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TIME DECAY OF SOLUTION FOR THE KDV EQUATION WITH MULTIPLICATIVE SPACE-TIME NOISE

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Dedicated to Professors Tetsuro Miyakawa and Kenji Yajima on the occasion of their 60th birthdays

Abstract. We consider the asymptotic behavior in large time of solutions for the KdV equation with multiplicative space-time noise. Under certain assumptions on the noise, we prove that the L^2 norm of solutions almost surely decays to zero as time goes to infinity. This phenomena is called stabilization by noise.

1. Introduction and theorems

In the present paper, we consider the time decay property of solutions for the KdV equation on a circle with multiplicative space-time noise:

$$du + (\partial_x^3 u + u\partial_x u)dt + u\Phi dW = 0, \quad t > 0, \quad x \in S^1 = \mathbf{R}/\mathbf{Z}, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad x \in S^1, \tag{1.2}$$

where u is a real-valued function. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We assume that each $\mathcal{F}_t$ includes the family $\{F \in \mathcal{F} : P(F) = 0\}$. Let W be a cylindrical Wiener process on $(\Omega, \mathcal{F}, P)$ adapted to $\{\mathcal{F}_t\}$. More precisely, W is expressed as follows.

$$W(t) = \sum_{k=-\infty}^{\infty} \beta_k(t) e^{2\pi i k x},$$

where $\{\beta_k\}_{k\geq 0}$ are mutually independent complex-valued Brownian motions adapted to $\{\mathcal{F}_t\}$, $\beta_0(t) = \overline{\beta_0}(t)$ and $\beta_k = \overline{\beta_{-k}}$ $(k = -1, -2, \cdots)$. Furthermore, Φ is a Hilbert-Schmidt operator from L^2 to L^2 such that $\overline{\Phi v} = \Phi \bar{v}$ for $v \in L^2$. Here and hereafter, we denote the complex conjugate of z by $\bar{z}$ for

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$z \in \mathbf{C}$ and we abbreviate $L^2(S^1)$ to L^2. We denote the inner product of L^2 by $(\cdot,\cdot)$. For $s \in \mathbf{R}$, we define the Sobolev space H^s by

$$H^s = \Big\{ u \in \mathcal{D}'(S^1) : \sum_{k=-\infty}^{\infty} (1+k^2)^s |\hat{u}(k)|^2 < \infty \Big\}$$

with the norm

$$\|u\|_{H^s} = \Big(\sum_{k=-\infty}^{\infty} (1+k^2)^s |\hat{u}(k)|^2 \Big)^{1/2},$$

where $\hat{u}(k)$ denotes the Fourier coefficient of the function u on S^1; that is, $\hat{u}(k) = (u, e^{2\pi i k x})$.

Many mathematicians have been studying the stability of steady solutions or the time decay of solutions for ordinary differential equations with stochastic perturbation from the viewpoints of dynamical system and control theory (see, e.g., [1], [12]-[14] for the multiplicative noise case and [18] for the additive noise case). For example, we consider the following stochastic ordinary differential equation

$$\begin{aligned} &du = audt + \sigma u d\beta, \qquad t > 0, \\ &u(0) = u_0, \end{aligned}$$

where a, $\sigma \in \mathbf{R}$ and β is a real Brownian motion. Itô's formula yields

