Differential and Integral Equations Volume 21, Numbers 7-8 (2008), 601–622

PSEUDO-RADIAL SOLUTIONS OF SEMI-LINEAR ELLIPTIC EQUATIONS ON SYMMETRIC DOMAINS

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(Submitted by: J.A. Goldstein)

Abstract. In this paper we investigate existence and characterization of non-radial pseudo-radial (or separable) solutions of some semi-linear elliptic equations on symmetric 2-dimensional domains. The problem reduces to the phase plane analysis of a dynamical system. In particular, we give a full description of the set of pseudo-radial solutions of equations of the form $\Delta u = \pm a^2(|x|)u|u|^{q-1}$, with $q > 0$, $q \neq 1$. We also study such equations over spherical or hyperbolic symmetric domains.

1. Introduction

A very rich literature has been devoted during the last decades to the study of semi-linear elliptic equations of the form

$$\Delta u = \varepsilon u|u|^{q-1}, \tag{1.1}$$

over a symmetric Euclidean domain Ω, where $\varepsilon = \pm 1$ and q is a positive real number, $q \neq 1$. In particular, it is known that when the domain Ω is an Euclidean ball, then any positive solution of (1.1) with $q > 1$, satisfying Dirichlet boundary conditions, is radial (see [6], see also [7] for the case $\Omega = \mathbb{R}^n$), while on an annular domain $\Omega = \{0 < R < |x| < R + c\}$, the equation (1.1) admits positive non-radial solutions for $\varepsilon = -1$ and any $q > 1$ (see [3, 5, 14, 16]). See also the important work of Rabinowitz [13] where sufficient conditions are given for the existence of infinitely many non-necessarily radial solutions to such equations.

Accepted for publication: February 2008.
AMS Subject Classifications: 35J60, 58J05, 34D05; 35B99, 35C99.
[1]Supported by a grant from the Lebanese University.

The existence of non-radial solutions is generally based on minimization techniques (see [16] for a survey). Another natural approach toward the existence of non-radial solutions consists in searching solutions of the form $u(r,\theta) = h(r)w(\theta)$, where (r,θ) are polar coordinates and w is a 2π-periodic function. Such a solution is sometimes called *"separable"* or *"pseudo-radial"* (see for instance [9]). Of course, u is non-radial as far as w is non-constant. Moreover, we say that u is of *mode* $k \in \mathbb{N}$ if the least period of w is $\frac{2\pi}{k}$ (thus, radial solutions could be considered as being of mode $+\infty$). Notice that singular pseudo-radial solutions play an important role in the study and classification of singularities of solutions of semi-linear equations.

In the present work, we make use of this last approach in order to investigate the existence of possibly singular non-radial solutions to

(a) "weighted" equations of the form

$$\Delta u = \varepsilon a^2(r) u|u|^{q-1}, \tag{1.2}$$

on a rotationally symmetric (non-necessarily bounded) Euclidean domain $\Omega \subset \mathbb{R}^2$ (see for instance [10, 11, 12] for results concerning radial solutions),

(b) equation (1.1) over rotationally symmetric non-necessarily flat domains like geodesic discs or annular domains of the sphere $\mathbb{S}^2$, the hyperbolic plane $\mathbb{H}^2$, a revolution surface, etc.

Notice that our study includes the "sublinear" case $0 < q < 1$, which is of particular interest since it is not much treated in the literature.

The natural general setting in which these topics can be put is the following. We consider Equation (1.1) on $I \times \mathbb{S}^1$, where I is an interval, $I \subset (0,\infty)$, endowed with a Riemannian metric of the form

$$g = a^2(r)dr^2 + b^2(r)d\theta^2,$$

where a and b are two positive differentiable functions on I. The Laplace operator associated with the metric g is given by

$$\Delta^g = \frac{1}{a^2}\frac{\partial^2}{\partial r^2} + \frac{1}{ab}\left(\frac{b}{a}\right)'\frac{\partial}{\partial r} + \frac{1}{b^2}\frac{\partial^2}{\partial \theta^2}. \tag{1.3}$$

Recall that a rotationally symmetric domain $\Omega \subset \mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$ can be identified, using appropriate polar coordinates (r,θ), with a cylinder $I \times \mathbb{S}^1$ endowed with the Riemannian metric $g = dr^2 + b^2(r)d\theta^2$, with

$$b(r) = \begin{cases} r & \text{in the Euclidean case} \\ \sin r & \text{in the spherical case} \\ \sinh r & \text{in the hyperbolic case.} \end{cases}$$

Here, the r variable represents in each case the geodesic distance to the center of the domain. On the other hand, the weighted equation $\Delta u = \varepsilon a^2(r)u|u|^{q-1}$ of item (a) above is equivalent to the equation

$$\Delta^g u = \varepsilon u|u|^{q-1},$$

where $g = a^2(r)g_{\mathbb{R}^2} = a^2(r)(dr^2 + r^2 d\theta^2)$ is a Riemannian metric conformal to the Euclidean one $g_{\mathbb{R}^2}$.

Hence, we consider for any Riemannian metric $g = a^2(r)dr^2 + b^2(r)d\theta^2$ on $I \times \mathbb{S}^1$, the PDE

$$\Delta^g u = \varepsilon u|u|^{q-1}, \tag{1.4}$$

where $\varepsilon = \pm 1$, and look for non-radial pseudo-radial solutions. We first show (Theorem 1) that a necessary and sufficient condition for equation (1.4) to admit such a non-radial pseudo-radial solution consists in the existence of a real number μ such that:

(i) (condition on the metric g)

$$\left(a^{-1}b^{\frac{2}{1-q}}b'\right)' = \mu\frac{(1-q)}{2}ab^{\frac{1+q}{1-q}}, \tag{1.5}$$

(ii) the following ODE admits a nonconstant 2π-periodic solution

$$w''(\theta) + \mu w(\theta) = \varepsilon w(\theta)|w(\theta)|^{q-1}. \tag{1.6}$$

Moreover, when (i) and (ii) are satisfied, then $u(r,\theta) = b^{\frac{2}{1-q}}(r)w(\theta)$ is a solution of (1.4). Notice that a more general version is actually given in Theorem 1.

For example, condition (i) is satisfied for the Euclidean metric ($a = 1$ and $b = r$) with $\mu = \frac{4}{(1-q)^2}$, the spherical metric ($a = 1$ and $b = \sin r$) with $\mu = 1$ and $q = 3$, and the hyperbolic metric ($a = 1$ and $b = \sinh r$) with $\mu = 1$ and $q = 3$. On the other hand, in the conformal case (i.e. $b(r) = ra(r)$), a consequence of condition (i) is that, if the weighted equation $\Delta u = \varepsilon a^2(r)u|u|^{q-1}$ admits a non-radial pseudo-radial solution in a rotationally symmetric $\Omega \subset \mathbb{R}^2$, then the function $a(r)$ has one of the following forms (Theorem 2):

$$a(r) = r^{-1}(Mr^\alpha + Nr^{-\alpha})^{\frac{1-q}{2}}, \qquad a(r) = r^{-1}(M + N\ln r)^{\frac{1-q}{2}}, \quad \text{or}$$

$$a(r) = r^{-1}[M\cos(\alpha\ln r) + N\sin(\alpha\ln r)]^{\frac{1-q}{2}},$$

with M, $N \in \mathbb{R}$ and $\alpha > 0$.

