\big\{l^2+2(n+4)l+4(2n+1)\big\}}{4(l+2)\{l+2(n-1)\}}.$$

From (4.14) and (4.15), it is sufficiently to show that $f_1(p)$ does not have any local minimum in $(1,(n+2+2l)/(n-2))$. So consider the sign of $f_1''(\tilde{p})$, where $\tilde{p}$ satisfies $f_1'(\tilde{p}) = 0$. Using

$$f_1'(p) = -3\lambda p^2 + 2\Big(3n-5\lambda-1+\frac{l}{2}\Big)p - 3(\lambda-2+l)$$

and

$$f_1''(p) = -6\lambda p + 2\Big(3n-5\lambda-1+\frac{l}{2}\Big),$$

we obtain

$$\tilde{p}f_1''(\tilde{p}) = -6\lambda\tilde{p}^2 + 2\Big(3n - 5\lambda - 1 + \frac{l}{2}\Big)\tilde{p} = -3\lambda(\tilde{p}^2 - 1) + 3(l-2) < 0.$$

Thus, if $f_1(p)$ has an extremum, then it must be a local maximum and we conclude

$$\eta(l,p,n,\lambda) = f_1(p) \geq \min\left\{ f_1(1), f_1\Big(\frac{n+2+2l}{n-2}\Big)\right\} > 0.$$

Since $\eta(l,p,n,\lambda) > 0$, the quadratic equation $\Theta'(x) = 0$ does not have two solutions in $(0,\infty)$; so $\Theta(x)$ has at most one extremum. Therefore, either $\Theta(x)$ is increasing in $[0,\infty)$, or $\Theta(x)$ has a local minimum at $\hat{x}_1 \in (0,\infty)$ such that $\Theta(x)$ is decreasing in $[0,\hat{x}_1)$ and increasing in $(\hat{x}_1,\infty)$. Thus, from (4.13) and $\Theta(0) < 0$ we can conclude that $\Theta(x)$ has exactly one zero in $(0,\infty)$. ☐

Now, for each $l \in (-2,1)$, we set

$$p^\sharp(l) := 1 + \frac{4(2+l)}{3(n-1)}.$$

It is easily seen that $1 + (2+l)/n < p^\sharp(l) < (n+2+2l)/(n-2)$ holds for any $n \geq 3$ and $l \in (-2,1)$. Moreover, we obtain the following result.

Lemma 4.4. *Let $n = 3$. For each $l \in (-2,1)$, if $1 + \frac{2+l}{3} < p \leq p^\sharp(l)$, then $\Theta(x)$ has exactly one zero in $(0,\infty)$.*

Proof. Noting that

$$l - (p-1) < l - \frac{2+l}{3} = \frac{2(l-1)}{3} < 0,$$

and $p \leq p^\sharp(l)$ is equivalent to $\lambda \geq 3/4$, we get

$$\begin{aligned}
\kappa(l,p,3,\lambda) &= -3(p-1)l^2 + \left\{3(p^2 - 10p - 7) + 4\lambda(p+3)^2\right\} l \\
&\quad + 4(p-1)\left\{-\lambda p^2 + 6(1-\lambda)p + 3(2-3\lambda)\right\} \\
&= 4(p+3)^2\{l - (p-1)\}\lambda \\
&\quad - 3(p-1)l^2 + 3(p^2 - 10p - 7)l + 24(p+1)(p-1) \\
&\leq 4(p+3)^2\{l - (p-1)\}\frac{3}{4} \\
&\quad - 3(p-1)l^2 + 3(p^2 - 10p - 7)l + 24(p+1)(p-1) \\
&= -3(p-1)l^2 + 3\left\{(p+3)^2 + (p^2 - 10p - 7)\right\} l \\
&\quad - 3(p-1)\left\{(p+3)^2 - 8(p+1)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= -3(p-1)^3 + 6(p-1)^2 l - 3(p-1)l^2 \\
&= -3(p-1)\{(p-1)-l\}^2 < 0.
\end{aligned}
$$

Therefore, the quadratic equation $\Theta'(x) = 0$ has exactly one solution in $(0,\infty)$; so $\Theta(x)$ has a local minimum at $\hat{x}_2 \in (0,\infty)$ such that $\Theta(x)$ is decreasing in $[0,\hat{x}_2)$ and increasing in $(\hat{x}_2,\infty)$. Thus, it follows from (4.13) and $\Theta(0) < 0$ that $\Theta(x)$ has a unique zero in $(0,\infty)$. □

Next, we will show the following lemma.

