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MULTIPLE SOLUTIONS FOR OPERATOR EQUATIONS INVOLVING DUALITY MAPPINGS ON ORLICZ-SOBOLEV SPACES

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Abstract. Let X be a real reflexive, smooth and separable Banach space having the Kadeč-Klee property and compactly imbedded in the real Banach space Y and let $G : Y \to \mathbb{R}$ be a differentiable functional. By using the "fountain theorem" (Bartsch [3]), we will study the multiplicity of solutions for the operator equation

$$J_\varphi u = G'(u),$$

where J_φ is the duality mapping on X, corresponding to the gauge function φ. Equations having the above form with J_φ a duality mapping on Orlicz-Sobolev spaces are considered as applications. As particular cases of the latter results, some multiplicity results concerning duality mappings on Sobolev spaces are derived.

1. Introduction

This paper is concerned with multiplicity results for equations of the type

$$J_\varphi u = G'(u), \tag{1.1}$$

where

(i) X is a real reflexive, smooth and separable Banach space having the Kadeč-Klee property and compactly imbedded in the real Banach space Y;
(ii) $J_\varphi : X \to X^*$ is a duality mapping corresponding to the gauge function φ (see Definition 1, below);

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(iii) $G' : Y \to Y^*$ is the differential of the functional $G : Y \to \mathbb{R}$.

As usual, X^* (respectively Y^*) denotes the dual space of X (respectively Y) and $\langle \cdot, \cdot \rangle_{X,X^*}$ (respectively $\langle \cdot, \cdot \rangle_{Y,Y^*}$) denotes the duality pairing between X^* and X (respectively Y^* and Y).

Often, we shall omit indicating the spaces in duality and, simply, we shall write $\langle \cdot, \cdot \rangle$. Our approach is a variational one, the so called "fountain theorem" (Bartsch [3], see also Willem [19]) being the basic ingredient which is used.

Equations having the form (1.1) with J_φ a duality mapping on Orlicz-Sobolev spaces are considered as applications. As particular cases of these results, some multiplicity results concerning duality mappings on Sobolev spaces are derived.

More particularly, these results apply to many differential operators which are, in fact, duality mappings on some appropriate spaces of functions (for example, if Δ_p, $1 < p < \infty$, is the so called p-Laplacian, then $-\Delta_p$ is the duality mapping on $W_0^{1,p}(\Omega)$ corresponding to the gauge function $\varphi(t) = t^{p-1}$, $t \geq 0$).

2. THE MAIN RESULT

Let X be a real reflexive and separable Banach space. It is well known that there are $E = \{e_1, ..., e_n, ...\} \subset X$ and $F = \{f_1, ..., f_n, ...\} \subset X^*$ such that $X = \overline{Sp(E)}$, $X^* = \overline{Sp(F)}$ and

$$\langle f_i, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For what follows, we shall note

$$X_j = Sp\left(\{e_j\}\right), \, Y_k = \bigoplus_{j=1}^{k} X_j, \, Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{2.1}$$

Theorem 1. *Let X be a real reflexive, smooth and separable Banach space having the Kadeč-Klee property and compactly imbedded in the real Banach space Y. Let $H \in \mathcal{C}^1(X, \mathbb{R})$ be an even functional having the form*

$$H = \Psi - G, \tag{2.2}$$

where

(*i*) *at any $u \in X$,*

$$\Psi(u) = \Phi(\|u\|),$$

with

$$\Phi(t) = \int_0^t \varphi(\xi) d\xi, \ \forall t \geq 0, \tag{2.3}$$

$\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *being a gauge function which satisfies*

$$p^* = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty;$$

(ii) $G : Y \to \mathbb{R}$ *satisfies:*

$(ii)_1$ $G' : Y \to Y^*$ *is demicontinuous;*

$(ii)_2$ *there is a constant* $\theta > p^*$ *such that, at any* $y \in Y$,

$$\left\langle G'(y), y \right\rangle_{Y,Y^*} - \theta G(y) \geq C = const.; \tag{2.4}$$

(iii) *for any* $u \in X$, *with* $\|u\|_X > 1$, *one has*

$$H(u) \geq c_1 \|u\|_X^p - c_2 \|i(u)\|_Y^q - d, \tag{2.5}$$

where i *stands for the compact injection of* X *in* Y *while* $q > p > 0$, $c_1 > 0$, $c_2 > 0$ *and* d *are real constants;*

(iv) *for any* $k \in \mathbb{N}^*$ *and* $u \in Y_k$, *with* $\|u\|_X > 1$, *one has*

$$H(u) \leq c_3 \|u\|_X^r - c_4 \|u\|_X^s + c_5, \tag{2.6}$$

where $s > 0$, $r < s$, $c_4 > 0$, c_3 *and* c_5 *are real constants* (*depending on* k).