$$u(t) = u_0 \exp\big((a - \tfrac{1}{2}\sigma^2)t + \sigma\beta(t)\big).$$

Because Brownian motion $\beta(t)$ grows at most like $\sqrt{2t \log\log t}$ almost surely as $t \to \infty$, $u(t)$ tends to zero almost surely in large time if $2a < \sigma^2$. This simple observation suggests that a multiplicative noise might make the zero solution asymptotically stable, if it is non-asymptotically stable without stochastic perturbation, where we mean by non-asymptotical stability we mean that a steady solution in question such as the zero solution is not asymptotically stable but Lyapunov stable. This phenomenon is often called stabilization by noise. But it is not always the case with systems of stochastic ordinary differential equations unlike the single stochastic differential equation (see Examples 3 and 4 on pages 177-179 in [12]). Actually, in [12] it is shown that a non-asymptotically stable deterministic system may become unstable when driven by white noise. There are many stochastic nonlinear partial differential equations whose zero solutions are non-asymptotically stable. The KdV equation with multiplicative noise is one of the important examples among those stochastic equations. The stochastic KdV equation appears in various fields of mathematical physics (see, e.g., [15]). So, it seems very interesting to investigate what kind of influence a multiplicative noise has on the asymptotic behavior of solutions for the KdV equation. In this paper, we present sufficient conditions on the operator Φ, which ensures that the solution of (1.1)-(1.2) decays to zero almost surely as $t \to \infty$. The operator $\Phi\Phi^*$ is called a covariance operator, which denotes the covariance of the noise ΦW. It is natural to express the sufficient condition for the time decay of solutions in terms of Φ.

Let $U(t) = e^{-t\partial_x^3}$. We first state the following theorem concerning the global existence of mild solutions for (1.1)-(1.2).

Theorem 1.1. *Let $s > 5/2$. Assume that Φ is a bounded linear operator from L^2 to H^s. Then, for any $u_0 \in H^2$, there exists a unique global mild solution u of* (1.1)-(1.2) *such that, for all $T > 0$,*

$$u \in L^2(\Omega; L^\infty(0,T;H^2)) \cap L^6(\Omega; C([0,T];L^2)), \tag{1.3}$$

$$\begin{aligned} u(t) = U(t)u_0 &- \int_0^t U(t-\tau)(u\partial_x u)(\tau)\, d\tau \\ &- \int_0^t U(t-\tau)u(\tau)\Phi dW(\tau), \quad t \in [0,T], \end{aligned} \tag{1.4}$$

where the third term on the right-hand side of (1.4) *is interpreted as the Itô integral.*

Theorem 1.1 follows immediately from the standard energy method together with a truncation technique and a contraction argument (see, e.g., [2]). Here, instead of the proof, we mention several remarks on Theorem 1.1 and an example of an operator Φ satisfying all the assumptions in Theorem 1.1.

Remark 1.1 (i) We note that if $s > 5/2$ and Φ is a bounded linear operator from L^2 to H^s, then Φ is Hilbert-Schmidt from L^2 to H^2. In that case, the Itô integral on the right-hand side of (1.4) makes sense in H^2. The Hilbert-Schmidt property of the operator Φ is required for the mathematically rigorous formulation of the Itô integral in an infinite-dimensional space (see [10, Chapter 4]). This requirement makes it difficult to handle physically feasible noises such as space-time white noise in partial differential equations.

(ii) The H^2 regularity of the solution might be too strong. In fact, it is known that (1.1)-(1.2) without stochastic perturbation has a unique global solution in $H^{-1/2}$ (see [5] and [9]). When a stochastic perturbation is additive noise, the Fourier restriction method introduced in [5] can be applied to the stochastic case (see [3] and [4]). But it still remains open whether the Fourier restriction method is applicable to the stochastic problem with multiplicative noise on S^1 (for the $\mathbf{R}$ case, see [2]).

Example 1.1. For $v = \sum_{k=-\infty}^{\infty} \hat{v}(k)e^{2\pi ikx}$, we define an operator Φ as follows.

$$\Phi v = \hat{v}(0) + \sum_{k\in\mathbf{Z},k\neq 0} \frac{\hat{v}(k)}{i|k|^{s-1}k} e^{2\pi ikx}.$$

In this case, we easily see that if $s > 5/2$, Φ is a bounded linear operator from L^2 to H^s. When Φ is the identity operator, W is called the space-time white noise, which is the most important noise. But Theorem 1.1 can not cover the space-time white noise.