Condition (ii) leads us to study the ODE (1.6) and seek its 2π-periodic solutions, according to the values of ε, q and μ. Actually, this study constitutes the main part of this paper. The case $\varepsilon = +1$, $0 < q < 1$ and $\mu > 0$

is the more interesting and novel one because of the lack of regularity of the non-linear term at the origin. Notice that some earlier results concerning the ODE (1.6) have been obtained for particular values of ε, q and μ (see for instance [2, 4, 17]).

In particular, for the weighted equation

$$\Delta u = \varepsilon r^{-2}(Mr^{\alpha} + Nr^{-\alpha})^{1-q}u|u|^{q-1}, \tag{1.7}$$

in $\mathbb{R}^2 \setminus \{O\}$, the corresponding μ is equal to α^2 and we obtain (Theorem 3), as an application of the ODE analysis, that

- if $\varepsilon = -1$, then, for every $q \neq 1$, $\alpha > 0$, equation (1.7) admits for any integer $k > \alpha$, a unique (up to sign) pseudo-radial solution of mode k which is sign changing,
- if $\varepsilon = +1$, $q > 1$ and $\alpha > 1$, then for every $k \geq 1$, equation (1.7) admits a unique (up to sign) pseudo-radial solution of mode k which is sign changing,
- if $\varepsilon = +1$, $0 < q < 1$ and $\alpha > 1$, then equation (1.7) admits, for any integer $k \in (\frac{1-q}{2}\alpha, \alpha)$, a unique (up to sign) sign changing pseudo-radial solution of mode k, and, for any possible integer $k \in ((1-q)\alpha, \sqrt{1-q}\,\alpha)$, a unique positive pseudo-radial solution of mode k. Moreover, if $(1-q)\alpha$ is an integer, then there exists a unique non-negative pseudo-radial solution of mode $(1-q)\alpha$,
- if $\varepsilon = +1$ and $0 < \alpha \leq 1$, then any pseudo-radial solution of (1.7) is radial.

In all cases, the radial part of pseudo-radial solutions is given by $h(r) = Mr^{\alpha} + Nr^{-\alpha}$.

In a similar manner, we describe the set of non-radial pseudo-radial solutions of the weighted equations (see Theorems 4 and 5)

$$\Delta u = \varepsilon r^{-2}[M + N\ln r]^{1-q}u|u|^{q-1},$$

and

$$\Delta u = \varepsilon r^{-2}[M\cos(\alpha \ln r) + N\sin(\alpha \ln r)]^{1-q}u|u|^{q-1}.$$

Concerning the equation $\Delta_{\mathbb{S}^2} u = \varepsilon u|u|^{q-1}$ (resp. $\Delta_{\mathbb{H}^2} u = \varepsilon u|u|^{q-1}$) on a rotationally symmetric domain of the sphere (resp. the hyperbolic plane), it turns out that any pseudo-radial solution is radial unless for $\varepsilon = +1$ and $q = 3$ where, for any integer $k \geq 2$, there exists a unique (up to sign) pseudo-radial solution of mode k which is sign changing.

The paper is organized as follows. In the second section, we prove Theorem 1 giving necessary and sufficient conditions for equation (1.4) to admit a non-radial pseudo-radial solution. The third section is devoted to the study of the resulting ODE (1.6). In the last section, we apply our results to answer completely the question of existence of positive and sign-changing non-radial pseudo-radial solutions in the conformal case as well as the case of symmetric domains of standard spaces.

2. A general result

Let $\Omega \subset \mathbb{R}^N$ be a rotationally symmetric domain (e.g. a ball, a spherical shell, $\mathbb{R}^N \backslash \{O\}$, etc.) that we parametrize by spherical coordinates $(r, \sigma) \in I \times \mathcal{S}^{N-1} \mapsto r\sigma \in \Omega$. We endow Ω (or, equivalently, $I \times \mathcal{S}^{N-1}$) with the Riemannian metric

$$g = a^2(r)dr^2 + b^2(r)d\sigma^2,$$

where a and b are two positive differentiable functions on I. The Laplace-Beltrami operator associated with the metric g is given by

$$\Delta^g = \frac{1}{b^2}\left[c^2\frac{\partial^2}{\partial r^2} + (N-1)cc'\frac{\partial}{\partial r} + \Delta_{\mathcal{S}^{N-1}}\right], \tag{2.1}$$

where $c^2 = \frac{b^2}{a^2}$ and $\Delta_{\mathcal{S}^{N-1}}$ stands for the Laplace-Beltrami operator of the standard sphere. Consider the equation

$$\Delta^g u = \varepsilon u|u|^{q-1}, \tag{2.2}$$

where $\varepsilon = \pm 1$ and q is a positive real number, $q \neq 1$. A solution u in Ω is said to be "pseudo-radial" if it can be written, with respect to the (r, σ) coordinates, as $u(r, \sigma) = h(r)w(\sigma)$.

Theorem 1. *Equation* (2.2) *admits a non-radial pseudo-radial solution if and only if there exists* $\mu \in \mathbb{R}$ *such that*

(i) (*Condition on the metric* g)

$$c^2(b^{\frac{2}{1-q}})'' + (N-1)cc'(b^{\frac{2}{1-q}})' = \mu b^{\frac{2}{1-q}}. \tag{2.3}$$

(ii) *The following equation admits a nonconstant solution on* $\mathbb{S}^{N-1}$

$$\Delta_{\mathbb{S}^{N-1}}w + \mu w = \varepsilon w|w|^{q-1}. \tag{2.4}$$

Moreover, when (i) and (ii) are satisfied, then any non-radial pseudo-radial solution of (2.2) *is of the form* $u(r, \sigma) = b^{\frac{2}{1-q}}(r)w(\sigma)$, *where* w *is a non-constant solution of* (2.4).

Notice that, in dimension N=2, the condition on the metric (2.3) reads

$$\left(a^{-1}b^{\frac{2}{1-q}}b'\right)' = \frac{\mu(1-q)}{2}ab^{\frac{1+q}{1-q}}, \tag{2.5}$$

while the equation (2.4) reduces to the ODE (1.6)

$$w''(\theta) + \mu w(\theta) = \varepsilon w(\theta)|w(\theta)|^{q-1},$$

where w is a 2π-periodic function.

The proof of Theorem 1 relies on the following elementary lemma.

Lemma 1. *Let a, b and c be three non-trivial real-valued functions on a set X and let α, β and γ be three differentiable functions on an interval $I \subset \mathbb{R}$ such that γ admits no zeros in I. Assume that, for every $(x, y) \in I \times X$,*

$$\alpha(x)a(y) + \beta(x)b(y) = \gamma(x)c(y), \tag{2.6}$$

then

- *either α and β are proportional to γ on I,*
- *or a, b and c are mutually proportional on X.*

Proof. Dividing (2.6) by $\gamma(x)$ and, then, differentiating with respect to the x variable, one obtains, for every $(x, y) \in I \times X$,

$$\bar{\alpha}(x)a(y) + \bar{\beta}(x)b(y) = c(y), \tag{2.7}$$

and

$$\bar{\alpha}'(x)a(y) = -\bar{\beta}'(x)b(y), \tag{2.8}$$

where $\bar{\alpha} := \frac{\alpha}{\gamma}$ and $\bar{\beta} := \frac{\beta}{\gamma}$. Two cases are to be considered.

(1) Assume that $ab \equiv 0$. Multiplying (2.8) by a and, then, by b, we deduce that $\bar{\alpha}' = \bar{\beta}' \equiv 0$. Hence, $\bar{\alpha}$ and $\bar{\beta}$ are constants which means that α and β are proportional to γ on I.