Then, the functional H *has a sequence of critical positive values which converges to* $+\infty$.

Before proceeding to the proof of Theorem 1, we list some results we need.

First, we recall that a real Banach space X is said to be *smooth* if it has the following property: for any $x \in X$, $x \neq 0$, there exists a unique $u^*(x) \in X^*$ such that $\langle u^*(x), x \rangle = \|x\|_X$ and $\|u^*(x)\|_{X^*} = 1$. It is well known (see, for instance, Diestel [8], Zeidler [20]) that the smoothness of X is equivalent with the Gâteaux differentiability of the norm. Consequently, if $(X, \|\cdot\|_X)$ is smooth, then, for any $x \in X$, $x \neq 0$, the only element $u^*(x) \in X^*$ with the properties $\langle u^*(x), x \rangle = \|x\|_X$ and $\|u^*(x)\|_{X^*} = 1$ is $u^*(x) = \|\cdot\|'_X (x)$ (where $\|\cdot\|'_X (x)$ denotes the Gâteaux gradient of the $\|\cdot\|_X$-norm at x).

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *gauge* function if φ is continuous and strictly increasing, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.

Definition 1. *If* X *is a real, smooth Banach space and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a gauge function, the duality mapping on* X *corresponding to* φ *is the mapping* $J_\varphi : X \to X^*$ *defined by*

$$J_\varphi 0 = 0, \quad J_\varphi x = \varphi\left(\|x\|_X\right) \|\cdot\|'_X (x), \ \textit{if } x \neq 0. \tag{2.7}$$

The following metric properties are consequences:

$$\|J_\varphi x\|_{X^*} = \varphi\left(\|x\|_X\right), \quad \langle J_\varphi x, x\rangle = \varphi\left(\|x\|_X\right)\|x\|_X, \forall x \in X. \tag{2.8}$$

Definition 2. *Let X be a real Banach space. The operator $T : X \to X^*$ is said to satisfy condition $(S)_2$ if and only if, as $n \to \infty$, the following holds:*

$$x_n \rightharpoonup x, \quad Tx_n \to Tx \text{ implies } x_n \to x.$$

We have denoted by "$\rightharpoonup$" (respectively "$\to$") the convergence in the weak (respectively strong) topology.

With respect to other conditions invoked in the theory of nonlinear operators as, e.g. $(S)_+$, (S), $(S)_0$ and $(S)_1$ (see Zeidler [20, page 583]), condition $(S)_2$ is stated as $(S)_+ \Rightarrow (S) \Rightarrow (S)_0 \Rightarrow (S)_1 \Rightarrow (S)_2$.

Definition 3. *A real Banach space has the Kadeč-Klee property if it is strictly convex and*

$$[\ x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\|\ ] \Rightarrow x_n \to x.$$

Remark 1. Any locally uniformly convex Banach space (in particular, any uniformly convex Banach space) has the Kadeč-Klee property. For the proof, we refer to Diestel [8].

We have the following three propositions. The first one is standard in the theory of monotone operators (see, e.g. Browder [5], Zeidler [20]).

Proposition 1. *Let X be a real, reflexive, and smooth Banach space. Then any duality mapping $J_\varphi : X \to X^*$ is*

a) monotone ($\langle J_\varphi u - J_\varphi v, u - v\rangle \geq 0$, for all $u, v \in X$);

b) demicontinuous ($x_n \to x \Rightarrow J_\varphi x_n \rightharpoonup J_\varphi x$);

c) coercive ($\frac{\langle J_\varphi u, u\rangle}{\|u\|} \to \infty$ as $\|u\| \to \infty$).

Remark 2. According to a well-known surjectivity result due to Browder (see, e.g. Browder [5]), it follows that any duality mapping on a real, reflexive, and smooth Banach space is surjective.