We express the operator Φ in terms of the integral kernel as follows.

$$\Phi v = \int_{S^1} K(x,y)v(y)\ dy \qquad K \in L^2(S^1 \times S^1).$$

It is always possible to express Φ in this form, because Φ is Hilbert-Schmidt. Then, we have the following theorem concerning the time decay of the solution given by Theorem 1.1.

Theorem 1.2. *Let Φ and u_0 satisfy the same assumptions as in Theorem* 1.1. *In addition, we assume that the operator Φ satisfies the following two properties.*

$$\sum_{k=-\infty}^{\infty} \Big[\int_{S^1} \mathrm{Re}\ \big(K(x,\cdot), e^{2\pi ik\cdot}\big)|v|^2\ dx\Big]^2 \geq \alpha^2 \|v\|_{L^2}^4, \quad v \in L^2, \tag{1.5}$$

$$2\alpha^2 > \beta^2, \tag{1.6}$$

where α is a positive constant and

$$\beta = \Big[\sup_{x\in S^1} \int_{S^1} |K(x,y)|^2\ dy\Big]^{1/2}.$$

Let $u(t,x;\omega)$ be a solution of (1.1)-(1.2) *given by Theorem* 1.1. *Then,*

$$\|u(t)\|_{L^2} \longrightarrow 0 \quad (t \to 0) \quad a.s.\ \omega \in \Omega.$$

Here, we give an example of an operator Φ satisfying (1.5) and (1.6).

Example 1.2. Let a, $b \in \mathbf{R}$ and $s > 5/2$. For $v = \sum_{k=-\infty}^{\infty} \hat{v}(k)e^{2\pi ikx}$, we define an operator Φ as follows.

$$\Phi v = a\hat{v}(0) + b \sum k \in \mathbf{Z},\ k \neq 0 \frac{\hat{v}(k)}{i|k|^{s-1}k} e^{2\pi ikx}.$$

In this case, simple calculations yield

$$K(x,y) = K(y,x) = a + b \sum_{k\in\mathbf{Z},\ k\neq 0} \frac{e^{2\pi ikx}}{i|k|^{s-1}k} e^{2\pi ikx},$$

$$\sum_{k=-\infty}^{\infty} \Big[\int_{S^1} \mathrm{Re}\, \big(K(x,\cdot), e^{2\pi ik\cdot}\big)|v|^2\ dx\Big]^2$$

$$= b^2 \sum_{k\neq 0} \frac{1}{|k|^{2s}} \Big(\int_{S^1} |v|^2 \sin 2\pi kx\ dx\Big)^2 + a^2\|v\|_{L^2}^4 \geq a^2 \|v\|_{L^2}^4,$$

$$\beta = \sup_{x\in S^1} \Big[\int_{S^1} |K(x,y)|^2\ dy\Big] = a^2 + b^2 \sum_{k\neq 0} \frac{1}{|k|^{2s}}.$$

Therefore, if we take $\alpha = a$ and the constants a and b satisfy

$$a^2 > b^2 \sum_{k\neq 0} \frac{1}{|k|^{2s}},$$

then the relation $2\alpha^2 > \beta^2$ holds.

Remark 1.3. A simple sufficient condition for the relation $2\alpha^2 > \beta^2$ is that the strength of the zero Fourier mode of the noise is larger than that of the rest.

Remark 1.4. It follows by Itô's formula and (1.1)-(1.2) that

$$E\big[\|u(t)\|_{L^2}^2\big] = E\big[\|u_0\|_{L^2}^2\big] + \int_0^t \sum_{k\in\mathbf{Z}} E\big[\|u(\tau)\Phi e^{2\pi ikx}\|_{L^2}^2\big]\ d\tau, \qquad t > 0,$$

where E denotes the expectation with respect to the probability measure P on $(\Omega, \mathcal{F})$. Accordingly, the second moment of the L^2 norm of the solution is increasing with respect to the time variable t, while Theorem 1.2 shows that the L^2 norm of the solution itself decays to zero almost surely as $t \to \infty$.