(2) Assume that there exists $y_0 \in X$ such that $a(y_0)b(y_0) \neq 0$. Setting $K = b(y_0)/a(y_0)$, we deduce from (2.8) that $\bar{\alpha}'(x) = -K\bar{\beta}'(x)$, that is $\bar{\alpha}(x) = -K\bar{\beta}(x) + C$ for some $C \in \mathbb{R}$, and, either $\bar{\alpha}' = \bar{\beta}' \equiv 0$ or $b(y) = Ka(y)$. Therefore, using (2.7), either α and β are proportional to γ on I, or b and c are proportional to a on X. □

Proof of Theorem 1. If $u(r, \sigma) = h(r)w(\sigma)$ is a non-radial pseudo-radial solution of (2.2), then

$$\begin{aligned} &\left[c^2 h''(r) + (N-1)cc'h'(r)\right] w(\sigma) + h(r)\Delta_{\mathcal{S}^{N-1}} w(\sigma) \\ &= \varepsilon b^2(r)h(r)|h(r)|^{q-1} w(\sigma)|w(\sigma)|^{q-1}. \end{aligned} \tag{2.9}$$

Let us apply the last lemma on $I_1 \times \mathbb{S}^{N-1}$, where I_1 is a subinterval on which the function h has no zeros. Since w is not constant (u is assumed to be non-radial) and $q \neq 1$, the function $w|w|^{q-1}$ can never be proportional to w on $\mathbb{S}^{N-1}$. Hence, there exist two constants λ and μ such that

$$h(r) = \lambda b^2(r)h(r)|h(r)|^{q-1}, \tag{2.10}$$

$$c^2h'' + (N-1)cc'h' = \mu h, \tag{2.11}$$

and, hence,

$$\Delta_{\mathcal{S}^{N-1}} w(\sigma) + \mu w(\sigma) = \frac{\varepsilon}{\lambda} w(\sigma)|w(\sigma)|^{q-1}. \tag{2.12}$$

From (2.10) we have $|h(r)| = (\lambda b^2)^{-\frac{1}{q-1}}$. Hence, the function $|h|$ is proportional to $b^{-\frac{2}{q-1}}$ as long as it does not vanish. For continuity reasons, this implies that $|h(r)| = (\lambda b^2)^{-\frac{1}{q-1}}$ on the whole I. In particular, h does not vanish on I and one can assume, without loss of generality, that $h = b^{-\frac{2}{q-1}}$. Indeed, it is clear that if u is a solution of (1.4) then $-u$ is also a solution. On the other hand, replacing h by $\lambda^{-\frac{1}{q-1}} h$ and w by $\lambda^{\frac{1}{q-1}} w$, the solution u remains unchanged and the PDE (2.12) reduces to (2.4). Finally, replacing h by $b^{-\frac{2}{q-1}}$ in (2.11) one gets (2.5).

Conversely, it is easy to check that if conditions (i) and (ii) are satisfied, then the function $u(r,\sigma) = b^{\frac{2}{1-q}}(r)w(\sigma)$, where w is a non-constant solution of (2.4), is a solution of (2.2). □

3. Study of the ODE

In this section, we investigate the existence of 2π-periodic solutions of the ODE (1.6)

$$w''(\theta) + \mu w(\theta) = \varepsilon w(\theta)|w(\theta)|^{q-1},$$

according to the values of the parameters ε, q and μ.

In order to transform the ODE into a dynamical system, we put $x = w$ and $y = w'$. We then get

$$(\mathcal{S}) \begin{cases} x' = P(y) := y, \\ y' = Q(x) := -\mu x + \varepsilon x|x|^{q-1}. \end{cases}$$

The origin is either a critical point ($q > 1$), or the only singular point ($0 < q < 1$) of the system ($\mathcal{S}$). Notice that solutions of (1.6) satisfy

$$w'^2(t) = w'^2(0) - \mu\left(w^2(t) - w^2(0)\right) + \frac{2\varepsilon}{q+1}\left[|w(t)|^{q+1} - |w(0)|^{q+1}\right]. \tag{3.1}$$

Equivalently, the orbit of ($\mathcal{S}$) passing through the point (x_0, y_0) is given by the equation

$$y^2 - y_0^2 = -\mu\left(x^2 - x_0^2\right) + \frac{2\varepsilon}{q+1}\left[|x|^{q+1} - |x_0|^{q+1}\right].$$

If an orbit of ($\mathcal{S}$) intersects one of the coordinates axes, then the intersection occurs perpendicularly (indeed, if $(x(t), y(t))$ satisfies ($\mathcal{S}$), then $x(t_0) = 0$

implies $y'(t_0) = 0$ and $y(t_0) = 0$ implies $x'(t_0) = 0$). Moreover, the system is clearly invariant under the transformations $\Phi_x : (t, x, y) \mapsto (-t, -x, y)$ and $\Phi_y : (t, x, y) \mapsto (-t, x, -y)$. The following lemma is a quasi-immediate consequence of these observations.

Lemma 2. *The coordinates axes are axes of symmetry of the dynamical system $(\mathcal{S})$ and the origin is a center of symmetry. Any orbit which intersects both the two coordinates axes is necessarily closed.*

3.1. **Case $q > 1$ and $\varepsilon = -1$.**

3.1.1. *Assume* $\mu \geq 0$. A classical study of the phase plane of the system (S) in this case shows that all solutions are periodic that turn around the origin. This can be seen considering for example the following Lyapounov function

$$E(w, w') := \mu w^2 + \frac{2}{q+1}|w|^{q+1} + w'^2.$$

Moreover, periodic solutions are given by the equation

$$E(x, y) = y_0^2,$$

where $y_0 = y(0) = w'(0)$.

Lemma 3. *The period function $y_0 \in (0, \infty) \mapsto T(y_0)$, where $T(y_0)$ is the period of the solution of $(\mathcal{S})$ passing through the point $(0, y_0)$, is decreasing with*

$$T(0, \infty) = \begin{cases} (0, \frac{2\pi}{\sqrt{\mu}}) & \text{if } \mu > 0 \\ (0, \infty) & \text{if } \mu = 0. \end{cases}$$

Proof. Notice that P is homogeneous and Q is sub homogeneous of degree 1 (indeed, $Q(\nu x) - \nu Q(x) = (\nu - \nu^q)x^q < 0$ for all $x > 0$ and all $\nu > 1$). Applying [8, Theorem 3], we get the monotony of the period function.