Proposition 2. *If X is a real, smooth Banach space having the Kadeč-Klee property, then any duality mapping $J_\varphi : X \to X^*$ satisfies condition $(S)_2$.*

Proof. One has

$$[x_n \rightharpoonup x \text{ and } J_\varphi x_n \to J_\varphi x] \Rightarrow [x_n \rightharpoonup x \text{ and } \|J_\varphi x_n\| \to \|J_\varphi x\|] \Rightarrow$$
$$\Rightarrow [x_n \rightharpoonup x \text{ and } \varphi(\|x_n\|) \to \varphi(\|x\|)] \Rightarrow$$
$$\Rightarrow [x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\|] \Rightarrow x_n \to x. \qquad \square$$

Remark 3. It may be shown that, under the hypotheses of Proposition 2, J_φ satisfies the stronger condition $(S)_+$

$$\left[u_n \rightharpoonup u \text{ and } \limsup_{n\to\infty} \langle J_\varphi u_n, u_n - u\rangle \le 0\right] \Rightarrow u_n \to u.$$

Proposition 3. *Let X be a real reflexive Banach space. We assume the following:*

$(H)_1$ *The operator $S : X \to X^*$ is monotone:*

$$\langle Su - Sv, u - v\rangle \ge 0 \textit{ for all } u, v \in X,$$

hemicontinuous:

$$\langle S(u + \lambda v), w\rangle \to \langle Su, w\rangle \;\; \textit{as } \lambda \searrow 0 \textit{ for all } u, v, w \in X,$$

and satisfies condition $(S)_2$.

$(H)_2$ *The operator $K : X \to X^*$ is compact.*

Then, any bounded sequence $(u_n) \subset X$ with $Su_n - Ku_n \to 0$ contains a convergent subsequence.

Proof. Passing to corresponding subsequences, we may assume:

$u_n \rightharpoonup u_0$ (by boundedness of (u_n) and reflexivity of X),

$Ku_n \to f$ (by boundedness of (u_n) and compactness of K),

$Su_n \to f$ (since $Su_n - Ku_n \to 0$). Let $u \in X$ be arbitrarily chosen. The operator S being monotone, we have

$$\langle Su - Su_n, u - u_n\rangle \ge 0 \text{ for all } n \in \mathbb{N}$$

and, by making $n \to \infty$, we obtain

$$\langle Su - f, u - u_0\rangle \ge 0.$$

By taking $u = u_0 + \lambda v$, with $\lambda > 0$ and $v \in X$ arbitrarily chosen, we derive that

$$\langle S(u_0 + \lambda v) - f, v\rangle \ge 0.$$

By making $\lambda \searrow 0$ and using the hemicontinuity of S, one obtains

$$\langle Su_0 - f, v\rangle \ge 0, \text{ for all } v \in X,$$

which implies $f = Su_0$. Now, condition $(S)_2$ together with $u_n \rightharpoonup u_0$ and $Su_n \to Su_0$ yields $u_n \to u_0$. □

Remark 4. Concerning the concepts of demicontinuity and hemicontinuity appearing in Proposition 1 and 3 respectively, let us remark that, generally, demicontinuity implies hemicontinuity. A situation when the two concepts are equivalent is described by the following result (see, e.g. Showalter [17]): let X be a real reflexive Banach space and $T : D(T) \subset X \longrightarrow X^*$ be a monotone operator. Then, at any $x \in intD(T)$ one has: T is demicontinuous at x if and only if T is hemicontinuous at x.

In order to state the next corollary, we recall that, if X is a real Banach space and $H \in \mathcal{C}^1(X, \mathbb{R})$, we say that H satisfies the *Palais-Smale condition* on X ((PS)-condition, for short), if any sequence $(u_n) \subset X$ with $(H(u_n))$ bounded and $H'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence.

In what follows, a sequence $(u_n) \subset X$ with $(H(u_n))$ bounded and $H'(u_n) \to 0$ as $n \to \infty$, will be called a *Palais-Smale sequence* ((PS)-sequence, for short) for H.

