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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR A NONLINEAR ELLIPTIC PROBLEM WITH THE CRITICAL EXPONENT AND A HARDY TERM

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Abstract. In this paper, we show that if $\mu > 0$ is small enough, the problem

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $\operatorname{cat}\Omega - 1$ positive solutions, where Ω is a noncontractible, bounded domain in $\mathbb{R}^N (N \geq 4)$ such that its boundary is smooth and $0 \in \Omega$.

1. Introduction

In this paper, we study the existence of multiple positive solutions of

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$, $2^* = 2N/(N-2)$, $0 \in \Omega$ and $\mu > 0$. Problems of this kind have been studied by many authors; see [3, 6, 7, 8, 9, 14, 15] and others. By the invariance of scaling, we can easily see that the corresponding functional to problem (1.1) has no minimizer on the so-called Nehari manifold except in the case of the entire domain. In the case of the entire domain, Terracini [17] obtained all positive solutions belonging to $\mathcal{D}^{1,2}(\mathbb{R}^N)$. By the Pohožaev type inequality [6, Lemma 3.7], we also know that problem (1.1) has no solution if Ω is star-shaped with respect

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to 0. In our recent paper [9], we showed that if $N \geq 4$, Ω is noncontractible, and $\mu > 0$ is small enough, problem (1.1) has at least one positive solution. In the case of $\mu = 0$, similar phenomena can be observed. That is, there is no minimizer on the Nehari manifold except in the case of the entire domain, all positive solutions belonging to $\mathcal{D}^{1,2}(\mathbb{R}^N)$ are known, if Ω is star-shaped then there is no solution by the Pohožaev inequality [12], and Bahri and Coron [1] showed that problem (1.1) has at least one positive solution if the topology of Ω is nontrivial. On the other hand, for Brézis and Nirenberg's problem [2], i.e., problem (1.1) such that $\mu = 0$ and the term $-\mu u/|x|^2$ is replaced by $-\lambda u$ with $\lambda > 0$, Rey [13] and Lazzo [10] showed that $\operatorname{cat} \Omega$ is a lower estimate of the number of positive solutions. Passaseo [11] improved their results by showing that if Ω is noncontractible then there are at least $\operatorname{cat} \Omega + 1$ positive solutions. In order to show the estimations given by Rey, Lazzo and Passaseo, the existence of the minimizer on the Nehari manifold is crucial. However, as we already mentioned, the corresponding functional to our problem (1.1) does not have a minimizer on the Nehari manifold.

In this paper, we show that $\operatorname{cat} \Omega - 1$ is a lower estimate of the number of positive solutions for (1.1) by using the relative Lusternik-Schnirelmann category developed by Fournier and Willem [5], Szulkin [16] and others. Our result is the following:

Theorem. *Let $N \geq 4$, and let $\Omega \subset \mathbb{R}^N$ be a noncontractible, bounded domain such that its boundary $\partial\Omega$ is smooth and $0 \in \Omega$. Then there exists $\mu_0 > 0$ such that for each $\mu \in (0, \mu_0)$, problem (1.1) has at least $\operatorname{cat} \Omega - 1$ positive solutions.*

In the next section, we recall the definition of relative category and its properties. In Section 3, we give the proof of our theorem.

2. Relative category

In this section, basically following [18, Section 5], we give a definition of relative category, and we show some properties. Let X be a topological space, let Y be a closed subset of X and let A, B be subsets of X. We say $A \prec_Y B$ in X if $Y \subset A \cap B$ and there exists $h \in C([0,1] \times A, X)$ such that

(i) $h(0, u) = u$, $h(1, u) \in B$ for each $u \in A$; and
(ii) $h(t, Y) \subset Y$ for each $t \in [0, 1]$.

Assume $Y \subset A$. We define $\operatorname{cat}_{X,Y} A$ as the least integer n such that there exists $n + 1$ closed subsets $A_0, A_1, \ldots, A_n$ of X such that

(i) $A = \bigcup_{i=0}^{n} A_i$;

(ii) $A_0 \prec_Y Y$ in X; and
(iii) $A_1, \dots, A_n$ are contractible in X;

and if such an integer does not exist, we define $\mathrm{cat}_{X,Y} A = \infty$. In the case $Y = \emptyset$, we write $\mathrm{cat}_X A$ instead of $\mathrm{cat}_{X,\emptyset} A$. In the case of $Y = \emptyset$ and $A = X$, we write $\mathrm{cat}\, X$ instead of $\mathrm{cat}_X X$ or $\mathrm{cat}_{X,\emptyset} X$.

