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STURM-LIOUVILLE PROBLEMS FOR AN ABSTRACT DIFFERENTIAL EQUATION OF ELLIPTIC TYPE IN UMD SPACES

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Abstract. In this paper we give some new results on Sturm-Liouville abstract problems of second-order differential equations of elliptic type in UMD spaces. Existence, uniqueness and maximal regularity of the strict solution are proved using the celebrated Dore-Venni theorem. This work completes the problems studied by Favini, Labbas, Maingot, Tanabe and Yagi under Dirichlet boundary conditions, see [6].

1. Introduction and hypotheses

In this paper, X is a complex Banach space, f belongs to $L^p((0,1);X)$ where $1 < p < \infty$ and d_0, d_1, u_1 are given elements of X, A is a closed linear operator in X with domain $D(A)$ and ω is some large positive number.

Let us first consider the second-order abstract differential equation in X

$$u''(x) + Au(x) - \omega u(x) = f(x), \quad \text{a.e } \ x \in (0,1), \tag{1.1}$$

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with the classical boundary conditions of Sturm-Liouville type

$$\begin{cases} a_0u(0) - b_0u'(0) = d_0 \\ a_1u(1) + b_1u'(1) = d_1, \end{cases}$$

where $a_j, b_j \geqslant 0$ and $a_j + b_j > 0$, $j = 0, 1$.

A generalization of this situation can be given by the study of (1.1) together with the abstract boundary conditions of Sturm-Liouville type

$$\begin{cases} hu'(0) - Hu(0) = d_0 \\ ku'(1) + Ku(1) = d_1, \end{cases}$$

where h, H, k, K are closed linear operators in X.

Here, for simplicity, we consider (1.1) with the boundary conditions

$$u'(0) - Hu(0) = d_0, \; u(1) = u_1. \tag{1.2}$$

Generally, more regularity is required on f to solve this problem, unless X has some particular geometrical properties. This is why we will assume in all this paper that

$$X \text{ is a UMD space.} \tag{1.3}$$

We recall that a Banach space X is a UMD space if and only if for some $p > 1$ (and thus for all $p > 1$) the Hilbert transform is continuous from $L^p(\mathbb{R}; X)$ into itself (see Burkholder [2]).

Some techniques used here are inspired by those used in [6]. The novelty in this work lies in the boundary condition containing the operator H, which can be unbounded. This difficulty will require a precise study of a sum of two linear operators. To this end, we will show that the Dore-Venni sum theory applies (see [4]).

We consider some fixed $\omega_0 \geqslant 0$ and we set, for $\omega \geqslant \omega_0$,

$$A_\omega = A - \omega I;$$

then our main ellipticity assumption is the following:

$$\begin{cases} A_{\omega_0} \text{ is a linear closed operator in } X, \; [0, +\infty[\subset \rho(A_{\omega_0}) \text{ and} \\ \sup\limits_{\lambda \geqslant 0} \|\lambda \left(A_{\omega_0} - \lambda I\right)^{-1}\|_{L(X)} < +\infty. \end{cases} \tag{1.4}$$

It follows that for $\omega \geqslant \omega_0$ the operator $-(-A_\omega)^{1/2}$ is the infinitesimal generator of an analytic semigroup on X, see for instance Balakrishnan [1].

We then search for a strict solution u to (1.1)-(1.2); i.e., a function u such that

$$u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A)), \tag{1.5}$$

and u satisfies $u(0) \in D(H)$ and (1.1)-(1.2).

Note that if $u \in W^{2,p}(0,1;X)$, then by the J.L. Lions theorem on traces, u can be seen as an element of $C^1([0,1];X)$. On the other hand, let $u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A))$, then

$$\begin{cases} u''(.) + Au(.) \in L^p(0,1;X), \text{ a.e. } \ x \in (0,1), \\ u(0) = u_0, \ u(1) = u_1, \end{cases}$$

from which we deduce that

$$u(0), u(1) \in \ (D\,(A)\,, X)_{\frac{1}{2p},p}\,, \tag{1.6}$$

see [6], Remark 1, statement 6.

We will suppose that

$$\begin{cases} H \text{ is a linear closed operator in } X, \, \mathbb{R}_- \subset \rho(H) \text{ and} \\ \sup\limits_{\zeta \geqslant 0} \left\| \zeta(H+\zeta I)^{-1} \right\|_{L(X)} < +\infty, \end{cases} \tag{1.7}$$

A and H are commuting in the resolvent sense; i.e.,

$$(A-\lambda I)^{-1}(H+\zeta I)^{-1} = (H+\zeta I)^{-1}(A-\lambda I)^{-1}, \ \lambda \geqslant \omega_0, \zeta \geqslant 0, \tag{1.8}$$

$$\begin{cases} \text{for any } s \in \mathbb{R}, \, (-A_{\omega_0})^{is} \in L(X) \text{ and there exists } \theta_A \in]0,\pi[\\ \text{such that: } \sup\limits_{s\in\mathbb{R}} \left\| e^{-\theta_A|s|}(-A_{\omega_0})^{is} \right\| < +\infty, \end{cases} \tag{1.9}$$

$$\begin{cases} (H)^{is} \in L(X) \text{ for any } s \in \mathbb{R} \text{ and there exists } \theta_H \in]0,\pi[\\ \text{such that: } \sup\limits_{s\in\mathbb{R}} \| e^{-\theta_H|s|}\,(H)^{is}\,\| < +\infty, \end{cases} \tag{1.10}$$

and

$$\frac{\theta_A}{2} + \theta_H \in]0,\pi[. \tag{1.11}$$

Remark 1. (1.3) implies that X is reflexive, moreover the operators A and H are sectorial, thus $D(A), D(H)$ are dense in X (see Haase [9], Proposition 1.1, page 18).

(2) Let $\omega \geqslant \omega_0$. We have $(D\,(A)\,, X)_{\frac{1}{2p},p} = (D\,(A_\omega)\,, X)_{\frac{1}{2p},p}$. On the other hand, (1.9) gives

$$\begin{cases} \text{for any } s \in \mathbb{R}, \, (-A_{\omega})^{is} \in L(X) \text{ and there exists } C \geqslant 1 \\ \text{such that: } \sup\limits_{s\in\mathbb{R}} \left\| e^{-\theta_A|s|}(-A_{\omega})^{is} \right\| \leqslant C, \end{cases} \tag{1.12}$$

where C can be chosen independent of ω (see Prüss-Sohr [15], Theorem 2, page 437), from which we deduce that

$$\left\| \left(\sqrt{-A_\omega}\right)^{is} \right\| \leqslant Ce^{(\theta_A/2)|s|},$$

for any $s \in \mathbb{R}$, (see Haase [9], Proposition 2.18, page 64).

(3) It is well known that, from (1.7), there exist $\nu_H > 0, r_0 > 0, C > 0$ such that

$$S_{\nu_H, r_0} \subset \rho(-H) \text{ and } \sup_{\lambda \in S_{\nu_H, r_0}} \left\| \lambda (H + \lambda I)^{-1} \right\|_{L(X)} < +\infty,$$

where $S_{\nu_H, r_0} = \{z \in \mathbb{C}^* : |\arg(z)| < \nu_H \} \cup B(0, r_0)$.

(4) Under (1.3)∼(1.11) and by a direct application of the Dore-Venni sum theory [4], for any $\omega \geqslant \omega_0$, the operator $(-Q_\omega) + H$, where

$$Q_\omega = -\sqrt{-A_\omega}, \tag{1.13}$$

is closed and boundedly invertible.