Corollary 1. *Let X be a real reflexive Banach space compactly imbedded in the real Banach space Y and $H \in \mathcal{C}^1(X, \mathbb{R})$ be such that*

$$H'(u) = Su - Nu, \tag{2.9}$$

where $S : X \longrightarrow X^*$ is monotone, hemicontinuous, satisfies condition $(S)_2$ and $N : Y \longrightarrow Y^*$ is demicontinuous. Assume that any Palais-Smale sequence for H is bounded. Then H satisfies the (PS)-condition.

Proof. Equality (2.9) is understood in the sense of X^*; i.e. $H'(u) = Su - (i^* \circ N \circ i)u$, for all $u \in X$, where $i : X \longrightarrow Y$ stands for the compact injection of X in Y and $i^* : Y^* \longrightarrow X^*$ is its adjoint. Thus, H' having the form $H'(u) = Su - Ku$ with $K = i^* \circ N \circ i : X \longrightarrow X^*$ compact, the result follows by Proposition 3. □

Corollary 2. *Let X be a real, reflexive, and smooth Banach space having the Kadeč-Klee property and compactly imbedded in the real Banach space Y. Let $H \in \mathcal{C}^1(X, \mathbb{R})$ be a functional having the form*

$$H = \Psi - G, \tag{2.10}$$

where

(i) at any $u \in X$,

$$\Psi(u) = \Phi(\|u\|)$$

with

$$\Phi(t) = \int_0^t \varphi(s)ds, \ \forall t \geq 0$$

where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a gauge function which satisfies

$$\sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} = p^* < \infty;$$

(ii) $G : Y \longrightarrow \mathbb{R}$ satisfies:

$(ii)_1$ $G' : Y \longrightarrow Y^*$ is demicontinuous;

$(ii)_2$ *there is a constant* $\theta > p^*$ *such that*

$$\left\langle G'(y), y\right\rangle_{Y,Y^*} - \theta G(y) \geq C = const.\ \forall y \in Y. \tag{2.11}$$

Then H *satisfies the* (PS)*-condition.*

Proof. It suffices to prove that the hypotheses of Corollary 1 are fulfilled with $S = J_\varphi$ and $N = G'$. Indeed, according to Asplund's theorem ([2]) $\Psi' = J_\varphi$ and, by Proposition 1 and 2, J_φ is monotone, demicontinuous and satisfies condition $(S)_2$. The demicontinuity of G' is assumed by $(ii)_1$. It remains to be proved that any Palais-Smale sequence for H is bounded.

Let $(u_n) \subset X$ be a Palais-Smale sequence for H. By putting $\varepsilon_n = H'(u_n)$ and taking into account the boundedness of $H(u_n)$ one has:

$$H(u_n) - \frac{1}{\theta}\left\langle H'(u_n), u_n\right\rangle_{X,X^*} \leq M + \frac{\varepsilon_n}{\theta}\left\|u_n\right\|,\ M = const. \tag{2.12}$$

On the other hand, since, at any $u \in X$, $H(u) = \Psi(u) - G(i(u))$, one has

$$H'(u) = \Psi'(u) - (i^* \circ G' \circ i)(u) = J_\varphi u - (i^* \circ G' \circ i)(u),$$

where, as usual, i stands for the injection of X in Y and i^* is its adjoint.

Consequently,

$$H(u_n) - \frac{1}{\theta}\left\langle H'(u_n), u_n\right\rangle_{X,X^*} = \Phi(\|u_n\|) - G(i(u_n)) -$$

$$-\frac{1}{\theta}\left\langle J_\varphi u_n - (i^* \circ G' \circ i)(u_n), u_n\right\rangle_{X,X^*} = \left[\Phi(\|u_n\|) - \frac{1}{\theta}\varphi(\|u_n\|)\left\|u_n\right\|\right] +$$

$$+\frac{1}{\theta}\left[\left\langle G'(i(u_n)), i(u_n)\right\rangle_{Y,Y^*} - \theta G(i(u_n))\right].$$

From the definition of p^*, $\varphi(\|u_n\|)\left\|u_n\right\| \leq p^*\Phi(\|u_n\|)$ so that, taking into account (2.11), one obtains

$$H(u_n) - \frac{1}{\theta}\left\langle H'(u_n), u_n\right\rangle_{X,X^*} \geq \left(1 - \frac{p^*}{\theta}\right)\Phi(\|u_n\|) + \frac{C}{\theta}. \tag{2.13}$$

By comparing (2.12) and (2.13), we infer that

$$\left(1 - \frac{p^*}{\theta}\right)\Phi(\|u_n\|) \leq M_1 + \frac{\varepsilon_n}{\theta}\left\|u_n\right\|,\ M_1 = M - \frac{C}{\theta}.$$

Since $\frac{\Phi(t)}{t} \to \infty$ as $t \to \infty$ and $\varepsilon_n \to 0$ as $n \to \infty$, this inequality implies the boundedness of (u_n). □

Next we state the basic result we need for proving Theorem 1, the so-called "fountain theorem."