Lemma 4.5. *Let $n \geq 4$. For each $l \in (-2,0]$, if $1+(2+l)/n < p \leq p^\sharp(l)$, $\Theta(x)$ has exactly one zero in $(0,\infty)$*

Proof. Note that

$$2l-(n-1)(p-1) < 0 \tag{4.16}$$

holds since $p > 1+(2+l)/n$, $n \geq 4$ and $l < 1$. Moreover, since $p \leq p^\sharp(l)$ is equivalent to $\lambda \geq 3(n-1)/8$, we obtain

$$
\begin{aligned}
&\kappa(l,p,n,\lambda) = 2\lambda(p+3)^2\{2l-(n-1)(p-1)\} - 3(p-1)l^2 \\
&\quad + \left\{(2n-3)p^2 - 6(2n-1)p - 3(2n+1)\right\} l + 6(p+1)(p-1)(n-1)^2 \\
&\leq \tfrac{3}{4}(n-1)(p+3)^2\{2l-(n-1)(p-1)\} - 3(p-1)l^2 \\
&\quad + \left\{(2n-3)p^2 - 6(2n-1)p - 3(2n+1)\right\} l + 6(p+1)(p-1)(n-1)^2 \\
&= -3(p-1)l^2 + \left\{\tfrac{3}{2}(n-1)(p+3)^2 + (2n-3)p^2 - 6(2n-1)p - 3(2n+1)\right\} l \\
&- \frac{3}{4}(n-1)^2(p+3)^2(p-1) + 6(n-1)^2(p+1)(p-1) \\
&= -3(p-1)l^2 + \frac{1}{2}\left\{(7n-9)p^2 - 6(n+1)p + 3(5n-11)\right\} l \\
&- \frac{3}{4}(n-1)^2(p-1)^3 \\
&= -3(p-1)l^2 + \frac{1}{2}\left\{(7n-9)(p-1)^2 + 8(n-3)(p-1) + 16(n-3)\right\} l \\
&- \frac{3}{4}(n-1)^2(p-1)^3.
\end{aligned}
$$

Thus, if $-2 < l \leq 0$, then $\kappa(l,p,n,\lambda) < 0$, which implies the quadratic equation $\Theta'(x) = 0$ has exactly one solution in $(0,\infty)$. Therefore, $\Theta(x)$ has a local minimum at $\hat{x}_3 \in (0,\infty)$ such that $\Theta(x)$ is decreasing in $[0,\hat{x}_3)$ and increasing in $(\hat{x}_3,\infty)$. Thus, it follows from (4.13) and $\Theta(0) < 0$ that $\Theta(x)$ has a unique zero in $(0,\infty)$. □

Moreover, we will show the following lemma.

Lemma 4.6. *Inequality* $p^\natural(l) \le p^\sharp(l)$ *holds for each* $l \in (-2, \sqrt{3}-1]$ *if* $n = 3$ *and* $l \in (-2, 0]$ *if* $n \ge 4$.

Proof. From

$$p^\sharp(l) - p^\natural(l) = -\frac{2(l+2)\left\{(n+1)l^2 + 2(n^2-7n+16)l - 4(n^2-7)\right\}}{3(n-1)(n-2)\left\{l^2+2(n+4)l+4(2n+1)\right\}},$$

it is sufficiently to investigate the sign of $f_2(l,n) := (n+1)l^2 + 2(n^2-7n+16)l-4(n^2-7)$. In case of $n = 3$, it is easily seen that $f_2(l,3) = 4(l^2+2l-1) \le 0$ if $-2 < l \le \sqrt{3}-1$. Moreover, if $n \ge 4$ and $-2 < l \le 0$, then

$$\begin{aligned} f_2(n,l) &< (n+1)\cdot 4 + 2(n^2-7n+16)\cdot 0 - 4(n^2-7) \\ &= -4(n^2-n-8) < 0, \end{aligned}$$

by noting $n^2 - 7n + 16 > 0$. Therefore, $p^\natural(l) \le p^\sharp(l)$ holds. □