(5) Under (1.4)∼(1.8), we get, for any $\omega \geqslant \omega_0, \xi \in D(H)$ and $x \geqslant 0$,

$$He^{xQ_\omega}\xi = e^{xQ_\omega} H\xi.$$

(6) We know that for any $\omega \geqslant \omega_0$, $\left(D\left(A_\omega\right), X\right)_{1/2,1} \subset D\left(Q_\omega\right)$, (see Grisvard, [8], page 667), but for $\theta_1 < \theta_2$ and $p_1, p_2 \in [1, +\infty]$ we have

$$(D(A), X)_{\theta_1, p_1} \subset (D(A), X)_{\theta_2, p_2},$$

(see Grisvard, [8], p 674), hence

$$\left(D\left(A_\omega\right), X\right)_{\frac{1}{2p}, p} \subset \left(D\left(A_\omega\right), X\right)_{1/2,1} \subset D\left(Q_\omega\right).$$

Remark 2. A connection between $D(A)$ and $D(H)$ lies in assumption (1.8). On the other hand, assumptions (1.3)∼(1.11) will allow us to use the fact that for any $\omega \geqslant \omega_0$ and ϕ in X, $\left(Q_\omega - H\right)^{-1}(\phi) \in D(H) \cap D(Q_\omega)$, and this will help us to construct a solution u which satisfies $u(0) \in D(H)$.

The plan of this paper is as follows. In Section 2, some preliminary technical results are proved. In Section 3, we give a representation formula of the solution u by using the reduction order method. Section 4 is devoted to the case when the boundary conditions are homogeneous. Section 5 completes our work in the general case. Finally, in Section 6, we give some examples of applications to partial differential equations.

2. Technical results

Lemma 3. *Due to assumptions* (1.3), (1.4) *and* (1.9) *and the previous remark, Statement* 2, *for $f \in L^p(0,1;X), 1 < p < \infty$, and $\omega \geqslant \omega_0$, we have*

(1)

$$x \longmapsto L_\omega(x,f) = Q_\omega \int_0^x e^{(x-s)Q_\omega} f(s)ds \in L^p(0,1;X), \tag{2.1}$$

(2)

$$x \longmapsto L_\omega\left(1-x, f(1-\cdot)\right) = Q_\omega \int_x^1 e^{(s-x)Q_\omega} f(s)ds \in L^p(0,1;X), \tag{2.2}$$

(3)

$$x \longmapsto \mathcal{L}_\omega(x,f) = Q_\omega \int_0^1 e^{(x+s)Q_\omega} f(s)ds \in L^p(0,1;X). \tag{2.3}$$

Proof. The first and second assertions are a consequence of the Dore-Venni theorem [4]. For Statement 3, one writes, for $x \in (0,1)$,

$$\begin{aligned}
\mathcal{L}_\omega(x,f) &= Q_\omega \int_0^1 e^{(x+s)Q_\omega} f(s)ds \\
&= Q_\omega \int_0^x e^{(x+s)Q_\omega} f(s)ds + Q_\omega \int_x^1 e^{(x+s)Q_\omega} f(s)ds \\
&= Q_\omega \int_0^x e^{(x-s)Q_\omega} e^{2sQ_\omega} f(s)ds + e^{2xQ_\omega} Q_\omega \int_x^1 e^{(s-x)Q_\omega} f(s)ds \\
&= L_\omega(x, e^{2\cdot Q_\omega} f) + e^{2xQ_\omega} L_\omega(1-x, f(1-\cdot)),
\end{aligned}$$

and we apply Statements 1, 2. □

We have the following lemmas as in [6].

Lemma 4. *Assume (1.4) and let* $p \in]1,\infty[$, $\omega \geqslant \omega_0$. *Then*

(1) $A_\omega e^{\cdot Q_\omega}\varphi \in L^p(0,1;X)$ *if and only if* $\varphi \in \left(D\left(A\right), X\right)_{\frac{1}{2p},p}$; *and*

(2) $Q_\omega e^{\cdot Q_\omega}\varphi \in L^p(0,1;X)$ *if and only if* $\varphi \in \left(D\left(A\right), X\right)_{\frac{1}{2p}+\frac{1}{2},p}$.

Proof. Let us recall that if $m \in \mathbb{N}^*$ and C generates an analytic semigroup, then $\phi \in (D(C^m), X)_{\frac{1}{mp},p}$ if and only if $C^m e^{\cdot C}\phi \in L^p(0,1;X)$; in fact

$$\begin{aligned}
\int_0^1 \left\|C^m e^{xC}\phi\right\|^p dx &\leqslant \int_0^\infty \left\|x^{m\left(1-(1-\frac{1}{mp})\right)} C^m e^{xC}\phi\right\|^p \frac{dx}{x} \\
&\leqslant K \left\|\phi\right\|_{(D(C^m),X)_{\frac{1}{mp},p}},
\end{aligned}$$

(see Triebel [17], theorem on page 96). Thus $A_\omega e^{\cdot Q_\omega}\varphi \in L^p(0,1;X)$ if and only if

$$\varphi \in \left(D\left(Q_\omega^2\right), X\right)_{\frac{1}{2p},p} = \left(D\left(A_\omega\right), X\right)_{\frac{1}{2p},p} = \left(D\left(A\right), X\right)_{\frac{1}{2p},p}.$$

Similarly, $Q_\omega e^{\cdot Q_\omega}\varphi \in L^p(0,1;X)$ if and only if $\varphi \in \left(D\left(Q_\omega\right), X\right)_{\frac{1}{p},p}$. We conclude by using the reiteration property in Lions-Peetre [12]:

$$\begin{aligned}\left(D\left(A_\omega\right), X\right)_{\frac{1}{2p}+\frac{1}{2},p} &= \left(X, D\left(Q_\omega^2\right)\right)_{\frac{1}{2}-\frac{1}{2p},p}\\ &= \left(X, D\left(Q_\omega\right)\right)_{1-\frac{1}{p},p} = \left(D\left(Q_\omega\right), X\right)_{\frac{1}{p},p}.\end{aligned}$$

□

Remark 5. Assume (1.3), (1.4) and (1.9). Let $p \in (1,\infty)$, $\omega \geqslant \omega_0$. Then

$$A_\omega e^{\cdot Q_\omega} Q_\omega^{-1}\int_0^1 e^{sQ_\omega} f(s)ds = -Q_\omega e^{\cdot Q_\omega}\int_0^1 e^{sQ_\omega} f(s)ds = -\mathcal{L}_\omega(\cdot, f),$$

so, from Lemma 3, we get

$$A_\omega e^{\cdot Q_\omega} Q_\omega^{-1}\int_0^1 e^{sQ_\omega} f(s)ds \in L^p(0,1;X),$$

and by Lemma 4, we deduce

$$Q_\omega{}^{-1}\int_0^1 e^{sQ_\omega} f(s)ds \in \left(D\left(A_\omega\right), X\right)_{\frac{1}{2p},p}.$$

Proposition 6. *Assume* (1.4). *For any $\omega \geqslant \omega_0$, the operator $I - e^{2Q_\omega}$ has a bounded inverse and*

$$\left(I - e^{2Q_\omega}\right)^{-1} = \frac{1}{2\pi i}\int_{\gamma_\#}\frac{e^{2z}}{1-e^{2z}}\left(zI - Q_\omega\right)^{-1}dz + I,$$

where $\gamma_\#$ is a suitable curve in the complex plane.