Let $\mathcal{O}$ be the orbit of a periodic solution w such that $w(0) = 0$ and $w'(0) = s$, with $s > 0$. For symmetry reasons, it suffices to work with the quarter of the orbit lying in the region $(x \geq 0) \cap (y \geq 0)$. In this region, one has, from (3.1), $w'(t) = \sqrt{s^2 - U(w(t))}$, where $U(x) = \mu x^2 + \frac{2}{q+1}x^{q+1}$. Denoting by $(x(s), 0)$, the first intersection point of the orbit with the x-axis, we have $w(T(s)/4) = x(s)$, $U(x(s)) = s^2$ and, then,

$$\begin{aligned} T(s) &= 4\int_0^{x(s)} \frac{dx}{\sqrt{s^2 - U(x)}} = 4\int_0^1 \frac{x(s)d\tau}{\sqrt{U(x(s)) - U(x(s)\tau)}} \\ &= 4\int_0^1 \frac{d\tau}{\sqrt{(1-\tau^2)\mu + \frac{2}{q+1}x(s)^{q-1}(1-\tau^{q+1})}}. \end{aligned}$$

Since $x(s) \to 0$ as $s \to 0$, and $\int_0^1 \frac{d\tau}{\sqrt{(1-\tau^2)}} = \frac{\pi}{2}$, we deduce using standard convergence results,

$$\lim_{s\to 0} T(s) = \begin{cases} \frac{2\pi}{\sqrt{\mu}} & \text{if } \mu > 0 \\ \infty & \text{if } \mu = 0. \end{cases}$$

Since $x(s) \to \infty$ as $s \to \infty$, we obtain using the same calculations,

$$\lim_{s\to\infty} T(s) = 0. \qquad \square$$

A direct consequence of Lemma 3 is the following:

Proposition 1. *Assume that $q > 1$, $\mu \geq 0$ and $\varepsilon = -1$. Then, for all integers $k > \sqrt{\mu}$, the ODE* (1.6) *admits a unique (up to sign) $\frac{2\pi}{k}$-periodic solution. Moreover, all the solutions are sign changing.*

3.1.2. *Assume $\mu < 0$.* This case was studied by Bidaut-Véron and Bouhar [2]. They obtained the following

Proposition 2. ([2, Lemma 1.1 and Lemma 1.2]) *Assume that $\mu < 0$, $q > 1$ and $\varepsilon = -1$. Then,*

(1) *For all positive integer k there exists a unique (up to sign) sign changing $\frac{2\pi}{k}$-periodic solution of the ODE* (1.6).
(2) *The ODE* (1.6) *admits positive 2π-periodic solutions if and only if $-\mu(q-1) > 1$. Moreover, in this case, for any integer $1 < k < \sqrt{-\mu(q-1)}$, the ODE* (1.6) *admits a unique positive $\frac{2\pi}{k}$-periodic solution.*

3.2. **Case $q > 1$ and $\varepsilon = +1$.**

3.2.1. *Assume $\mu \leq 0$.* This is an obvious case since the Laplace operator $w \mapsto w''$ is non-positive on the circle. Therefore, the only periodic solution of (1.6) is the trivial one $w = 0$.

3.2.2. *Assume $\mu > 0$.* This case was studied by Chafee and Infante [4]. Here, we give a different approach based on the analysis of the dynamical system (S).

A direct calculation shows that the critical points of $(\mathcal{S})$ are the origin and the two points $(-c, 0)$ and $(c, 0)$, where $c = \mu^{\frac{1}{q-1}}$. The origin is a center while the two others are saddle points.

A classical study of the dynamical system gives the following:

Lemma 4. *Assume that $\mu > 0$, $q > 1$ and $\varepsilon = +1$. Then, the dynamical system (S) satisfies the following properties:*

(1) *There exists a unique heteroclinic orbit emanating from $(-c, 0)$ which tends to $(c, 0)$ as $t \to \infty$ in the upper half plane and one heteroclinic orbit emanating from $(c, 0)$ which tends to $(-c, 0)$ as $t \to \infty$ in the lower half plane. The equations of these orbits are given by*

$$y^2 = -\mu\left(x^2 - c^2\right) + \frac{2}{q+1}\left[|x|^{q+1} - c^{q+1}\right].$$

(2) *Every point in the open bounded region delimited by these two heteroclinic orbits belongs to a periodic orbit which turns around the origin.*

(3) *The period function $y_0 \in (0, \gamma) \mapsto T(y_0)$, where $T(y_0)$ is the period of the solution passing through the point $(0, y_0)$ and $\gamma = \sqrt{\mu c^2 - \frac{2}{q+1} c^{q+1}}$, is increasing with $T(0, \gamma) = \left(\frac{2\pi}{\sqrt{\mu}}, \infty\right)$.*

Proof. By classical arguments, one can show that for $y_0 > 0$ small enough, orbits emanating from $(0, y_0)$ turn around the origin, while for $y_0 > 0$ large enough, they still contained in a half plane $\{y > \alpha\}$ for some $\alpha > 0$. This ensures the existence, and by regularity, the uniqueness, of the heteroclinic orbits described above.

To study the period function, one follows the same arguments as in the proof of Lemma 3 (in this case, the expression of U must be $U(x) = \mu x^2 - \frac{2}{q+1} x^{q+1}$). We get

$$\begin{aligned} T(s) &= 4 \int_0^{x(s)} \frac{dx}{\sqrt{s^2 - U(x)}} = 4 \int_0^1 \frac{x(s)d\tau}{\sqrt{U(x(s)) - U(x(s)\tau)}} \\ &= 4 \int_0^1 \frac{d\tau}{\sqrt{\mu(1-\tau^2) - \frac{2}{q+1} x(s)^{q-1}(1-\tau^{q+1})}}. \end{aligned}$$

Hence,

$$\lim_{s \to 0} T(s) = \frac{4}{\sqrt{\mu}} \int_0^1 \frac{d\tau}{\sqrt{1-\tau^2}} = \frac{2\pi}{\sqrt{\mu}}.$$

Since $x(s) \to c = \mu^{\frac{1}{q-1}}$ as $s \to \gamma = \sqrt{U(c)}$ (recall that $U(x(s)) = s^2$) and that the integral

$$\int_0^1 \frac{d\tau}{\sqrt{\mu(1-\tau^2) - \frac{2\mu}{q+1}(1-\tau^{q+1})}},$$

is infinite, we deduce that $\lim_{s \to \gamma} T(s) = \infty$. □

As a consequence, we have

Proposition 3. *Assume that* $q > 1$, $\mu > 0$ *and* $\varepsilon = +1$. *The ODE* (1.6) *admits* 2π*-periodic solutions if and only if* $\mu > 1$. *Moreover, in this case, for all positive integer* $k < \sqrt{\mu}$, *the ODE* (1.6) *admits a unique* (*up to sign*) $\frac{2\pi}{k}$*-periodic solution which is sign changing.*

3.3. **Case** $0 < q < 1$ **and** $\varepsilon = -1$.

3.3.1. *Assume* $\mu \geq 0$. The system (S) is of class C^1 in $\mathbb{R}^2 \setminus \{O\}$, the origin being its unique singular point. However the analysis is almost the same as for $q > 1$. Indeed, using Bendixson-Poincaré theory, closed orbits must turn around the origin.

Using arguments like in the proof of Lemma 3 and the Lyapounov function

$$E(w, w') = \mu w^2 + \frac{2}{q+1} w^{q+1} + w'^2,$$

one can show the following

Lemma 5. *The period function* $y_0 \in (0, \infty) \mapsto T(y_0)$, *where* $T(y_0)$ *is the period of the solution of* $(\mathcal{S})$ *passing through the point* $(0, y_0)$, *is increasing with*

$$T(0,\infty) = \begin{cases} (0, \frac{2\pi}{\sqrt{\mu}}) & \text{if } \mu > 0 \\ (0, \infty) & \text{if } \mu = 0. \end{cases}$$

Consequently, one obtains the following:

Proposition 4. *Assume that* $0 < q < 1$, $\mu \geq 0$ *and* $\varepsilon = -1$. *Then, for all integers* $k > \sqrt{\mu}$, *the ODE* (1.6) *admits a unique* (*up to sign*) $\frac{2\pi}{k}$*-periodic solution. Moreover, all these solutions are sign changing.*

3.3.2. *Assume* $\mu < 0$. A direct calculation shows that the critical points of $(\mathcal{S})$ are the two saddle points $(-c, 0)$ and $(c, 0)$, where $c = (-\mu)^{\frac{1}{q-1}}$. The origin is a singular point.