There are many papers which study sufficient conditions for solutions of stochastic partial differential equations to decay to zero almost surely in large time (see, e.g., [6], [7], [14] and [17]). We now introduce two notions of "asymptotic stability" to explain the results in the previous papers. We say that the zero solution is mean square asymptotically stable if all solutions $u(t)$ satisfy

$$E\big[\|u(t)\|_{L^2}^2\big] \leq CE\big[\|u_0\|_{L^2}^2\big]e^{-\gamma t}, \qquad t > 0 \qquad (MS)$$

for some C, $\gamma > 0$, where u_0 denotes the initial data of $u(t)$, and that the zero solution is almost sure pseudo-asymptotically stable if all solutions $u(t)$

satisfy

$$\|u(t)\|_{L^2}^2 \longrightarrow 0 \quad (t \to 0) \quad a.s. \tag{AS}$$

Here, we note that (AS) does not imply the usual asymptotic stability, because we do not impose the Lyapunov stability on the zero solution. This is the reason why the property (AS) is called "pseudo-asymptotic stability". Haussman [14] and Caraballo and Liu [6] show properties (MS) and (AS) for linear and nonlinear parabolic equations with multiplicative noise under certain conditions (for recent results on the stabilization by noise in reaction-diffusion equations, see [8] and the references therein). The strategy of proofs in [14] and [6] is based on the argument by Kozin [16]; that is, they first prove (MS) and next derive (AS) from (MS) by using probabilistic techniques. Consequently, the proofs in [14] and [6] are not applicable to conservative systems such as the KdV equation. On the other hand, in [17], Liu gives a new proof using the exponential martingale inequality to show (AS) without (MS). But, in [17], the assumptions on stochastic terms seem much too restrictive (for example, see (2) in Theorems 2.1 and 2.2 of [17]). In [7], Caraballo, Liu and Mao develop the argument combining the Lyapunov function of logarithmic type and the exponential martingale inequality to prove (AS) for a larger class of stochastic differential equations than in [17]. But they only treat a time-dependent and space-independent noise in [7]. In the present paper, we employ the argument by Caraballo, Liu and Mao [7] to investigate what a sufficient condition on the operator Φ for (AS) is like in the case of the KdV equation with multiplicative space-time dependent noise.

The plan of this paper is as follows. In Section 2, we mention several lemmas needed for the proof of Theorem 1.2. In Section 3, we prove Theorem 1.2 by using lemmas in Section 2.

2. LEMMAS

In this section, for the reader unfamiliar with the probability theory, we summarize one measure theoretic lemma and two probability theoretic lemmas, which will be useful for the proof of Theorem 1.2. We omit the proofs of those lemmas.

We begin with the well-known Borel-Cantelli lemma.

Lemma 2.1. *Let $\{A_n\}$ be a sequence of measurable sets of Ω. Assume that*

$$\sum_{n=1}^{\infty} P(A_n) < \infty.$$

Then, we have

$$P\Big(\lim_{m\to\infty}\bigcup_{n=m}^{\infty}A_n\Big)=0.$$

Or equivalently,

$$P\big(\{\omega\in\Omega:\omega\ belongs\ to\ infinitely\ many\ A_n\}\big)=0.$$

We next state the exponential martingale inequality, which is a variant of Doob's inequality for the exponential martingale. Let n be a positive integer. Let $\mathcal{L}^2_{exp}(0,\infty)$ denote the set of all $\mathbf{R}^n$-valued progressively measurable $\{\mathcal{F}_t\}$-adapted processes $f(t)=(f_1(t),\cdots,f_n(t))$ such that, for any b, $T>0$,

$$E\Big[\exp\Big(\frac{b^2}{2}\int_0^T\sum_{k=1}^n|f_k(t)|^2\ dt\Big)\Big]<\infty. \tag{2.1}$$