Theorem 2. (Bartsch [3]; see also Willem [19, Theorem 3.6]) *Let X be a real, reflexive, and separable Banach space and let X_k, Y_k, Z_k be the subspaces of X given by (2.1). Let $H \in \mathcal{C}^1(X, \mathbb{R})$ be an even functional satisfying the following hypotheses:*

$(H)_1$ *H satisfies the $(PS)_c$-condition, for any $c > 0$;*

$(H)_2$ *For any $k \in \mathbb{N}^*$ there exists $\rho_k > r_k > 0$ such that*

$$a_k = \max_{u \in Y_k,\ \|u\|_X = \rho_k} H(u) \le 0 \tag{2.14}$$

and

$$b_k = \inf_{u \in Z_k,\ \|u\|_X = r_k} H(u) \to \infty \text{ as } k \to \infty. \tag{2.15}$$

Then, H has a sequence of critical positive values which converges to $+\infty$.

In connection with the statement of Theorem 2, recall that if X is a real Banach space and $H \in \mathcal{C}^1(X, \mathbb{R})$, we say that H satisfies the *Palais-Smale condition at level c* on X ($(PS)_c$*-condition*, for short), if any sequence $(u_n) \subset X$ for which $H(u_n) \to c$ and $H'(u_n) \to 0$, as $n \to \infty$, possesses a convergent subsequence. Obviously, if H satisfies the (PS)-condition on X, then H satisfies the $(PS)_c$-condition on X for any $c \in \mathbb{R}$.

Proof of Theorem 1. We shall prove that the hypotheses of the "fountain theorem" (Theorem 2) are satisfied and then the result will follow by this theorem. Indeed, according to Corollary 2, H satisfies the (PS)-condition, therefore H satisfies the $(PS)_c$-condition for any $c > 0$. Thus, hypothesis $(H)_1$ of Theorem 2 is satisfied.

We split into two steps the proof of the fact that hypothesis $(H)_2$ of Theorem 2 is also satisfied.

Step 1. Define

$$\alpha_k = \sup\{\|i(u)\|_Y : u \in Z_k,\ \|u\|_X = 1\},\ k \in \mathbb{N}^*$$

and show that

a) $0 < \alpha_{k+1} \le \alpha_k$, $\forall k \in \mathbb{N}^*$ and $\alpha_k \to 0$ as $k \to \infty$;

b)

$$\|i(u)\|_Y \le \alpha_k \|u\|_X,\ \forall u \in Z_k,\ k \in^*, \tag{2.16}$$

where i stands for the compact injection of X in Y.

Indeed, let $C = const. > 0$ be such that

$$\|i(u)\|_Y \le C \|u\|_X,\ \forall u \in X.$$

Since for any $u \in Z_k$, with $\|u\|_X = 1$, one has $\|i(u)\|_Y \le C$, we deduce that $\alpha_k \le C$. Since $Z_{k+1} \subset Z_k$ one deduces that $\alpha_{k+1} \le \alpha_k$. Since $i(u) \ne i(0) = 0$

for any $u \in X$, $u \neq 0$, one deduces that $\|i(u)\|_Y > 0$ for any $u \in Z_k$, with $\|u\|_X = 1$. Consequently, $\alpha_k > 0$.

By the definition of α_k, there is $u_k \in Z_k$, with $\|u_k\|_X = 1$ such that

$$0 \leq \alpha_k - \|i(u_k)\|_Y < \frac{1}{k},\ k \in \mathbb{N}^*. \tag{2.17}$$

We shall prove that $u_k \rightharpoonup 0$ (in X). Since X is reflexive and (u_k) is bounded, it suffices to show that zero is the unique weak cluster point of (u_k).