We give some basic properties of relative category. Although we do not assume A and B are closed, the proof of [18, Proposition 5.6] works. So we omit the proof.

Lemma 1. *Let X be a topological space. Let Y be a closed subset of X and let A, B be subsets of X such that $Y \subset A$. Then the following hold:*

(i) $\mathrm{cat}_{X,Y} Y = 0$;
(ii) *if $A \subset B$, then* $\mathrm{cat}_{X,Y} A \le \mathrm{cat}_{X,Y} B$;
(iii) $\mathrm{cat}_{X,Y}(A \cup B) \le \mathrm{cat}_{X,Y} A + \mathrm{cat}_X B$;
(iv) *if there exists $\eta \in C([0,1] \times X, X)$ which satisfies $\eta(0,u) = u$ for each $u \in A$, $\eta(t, Y) \subset Y$ for each $t \in [0,1]$ and $Y \subset \eta(1, A)$, then* $\mathrm{cat}_{X,Y} A \le \mathrm{cat}_{X,Y}\, \eta(1, A)$.

Remark 1. Although we defined $A \prec_Y B$, we only use the case $B = Y$. The reason why $A \prec_Y B$ is defined can be found in [18, Proposition 5.17].

We say a metric space X is an absolute neighborhood extensor (abbreviated as ANE) if for each metric space E, closed subset D of E, and continuous function $f : D \to X$, there exist a neighborhood U of D and a continuous extension of f from U into X.

Lemma 2. *Let X be a metric space. Assume that X is an ANE. Then, for each subset A of X, there exists a closed neighborhood B of A such that* $\mathrm{cat}_X B = \mathrm{cat}_X A$.

For the proof, see [18, Proposition 5.9].

3. Proof of Theorem

We basically follow the notation and arguments in [9]. Let $N \ge 3$. We denote by $\mathcal{D}^{1,2}(\mathbb{R}^N)$ the Hilbert space $\{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in (L^2(\mathbb{R}^N))^N\}$ with respect to the inner product $\int_{\mathbb{R}^N} \nabla u \nabla v \, dx$. For each nonempty open set $U \subset \mathbb{R}^N$, we denote by $\mathcal{D}_0^{1,2}(U)$ the closure of $C_0^\infty(U)$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. We set $\overline{\mu} = (N-2)^2/4$. We recall Hardy's inequality:

$$\overline{\mu} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx \le \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \quad \text{for each } u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

For each nonempty open set $U \subset \mathbb{R}^N$ and $\mu \in [0, \overline{\mu})$, we define

$$I^{(U,\mu)}(u) = \int_U \left(\frac{1}{2}|\nabla u|^2 - \frac{\mu}{2}\frac{|u|^2}{|x|^2} - \frac{1}{2^*}|u^+|^{2^*} \right) dx \quad \text{for } u \in \mathcal{D}_0^{1,2}(U).$$

We know that $I^{(U,\mu)}$ is well defined by Hardy's inequality and that each nontrivial critical point of $I^{(\Omega,\mu)}$ corresponds to a positive solution of (1.1). For each $\mu \in [0, \overline{\mu})$ and nonempty open set $U \subset \mathbb{R}^N$, we define

$$\mathcal{S}_\mu(U) = \left\{ u \in \mathcal{D}_0^{1,2}(U) \setminus \{0\} : \int_U \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right) dx = \int_U |u^+|^{2^*} dx \right\}$$

and

$$c_\mu = \inf\{ I^{(U,\mu)}(u) : u \in \mathcal{S}_\mu(U) \}.$$

We know that the value c_μ does not depend on a nonempty open set U. For each $\mu \in [0, \overline{\mu})$ and $(z, \varepsilon) \in \mathbb{R}^N \times \mathbb{R}^+$, we set

$$u^{(\mu)}_{(z,\varepsilon)}(x) = \frac{\left(N(N-2)\nu_\mu^2 \varepsilon^{2\nu_\mu} \right)^{\frac{N-2}{4}}}{|x-z|^{1-\nu_\mu} \left(\varepsilon^{2\nu_\mu} + |x-z|^{2\nu_\mu} \right)^{\frac{N-2}{2}}} \quad \text{for } x \in \mathbb{R}^N,$$