Proof. Since the imaginary axis is contained in the resolvent set $\rho\left(Q_\omega\right)$, we then can adapt the complete proof of Lunardi [13], page 59 by choosing an appropriate curve $\gamma_\#$ which takes into account the fact that Q_ω generates an analytic semigroup. In fact we only need the result for ω large enough and we will show later, see (2.7), that there exists $\omega_1 \geqslant \omega_0$, such that for $\omega \geqslant \omega_1$

$$\| e^{2Q_\omega} \|_{L(X)} < 1,$$

from which we can deduce that $I - e^{2Q_\omega}$ has a bounded inverse, without using Lunardi's method quoted above. □

Proposition 7. *Assume* (1.3) $\sim$ (1.11). *Then there exists a constant* $C>0$, *such that for all* $\omega \geqslant \omega_0$

$$\| (Q_\omega - H)^{-1} \|_{L(X)} \leqslant \frac{C}{\omega^{1/4}}.$$

Proof. Let $\omega \geqslant \omega_0$. Then in virtue of the Dore-Venni theorem (see [4], Theorem 2.1, page 3), one has

$$\begin{aligned}(Q_\omega - H)^{-1} &= -\left(\sqrt{-A_\omega} + H\right)^{-1} \\ &= \frac{-1}{2i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\left((-A_\omega)^{1/2}\right)^{-z} H^{z-1}}{\sin(\pi z)} dz.\end{aligned} \tag{2.4}$$

Set $z = 1/2 + i\beta$, $\beta \in \mathbb{R}$. By the well-known complex powers calculus of linear operators, we have

$$\begin{cases} (-A_\omega)^{-i\beta/2}(-A_\omega)^{-1/4} \subset (-A_\omega)^{-z/2} \\ \left((-A_\omega)^{1/2}\right)^{-z} = (-A_\omega)^{-z/2}, \end{cases}$$

(see Haase [9], Proposition 2.18 page 64, Statements a) and b)). But due to (1.4) and (1.9), $(-A_\omega)^{-i\beta/2}$, $(-A_\omega)^{-1/4} \in L(X)$, therefore,

$$(-A_\omega)^{-i\beta/2}(-A_\omega)^{-1/4} = (-A_\omega)^{-z/2}.$$

Now, using the following well-known formulas

$$\begin{cases} (-A_\omega)^{-1/4} = \frac{\sin(\pi/4)}{\pi}\int_0^\infty s^{-1/4}(s+\omega-A)^{-1}\,ds \\ \dfrac{\pi}{\sin(\pi/4)} = \int_0^\infty \dfrac{\tau^{-1/4}}{1+\tau}d\tau, \end{cases}$$

(see Krein [11], page 112), we obtain

$$\begin{aligned}&\left\|(-A_\omega)^{-1/4}\right\| \leqslant \frac{\sin(\pi/4)}{\pi}\int_0^\infty s^{-1/4}\left\|(s+\omega-A)^{-1}\right\| ds \\ &\leqslant \frac{C\sin(\pi/4)}{\pi}\int_0^\infty \frac{s^{-1/4}}{1+s+\omega}ds = \frac{C\sin(\pi/4)}{\pi}\omega^{-1/4}\int_0^\infty \frac{\omega\tau^{-1/4}}{1+\omega(\tau+1)}d\tau \\ &\leqslant \frac{C\sin(\pi/4)}{\pi}\omega^{-1/4}\int_0^\infty \frac{\tau^{-1/4}}{\tau+1}d\tau = C\omega^{-1/4},\end{aligned}$$

and finally

$$\left\|(-A_\omega)^{-z/2}\right\| \leqslant \left\|(-A_\omega)^{-i\beta/2}\right\| \left\|(-A_\omega)^{-1/4}\right\| \leqslant \frac{C}{\omega^{1/4}}e^{|\beta|.\theta_A/2}.$$

Taking into account (1.11), we then deduce that

$$\| (Q_\omega - H)^{-1} \|_{L(X)} \leqslant \frac{C}{\omega^{1/4}} \int_{-\infty}^{+\infty} \frac{e^{(\theta_A/2+\theta_H)|\beta|}}{e^{\pi|\beta|}} d\beta \leqslant \frac{C}{\omega^{1/4}}.$$

(We have used the fact that for $z = \frac{1}{2} + i\beta,\ \beta \in \mathbb{R},\ |\sin(\pi z)| = \cosh(\pi\beta) = \cosh(\pi|\beta|) \geqslant \frac{e^{\pi|\beta|}}{2}$.) $\square$

Lemma 8. *Assume* (1.3) $\sim$ (1.11) *and consider, for* $\omega \geqslant \omega_0$*, the linear operator*

$$\Lambda_\omega = (Q_\omega - H) + e^{2Q_\omega} (Q_\omega + H), \tag{2.5}$$

of domain $D(\Lambda_\omega) = D(Q_\omega) \cap D(H)$*. Then there exists* $\omega^* \geqslant \omega_0$ *such that, for all* $\omega \geqslant \omega^*$*,* Λ_ω *is closed and boundedly invertible and*

$$\Lambda_\omega^{-1} = (Q_\omega - H)^{-1} \left[I + 2(I - e^{2Q_\omega})^{-1} Q_\omega e^{2Q_\omega} (Q_\omega - H)^{-1} \right]^{-1} (I - e^{2Q_\omega})^{-1}. \tag{2.6}$$

Proof. Let $\omega \geqslant \omega_0$. Due to Remark 1, Statement 4 and Proposition 6, $Q_\omega - H$ and $I - e^{2Q_\omega}$ are boundedly invertible, so we can write, on $D(Q_\omega) \cap D(H)$,

$$\begin{aligned} \Lambda_\omega &= (Q_\omega - H) - e^{2Q_\omega} (Q_\omega - H - 2Q_\omega) \\ &= \left(I - e^{2Q_\omega}\right)(Q_\omega - H) + 2e^{2Q_\omega} Q_\omega \\ &= \left[\left(I - e^{2Q_\omega}\right) + 2e^{2Q_\omega} Q_\omega (Q_\omega - H)^{-1}\right](Q_\omega - H) \\ &= \left(I - e^{2Q_\omega}\right)\left[I + 2(I - e^{2Q_\omega})^{-1} Q_\omega e^{2Q_\omega} (Q_\omega - H)^{-1}\right](Q_\omega - H). \end{aligned}$$

Now, due to Dore and Yakubov [5], Lemma page 103, for $\alpha \in \mathbb{R}$ there exist constants $C, k > 0$ (which do not depend on ω) such that for any $x \geqslant 1$

$$\| (-A + \omega I)^\alpha e^{-x(-A+\omega I)^{1/2}} \| \leqslant C e^{-kx\sqrt{\omega}};$$

in particular,

$$\| Q_\omega e^{2Q_\omega} \| \leqslant C e^{-2k\sqrt{\omega}} \text{ and } \| e^{2Q_\omega} \| \leqslant C e^{-2k\sqrt{\omega}}, \tag{2.7}$$

so, from Proposition 7, for ω large enough,

$$\| 2(I - e^{2Q_\omega})^{-1} Q_\omega e^{2Q_\omega} (Q_\omega - H)^{-1} \| \leqslant \frac{2C}{\omega^{1/4}} \frac{\| Q_\omega e^{2Q_\omega} \|}{1 - \|e^{2Q_\omega}\|},$$

which implies the existence of $\omega^* \geqslant \omega_0$ such that for any $\omega \geqslant \omega^*$

$$\left\| 2(I - e^{2Q_\omega})^{-1} Q_\omega e^{2Q_\omega} (Q_\omega - H)^{-1} \right\| < 1.$$

Hence, for $\omega \geqslant \omega^*$,

$$I + 2(I - e^{2Q_\omega})^{-1} Q_\omega e^{2Q_\omega} (Q_\omega - H)^{-1},$$

is boundedly invertible, from which we deduce that $\Lambda_\omega^{-1} \in L(X)$. □

We will now only consider $\omega \geqslant \omega^*$ where ω^* is defined in Lemma 8.