Lemma 2.2. *Assume that* $f=(f_1,\cdots,f_n)\in\mathcal{L}^2_{exp}(0,\infty)$ *and that* $\{\beta_k\}_{k=1}^n$ *are mutually independent real-valued Brownian motions adapted to* $\{\mathcal{F}_t\}$. *Let* a, b *and* T *be any positive numbers. Then,*

$$P\Big(\Big\{\sup_{0\le t\le T}\Big[\sum_{k=1}^n\int_0^t f_k(\lambda)\ d\beta_k(\lambda)-\frac{b}{2}\sum_{k=1}^n\int_0^t|f_k(\lambda)|^2\ d\lambda\Big]>a\Big\}\Big)\le e^{-ab}.$$

For the proof of Lemma 2.2, see [10, Theorem 6.5 on page 87 and Theorem 7.5 on page 93]. In [10], Lemma 2.2 is proved under a weaker assumption than (2.1).

Remark 2.1. The condition (2.1) is called the Novikov condition, which ensures that the following process is martingale:

$$\exp\Big[b\sum_{k=1}^n\int_0^t f_k(\lambda)\ d\beta_k(\lambda)-\frac{b^2}{2}\sum_{k=1}^n\int_0^t|f_k(\lambda)|^2\ d\lambda\Big],$$

where b is a positive constant. If $n=\infty$, we need to replace (2.1) by the following condition on the convergence of the series:

$$E\Big[\exp\Big(\frac{b^2}{2}\int_0^T\sum_{k=1}^\infty|f_k(t)|^2\ dt\Big)\Big]<\infty \tag{2.2}$$

for any b, $T>0$. When we apply Lemma 2.2 to our problem, this convergence property (2.2) holds, because the operator Φ is Hilbert-Schmidt.

We finally state Itô's formula. Let H, U_0, and U_1 be separable Hilbert spaces with continuous embedding $Q:U_0\to U_1$. We assume that Q is a bounded nonnegative self-adjoint operator on U_0. Let Φ be a Hilbert-Schmidt operator from U_1 to H and let W be a cylindrical Q-Wiener process with incremental covariance operator Q. We assume that $X(0)$ is an $\mathcal{F}_0$-measurable H-valued random variable and that $f(t)$ is an H-valued predictable process which is Bochner integrable on $[0,T]$ almost surely, where T is a positive constant. We put

$$X(t) = X(0) + \int_0^t f(s)\ ds + \int_0^t \Phi dW(s), \qquad t \in [0,T].$$

Furthermore, we assume that a function $F : [0,T] \times H \to \mathbf{R}$ and its partial derivatives F_t, F_x, F_{xx} are uniformly continuous on bounded subsets of $[0,T] \times H$.

Lemma 2.3. *Under the above assumptions, for all $t \in [0,T]$, we have*

$$\begin{aligned} F(t, X(t)) = F(0, X(0)) &+ \int_0^t \langle F_x(s, X(s)), \Phi dW(s)\rangle \\ &+ \int_0^t \Big\{ F_t(s, X(s)) + \langle F_x(s, X(s)), f(s)\rangle \\ &+ \frac{1}{2}\mathrm{Tr}\Big[F_{xx}(s, X(s))(\Phi Q^{1/2})(\Phi Q^{1/2})^*\Big]\Big\}\ ds \quad a.s. \end{aligned}$$

For the infinite-dimensional version of Itô's formula, see, e.g., [10, Theorem 4.17 on page 105]).

Remark 2.2. (i) Let H be a separable Hilbert space and $\mathcal{B}(H)$ be the Borel σ-field of H. The mapping $f(t) : [0,T] \times \Omega \to H$ is said to be progressively measurable if f is $\mathcal{B}([0,t]) \times \mathcal{F}_t - \mathcal{B}(H)$ measurable for any $t \in [0,T]$. The predictable property is more restrictive than the progressible measurability. But it can be shown that the solution given by Theorem 1.1 has a predictable version. For details, see, e.g., [10, Section 3.3 in Chapter 3 and Section 6.1 in Chapter 6]. In fact, Itô's formula, that is, Lemma 2.3, holds if f is progressively measurable.