where

$$\nu_\mu = \left(1 - \frac{4\mu}{(N-2)^2} \right)^{\frac{1}{2}}.$$

In [17], Terracini showed that, for each $\mu \in (0, \overline{\mu})$ and $\varepsilon > 0$, $u^{(\mu)}_{(0,\varepsilon)}$ is a solution of (1.1) with $\Omega = \mathbb{R}^N$ which satisfies $c_\mu = I^{(\mathbb{R}^N,\mu)}(u^{(\mu)}_{(0,\varepsilon)})$. It is also known that, for each $z \in \mathbb{R}^N$ and $\varepsilon > 0$, $u^{(0)}_{(z,\varepsilon)}$ is a solution of (1.1) with $\Omega = \mathbb{R}^N$ and $\mu = 0$ which satisfies $c_0 = I^{(\mathbb{R}^N,0)}(u^{(0)}_{(z,\varepsilon)})$.

Now, we show the outline of the proof of Theorem. Let $\mu_0 \in (0, \overline{\mu})$ which will be precisely chosen later, and let $\mu \in (0, \mu_0)$. Then we may assume that there exist $a_0, b_0 \in (c_\mu, c_0)$ such that $a_0 < b_0$ and

$$\{ u \in \mathcal{S}_\mu(\Omega) : I^{(\Omega,\mu)}(u) \in (-\infty, a_0] \cup [b_0, c_0),\ \nabla I^{(\Omega,\mu)}(u) = 0 \} = \emptyset;$$

otherwise there is nothing to prove. Let $a \in (c_\mu, a_0)$ and $b \in (b_0, c_0)$, which will be also precisely chosen later. We set

$$X = \{ u \in \mathcal{S}_\mu(\Omega) : I^{(\Omega,\mu)}(u) < b \} \quad \text{and} \quad Y = \{ u \in \mathcal{S}_\mu(\Omega) : I^{(\Omega,\mu)}(u) \le a \}.$$

For each $j \in \mathbb{N}$, we define

$$\mathcal{A}_j = \{A \subset X : A \supset Y, \operatorname{cat}_{X,Y} A \geq j\};$$

$$\mathfrak{c}_j = \begin{cases} \inf_{A \in \mathcal{A}_j} \sup_{u \in A} I^{(\Omega,\mu)}(u) & \text{if } \mathcal{A}_j \neq \emptyset, \\ \infty & \text{if } \mathcal{A}_j = \emptyset. \end{cases}$$

By $c_\mu < a < a_0$, we have $a < \mathfrak{c}_j$ for each $j \in \mathbb{N}$. We also define

$$K_c = \{u \in \mathcal{S}_\mu(\Omega) : \nabla I^{(\Omega,\mu)}(u) = 0,\ I^{(\Omega,\mu)}(u) = c\} \quad \text{for each } c \in \mathbb{R}.$$

We will show the following lemma and propositions:

Proposition 1. *Let $k \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ such that*

$$(a <)\mathfrak{c} \equiv \mathfrak{c}_k = \cdots = \mathfrak{c}_{k+m} < b.$$

Then $\operatorname{cat}_X K_{\mathfrak{c}} \geq m + 1$.

Lemma 3. *If $\operatorname{cat}_{X,Y} X \geq j$, then $\mathfrak{c}_j < b$.*

Proposition 2. $\operatorname{cat}_{X,Y} X \geq \operatorname{cat} \Omega - 1$.

Using these lemma and propositions, we can give the proof of our theorem as follows.

Proof of Theorem. We set $j = \operatorname{cat} \Omega - 1$. By Lemma 3 and Proposition 2, we have

$$(a <)\mathfrak{c}_1 \leq \mathfrak{c}_2 \leq \cdots \leq \mathfrak{c}_j < b.$$

By Proposition 1, we can infer that $I^{(\Omega,\mu)}$ has at least j critical points in X. This completes the proof. □

Now, we come back to the detailed proof. We choose $\mu' \in (0, \overline{\mu})$ such that $2c_\mu > c_0$ for each $\mu \in (0, \mu')$. By [9, Lemma 2.2], we can infer the following.