The proof of the following lemma is a slight modification of that of Lemma 4.

Lemma 6. *Assume that* $\mu < 0$, $0 < q < 1$ *and* $\varepsilon = -1$. *Then the dynamical system* (S) *satisfies the following properties:*

(1) *There exists a unique heteroclinic orbit emanating from* $(-c, 0)$ *which tends to* $(c, 0)$ *as* $t \to \infty$ *in the upper half plane and one heteroclinic orbit emanating from* $(c, 0)$ *which tends to* $(-c, 0)$ *as* $t \to \infty$ *in the lower half plane. The equations of these orbits are given by*

$$y^2 = -\mu\left(x^2 - c^2\right) + \frac{2}{q+1}\left[x|x|^q - c^{q+1}\right].$$

(2) *Every point, except the origin, in the bounded region delimited by these two heteroclinic orbits belongs to a periodic orbit which turns around the origin.*

(3) *The period function* $y_0 \in (0,\gamma) \mapsto T(y_0)$*, where* $T(y_0)$ *is the period of the solution passing through the point* $(0, y_0)$ *and* $\gamma = \sqrt{\mu c^2 + \frac{2}{q+1} c^{q+1}}$*, is increasing with* $T\,(0,\gamma) = (0,\infty)$*.*

Proof. The existence and uniqueness of the heteroclinic orbits rely on the same observation as in the proof of Lemma 4. Moreover, using the same arguments as in the proof of Lemma 3, one obtains for the period function (here, we have $U(x) = \mu x^2 + \frac{2}{q+1} x^{q+1}$),

$$\begin{aligned} T(s) &= 4\int_0^{x(s)} \frac{dx}{\sqrt{s^2 - U(x)}} = 4\int_0^1 \frac{x(s)d\tau}{\sqrt{U(x(s)) - U(x(s)\tau)}} \\ &= 4x(s)^{\frac{1-q}{2}} \int_0^1 \frac{d\tau}{\sqrt{\mu x(s)^{1-q}(1-\tau^2) + \frac{2}{q+1}(1-\tau^{q+1})}}. \end{aligned}$$

As before, we deduce that $\lim_{s\to 0} T(s) = 0$ and, since $x(s) \to c = (-\mu)^{\frac{1}{q-1}}$ as $s \to \gamma = \sqrt{U(c)}$,

$$\lim_{s\to\gamma} T(s) = \frac{4}{\sqrt{-\mu}} \int_0^1 \frac{d\tau}{\sqrt{\frac{2}{q+1}(1-\tau^{q+1}) - (1-\tau^2)}} = +\infty. \qquad \square$$

As a consequence we have:

Proposition 5. *Assume that* $\mu < 0$, $0 < q < 1$ *and* $\varepsilon = -1$*. For all positive integers* k*, the ODE* (1.6) *admits a unique (up to sign)* $\frac{2\pi}{k}$*-periodic solution. Moreover, all these solutions are sign changing.*

3.4. Case $0 < q < 1$ and $\varepsilon = +1$.

3.4.1. *Assume* $\mu \leq 0$. As in subsection 3.2.1, in this case the only periodic solution is the trivial one.

3.4.2. *Assume* $\mu > 0$. The critical points of the system $(\mathcal{S})$ are the two points $(-c, 0)$ and $(c, 0)$, with $c = \mu^{\frac{1}{q-1}}$, and they are centers.

Remark 1. Notice that $Q'(x) = -\mu + q|x|^{q-1}$ is negative for $x > c$. Hence, $Q(x)$ is positive for $0 < x < c$ and negative and decreasing for $x > c$.

Lemma 7. *The orbits of the dynamical system* (S) *never admit horizontal or vertical asymptotes.*

Proof. The existence of a vertical asymptote for $(x(t), y(t))$ means that there exists $a \in \mathbb{R}$ such that $x(t) \to a$ and $y(t) \to \pm\infty$ as $t \to \infty$, which contradicts the equation $x'(t) = y(t)$.

In a similar manner, we can exclude horizontal asymptotes by noticing that $y'(t) = Q(x(t))$ with $Q(x) \to \pm\infty$ as $x \to \pm\infty$. □

Lemma 8. (1) *Every orbit which intersects the y-axis at $y_0 \neq 0$ is periodic and turns around the origin and the two critical points.*

(2) *Every orbit which intersects the x-axis at $x_0 \in (-c, c)$, $x_0 \neq 0$ is periodic and turns around the critical point $(c, 0)$ if $x_0 > 0$, and around the point $(-c, 0)$ if $x_0 < 0$.*

Proof. (1) Using time shift invariance and symmetry, one can assume that the orbit starts at $(0, y_0)$ with $y_0 > 0$. From $y' = Q(x)$ and Lemma 7, one deduces that y increases for $0 < x < c$, then decreases for $x > c$ and intersects the x-axis at finite time. Applying Lemma 2 the orbit is periodic.

(2) By the same analysis, every orbit which starts at $(x_0, 0)$ with $0 < x_0 < c$ must intersect the x-axis at finite time and, hence, is periodic. By symmetry the same result holds for $-c < x_0 < c$. □

Lemma 9. *There exists exactly one homoclinic-like orbit emanating from the origin in the right (resp. left) half-plane and enclosing the critical point $(c, 0)$ (resp. $(-c, 0)$). Moreover, the equation of this orbit is given by*

$$y^2 + \mu x^2 = \frac{2}{1+q}|x|^{1+q}. \tag{3.2}$$

Remark 2. In the particular case $\mu = \frac{4}{(1-q)^2}$, which corresponds to the Euclidean metric, the orbit (3.2) could be parametrized by $(x(t), x'(t))$ where

$$x(t) := x_*|\sin(t)|^{\sqrt{\mu}}, \tag{3.3}$$

with $x_* := \left(\frac{2}{(1+q)\mu}\right)^{\frac{1}{1-q}}$.

Proof. Existence. Integrating the differential equation

$$\frac{dy}{dx} = \frac{-\mu x + x^q}{y},$$

from 0 to t, we get

$$y^2 + U(x) = y^2(0) + U(x(0)), \tag{3.4}$$

where $U(x) := \mu x^2 - \frac{2}{q+1}x^{q+1}$. Thus the equation of the orbit passing through the origin is given by

$$y^2 = -U(x) = -\mu x^2 + \frac{2}{q+1}x^{q+1}.$$

Uniqueness. Since the origin is a singular point, we cannot use classical theorems to deduce uniqueness of the special homoclinic orbits given by (3.2). To show uniqueness, consider the change of variables $X := \varphi(x)$ and $Y := y$, where $y = \varphi(x)$ is the equation of the upper part of the positive special homoclinic orbit defined by (3.2). Then, $X' = \frac{d}{dt}(\varphi(x)) = x'\varphi'(x) = y\varphi'(x)$ with $\varphi'(x) = Q(x)/\varphi(x)$. On the other hand, $Y' = y' = Q(x)$. Thus $X' = Q(x)y/\varphi(x) = Y'Y/X$, i.e. $\frac{d}{dt}\left(X^2\right) = 2X'X = 2Y'Y = \frac{d}{dt}\left(Y^2\right)$. Therefore, $X^2(t) - Y^2(t) = X^2(0) - Y^2(0)$ which gives the uniqueness. □