3. Representation of the solution

We assume here that (1.3)∼(1.11) hold.

In order to solve problem (1.1)-(1.2), we use the well-known Krein's reduction order method (see Krein [11]). Suppose that the problem has a strict solution u; that is, $u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A))$, $u(0) \in D(H)$ and (1.1)-(1.2) is satisfied. For almost every $x \in (0,1)$, set

$$v = -Q_\omega^{-1} u'(x), \quad y = (u(x) - v(x))/2, \quad z = (u(x) + v(x))/2,$$

then

$$y'(x) = (u'(x) - v'(x))/2 = (-Q_\omega v(x) + Q_\omega^{-1} u''(x))/2.$$

Using (1.1) we obtain

$$\begin{cases} y'(x) = Q_\omega y(x) + \frac{1}{2} Q_\omega^{-1} f(x), \text{ a.e } x \in (0,1), \\ y(0) = y_0, \end{cases}$$

where

$$y_0 = \tfrac{1}{2}(u(0) - v(0)) = \tfrac{1}{2}(u(0) + Q_\omega^{-1} u'(0)).$$

Similarly we have

$$\begin{cases} z'(x) = -Q_\omega z(x) - \frac{1}{2} Q_\omega^{-1} f(x), \text{ a.e. } x \in (0,1), \\ z(1) = z_1, \end{cases}$$

where $z_1 = \frac{1}{2}\left(u_1 - Q_\omega^{-1} u'(1)\right)$. Therefore, it follows that, for almost every $x \in (0,1)$,

$$\begin{cases} y(x) = e^{xQ_\omega} y_0 + \frac{1}{2}\int_0^x e^{(x-s)Q_\omega} Q_\omega^{-1} f(s) ds \\ z(x) = e^{(1-x)Q_\omega} z_1 + \frac{1}{2}\int_x^1 e^{(s-x)Q_\omega} Q_\omega^{-1} f(s) ds, \end{cases} \tag{3.1}$$

and

$$u(x) = y(x) + z(x) = e^{xQ_\omega} y_0 + e^{(1-x)Q_\omega} z_1 + I_x + J_x,$$

where

$$\begin{cases} I_x = \frac{1}{2}\int_0^x e^{(x-s)Q_\omega} Q_\omega^{-1} f(s) ds \\ J_x = \frac{1}{2}\int_x^1 e^{(s-x)Q_\omega} Q_\omega^{-1} f(s) ds. \end{cases}$$

Now, to obtain the final representation of u, it is enough to compute y_0 and z_1 with respect to the data d_0, u_1, f, H and A_ω.

One has that $y_0, z_1 \in D(Q_\omega)$ due to (1.6) and Remark 1, Statement 6, so by virtue of Lemma 3, we have, for almost every $x \in (0,1)$,

$$u'(x) = e^{xQ_\omega} Q_\omega y_0 - e^{(1-x)Q_\omega} Q_\omega z_1 + Q_\omega I_x - Q_\omega J_x, \tag{3.2}$$

but due to Lemma 8 $\Lambda_\omega^{-1}(X) \subset D(H)$, so, from (3.1), (3.2), and (1.9), we can deduce

$$\begin{cases} H\Lambda_\omega^{-1} u(0) = H\Lambda_\omega^{-1} y_0 + He^{Q_\omega}\Lambda_\omega^{-1} z_1 + H\Lambda_\omega^{-1} J_0 \\ \Lambda_\omega^{-1} u'(0) = Q_\omega \Lambda_\omega^{-1} y_0 - e^{Q_\omega} Q_\omega \Lambda_\omega^{-1} z_1 - Q_\omega \Lambda_\omega^{-1} J_0. \end{cases}$$

Then

$$\begin{aligned} \Lambda_\omega^{-1} d_0 &= \Lambda_\omega^{-1}\left[u'(0) - Hu(0)\right] \\ &= \Lambda_\omega^{-1} u'(0) - H\Lambda_\omega^{-1} u(0) \\ &= (Q_\omega - H)\Lambda_\omega^{-1} y_0 - (Q_\omega + H)e^{Q_\omega}\Lambda_\omega^{-1} z_1 - (Q_\omega + H)\Lambda_\omega^{-1} J_0. \end{aligned}$$

Now, due to (3.2) and (3.1), we have $z_1 = u(1) - y(1) = u_1 - e^{Q_\omega} y_0 - I_1$, hence,

$$\begin{aligned} \Lambda_\omega^{-1} d_0 &= \left[(Q_\omega - H) + (Q_\omega + H)e^{2Q_\omega}\right]\Lambda_\omega^{-1} y_0 \\ &\quad - (Q_\omega + H)\Lambda_\omega^{-1}\left(e^{Q_\omega} u_1 - e^{Q_\omega} I_1 + J_0\right) \\ &= y_0 - (Q_\omega + H)\Lambda_\omega^{-1}\left(e^{Q_\omega} u_1 - e^{Q_\omega} I_1 + J_0\right), \end{aligned}$$

or

$$y_0 = \Lambda_\omega^{-1} d_0 + (Q_\omega + H)\Lambda_\omega^{-1}\left(e^{Q_\omega} u_1 - e^{Q_\omega} I_1 + J_0\right), \tag{3.3}$$

and

$$z_1 = u_1 - I_1 - e^{Q_\omega}\Lambda_\omega^{-1} d_0 - e^{Q_\omega}(Q_\omega + H)\Lambda_\omega^{-1}\left(e^{Q_\omega} u_1 - e^{Q_\omega} I_1 + J_0\right). \tag{3.4}$$

Finally, from (3.2), (3.3), (3.4), we deduce that u is given formally by

$$\begin{aligned} u(x) &= e^{xQ_\omega}\left[\Lambda_\omega^{-1} d_0 + (Q_\omega + H)\Lambda_\omega^{-1} e^{Q_\omega} u_1\right] \\ &\quad + \frac{1}{2} e^{xQ_\omega}(Q_\omega + H)\Lambda_\omega^{-1} Q_\omega^{-1} \int_0^1 e^{sQ_\omega} f(s)\,ds \\ &\quad - \frac{1}{2} e^{xQ_\omega}(Q_\omega + H)\Lambda_\omega^{-1} e^{Q_\omega} Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega} f(s)\,ds \\ &\quad + e^{(1-x)Q_\omega}\left[(I - (Q_\omega + H)\Lambda_\omega^{-1} e^{2Q_\omega})u_1 - \Lambda_\omega^{-1} e^{Q_\omega} d_0\right] \\ &\quad - \frac{1}{2} e^{(1-x)Q_\omega}(Q_\omega + H)\Lambda_\omega^{-1} e^{Q_\omega} Q_\omega^{-1} \int_0^1 e^{sQ_\omega} f(s)\,ds \end{aligned}$$

$$-\frac{1}{2}e^{(1-x)Q_\omega}(I-(Q_\omega+H)\Lambda_\omega^{-1}e^{2Q_\omega})Q_\omega^{-1}\int_0^1 e^{(1-s)Q_\omega}f(s)ds$$
$$+\frac{1}{2}Q_\omega^{-1}\int_0^x e^{(x-s)Q_\omega}f(s)ds+\frac{1}{2}Q_\omega^{-1}\int_x^1 e^{(s-x)Q_\omega}f(s)ds.$$