(ii) For the proof of Theorem 1.2 described in Section 3 below, we use Lemma 2.3 in the following setting (see (3.1) in Section 3 and a remark after it).

$$F = \log(\|u(t)\|_{L^2}^2 + \varepsilon) \quad (\varepsilon > 0), \qquad H = U_0 = U_1 = L^2,$$

and Q is a canonical injection from L^2 into L^2, that is, the identity mapping.

3. Proof of Theorem 1.2

In this section, we employ the argument by Caraballo, Liu and Mao [7] to prove Theorem 1.2.

Le u be a solution of (1.1)-(1.2) given by Theorem 1.1. We put

$$V(t;\omega) = \|u(t;\omega)\|_{L^2}^2.$$

We choose $\log V(t)$ as a Lyapunov functional. Let $\omega \in \Omega$ be fixed. If there exists $t_0 \geq 0$ such that $V(t_0;\omega) = 0$, we have, by the pathwise uniqueness of the solution,

$$u(t;\omega) = 0 \qquad (t > t_0).$$

In this case, the proof is complete. We next define the first hitting time $\sigma(\omega)$ by

$$\sigma(\omega) = \inf\{t > 0 : V(t;\omega) = 0\}.$$

Here we interpret $\sigma = \infty$ if the set $\{t > 0 : V(t;\omega) = 0\}$ is empty. We set

$$D(t) = \{\omega \in \Omega : \sigma(\omega) > t\}$$

for $t > 0$ and let $\omega \in D(t)$. Then, Itô's formula (Lemma 2.3) yields

$$\begin{aligned} \log V(t) = \log V(0) &+ \sum_{k=-\infty}^{\infty} \Big[2 \int_0^t \frac{\mathrm{Re}\big(u\Phi e^{2\pi ikx}, u\big)}{V}\, d\beta_k(\tau) \\ &- 2 \int_0^t \frac{\big\{\mathrm{Re}\ \big(u\Phi e^{2\pi ikx}, u\big)\big\}^2}{V^2}\, d\tau + \int_0^t \frac{\big(u\Phi e^{2\pi ikx}, u\Phi e^{2\pi ikx}\big)}{V}\, d\tau \Big], \\ &a.s.\ \omega \in D(t), \quad t > 0. \end{aligned} \tag{3.1}$$

This computation can be justified if we consider $V_\varepsilon = V + \varepsilon$ for $\varepsilon > 0$ and pass to the limit as $\varepsilon \to 0^+$. In order to estimate the first term in the summation on the right-hand side of (3.1), we use the exponential martingale inequality. It is easy to verify the Novikov condition (2.2), since the Hilbert-Schmidt property of the operator Φ from L^2 to H^2 implies that

$$\sum_{k=-\infty}^{\infty} \|\Phi e^{2\pi ikx}\|_{L^\infty}^2 \leq C \sum_{k=-\infty}^{\infty} \|\Phi e^{2\pi ikx}\|_{H^2}^2 < \infty$$

and Φ is independent of ω. For a progressively measurable $\{\mathcal{F}_t\}$-adapted process $f(t)$, we put

$$\big[f(t;\omega)\big]_{D(t)} = \begin{cases} f(t;\omega), & \omega \in D(t), \quad t \geq 0, \\ 0, & \omega \notin D(t), \quad t \geq 0. \end{cases}$$

Thus, by the exponential martingale inequality (Lemma 2.2) and the non-increasing property in t of $D(t)$, we have, for any a, b, $T>0$,

$$P\Big(\Big\{\omega:\sup_{0\le t\le T}\Big[\sum_{k=-\infty}^{\infty}\Big(\int_0^t\Big[\frac{\mathrm{Re}\,\big(u\Phi e^{2\pi ikx},u\big)}{V}\Big]_{D(\tau)}\,d\beta_k(\tau)$$
$$-\frac{b}{2}\int_0^t\Big[\frac{\mathrm{Re}\,\big(u\Phi e^{2\pi ikx},u\big)}{V}\Big]^2_{D(\tau)}\,d\tau\Big)\Big]>\frac{a}{2}\Big\}\Big)\le e^{-\frac{1}{2}ab}.$$