Lemma 4. *Let $\mu \in (0, \mu')$. Let $\{v_n\} \subset \mathcal{S}_\mu(\Omega)$ such that $I^{(\Omega,\mu)}(v_n) \to c < c_0$ and $\nabla I^{(\Omega,\mu)}(v_n) \to 0$. Then one of the following holds:*

(i) *there exist a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ and a solution $v \in \mathcal{S}_\mu(\Omega)$ of* (1.1) *such that $\|v_{n_i} - v\| \to 0$;*

(ii) *there exist a subsequence of $\{v_{n_i}\}$ of $\{v_n\}$, $\{\varepsilon_i\} \subset (0, \infty)$ and $\{x_i\} \subset \mathbb{R}^N$ such that $|x_i| \to 0$, $\varepsilon_i \to 0$ and $\|v_{n_i} - u^{(\mu)}_{(x_i,\varepsilon_i)}\| \to 0$.*

Since $\{v \in \mathcal{S}_\mu(\Omega) : I^{(\Omega,\mu)}(v) = c_\mu\} = \emptyset$, by the lemma above and Ekeland's variational principle, we have the following.

Lemma 5. *Let $\mu \in (0, \mu')$. Then the following hold:*

(i) $I^{(\Omega,\mu)}$ *satisfies* $(\mathrm{PS})_c$ *for each* $c \in (c_\mu, c_0)$;

(ii) *for each* $\{v_n\} \subset \mathcal{S}_\mu(\Omega)$ *with* $I^{(\Omega,\mu)}(v_n) \to c_\mu$, *there exist* $\{v_{n_i}\} \subset \{v_n\}$, $\{\varepsilon_i\} \subset \mathbb{R}^+$ *and* $\{x_i\} \subset \mathbb{R}^N$ *such that* $|x_i| \to 0$, $\varepsilon_i \to 0$ *and* $\|v_{n_i} - u^{(\mu)}_{(x_i,\varepsilon_i)}\| \to 0$.

Here, we give the proofs of Lemma 3 and Proposition 1.

Proof of Lemma 3. It is sufficient to show that there exists $Z \in \mathcal{A}_j$ such that $\sup_{u\in Z} I^{(\Omega,\mu)}(u) < b$. By the fact that $K_b = \emptyset$ and Lemma 5, we can infer that there exists $\eta \in C([0,1]\times X, X)$ such that $\sup_{u\in\eta(1,X)} I^{(\Omega,\mu)}(u) < b$. Since we have

$$\operatorname{cat}_{X,Y} X \le \operatorname{cat}_{X,Y} \eta(1,X) \le \operatorname{cat}_{X,Y} X$$

by Lemma 1, we obtain $\operatorname{cat}_{X,Y} \eta(1,X) \ge j$. This completes the proof. □

We note that X is an ANE by [4, Proposition 27.6]. For the sake of completeness, we remark that ANE is called ANR in [4].

Proof of Proposition 1. Suppose that the conclusion does not hold; i.e., $\operatorname{cat}_X K_{\mathfrak{c}} \le m$. By Lemma 2, there exists a closed neighborhood B of $K_{\mathfrak{c}}$ in X such that $\operatorname{cat}_X K_{\mathfrak{c}} = \operatorname{cat}_X B$. Choose small $\varepsilon > 0$ and $A \in \mathcal{A}_{k+m}$ such that $\sup_{u\in A} I^{(\Omega,\mu)}(u) \le c + \varepsilon$. By $(\overline{A \setminus B}) \cap K_{\mathfrak{c}} = \emptyset$ and Lemma 5, there exists $\eta \in C([0,1] \times X, X)$ which satisfies

$$\eta(t,Y) \subset Y \quad \text{for all } t \in [0,1] \qquad \text{and} \qquad I^{(\Omega,\mu)}(\eta(1, A \setminus B)) \le \mathfrak{c} - \varepsilon.$$

By Lemma 1, we obtain

$$\begin{aligned} k + m &\le \operatorname{cat}_{X,Y} A \le \operatorname{cat}_{X,Y} (A \setminus B) + \operatorname{cat}_X B \le \operatorname{cat}_{X,Y} \eta(1, A\setminus B) + m \\ &\le \operatorname{cat}_{X,Y} \{u \in X : I^{(\Omega,\mu)}(u) \le \mathfrak{c} - \varepsilon\} + m \le k - 1 + m, \end{aligned}$$

which is a contradiction. Thus we have shown $\operatorname{cat}_X K_{\mathfrak{c}} \ge m + 1$. □

In the rest of this paper, we show how to choose μ_0, a and b, and we give the proof of Proposition 2.