Variation of the period. *Sign changing solutions.*

Lemma 10. *The period function $y_0 \in (0, \infty) \mapsto T(y_0)$, where $T(y_0)$ is the period of the solution passing through the point $(0, y_0)$, is decreasing and $T(0, \infty) = \left(\frac{2\pi}{\sqrt{\mu}}, \frac{4\pi}{(1-q)\sqrt{\mu}}\right)$.*

Proof. We follow the same arguments as in the proof of Lemma 3, we obtain, with the same notations. In this case, the expression of U is $U(x) = \mu x^2 - \frac{2}{q+1}x^{q+1}$. We get

$$\begin{aligned} T(s) &= 4\int_0^{x(s)} \frac{dx}{\sqrt{s^2 - U(x)}} = 4\int_0^1 \frac{x(s)d\tau}{\sqrt{U(x(s)) - U(x(s)\tau)}} \\ &= 4\int_0^1 \frac{d\tau}{\sqrt{\mu(1-\tau^2) - \frac{2}{q+1}x(s)^{q-1}(1-\tau^{q+1})}}. \end{aligned}$$

Since $x(s) \to \infty$ as $s \to \infty$, we have

$$\lim_{s\to\infty} T(s) = \frac{4}{\sqrt{\mu}}\int_0^1 \frac{d\tau}{\sqrt{1-\tau^2}} = \frac{2\pi}{\sqrt{\mu}}.$$

On the other hand, when s goes to 0, $x(s)$ tends to x_*, the positive solution of $U(x_*) = 0$ (i.e. $x_*^{q-1} = \frac{q+1}{2}\mu$). Therefore,

$$\lim_{s\to 0} T(s) = 4\int_0^{x_*} \frac{dx}{\sqrt{-U(x)}},$$

and, using the change of variable $x = x_* \sin^{\frac{2}{1-q}} t$, we get

$$\lim_{s\to 0} T(s) = \frac{8}{1-q}\int_0^{\frac{\pi}{2}} \frac{\cos t\, dt}{\sqrt{\frac{2}{q+1}x_*^{q-1} - \mu \sin^2 t}}$$
$$= \frac{8}{(1-q)\sqrt{\mu}}\int_0^{\frac{\pi}{2}} dt = \frac{4\pi}{(1-q)\sqrt{\mu}}. \qquad \square$$

Positive solutions. Observe that the homoclinic-like solution given by (3.3) is $\frac{2\pi}{(1-q)\sqrt{\mu}}$-periodic. On the other hand, the linearized equation at $(c, 0)$ is

$$w'' + \mu(1-q)w = 0.$$

Roughly speaking the period function takes the value $\frac{2\pi}{\sqrt{\mu(1-q)}}$ at the point orbit $(c, 0)$ and $\frac{2\pi}{(1-q)\sqrt{\mu}}$ at the homoclinic-like orbit.

Lemma 11. *The period function $s \in (0,\gamma) \mapsto T(s)$, where $T(s)$ is the period of the solution passing through the point (c, s) and $\gamma > 0$ such that $\gamma^2 = -\mu c^2 + \frac{2}{q+1}c^{q+1}$, is increasing with $T(0,\gamma) = \left(\frac{2\pi}{\sqrt{\mu(1-q)}}, \frac{2\pi}{(1-q)\sqrt{\mu}}\right)$.*

Proof. For any $s \in (0,\gamma)$ we denote by $(y(s), 0)$ and $(z(s), 0)$, with $0 < y(s) < c < z(s) < x_*$, the intersection points of the orbit passing through the point (c, s) with the x-axis. Following an idea of [1] (see also [2]), since $y^2 = s^2 - U(x) + U(c)$, we have

$$\begin{aligned} T(s)/2 &= \int_c^{z(s)} \frac{du}{\sqrt{s^2 - U(u) + U(c)}} - \int_c^{y(s)} \frac{du}{\sqrt{s^2 - U(u) + U(c)}} \\ &= \int_0^1 (z'(st) - y'(st))\frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

In particular, since $\gamma^2 = -U(c)$,

$$\lim_{s\to\gamma} T(s) = \int_0^{x_*} \frac{2du}{\sqrt{\gamma^2 - U(u) + U(c)}} = \int_0^{x_*} \frac{2du}{\sqrt{-U(u)}} = \frac{2\pi}{(1-q)\sqrt{\mu}},$$

(see the end of the proof of Lemma 10). On the other hand, since $y(s)$ and $z(s)$ satisfy

$$0 = U(y(s)) - U(c) - s^2 = U(z(s)) - U(c) - s^2,$$

we easily get

$$\lim_{s\to 0} \left(z'(s) - y'(s)\right) = \frac{2}{\sqrt{\mu(1-q)}}.$$

Therefore,

$$\lim_{s\to 0} T(s) = \frac{2\pi}{\sqrt{\mu(1-q)}}.$$

Now, to show the monotony of the function T, we first observe that

$$\frac{d}{ds}T(s) = 2\int_0^1 \left(z''(st) - y''(st)\right)\frac{tdt}{\sqrt{1-t^2}}.$$

Thus, it suffices to show that $z'' - y''$ is non-negative. From the equation $U(z(s)) - U(c) - s^2 = 0$ we get $z'(s) = 2s/U'(z(s))$ and, hence,

$$z''(s) = 2\frac{U'^2(z(s)) + 2(U(c) - U(z(s)))U''(z(s))}{U'^3(z(s))}.$$

Since $z(s) \to c$ as $s \to 0$, U is smooth, $U'(c) = 0$ and $U''(c) \neq 0$, expanding U up to order 4 in the neighborhood of c shows that

$$\lim_{s\to 0} 2\frac{U'^2(z(s)) + 2(U(c) - U(z(s)))U''(z(s))}{U'^3(z(s))} = -\frac{2U^{(3)}(c)}{U''^2(c)}.$$

Thus,

$$\lim_{s\to 0} z''(0) = -\frac{2U^{(3)}(c)}{3U''^2(c)} = \lim_{s\to 0} y''(0).$$

Let us show that

$$z''(s) \geq -\frac{2U^{(3)}(c)}{U''^2(c)} \geq y''(s). \tag{3.5}$$

For the first inequality, it suffices to show that

$$F := U'^2 + 2U''[U(c) - U(z)] + \frac{U'^3U^{(3)}(c)}{3U''^2(c)} \geq 0,$$

on (c, x_*). In fact we will show that this is true even on $(0, +\infty)$. Indeed, notice first that

$$F' = 2U^{(3)}[U(c) - U] + \frac{U^{(3)}(c)}{U''^2(c)}U'^2U'' \leq 0,$$

on $(0, q^{\frac{1}{1-q}}c)$. On the other hand, set $H := -\frac{F}{U''}$ on $(q^{\frac{1}{1-q}}c, \infty)$, then we have $H' = -K\frac{U'^2U^{(3)}}{U''^2}$, where

$$K = -1 + \frac{U^{(3)}(c)}{3U''^2(c)}\left[3\frac{U''^2}{U^{(3)}} - U'\right].$$

Now,

$$K' = \frac{U^{(3)}(c)U''}{3U''^2(c)(U^{(3)})^2}\left[5(U^{(3)})^2 - 3U^{(4)}U''\right] > 0,$$

with $K(c) = 0$, then K is non-positive on $(q^{\frac{1}{1-q}}c, c)$ and non-negative on (c, ∞). Hence, H' is non-negative on $(q^{\frac{1}{1-q}}c, c)$ and non-positive on (c, ∞) and since $H(c) = 0$ we get $H \leq 0$. Thus $F \geq 0$ on $(q^{\frac{1}{1-q}}c, \infty)$, Therefore, $F \geq 0$ on $(0, \infty)$. In a similar manner, one can show the second inequality in (3.5). $\square$

The following proposition summarizes the consequences of Lemmas 9, 10 and 11.