Now let θ be a positive constant with $0<\theta<1$ to be determined later. We take

$$a=2\theta^{-1}\log m,\qquad b=2\theta,\qquad T=m,$$

for a positive integer m. Then, the Borel-Cantelli lemma (Lemma 2.1) implies that for almost sure $\omega\in\Omega$, there exists a positive integer $m_0=m_0(\omega)$ such that

$$\sum_{k=-\infty}^{\infty}\int_0^t\Big[\frac{\mathrm{Re}\,\big(u\Phi e^{2\pi ikx},u\big)}{V}\Big]_{D(\tau)}d\beta_k(\tau) \tag{3.2}$$
$$\le 2\theta^{-1}\log m+\theta\sum_{k=-\infty}^{\infty}\int_0^t\Big[\frac{\mathrm{Re}\,\big(u\Phi e^{2\pi ikx},u\big)}{V}\Big]^2_{D(\tau)}d\tau,\quad 0\le t\le m,\ m\ge m_0.$$

Accordingly, (3.1) and (3.2) yield

$$\log V(t)\le\log V(0)+4\theta^{-1}\log m \tag{3.3}$$
$$-2(1-\theta)\int_0^t\frac{\sum_{k=-\infty}^{\infty}\big\{\mathrm{Re}\,\big(u\Phi e^{2\pi ikx},u\big)\big\}^2}{V^2}\,d\tau$$
$$+\int_0^t\frac{\sum_{k=-\infty}^{\infty}\big(u\Phi e^{2\pi ikx},u\Phi e^{2\pi ikx}\big)}{V}\,d\tau,$$
$$a.s.\ \omega\in D(t),\quad 0\le t\le m,\quad m\ge m_0.$$

On the other hand, it follows from the assumption (1.5) and the definition of β that

$$\sum_{k=-\infty}^{\infty}\big\{\mathrm{Re}(u\Phi e^{2\pi ikx},u)\big\}^2=\sum_{k=-\infty}^{\infty}\Big\{\int_{S^1}\mathrm{Re}\,\big(K(x,\cdot),e^{-2\pi ik\cdot}\big)|u|^2\,dx\Big\}^2\ge\alpha^2V^2,$$
$$\sum_{k=-\infty}^{\infty}\big(u\Phi e^{2\pi ikx},u\Phi e^{2\pi ikx}\big)=\int_{S^1}|u(x)|^2\sum_{k=-\infty}^{\infty}\big|\big(K(x,\cdot),e^{2\pi ik\cdot}\big)\big|^2\,dx\le\beta^2V.$$

Hence, (3.3) yields

$$\log \frac{V(t)}{V(0)} \leq 4\theta^{-1} \log m + \int_0^t \{-2(1-\theta)\alpha^2 + \beta^2\} \, d\tau$$
$$= 4\theta^{-1} \log m + \{\beta^2 - 2(1-\theta)\alpha^2\} t,$$
$$a.s.\ \omega \in D(t), \quad m-1 \leq t \leq m, \quad m \geq m_0.$$

Since $D(t) \subset D(s)$ for $0 < s < t$, we put $D_\infty = \lim_{t\to\infty} D(t)$. We note that $D_\infty \in \mathcal{F}$. Therefore, if we choose $\theta > 0$ so that $2(1-\theta)\alpha^2 > \beta^2$, then we obtain

$$\limsup_{t\to\infty} \left(\frac{1}{t} \log \frac{V(t)}{V(0)} \right) \leq -\{2(1-\theta)\alpha^2 - \beta^2\} < 0, \qquad a.s.\ \omega \in D_\infty.$$

This inequality implies that for $a.s.\ \omega \in D_\infty$, there exist two positive constants $\widetilde{T}$ and $\tilde{\gamma}$ such that

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} \, e^{-\tilde{\gamma} t}, \qquad t \geq \widetilde{T},$$

where $\widetilde{T}$ and $\tilde{\gamma}$ depend on ω. This completes the proof of Theorem 1.2, since the solution $u(t;\omega)$ vanishes in finite time for almost sure $\omega \in \Omega \setminus D_\infty$.