We then can write u as

$$u(x)=S_1(x,d_0,u_1)+S_2(x,f)+R(x,d_0,u_1,f), \tag{3.5}$$

where

$$S_1(x,d_0,u_1)=\Lambda_\omega^{-1}e^{xQ_\omega}d_0+e^{(1-x)Q_\omega}u_1,$$

$$\begin{aligned} S_2(x,f)=&\frac{1}{2}(Q_\omega+H)\Lambda_\omega^{-1}Q_\omega^{-1}e^{xQ_\omega}\int_0^1 e^{sQ_\omega}f(s)ds \\ &-\frac{1}{2}Q_\omega^{-1}e^{(1-x)Q_\omega}\int_0^1 e^{(1-s)Q_\omega}f(s)ds \\ &+\frac{1}{2}Q_\omega^{-1}\int_0^x e^{(x-s)Q_\omega}f(s)ds+\frac{1}{2}Q_\omega^{-1}\int_x^1 e^{(s-x)Q_\omega}f(s)ds), \end{aligned} \tag{3.6}$$

$$\begin{aligned} R(x,d_0,u_1,f)=&(Q_\omega+H)\Lambda_\omega^{-1}e^{xQ_\omega}e^{Q_\omega}u_1 \\ &-(Q_\omega+H)\Lambda_\omega^{-1}e^{(1-x)Q_\omega}e^{2Q_\omega}u_1-\Lambda_\omega^{-1}e^{(1-x)Q_\omega}e^{Q_\omega}d_0 \\ &-\frac{1}{2}(Q_\omega+H)\Lambda_\omega^{-1}Q_\omega^{-1}e^{xQ_\omega}e^{Q_\omega}\int_0^1 e^{(1-s)Q_\omega}f(s)ds \\ &-\frac{1}{2}(Q_\omega+H)\Lambda_\omega^{-1}Q_\omega^{-1}e^{(1-x)Q_\omega}e^{Q_\omega}\int_0^1 e^{sQ_\omega}f(s)ds \\ &+\frac{1}{2}(Q_\omega+H)\Lambda_\omega^{-1}Q_\omega^{-1}e^{(1-x)Q_\omega}e^{2Q_\omega}\int_0^1 e^{(1-s)Q_\omega}f(s)ds. \end{aligned} \tag{3.7}$$

The previous work shows that if problem (1.1)-(1.2) has a strict solution, then this solution is unique and determined by (3.5).

Lemma 9. *Assume* (1.3) $\sim$ (1.11) *and let* d_0, $u_1 \in X$, $f \in L^p((0,1);X)$,$1 < p < \infty$. *Then*

$$A_\omega R(\cdot,d_0,u_1,f)\in L^p(0,1;X).$$

Proof. For any $\xi \in X$, $k \in \mathbb{N}$, we have $e^{Q_\omega}\xi \in D\left(Q_\omega^k\right)$, so

$$A_\omega e^{\cdot Q_\omega}e^{Q_\omega}\xi=e^{\cdot Q_\omega}A_\omega e^{Q_\omega}\xi,$$

and $A_\omega e^{\cdot Q_\omega} e^{Q_\omega}\xi$ is bounded and thus in $L^p((0,1);X)$. To conclude it is enough to remark that $A_\omega R(\cdot, d_0, u_1, f)$ can be written as a sum of terms $PA_\omega e^{\cdot Q_\omega} e^{Q_\omega}\xi$, $PA_\omega e^{(1-\cdot)Q_\omega} e^{Q_\omega}\xi$, where $P \in L(X)$, $\xi \in X$. □

4. THE RESULTS IN THE CASE $d_0 = u_1 = 0$

In this case, our previous problem becomes

$$\begin{cases} u''(x) + Au(x) - \omega u(x) = f(x), & x \in (0,1), \\ u'(0) - Hu(0) = 0, \ u(1) = 0. \end{cases} \tag{4.1}$$

The main result in this section is the following:

Theorem 10. *Assume* (1.3) $\sim$ (1.11). *Let* $f \in L^p((0,1);X)$ *with* $1 < p < \infty$ *and* $\omega \geqslant \omega^*$. *Then problem* (4.1) *has a unique strict solution* u*; that is,*

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)),$$

$u(0) \in D(H)$, *and* u *satisfies (4.1). Moreover* u *is given by*

$$u(x) = S_2(x,f) + R(x,0,0,f), \quad x \in (0,1), \tag{4.2}$$

where S_2, R *are given by* (3.6), (3.7).

Proof. Consider u given by the representation formula (3.5) which reduces here to (4.2). Let us show that $A_\omega u \in L^p((0,1);X)$. In fact, due to Lemma 9, it is enough to prove that $A_\omega S_2(\cdot,f) \in L^p((0,1);X)$. From (3.6), one has for almost every $x \in (0,1)$

$$\begin{aligned} A_\omega S_2(x,f) &= T_\omega Q_\omega e^{xQ_\omega} \int_0^1 e^{sQ_\omega} f(s)ds \\ &\quad + \frac{1}{2} Q_\omega e^{(1-x)Q_\omega} \int_0^1 e^{(1-s)Q_\omega} f(s)ds \\ &\quad - \frac{1}{2} Q_\omega \int_0^x e^{(x-s)Q_\omega} f(s)ds - \frac{1}{2} Q_\omega \int_x^1 e^{(s-x)Q_\omega} f(s)ds, \end{aligned} \tag{4.3}$$

where $T_\omega = -\frac{1}{2}(Q_\omega + H)\Lambda_\omega^{-1} \in L(X)$. This can be written

$$\begin{aligned} A_\omega S_2(x,f) &= T_\omega \mathcal{L}_\omega(x,f) + \frac{1}{2}\mathcal{L}_\omega(1-x, f(1-.)) \\ &\quad -\frac{1}{2}L_\omega(x,f) - \frac{1}{2}L_\omega(1-x, f(1-.)), \end{aligned}$$

from which we deduce that $A_\omega S_2(\cdot,f) \in L^p((0,1);X)$ by Lemma 3.

Now we will conclude by showing that the function u given by (4.2) satisfies (4.1).