Thus, in view of Lemmas 4.3, 4.4, 4.5, 4.6 and (4.13), under the assumptions on l, n and p in Lemma 4.2, there exists a unique number $x_0 \in (0,\infty)$ such that

$$\begin{cases} \Theta(x) < 0 \quad \text{in} \quad (0,x_0), & \text{i.e.,} \quad \Psi'(\tilde r) < 0 \quad \text{if} \quad \tilde r \in (0,\sqrt{x_0}), \\ \Theta(x_0) = 0, & \text{i.e.,} \quad \Psi'(\tilde r) = 0 \quad \text{if} \quad \tilde r = \sqrt{x_0}, \\ \Theta(x) > 0 \quad \text{in} \quad (x_0,\infty), & \text{i.e.,} \quad \Psi'(\tilde r) > 0 \quad \text{if} \quad \tilde r \in (\sqrt{x_0},\infty). \end{cases}$$

Define $\xi := \{\tilde r \in (0,\infty) : \Psi(\tilde r) = 0\}$. Then $\Psi'(\xi) \ge 0$ from (4.6), which implies $\xi \ge \sqrt{x_0}$. Moreover, equality is not possible in this relation. In fact, $\Psi''(\sqrt{x_0})$ must be non-positive from (4.6) if $\Psi(\sqrt{x_0}) = 0$ with $\Psi'(\sqrt{x_0}) = 0$. But it is impossible by the following lemma.

Lemma 4.7. *Under the assumptions on* l, n *and* p *in Proposition* 1.1, *if* $\Psi(r) = \Psi'(r) = 0$ *for some* $r > 0$, *then* $\Psi''(r) > 0$.

Proof. Differentiating (4.9) and using (4.1), (4.10) and

$$X(\tilde r)^4 = \Big(\frac{2B(\tilde r)C(\tilde r)}{A^2} - \frac{B(\tilde r)^3}{A^3}\Big)X(\tilde r) + \Big(\frac{C(\tilde r)^2}{A^2} - \frac{B(\tilde r)^2C(\tilde r)}{A^3}\Big),$$

we obtain

$$\begin{aligned} &\Psi''(r)|_{\Psi(r)=0} \\ &\quad = \frac{1}{p+3}\Theta(r^2)\Big(-\frac{2}{r^2}X(r) - \frac{p+1}{p+3} + \frac{(-2n+3)p-2n+5+2l}{r^2}\Big) \\ &\qquad + \frac{1}{p+3}\left\{3(p+1)(p-1)r^4 + 4\eta(l,p,n,\lambda)r^2 + 2\kappa(l,p,n,\lambda)\right\}. \end{aligned}$$

Moreover, since $\Theta(r^2) = 0$ if r satisfies $\Psi(r) = \Psi'(r) = 0$, we have either

$$\Psi''(r)|_{\Psi(r)=\Psi'(r)=0}$$

$$
\begin{aligned}
&= \frac{1}{p+3}\left\{3(p+1)(p-1)r^4 + 4\eta(l,p,n,\lambda)r^2 + 2\kappa(l,p,n,\lambda)\right\} \\
&= \frac{1}{p+3}\big[2(p+1)(p-1)r^4 + 2\eta(l,p,n,\lambda)r^2 \\
&+ 2\left\{(p+1)(p-1)r^4/2 + \eta(l,p,n,\lambda)r^2 + \kappa(l,p,n,\lambda)\right\}\big] \\
&= \frac{1}{p+3}\big[2(p+1)(p-1)r^4 + 2\eta(l,p,n,\lambda)r^2 \\
&-4\{2(n-1)+l\}\{(n-2)p+n-4-l\}\{(n-2)p-(n+2)-2l\}r^{-2}\big],
\end{aligned}
$$

or

$$
\begin{aligned}
&\Psi''(r)|_{\Psi(r)=\Psi'(r)=0} \\
&= \frac{1}{p+3}\left\{3(p+1)(p-1)r^4 + 4\eta(l,p,n,\lambda)r^2 + 2\kappa(l,p,n,\lambda)\right\} \\
&= \frac{1}{p+3}\big[(p+1)(p-1)r^4 - 2\kappa(l,p,n,\lambda)r^2 \\
&+ 4\left\{(p+1)(p-1)r^4/2 + \eta(l,p,n,\lambda)r^2 + \kappa(l,p,n,\lambda)\right\}\big] \\
&= \frac{1}{p+3}\big[(p+1)(p-1)r^4 - 2\kappa(l,p,n,\lambda)r^2 \\
&-8\{2(n-1)+l\}\{(n-2)p+n-4-l\}\{(n-2)p-(n+2)-2l\}r^{-2}\big].
\end{aligned}
$$

Therefore, since we know the sign of η and κ from the proof of Lemmas 4.3, 4.4 and 4.5, by Lemma 4.6 we obtain $\Psi''(r)|_{\Psi(r)=\Psi'(r)=0} > 0$. □