Remark 3.1. Avoiding (t, ω) for which $V(t;\omega) = 0$, we have proceeded with the proof of Theorem 1.2 above. We do not know whether the pathwise solution $u(t;\omega)$ of (1.1)-(1.2) with $u_0(x;\omega) \neq 0$ will never vanish in finite time. The pathwise uniqueness of solutions cannot answer the question of whether the pathwise solution starting from non-zero initial data will ever keep its L^2 norm positive. If we had the backward pathwise uniqueness, we could affirmatively answer this question. In the case of the deterministic KdV equation, the initial-value problem can be solved backward and so the backward uniqueness is obvious. But it does not seem very clear in the case of the stochastic KdV equation.

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REFERENCES

[1] L. Arnold, H. Crauel, and V. Wihstutz, *Stabilization of linear systems by noise,* SIAM J. Control Optim. , 21 (1983), 451–461.

[2] A. de Bouard and A. Debussche, *The Korteweg-de Vries equation with multiplicative homogeneous noise,* in "Stochastic Differential Equations: Theory and Applications," edits. P.H. Baxendale and S.V. Lototsky, pp. 113–133,World Scientific (2007).

[3] A. de Bouard, A. Debussche, and Y. Tsutsumi, *White noise driven Korteweg-de Vries equation,* J. Funct. Anal. , 169 (1999), 532–558.

[4] A. de Bouard, A. Debussche, and Y. Tsutsumi, *Periodic solutions of the Korteweg-de Vries equation driven by white noise,* SIAM J. Math. Anal. , 36 (2004/2005), 815–855.

[5] J. Bourgain, *Fourier transformation restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations II,* Geom. Funct. Anal. , 3 (1993), 209–262..

[6] T. Caraballo and K. Liu, *On exponential stability criteria of stochastic partial differential equations,* Stoch. Proc. Appl. , 83 (1999), 289–301.

[7] T. Caraballo, K. Liu, and X. Mao, *On stabilization of partial differential equations by noise,* Nagoya Math. J. , 161 (2001), 155–170.

[8] S. Cerrai, *Stabilization by noise for a class of stochastic reaction-diffusion equations,* Probab. Theory Related Fields, 133 (2005), 190–214.

[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$*,* J. Amer. Math. Soc. , 16 (2003), 705–749.

[10] G. Da Prato and J. Zabczyk, "Stochastic Equations in Infinite Dimensions," Cambridge University Press, Cambridge (1992).

[11] A. Friedman, "Stochastic Differential Equations and Applications," Vol. 1 Academic Press, New York (1975).

[12] R. Z. Has'minskii, "Stochastic Stability of Differential Equations," Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands (1980).

[13] U. G. Haussmann, *Stabilization of linear systems with multiplicative noise,* Lecture Notes Math. , 294 (1972), 125–130.

[14] U. G. Haussmann, *Asymptotic stability of the linear Ito equation in infinite dimensions,* J. Math. Anal. Appl. , 65 (1978), 219–235.

[15] R. L. Herman, *The stochastic, damped KdV equation,* J. Phys. A: Math. Gen. , 23 (1990), 1063–1084.

[16] F. Kozin, *On almost sure asymptotic sample properties of diffusion process defined by stochastic differential equations,* J. Math. Kyoto Univ. , 4 (1965), 515–528.

[17] K. Liu, *On stability for a class of semilinear stochastic evolution equations,* Stoch. Proc. Appl. , 70 (1997), 219–241.

[18] M. Scheutzow, *Stabilization and destabilization by noise in the plane,* Stoch. Anal. Appl. , 11 (1993), 97–113.