Comparing with (4.3) we get, for almost every $x \in (0,1)$,

$$\begin{aligned} S_2''(x,f) &= \frac{1}{2}(Q_\omega + H)\Lambda_\omega^{-1} Q_\omega e^{xQ_\omega} \int_0^1 e^{sQ_\omega} f(s)ds \\ &- \frac{1}{2} Q_\omega e^{(1-x)Q_\omega} \int_0^1 e^{(1-s)Q_\omega} f(s)ds \\ &+ \frac{1}{2} Q_\omega \int_0^x e^{(x-s)Q_\omega} f(s)ds + \frac{1}{2} f(x) \\ &+ \frac{1}{2} Q_\omega \int_x^1 e^{(s-x)Q_\omega} f(s)ds + \frac{1}{2} f(x) = -A_\omega S_2(x,f) + f(x); \end{aligned}$$

similarly, $R''(x,0,0,f) = -A_\omega R(x,0,0,f)$, thus, u given by (4.2) satisfies (1.1). Finally,

$$\begin{aligned} u(1) &= \frac{1}{2} e^{Q_\omega}(Q_\omega + H)\Lambda_\omega^{-1} Q_\omega^{-1} \int_0^1 e^{sQ_\omega} f(s)ds \\ &- \frac{1}{2} e^{Q_\omega}(Q_\omega + H)\Lambda_\omega^{-1} e^{Q_\omega} Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega} f(s)ds \\ &- \frac{1}{2}(Q_\omega + H)\Lambda_\omega^{-1} e^{Q_\omega} Q_\omega^{-1} \int_0^1 e^{sQ_\omega} f(s)ds \\ &- \frac{1}{2}(I - (Q_\omega + H)\Lambda_\omega^{-1} e^{2Q_\omega}) Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega} f(s)ds \\ &+ \frac{1}{2} Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega} f(s)ds = 0. \end{aligned}$$

Moreover, applying Λ_ω^{-1} to (2.5) and using Remark 1, statement 5, we get

$$I - (Q_\omega + H)\,\Lambda_\omega^{-1} e^{2Q_\omega} = (Q_\omega - H)\,\Lambda_\omega^{-1}, \tag{4.4}$$

so

$$\begin{aligned} u'(0) &= \frac{1}{2}(Q_\omega + H)\Lambda_\omega^{-1} \int_0^1 e^{sQ_\omega} f(s)ds \\ &- \frac{1}{2}(Q_\omega + H)\Lambda_\omega^{-1} e^{Q_\omega} \int_0^1 e^{(1-s)Q_\omega} f(s)ds \\ &+ \frac{1}{2} e^{Q_\omega}(Q_\omega + H)\Lambda_\omega^{-1} e^{Q_\omega} \int_0^1 e^{sQ_\omega} f(s)ds \\ &+ \frac{1}{2} e^{Q_\omega}((Q_\omega - H)\,\Lambda_\omega^{-1}) \int_0^1 e^{(1-s)Q_\omega} f(s)ds - \frac{1}{2} \int_0^1 e^{sQ_\omega} f(s)ds, \end{aligned}$$

and then

$$\begin{aligned} u'(0) &= \frac{1}{2}\left[(Q_\omega + H)\Lambda_\omega^{-1} + (Q_\omega + H)\Lambda_\omega^{-1}e^{2Q_\omega} - I\right]\int_0^1 e^{sQ_\omega} f(s)ds \\ &\quad -H\Lambda_\omega^{-1}e^{Q_\omega}\int_0^1 e^{(1-s)Q_\omega} f(s)ds \\ &= H\Lambda_\omega^{-1}\int_0^1 e^{sQ_\omega} f(s)ds - H\Lambda_\omega^{-1}e^{Q_\omega}\int_0^1 e^{(1-s)Q_\omega} f(s)ds. \end{aligned}$$

Similarly, using again (4.4),

$$\begin{aligned} u(0) &= \frac{1}{2}(Q_\omega + H)\Lambda_\omega^{-1}Q_\omega^{-1}\int_0^1 e^{sQ_\omega} f(s)ds \\ &\quad -\frac{1}{2}(Q_\omega + H)\Lambda_\omega^{-1}e^{Q_\omega}Q_\omega^{-1}\int_0^1 e^{(1-s)Q_\omega} f(s)ds \\ &\quad -\frac{1}{2}e^{Q_\omega}(Q_\omega + H)\Lambda_\omega^{-1}e^{Q_\omega}Q_\omega^{-1}\int_0^1 e^{sQ_\omega} f(s)ds \\ &\quad -\frac{1}{2}e^{Q_\omega}((Q_\omega - H)\,\Lambda_\omega^{-1})Q_\omega^{-1}\int_0^1 e^{(1-s)Q_\omega} f(s)ds \\ &\quad +\frac{1}{2}Q_\omega^{-1}\int_0^1 e^{(s-x)Q_\omega} f(s)ds, \end{aligned}$$

then

$$\begin{aligned} u(0) &= \frac{1}{2}\left[(Q_\omega + H)\Lambda_\omega^{-1} - (Q_\omega + H)\Lambda_\omega^{-1}e^{2Q_\omega} + I\right]Q_\omega^{-1}\int_0^1 e^{sQ_\omega} f(s)ds \\ &\quad - \frac{1}{2}Q_\omega\Lambda_\omega^{-1}e^{Q_\omega}Q_\omega^{-1}\int_0^1 e^{(1-s)Q_\omega} f(s)ds \\ &= \Lambda_\omega^{-1}\int_0^1 e^{sQ_\omega} f(s)ds - \Lambda_\omega^{-1}e^{Q_\omega}\int_0^1 e^{(1-s)Q_\omega} f(s)ds, \end{aligned}$$

from which we deduce that $u(0) \in D(H)$ and

$$Hu(0) = H\Lambda_\omega^{-1}\int_0^1 e^{sQ_\omega} f(s)ds - H\Lambda_\omega^{-1}e^{Q_\omega}\int_0^1 e^{(1-s)Q_\omega} f(s)ds = u'(0).$$

5. The results in the general case

In this section, d_0 and u_1 are not supposed to be 0 and the main result in this paper is the following:

Theorem 11. *Assume* (1.3) $\sim$ (1.11). *Let* $f \in L^p((0,1);X)$ *with* $1 < p < \infty$ *and* $\omega \geqslant \omega^*$. *Then the following two assertions are equivalent.*

(1) $d_0 \in (D(A), X)_{\frac{1}{2p}+\frac{1}{2},p}$, $u_1 \in (D(A), X)_{\frac{1}{2p},p}$.

(2) *Problem* (1.1) *and* (1.2) *has a strict solution* u*; that is,*

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)),$$

$u(0) \in D(H)$ *and* u *satisfies* (1.1) *and* (1.2).

Moreover, in this case, u *is uniquely determined by* (3.5).

Proof. Suppose that $d_0 \in (D(A), X)_{\frac{1}{2p}+\frac{1}{2},p}$ and $u_1 \in (D(A), X)_{\frac{1}{2p},p}$. From (3.5), we get $u = \overline{u} + \overline{\overline{u}}$, where

$$\overline{u} = S_2(\cdot, f) + R(\cdot, 0, 0, f), \quad \overline{\overline{u}} = S_1(\cdot, d_0, u_1) + R(\cdot, d_0, u_1, 0).$$

We need to show that $A_\omega u \in L^p((0,1);X)$, but, due to Theorem 10, we have $A_\omega \overline{u} \in L^p((0,1);X)$; moreover, due to Lemma 9, $A_\omega R(\cdot, d_0, u_1, 0) \in L^p((0,1);X)$, so, it is enough to study $A_\omega S_1(\cdot, d_0, u_1)$. We have

$$\begin{aligned} A_\omega S_1(\cdot, d_0, u_1) &= A_\omega \Lambda_\omega^{-1} e^{\cdot Q_\omega} d_0 + A_\omega e^{(1-\cdot)Q_\omega} u_1 \\ &= -Q_\omega \Lambda_\omega^{-1} Q_\omega e^{\cdot Q_\omega} d_0 + A_\omega e^{(1-\cdot)Q_\omega} u_1. \end{aligned} \tag{5.1}$$