Therefore, in view of (4.6), there exists a unique number $\hat{r}$ satisfying $\Psi(\hat{r}) = 0$ with $\hat{r} > \sqrt{x_0}$, such that $\Psi(r) < 0$ in $[0, \hat{r})$ and $\Psi(r) > 0$ in $(\hat{r}, \infty)$. This completes the proof of Lemma 4.2. □

Remark 4.1. In view of the proof of Lemmas 4.5 and 4.6, there exists sufficiently small positive number ε such that $\kappa(n,l,p,\lambda) > 0$ for $l \in (-2, \varepsilon)$. So Theorems 1.2 and 1.3 also hold for $l \in (0, \varepsilon)$ if $n \geq 4$.

Appendix A. Proof of Theorem C

Here, we give the sketch of the proof of Theorem C.

In [9], the structure of positive solutions to the following initial value problem has been studied:

$$
\begin{cases} V''(t) + \dfrac{n-1}{t}V'(t) + \widetilde{K}(t)(V_+(t))^p = 0, \quad t > 0, \\ V(0) = \alpha > 0, \end{cases} \tag{A.1}
$$

where $V_+(t) := \max\{V(t), 0\}$ and $\widetilde{K}(t)$ satisfies the following conditions:

$$\begin{cases} \widetilde{K}(t) \in C(0,\infty), \quad \widetilde{K}(t) \geq 0 \quad \text{and} \quad K(r) \not\equiv 0 \quad \text{on} \quad (0,\infty), \\ t\widetilde{K}(t) \in L^1(0,1), \quad t^{n-1-(n-2)p}\widetilde{K}(t) \in L^1(1,\infty). \end{cases} \tag{A.2}$$

Initial value problem (A.1) has a unique solution $V(t) \in C([0,\infty)) \cap C^2((0,\infty))$ under the first, second and third condition in (A.2). We denote the unique solution by $V(t;\alpha)$. Then either

$$\lim_{t\to\infty} t^{n-2}V(t;\alpha) \in (0,\infty) \quad \text{or} \quad \lim_{t\to\infty} t^{n-2}V(t;\alpha) = \infty$$

holds. Moreover, define

$$\mathcal{G}(t) := \frac{1}{p+1}t^n\widetilde{K}(t) - \frac{n-2}{2}\int_0^t \tau^{n-1}\widetilde{K}(\tau)d\tau, \tag{A.3}$$

$$\mathcal{H}(t) := \frac{1}{p+1}t^{2-(n-2)p}\widetilde{K}(t) - \frac{n-2}{2}\int_t^\infty \tau^{1-(n-2)p}\widetilde{K}(\tau)d\tau, \tag{A.4}$$

and

$$t_{\mathcal{G}} := \inf\{t \in (0,\infty) \ : \ \mathcal{G}(t) < 0\}, \quad t_{\mathcal{H}} := \sup\{t \in (0,\infty) \ : \ \mathcal{H}(t) < 0\}.$$

Then the following result has been proved in [9].

Theorem D. ([9]) *Suppose that $G(r) \not\equiv 0$ on $(0,\infty)$. If*

$$0 < t_{\mathcal{H}} \leq t_{\mathcal{G}} < \infty, \tag{A.5}$$

then there exists a unique positive number α_0 such that the structure of solutions to (A.1) *is as follows.*

(a) *For every $\alpha \in (\alpha_0,\infty)$, $V(t;\alpha)$ has a zero in $(0,\infty)$.*
(b) *If $\alpha = \alpha_0$, then $V(t;\alpha) > 0$ on $[0,\infty)$ and*

$$0 < \lim_{t\to\infty} t^{n-2}V(t;\alpha) < \infty. \tag{A.6}$$

(c) *For every $\alpha \in (0,\alpha_0)$, $V(t;\alpha) > 0$ on $[0,\infty)$ and*

$$\lim_{t\to\infty} t^{n-2}V(t;\alpha) = \infty. \tag{A.7}$$

Now, we will show Theorem C by using Theorem D. Let $t = t(r)$ ($r \in (0,\infty)$) be a function defined by

$$t^{2-n} = (n-2)\int_r^\infty g(s)^{-1}ds. \tag{A.8}$$

Since $g(r)$ satisfies (g), $\lim_{r\to+0} t(r) = 0$, $\lim_{r\to\infty} t(r) = \infty$ and $t(r)$ is a strictly increasing function on $(0,\infty)$. So $t \ : \ (0,\infty) \longrightarrow (0,\infty)$ is a bijective function, namely, we can define an inverse function $r = r(t)$ $(t \in (0,\infty))$. Moreover, differentiating (A.8), we obtain