Proposition 6. *Assume that $\mu > 0$, $\varepsilon = +1$ and $0 < q < 1$. Then,*

(1) *The ODE* (1.6) *admits non-constant 2π-periodic sign changing solutions if and only if $\mu > 1$. Moreover, in this case, for all integers $k \in (\frac{1-q}{2}\sqrt{\mu}, \sqrt{\mu})$ the ODE* (1.6) *admits a unique* (*up to sign*) *sign changing $\frac{2\pi}{k}$-periodic solution.*

(2) *The ODE* (1.6) *admits non-constant 2π-periodic positive solutions if and only if $\big((1-q)\sqrt{\mu}, \sqrt{\mu(1-q)}\big) \cap \mathbb{N} \neq \emptyset$. Moreover, in this case, for all integers $(1-q)\sqrt{\mu} < k < \sqrt{\mu(1-q)}$, the ODE* (1.6) *admits a unique positive $\frac{2\pi}{k}$-periodic solution.*

(3) *There exists a unique $\frac{2\pi}{(1-q)\sqrt{\mu}}$-periodic non-negative solution of* (1.6). *It vanishes only once in a period and its orbit is given by* (3.2).

4. Applications

In this section we apply the results above to the existence problem of non-radial pseudo-radial solutions of the PDE (1.4) and its particular case (1.2).

4.1. **Case of a metric $g = a^2(r)g_{\mathbb{R}^2}$.** In this case, $b(r) = ra(r)$. Note that $\Delta^g u = \frac{1}{a^2(r)}\Delta u$, where Δ is the Euclidean Laplacian. Hence, the PDE

$$\Delta^g u = \varepsilon u|u|^{q-1},$$

is equivalent to the following one

$$\Delta u = \varepsilon a^2(|x|)u|u|^{q-1}, \tag{4.1}$$

that we consider on a disc or an annular domain $\Omega := \{0 \leq R_1 < |x| < R_2 \leq +\infty\}$ of $\mathbb{R}^2$.

Theorem 2. *If the equation* (4.1) *admits a non-radial pseudo-radial solution in Ω, then, the function $r \mapsto a(r)$ has one of the three following forms*

$$a(r) = r^{-1}\big[Mr^{\alpha} + Nr^{-\alpha}\big]^{\frac{1-q}{2}},$$

where $\alpha > 0$ *and* $M, N \in \mathbb{R}$ *are such that* $Mr^{\alpha} + Nr^{-\alpha} > 0$ *on* (R_1, R_2)*,*

$$a(r) = r^{-1}[M + N \ln r]^{\frac{1-q}{2}},$$

where $M, N \in \mathbb{R}$ *are such that* $M + N \ln r > 0$ *on* (R_1, R_2)*, or*

$$a(r) = r^{-1}[M \cos(\alpha \ln r) + N \sin(\alpha \ln r)]^{\frac{1-q}{2}},$$

where $\alpha > 0$ *and* $M, N \in \mathbb{R}$*.*

Proof. Applying Theorem 1, the equation (2.11) becomes (with $c = b/a = r$ and $h = b^{\frac{2}{1-q}}$)

$$(rh')' = rh'' + h' = \mu \frac{h}{r}. \tag{4.2}$$

The general solution of (4.2) is given by

$$h(r) = \begin{cases} Mr^{\sqrt{\mu}} + Nr^{-\sqrt{\mu}} & \text{if } \mu > 0, \\ M + N \ln r & \text{if } \mu = 0, \\ M \cos(\sqrt{-\mu} \ln r) + N \sin(\sqrt{-\mu} \ln r) & \text{if } \mu < 0. \end{cases} \qquad \square$$

Hence, for our purpose, the only relevant equations of the form $\Delta u = \varepsilon a^2(|x|)u|u|^{q-1}$ in Ω are

$$\Delta u = \varepsilon |x|^{-2} \left[M|x|^{\alpha} + N|x|^{-\alpha}\right]^{1-q} u|u|^{q-1}, \tag{4.3}$$

$$\Delta u = \varepsilon |x|^{-2} [M + N \ln |x|]^{1-q} u|u|^{q-1}, \tag{4.4}$$

and

$$\Delta u = \varepsilon |x|^{-2} [M \cos(\alpha \ln |x|) + N \sin(\alpha \ln |x|)]^{1-q} u|u|^{q-1}, \tag{4.5}$$

where α, M and N are as in Theorem 2.

For (4.3), the set of pseudo-radial non-radial solutions can be described in the following way:

Theorem 3. (1) *If* $\varepsilon = +1$ *and* $0 < \alpha \leq 1$*, then, any pseudo-radial solution of* (4.3) *is radial.*

(2) *Define, for* $\varepsilon = -1$ *or* $\alpha > 1$*, the subset* $\mathcal{A}(\varepsilon, q, \alpha) \subset \mathbb{N}$ *by*

$$\mathcal{A}(\varepsilon, q, \alpha) \quad = \quad \begin{cases} (\alpha, +\infty) \cap \mathbb{N}, & \text{if } \varepsilon = -1 \\ (0, +\infty) \cap \mathbb{N} & \text{if } \varepsilon = +1, q > 1 \text{ and } \alpha > 1 \\ (\alpha(1-q)/2, \alpha) \cap \mathbb{N} & \text{if } \varepsilon = +1, 0 < q < 1 \text{ and } \alpha > 1, \end{cases}$$

Then, the set of sign changing non-radial pseudo-radial solutions of (4.3) *is parameterized by* $\mathcal{A}(\varepsilon, q, \alpha)$ *in the sense that it consists in the functions*

$$u_k(x) = \pm[M|x|^{\alpha} + N|x|^{-\alpha}]w_k(\theta), \quad k \in \mathcal{A}(\varepsilon, q, \alpha),$$

where, for every $k \in \mathcal{A}(\varepsilon, q, \alpha)$*,* w_k *is the unique* (*up to sign*) *sign changing* $\frac{2\pi}{k}$*-periodic solution of* (1.6)*.*

(3) *Equation* (4.3) *admits positive non-radial pseudo-radial solutions if and only if* $\varepsilon = +1$, $0 < q < 1$ *and* $\mathcal{B}(+1, q, \alpha) := (\alpha(1-q), \alpha\sqrt{1-q}) \cap \mathbb{N} \neq \emptyset$. *Moreover, all these solutions are of the form*

$$u_k(x) = [M|x|^{\alpha} + N|x|^{-\alpha}]v_k(\theta),$$

where, for every $k \in \mathcal{B}(+1, q, \alpha)$, v_k *is the unique positive* $\frac{2\pi}{k}$*-periodic solution of* (1.6).

(4) *If* $(1-q)\alpha \in \mathbb{N}$, *then equation* (4.3) *admits a unique non-negative pseudo-radial solution given by*

$$u(x) = [M|x|^{\alpha} + N|x|^{-\alpha}]v(\theta),$$

where v *is the unique non-negative periodic solution of* (1.6) (*corresponding to the homoclinic-like orbit*) *which vanishes once in a period.*

Concerning (4.4), we have the following:

Theorem 4. (1) *If* $\varepsilon = +1$, *then, for every* $q > 0$, $q \neq 1$, *any pseudo-radial solution of* (4.4) *is radial.*

(2) *If* $\varepsilon = -1$, *then, for every* $q > 0$, $q \neq 1$, *the set of non-radial pseudo-radial solutions of* (4.4) *is parameterized by* $\mathbb{N}^*$ *and consists in the functions*

$$u_k(x) = \pm[M + N \ln |x|]w_k(\theta),$$

where, for every $k \in \mathbb{N}^*$, w_k *is the unique* (*up to sign*) *sign changing* $\frac{2\pi}{k}$*-periodic solution of* (1.6).