We choose a positive constant d such that Ω_d^+ is homotopy equivalent to Ω and $\overline{\Omega_d^-}$ is a deformation retract of Ω_d^+, where $\Omega_d^+ = \{x \in \mathbb{R}^N : d(x,\Omega) < d\}$ and $\Omega_d^- = \{x \in \Omega : d(x, \partial\Omega) > d\}$. We note that

$$\operatorname{cat}\Omega = \operatorname{cat}\Omega_d^+ = \operatorname{cat}_{\Omega_d^+} \overline{\Omega_d^-}.$$

We define

$$\beta(v) = \frac{\int_\Omega x|\nabla v(x)|^2\,dx}{\int_\Omega |\nabla v(x)|^2\,dx} \quad \text{for each } v \in \mathcal{D}_0^{1,2}(\Omega) \setminus \{0\}.$$

The following is obtained as [9, Lemma 2.4].

Lemma 6. *There exists $\mu_0 \in (0, \mu')$ such that, for each $\mu \in (0, \mu_0)$ and $v \in \mathcal{S}_\mu(\Omega)$ with $I^{(\Omega,\mu)}(v) < c_0$, there holds $\beta(v) \in \Omega_d^+$.*

We fix $\mu_0 > 0$ as in the previous lemma. We choose any $\mu \in (0, \mu_0)$. Since $\mu_0 < \mu'$, we have $c_0 < 2c_\mu$. We fix a radially symmetric C^∞-function $\varphi : \mathbb{R}^N \to [0, 1]$ which satisfies

$$\varphi(x) = 1 \quad \text{for } |x| \le d/3 \quad \text{and} \quad \varphi(x) = 0 \quad \text{for } |x| \ge 2d/3.$$

For each $(z, \varepsilon) \in \overline{\Omega_d^-} \times \mathbb{R}^+$, we set

$$v_{(z,\varepsilon)} = \varphi(\cdot - z) u_{(z,\varepsilon)}^{(0)}.$$

By the estimates due to Brézis and Nirenberg [2], we have

$$\int_\Omega |\nabla v_{(z,\varepsilon)}|^2\, dx = Nc_0 + O(\varepsilon^{N-2}) \left(= S^{N/2} + O(\varepsilon^{N-2})\right),$$
$$\int_\Omega |v_{(z,\varepsilon)}|^{2^*}\, dx = Nc_0 + O(\varepsilon^{N}) \left(= S^{N/2} + O(\varepsilon^{N})\right),$$
$$\int_\Omega |v_{(z,\varepsilon)}|^2\, dx \ge \begin{cases} C_1\varepsilon^2 & \text{if } N \ge 5, \\ C_2\varepsilon^2|\log\varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ C_3\varepsilon + O(\varepsilon^2) & \text{if } N = 3, \end{cases}$$

where S is the best Sobolev constant and each C_i is an appropriate positive constant. For each $(z, \varepsilon) \in \overline{\Omega_d^-} \times \mathbb{R}^+$, there exists unique $t_{(z,\varepsilon)} > 0$ such that

$$w_{(z,\varepsilon)} \equiv t_{(z,\varepsilon)} \varphi(\cdot - z) u_{(z,\varepsilon)}^{(0)} \in \mathcal{S}_\mu(\Omega).$$

Then we have the following which was obtained as [9, Lemma 2.5]. Here, we need the assumption $N \ge 4$.

Lemma 7. *Let $\varepsilon > 0$ be small enough. Then*

$$\sup\{I^{(\Omega,\mu)}(w_{(z,\varepsilon)}) : z \in \overline{\Omega_d^-}\} < c_0.$$

We fix $\varepsilon > 0$ which satisfies the property in the previous lemma. Using this ε, we define $\gamma : \overline{\Omega_d^-} \to X$ by $\gamma(z) = w_{(z,\varepsilon)}$ for each $z \in \overline{\Omega_d^-}$, and we choose $b \in (b_0, c_0)$ such that

$$\sup\{I^{(\Omega,\mu)}(\gamma(z)) : z \in \overline{\Omega_d^-}\} < b < c_0.$$

Since $0 \in \Omega$, we can choose $r_0 > 0$ with $B_{r_0}(0) \subset \Omega$. By Lemma 5, we have the following.