Using Lemma 4, $Q_\omega e^{\cdot Q_\omega} d_0$, $A_\omega e^{(1-\cdot)Q_\omega} u_1 \in L^p((0,1);X)$, and from (2.6) we get $Q_\omega \Lambda_\omega^{-1} \in L(X)$. We then deduce that $A_\omega S_1(\cdot, d_0, u_1) \in L^p((0,1);X)$. Now we will conclude by showing that u satisfies (1.1)-(1.2). One has

$$\begin{aligned} S_1''(\cdot, d_0, u_1) &= \Lambda_\omega^{-1} Q_\omega^2 e^{\cdot Q_\omega} d_0 + Q_\omega^2 e^{(1-\cdot)Q_\omega} u_1 = -A_\omega S_1(\cdot, d_0, u_1), \\ R''(\cdot, d_0, u_1, 0) &= (Q_\omega + H)\Lambda_\omega^{-1} Q_\omega^2 e^{\cdot Q_\omega} e^{Q_\omega} u_1 \\ &\quad - (Q_\omega + H)\Lambda_\omega^{-1} Q_\omega^2 e^{(1-\cdot)Q_\omega} e^{2Q_\omega} u_1 - \Lambda_\omega^{-1} e^{(1-\cdot)Q_\omega} Q_\omega^2 e^{Q_\omega} d_0 \\ &= -A_\omega R(\cdot, d_0, u_1, 0), \end{aligned}$$

so

$$\overline{\overline{u}}'' + A\overline{\overline{u}} - \omega \overline{\overline{u}} = 0.$$

Since, from Theorem 10, we already know that $\overline{u}'' + A\overline{u} - \omega \overline{u} = f$, then u satisfies (1.1).

Finally, from Theorem 10, we get $\overline{u}(1) = 0$, thus,

$$\begin{aligned} u(1) = \overline{\overline{u}}(1) &= e^{Q_\omega} \left[\Lambda_\omega^{-1} d_0 + (Q_\omega + H)\Lambda_\omega^{-1} e^{Q_\omega} u_1\right] \\ &\quad + \left[(I - (Q_\omega + H)\Lambda_\omega^{-1} e^{2Q_\omega}) u_1 - \Lambda_\omega^{-1} e^{Q_\omega} d_0\right] = u_1; \end{aligned}$$

moreover, due to (4.4), we have

$$\overline{\overline{u}}'(0) = Q_\omega \left[\Lambda_\omega^{-1} d_0 + (Q_\omega + H)\Lambda_\omega^{-1} e^{Q_\omega} u_1\right]$$

$$\begin{aligned}
& -Q_\omega e^{Q_\omega}\left[(Q_\omega - H)\,\Lambda_\omega^{-1}u_1 - \Lambda_\omega^{-1}e^{Q_\omega}d_0\right] \\
= \; & Q_\omega\Lambda_\omega^{-1}d_0 + Q_\omega e^{Q_\omega}\Lambda_\omega^{-1}e^{Q_\omega}d_0 + 2H\Lambda_\omega^{-1}Q_\omega e^{Q_\omega}u_1,
\end{aligned}$$

and also

$$\begin{aligned}
\overline{\overline{u}}(0) &= \Lambda_\omega^{-1}d_0 + (Q_\omega + H)\Lambda_\omega^{-1}e^{Q_\omega}u_1 \\
&\quad + e^{Q_\omega}\left[(I - (Q_\omega + H)\Lambda_\omega^{-1}e^{2Q_\omega})u_1 - \Lambda_\omega^{-1}e^{Q_\omega}d_0\right] \\
&= \Lambda_\omega^{-1}d_0 + (Q_\omega + H)\Lambda_\omega^{-1}e^{Q_\omega}u_1 \\
&\quad + e^{Q_\omega}\left[(Q_\omega - H)\,\Lambda_\omega^{-1})u_1 - \Lambda_\omega^{-1}e^{2Q_\omega}d_0\right] \\
&= \Lambda_\omega^{-1}d_0 - \Lambda_\omega^{-1}e^{2Q_\omega}d_0 + 2Q_\omega\Lambda_\omega^{-1}e^{Q_\omega}u_1.
\end{aligned}$$

We deduce that $\overline{\overline{u}}(0) \in D(H)$ and

$$H\overline{\overline{u}}(0) = H\Lambda_\omega^{-1}d_0 - H\Lambda_\omega^{-1}e^{2Q_\omega}d_0 + 2H\Lambda_\omega^{-1}Q_\omega e^{Q_\omega}u_1,$$

so

$$\begin{aligned}
\overline{\overline{u}}'(0) - H\overline{\overline{u}}(0) &= Q_\omega\Lambda_\omega^{-1}d_0 + Q_\omega e^{Q_\omega}\Lambda_\omega^{-1}e^{Q_\omega}d_0 \\
&\quad - H\Lambda_\omega^{-1}d_0 + H\Lambda_\omega^{-1}e^{2Q_\omega}d_0 \\
&= \left[Q_\omega - H + (Q_\omega + H)\,e^{2Q_\omega}\right]\Lambda_\omega^{-1}d_0 = d_0.
\end{aligned}$$

On the other hand, Theorem 10 gives $\overline{u}'(0) - H\overline{u}(0) = 0$, which implies $u'(0) - Hu(0) = d_0$. Conversely, assume that problem (1.1) and (1.2) has a strict solution u. Then, from (1.6), $u_1 \in (D(A), X)_{\frac{1}{2p},p}$. Moreover, by uniqueness $u = S_1(\cdot, d_0, u_1) + S_2(\cdot, f) + R(\cdot, d_0, u_1, f)$. Furthermore, due to Lemma 9 we have $A_\omega R(\cdot, d_0, u_1, f) \in L^p((0,1); X)$, and the proof of Theorem 10 asserts that $A_\omega S_2(\cdot, f) \in L^p(0,1; X)$, hence,

$$A_\omega S_1(\cdot, d_0, u_1) = A_\omega u - A_\omega S_2(\cdot, f) - A_\omega R(\cdot, d_0, u_1, f) \in L^p((0,1); X).$$

Now, from (5.1), we can write

$$Q_\omega e^{\cdot Q_\omega}d_0 = -\Lambda_\omega Q_\omega^{-1}A_\omega S_1(\cdot, d_0, u_1) + \Lambda_\omega Q_\omega^{-1}A_\omega e^{(1-\cdot)Q_\omega}u_1 \in L^p((0,1); X),$$

from which we deduce that $d_0 \in (D(A_\omega), X)_{\frac{1}{2p}+\frac{1}{2},p}$, in virtue of Lemma 4.

6. Examples

6.1. **The case of the strip.** Take $X = L^p(\mathbb{R}), 1 < p < \infty$. Define the operators A and H by

$$\begin{cases} D(A) = W^{2,p}(\mathbb{R}), \; Au = au'', \\ D(H) = W^{1,p}(\mathbb{R}), \; Hu = bu', \end{cases}$$

where $a, b > 0$.

Let $\omega_0 > 0$. A simple computation shows that (1.4), (1.7), (1.8) hold. Moreover, A satisfies (1.9), see Fuhrman [7] or Prüss-Sohr [16]. Assumptions (1.10) and (1.11) are also satisfied, see Fuhrman [7] or Prüss [14], Example 8.1, page 215, and we have exactly

$$\theta_A = \epsilon_1, \quad \theta_H = \pi/2 + \epsilon_2$$

with any $\epsilon_1 \in (0, \pi)$ and any $\epsilon_2 \in (0, \pi/2)$. Applying Theorem 11, we get the following.