Remark 3. A particular case of (4.3) is

$$\Delta u = \varepsilon |x|^p u|u|^{q-1}, \quad p \in \mathbb{R} \setminus \{-2\}. \tag{4.6}$$

One can apply Theorem 3 with $\alpha = \left|\frac{p+2}{q-1}\right|$, $(M, N) = (1, 0)$ if $\frac{p+2}{q-1} < 0$ and $(M, N) = (0, 1)$ if $\frac{p+2}{q-1} > 0$.

The case $p = -2$ corresponds to (4.4) with $M = 1$ and $N = 1$ and is covered by Theorem 4.

Concerning (4.5), we have the following

Theorem 5. (1) *If* $\varepsilon = +1$, *then, for every* $q > 0$, $q \neq 1$, *any pseudo-radial solution of* (4.5) *is radial.*

(2) *For* $\varepsilon = -1$, *the set of sign changing non-radial pseudo-radial solutions of* (4.5) *is parameterized by* $\mathbb{N}^*$ *and consists in the functions*

$$u_k(x) = \pm[M \cos(\alpha \ln |x|) + N \sin(\alpha \ln |x|)]w_k(\theta),$$

where, for every $k \in \mathbb{N}^*$, w_k *is the unique (up to sign) sign changing* $\frac{2\pi}{k}$*-periodic solution of* (1.6).

(3) *Equation* (4.5) *admits positive non-radial pseudo-radial solutions if and only if* $\varepsilon = -1$, $q > 1$ *and* $\alpha\sqrt{q-1} > 2$. *Moreover, all these solutions are of the form*

$$u_k(x) = [M \cos(\alpha \ln |x|) + N \sin(\alpha \ln |x|)] v_k(\theta),$$

where, for every $k \in (1, \alpha\sqrt{q-1}) \cap \mathbb{N}$, v_k *is the unique positive* $\frac{2\pi}{k}$*-periodic solution of* (1.6).

4.2. **The spherical case:** $a(r) = 1$ **and** $b(r) = \sin r$**.** In this case, condition (2.5) implies

$$\mu \frac{1-q}{2} = \frac{2}{1-q} \cos^2 r - \sin^2 r, \tag{4.7}$$

which is only possible if $q = 3$ and $\mu = 1$. Applying Propositions 1 and 3 we get:

Theorem 6. *Let* $\Omega \subset \mathbb{S}^2$ *be a rotationally symmetric domain of the standard sphere and consider in* Ω *the following equation:*

$$\Delta_{\mathbb{S}^2} u = \varepsilon u |u|^{q-1}, \tag{4.8}$$

where $\Delta_{\mathbb{S}^2}$ *is the standard Laplacian of* $\mathbb{S}^2$.

(*i*) *If* $\varepsilon = +1$ *or* $q \neq 3$, *then any pseudo-radial solution of* (4.8) *is radial.*

(*ii*) *The equation* $\Delta_{\mathbb{S}^2} u = -u^3$ *admits infinitely many non-radial pseudo-radial solutions in* Ω *which are all of the form*

$$u_k(x) = \frac{w_k(\theta)}{\sin r},$$

where, for any integer $k \geq 2$, w_k *is the unique* (*up to sign*) $\frac{2\pi}{k}$*-periodic solution of* (1.6).

4.3. **The hyperbolic metric case:** $a(r) = 1$ **and** $b(r) = \sinh r$**.** Here also, condition (2.5) is satisfied if and only if $q = 3$ and $\mu = 1$. Like in the spherical case, we get the following

Theorem 7. *Let* $\Omega \subset \mathbb{H}^2$ *be a rotationally symmetric domain of the hyperbolic plane and consider in* Ω *the following equation:*

$$\Delta_{\mathbb{H}^2} u = \varepsilon u |u|^{q-1}, \tag{4.9}$$

where $\Delta_{\mathbb{H}^2}$ *is the Laplacian of* $\mathbb{H}^2$.

i) *If* $\varepsilon = +1$ *or* $q \neq 3$, *then, any pseudo radial solution of* (4.9) *is radial.*

ii) *The equation* $\Delta_{\mathbb{H}^2} u = -u^3$ *admits infinitely many non-radial pseudo-radial solutions in* Ω *which are all of the form*

$$u_k(x) = \frac{w_k(\theta)}{\sinh r},$$

where, for any integer $k \geq 2$, w_k is the unique (up to sign) $\frac{2\pi}{k}$-periodic solution of(1.6).

4.4. **Case of a metric conformal to the cylindrical one: $a = b$.** The standard metric of the cylinder $C = (R_1, R_2) \times \mathbb{S}^1$ is the product metric $dr^2 + d\theta^2$, its Laplacian is given by $\Delta_C = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \theta^2}$. For a conformal metric $g = a^2(r)[dr^2 + d\theta^2]$, the associated Laplacian is $\Delta^g = a^{-2}(r)\Delta_C$. The equation

$$\Delta^S u = \varepsilon a(r)^2 u|u|^{q-1}, \tag{4.10}$$

is then equivalent to

$$\Delta^g u = \varepsilon u|u|^{q-1}.$$

Condition (2.5) gives, with $a = b$,

$$\left[a^{\frac{1+q}{1-q}} a'\right]' = \frac{\mu}{2}(1-q) a^{\frac{2}{1-q}}.$$

Setting $\gamma = \frac{1+q}{1-q}$, this last equation becomes

$$(a^{\gamma+1})'' = \mu a^{\gamma+1},$$

which gives

$$a^{\gamma+1}(r) = \begin{cases} A\cosh\sqrt{\mu}\,r + B\sinh\sqrt{\mu}\,r & \text{if } \mu > 0 \\ A + Br & \text{if } \mu = 0 \\ A\cos\sqrt{-\mu}\,r + B\sin\sqrt{-\mu}\,r & \text{if } \mu < 0, \end{cases}$$

where A and B are such that the right hand side is positive on (R_1, R_2). Therefore, for our purpose, the only relevant equations of the form (4.10) are

$$\Delta_C u = \varepsilon[A\cosh\alpha r + B\sinh\alpha r]^{1-q} u|u|^{q-1}, \tag{4.11}$$

$$\Delta u = \varepsilon[A + Br]^{1-q} u|u|^{q-1}, \tag{4.12}$$

and

$$\Delta u = \varepsilon[A\cos\alpha r + B\sin\alpha r]^{1-q} u|u|^{q-1}, \tag{4.13}$$

where $\alpha > 0$.

The description of the set of non-radial pseudo-radial solutions of these equations is the same as for equations (4.3), (4.4) and (4.5) respectively. Indeed, the statement of Theorem 3 remains valid for equation (4.11) provided that $M|x|^\alpha + N|x|^{-\alpha}$ is replaced by $A\cosh\alpha r + B\sinh\alpha r$. Similarly, Theorem 4 applies to (4.12) replacing $M + N\ln|x|$ by $A + Br$, and Theorem 5 applies to (4.13) replacing $M\cos(\alpha\ln|x|) + N\sin(\alpha\ln|x|)$ by $A\cos\alpha r + B\sin\alpha r$.

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