Lemma 8. *If $a \in (c_\lambda, a_0)$ is sufficiently close to c_λ, then*

$$\beta(Y) \subset B_{r_0}(0).$$

In the rest of this paper, we fix a which satisfies the property in the previous lemma. Now, we give the proof of Proposition 2.

Proof of Proposition 2. If $\mathrm{cat}_{X,Y} X = \infty$, there is nothing to prove. So we assume that $n \equiv \mathrm{cat}_{X,Y} X \in \mathbb{N} \cup \{0\}$. Then we can choose closed subsets $A_0, A_1, \dots, A_n$ of X such that

(i) $X \subset \bigcup_{i=0}^n A_i$;
(ii) $A_0 \prec_Y Y$ in X; and
(iii) for each $i = 1, \dots, n$, A_i is contractible in X.

Then $Y \subset A_0$ and there exist $h_i \in C([0,1] \times A_i, X)$ $(i = 0, 1, \dots, n)$ such that

$$\begin{cases} h_0(0,u) = u & \text{for each } u \in A_0, \\ h_0(1,u) \in Y & \text{for each } u \in A_0, \\ h_0(t,u) \in Y & \text{for each } (t,u) \in [0,1] \times Y, \end{cases}$$

and for each $i = 1, \dots, n$,

$$\begin{cases} h_i(0,u) = u & \text{for each } u \in A_i, \\ h_i(1,u) = h_i(1,v) & \text{for each } u, v \in A_i. \end{cases}$$

We set $\widetilde{A}_i = \gamma^{-1}(A_i)$ for each $i = 0, \dots, n$. It is easy to see $\overline{\Omega_d^-} = \widetilde{A}_0 \cup \dots \cup \widetilde{A}_n$. Since each $\widetilde{A}_i$ is a closed subset of $\overline{\Omega_d^-}$, it is also a closed subset of Ω_d^+. For each $i = 1, \dots, n$, we define $g_i \in C([0,1] \times \widetilde{A}_i, \Omega_d^+)$ by

$$g_i(t,z) = \beta(h_i(t, \gamma(z))) \quad \text{for} \quad (t,z) \in [0,1] \times \widetilde{A}_i.$$

Since we have

$$\begin{cases} g_i(0,z) = \beta(h_i(0,\gamma(z))) = \beta(\gamma(z)) = z & \text{for each } z \in \widetilde{A}_i, \\ g_i(1,z) = \beta(h_i(1,\gamma(z))) = \beta(h_i(1,\gamma(y))) = g_i(1,y) & \text{for each } z, y \in \widetilde{A}_i, \end{cases}$$

we can find that $\widetilde{A}_i$ is contractible in Ω_d^+. Next, we will show that $\widetilde{A}_0$ is also contractible in Ω_d^+. We define $g_0 \in C([0,1] \times \widetilde{A}_0, \Omega_d^+)$ by

$$g_0(t,z) = \begin{cases} \beta(h_0(2t, \gamma(z))) & \text{if } (t,z) \in [0,1/2] \times \widetilde{A}_0, \\ (2-2t)\beta(h_0(1,\gamma(z))) & \text{if } (t,z) \in [1/2,1] \times \widetilde{A}_0. \end{cases}$$

Since we have

$$\begin{cases} g_0(0,z) = \beta(h_i(0,\gamma(z))) = \beta(\gamma(z)) = z \quad \text{for each } z \in \widetilde{A}_0, \\ g_0(1/2,z) = \beta(h_0(1,\gamma(z))) \subset \beta(Y) \subset B_{r_0}(0) \subset \Omega \quad \text{for each } z \in \widetilde{A}_0, \\ g_0(1,z) = (2-2)\beta(h_0(1,\gamma(z))) = 0 \quad \text{for each } z \in \widetilde{A}_0, \end{cases}$$

we can find that g_0 is well defined and $\widetilde{A}_0$ is contractible in Ω_d^+. Thus we have

$$n+1 \geq \operatorname{cat}_{\Omega_d^+} \overline{\Omega_d^-} = \operatorname{cat} \Omega,$$

which yields

$$\operatorname{cat}_{X,Y} X = n \geq \operatorname{cat} \Omega - 1.$$

This completes the proof. □

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