$$t^{1-n}\frac{dt}{dr} = g(r)^{-1}, \tag{A.9}$$

and differentiating (A.9), we get

$$(1-n)t^{-n}\Big(\frac{dt}{dr}\Big)^2 + t^{1-n}\frac{d^2t}{dr^2} = -g(r)^{-2}g'(r) = -t^{1-n}\frac{dt}{dr}g(r)^{-1}g'(r),$$

namely,

$$\frac{g'(r)}{g(r)}\frac{dt}{dr} = \frac{n-1}{t}\Big(\frac{dt}{dr}\Big)^2 - \frac{d^2t}{dr^2}. \tag{A.10}$$

Now, set $V(t) = v(r)$. Then we have

$$v'(r) = V'(t)\frac{dt}{dr} \qquad \text{and} \qquad v''(r) = V''(t)\Big(\frac{dt}{dr}\Big)^2 + V'(t)\frac{d^2t}{dr^2}. \tag{A.11}$$

Substituting (A.11) into the equation of (1.12) and using (A.9) and (A.10), we obtain

$$V''(t) + \frac{n-1}{t}V'(t) + \widetilde{K}(t)V(t)^p = 0, \quad t > 0, \tag{A.12}$$

where

$$\widetilde{K}(t) := \Big(\frac{dt}{dr}\Big)^{-2}K(r(t)) = t^{2-2n}g(r(t))^2K(r(t)). \tag{A.13}$$

It follows from (g) and (K) that $\widetilde{K}(t)$ defined by (A.13) satisfies all conditions of (A.2). So, we can apply Theorem D to the initial value problem (A.12) with $V(0) = \alpha$. (Note that $\lim_{t\to+0} V(t) = \lim_{t\to+0} v(r(t)) = \alpha$ from $\lim_{t\to+0} r(t) = 0$.) Substituting (A.13) into (A.3) and (A.4), we obtain

$$\mathcal{G}(t) = \frac{1}{p+1}t^{2-n}g(r(t))^2K(r(t)) - \frac{n-2}{2}\int_0^t \tau^{1-n}g(r(\tau))^2K(r(\tau))d\tau \tag{A.14}$$

and

$$\begin{aligned}\mathcal{H}(t) \;\; = \;\; & \frac{1}{p+1}t^{(2-n)(p+2)}g(r(t))^2K(r(t)) \\ & - \frac{n-2}{2}\int_t^\infty \tau^{3-2n-(n-2)p}g(r(\tau))^2K(r(\tau))d\tau.\end{aligned} \tag{A.15}$$

Moreover, by setting $\xi = r(\tau)$, where the relation between ξ and τ is

$$\tau^{2-n} = (n-2)\int_\xi^\infty g(s)^{-1}ds, \quad \xi \in (0,\infty),$$

we have $\dfrac{d\xi}{d\tau} = \tau^{1-n} g(r(\tau))$. Therefore, we obtain

$$\mathcal{G}(t) = \frac{n-2}{2} G(r(t)) \quad \text{and} \quad \mathcal{H}(t) = \frac{n-2}{2} H(r(t)),$$

where $G(r)$ and $H(r)$ are functions of r defined by (1.14) and (1.15), respectively. Now, we suppose $0 < r_H \leq r_G < \infty$. Since $r = r(t)$ is strictly increasing function, there exists unique positive number $t_{\mathcal{G}}$ such that $r_G = r(t_{\mathcal{G}})$. Moreover, it follows from $r_G = \inf\{r \in (0, \infty) : G(r) < 0\}$ that $t_{\mathcal{G}} = \inf\{t \in (0, \infty) : \mathcal{G}(t) < 0\}$. Similarly, there exists unique positive number $t_{\mathcal{H}}$ such that $r_H = r(t_{\mathcal{H}})$ and $t_{\mathcal{H}} = \sup\{t \in (0, \infty) : \mathcal{H}(t) < 0\}$. Moreover, it is readily seen that $0 < t_{\mathcal{H}} \leq t_{\mathcal{G}} < \infty$ holds. So, we obtain the structure of positive solutions of (A.12) with $V(0) = \alpha$ from Theorem D. Finally, since (A.6) and (A.7) are equivalent to (1.18) and (1.19), respectively, we conclude Theorem C with noting $v(r) = V(t)$. $\square$

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