Proposition 12. *Let* $p \in (1, \infty)$, $f \in L^p((0,1); X)$, *and*

$$d_0 \in \left(W^{2,p}(\mathbb{R}), L^p(\mathbb{R})\right)_{\frac{1}{2p}+\frac{1}{2},p}, \quad u_1 \in \left(W^{2,p}(\mathbb{R}), L^p(\mathbb{R})\right)_{\frac{1}{2p},p}.$$

Then there exists $\omega^* > 0$ *such that, for all* $\omega \geqslant \omega^*$, *the problem*

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + a\dfrac{\partial^2 u}{\partial y^2}(x,y) - \omega u(x,y) = f(x,y), \ (x,y) \in]0,1[\times \mathbb{R}, \\ \dfrac{\partial u}{\partial x}(0,y) - b\dfrac{\partial u}{\partial y}(0,y) = d_0(y), \ y \in \mathbb{R}, \\ u(1,y) = u_1(y), \ y \in \mathbb{R}, \end{cases} \tag{6.1}$$

has a unique strict solution u; that is,

$$u \in W^{2,p}((0,1); L^p(\mathbb{R})) \cap L^p((0,1); W^{2,p}(\mathbb{R}))$$

and u satisfies (6.1).

Note that the interpolation spaces

$$\left(W^{2,p}(\mathbb{R}), L^p(\mathbb{R})\right)_{\frac{1}{2p}+\frac{1}{2},p}, \quad \left(W^{2,p}(\mathbb{R}), L^p(\mathbb{R})\right)_{\frac{1}{2p},p}$$

coincide respectively with the following well-known Besov spaces

$$B_{p,p}^{2(1-\frac{1}{2p}-\frac{1}{2})}(\mathbb{R}) = B_{p,p}^{1-\frac{1}{p}}(\mathbb{R}), \quad B_{p,p}^{2(1-\frac{1}{2p})}(\mathbb{R}) = B_{p,p}^{2-\frac{1}{p}}(\mathbb{R}),$$

which are completely described in Grisvard [8], page 680.

6.2. **A simple example.** Take $X = L^p(\mathbb{R}), 1 < p < \infty$. Define the operators A and H by

$$\begin{cases} D(A) = W^{2,p}(\mathbb{R}), \ A\varphi = a\varphi'' \\ D(H) = W^{2,p}(\mathbb{R}), \ H\varphi = -b\varphi'', \end{cases}$$

where $a, b > 0$.

Let $\omega_0 > 0$. A simple computation shows that (1.4), (1.7), and (1.8) hold. Moreover, A satisfies (1.9), see Fuhrman [7]. Assumptions (1.10) and (1.11) are also satisfied and we have

$$\theta_A = \epsilon_1, \quad \theta_H = \epsilon_2,$$

with any $\epsilon_1, \epsilon_2 \in (0,\pi)$. We therefore obtain the following.

Proposition 13. *Let $p \in (1,\infty)$, $f \in L^p((0,1);X)$, and*

$$d_0 \in B_{p,p}^{1-\frac{1}{p}}(\mathbb{R}), \quad u_1 \in B_{p,p}^{2-\frac{1}{p}}(\mathbb{R}).$$

Then there exists $\omega^* > 0$ such that, for all $\omega \geqslant \omega^*$, the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + a\dfrac{\partial^2 u}{\partial y^2}(x,y) - \omega u(x,y) = f(x,y), \ (x,y) \in (0,1)\times\mathbb{R}, \\ \dfrac{\partial u}{\partial x}(0,y) + b\dfrac{\partial^2 u}{\partial y^2}(0,y) = d_0(y), \ y \in \mathbb{R}, \\ u(1,y) = u_1(y), \ y \in \mathbb{R}, \end{cases}$$

has a unique strict solution u satisfying

$$u \in W^{2,p}(0,1;L^p(\mathbb{R})) \cap L^p(0,1;W^{2,p}(\mathbb{R})).$$

6.3. **The case of a rectangle.** Consider the operator T in $X = L^2(0,1)$ defined by periodic boundary conditions

$$\begin{cases} D(T) = \left\{f \in H^1(0,1) : f(0) = f(1)\right\} \\ Tf = if'. \end{cases} \tag{6.2}$$

Then it is well known that T is self-adjoint and its spectrum is $\sigma(T) = 2\pi\mathbb{Z}$. Introduce A_ω by

$$A_\omega = A - \omega I = -T^2 - \omega I, \ (\omega > 0).$$

Now, consider $H = -iT$. Then H generates a strongly continuous group, $-A_\omega$ and $\sqrt{-A_\omega}$ are positive self-adjoint operators and

$$\begin{cases} D(A) = \left\{v \in H^2(0,1) : f(0) = f(1), \ f'(0) = f'(1)\right\} \\ Af = f''. \end{cases} \tag{6.3}$$

H generates a strongly continuous group and then it admits a bounded $\mathcal{H}^\infty$ calculus. All our assumptions are verified; see for instance R. Denk, M. Hieber and J. Prüss [3].

Hence Theorem 11 enables us to solve the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \dfrac{\partial^2 u}{\partial y^2}(x,y) - \omega u(x,y) \\ = f(x,y),\ (x,y) \in (0,1)\times(0,1), \\ \dfrac{\partial u}{\partial x}(0,y) - \dfrac{\partial u}{\partial y}(0,y) = d_0(y),\ 0 < y < 1, \\ u(1,y) = u_1(y),\ 0 < y < 1, \\ u(x,0) = u(x,1), \dfrac{\partial u}{\partial y}(x,0) = \dfrac{\partial u}{\partial y}(x,1),\ 0 < x < 1, \end{cases} \tag{6.4}$$

provided that ω is large enough and

$$\begin{cases} f \in L^p(0,1;L^2(0,1)), \\ d_0 \in \left(D(A), L^2(0,1)\right)_{\frac{1}{2p}+\frac{1}{2},p}, \\ u_1 \in \left(D(A), L^2(0,1)\right)_{\frac{1}{2p},p}. \end{cases}$$

These last interpolation spaces coincide with some Besov spaces which are completely characterized in [8], page 680 and pages 708-709.

6.4. **The case of a bounded domain in $\mathbb{R}^n$.** Consider Ω a bounded domain in $\mathbb{R}^n$, $n \geqslant 1$, with a smooth boundary $\partial\Omega$ and $X = L^p(\Omega)$, $1 < p < \infty$. Take A, H defined in X by

$$\begin{cases} D(A) = \left\{u \in W^4(\Omega) : u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = 0\right\},\ Au = \alpha\Delta^2 u, \\ D(H) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),\ Hu = \beta\Delta, \end{cases}$$

with $\alpha, \beta < 0$.

Then (1.4)∼(1.9) hold (for details see [6], example 3, page 210). We can then solve the boundary-value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \alpha\Delta^2 u(x,y) - \omega u(x,y) \\ = f(x,y),\ (x,y) \in (0,1)\times\Omega, \\ \dfrac{\partial u}{\partial x}(0,y) - \beta\Delta u(0,y) = d_0(y),\ y \in \Omega, \\ u(1,y) = u_1(y),\ y \in \Omega, \\ u(x,\xi) = \Delta_y u(x,\xi) = 0,\ (x,\xi) \in (0,1)\times\partial\Omega, \end{cases} \tag{6.5}$$

provided that ω is large enough and

$$\begin{cases} f \in L^p((0,1);L^p(\Omega)) \\ d_0 \in (D(A), L^p(\Omega))_{\frac{1}{2p}+\frac{1}{2},p} \\ u_1 \in (D(A), L^p(\Omega))_{\frac{1}{2p},p}. \end{cases}$$

These interpolation spaces could be well characterized according to P. Grisvard, see [17], Theorem 4.3.3, page 321.

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