Consider a subsequence of (u_k) (still denoted by (u_k)) and an element $u \in X$ such that $u_k \rightharpoonup u$. We shall prove that $u = 0$. Let $p \in \mathbb{N}^*$ be fixed (but arbitrary chosen). One has $f_p(u_k) \to f_p(u)$ as $k \to \infty$. But, for any $k > p$, $f_p(u_k) = 0$ (that's because $u_k \in Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$, $X_j = Sp(\{e_j\})$ and $f_p(e_j) = 0$ for any $j \geq k$).

Consequently, $f_p(u) = 0$. Since $X^* = \overline{Sp(\{f_1, ..., f_n, ...\})}$, we deduce, by density, that $f(u) = 0$, for all $f \in X^*$, thus $u = 0$. Since $u_k \rightharpoonup 0$ (in X), the compactness of i implies $i(u_k) \to 0$ in Y and thus, from (2.17), $\alpha_k \to 0$. Clearly, b) directly follows by the definition of α_k.

Step 2. Define $r_k = (\frac{c_1}{2c_2\alpha_k^q})^{\frac{1}{q-p}}$ and $\rho_k = \max(r_k + 1, t_0)$, $t_0 > 0$ being such that $h(t) = c_3 t^r - c_4 t^s + c_5 \leq 0$ for $t \geq t_0$ (Since $h(t) \to -\infty$ as $t \to \infty$, such a t_0 exists). Clearly, one has $\rho_k > r_k > 0$. Moreover, we shall show that (2.14) and (2.15) hold.

Let $u \in Y_k$ with $\|u\|_X = \rho_k$. Since $\rho_k > 1$, it follows from (iv) that $H(u) \leq c_3\rho_k^r - c_4\rho_k^s + c_5 = h(\rho_k)$ and, since $\rho_k \geq t_0$, it follows that $h(\rho_k) \leq 0$, thus, (2.14) holds.

Let k_0 be such that $r_k > 1$ for any $k \geq k_0$ (since $r_k \to \infty$ as $k \to \infty$, such a k_0 exists). Since $\|i(u)\|_Y \leq \alpha_k \|u\|_X$, for any $u \in Z_k$ (see (2.16)), we deduce from (2.5) that, for $k \geq k_0$ and $u \in Z_k$ satisfying $\|u\|_X = r_k$,

$$H(u) \geq c_1 \|u\|_X^p - c_2\alpha_k^q \|u\|_X^q - d = c_1 r_k^p - c_2\alpha_k^q r_k^q - d = \frac{c_1}{2} r_k^p - d.$$

Consequently, for $k \geq k_0$

$$\inf_{u \in Z_k,\ \|u\|_X = r_k} H(u) \geq \frac{c_1}{2} r_k^p - d \to \infty \text{ as } k \to \infty.$$

Consequently, (2.15) holds as well and the proof is complete. □

3. Applications to Orlicz-Sobolev spaces

First, we recall some basic definitions and results concerning the Orlicz-Sobolev spaces we need. All of them may also be found (with more details) in [10, Section 2].

Throughout this section Ω denotes a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. Let $a : \mathbb{R} \to \mathbb{R}$ be a strictly increasing odd continuous function with $\lim_{t\to+\infty} a(t) = +\infty$.

The *N-function* generated by a is defined as follows:

$$A(t) = \int_0^t a(s)\mathrm{d}s, \tag{3.1}$$

for all $t \in \mathbb{R}$. The N-function given by

$$\overline{A}(u) = \int_0^u a^{-1}(s)\mathrm{d}s,$$

is called the *complementary N-function* to A.

If A and B are two N-functions, we say that B *dominates* A *near infinity* if there exist positive constants k and t_0 such that $A(t) \leq B(kt)$, for all $t \geq t_0$. The two N-functions A and B are *equivalent near infinity* if each dominates the other near infinity. If B dominates A near infinity and A and B are not equivalent near infinity, then we say that A *increases essentially more slowly than* B *near infinity.* This is the case if and only if, for every $k > 0$,

$$\lim_{t\to\infty} \frac{A(kt)}{B(t)} = 0.$$

If the N-functions A and B are equivalent near infinity, then A and B define the same Orlicz space ([1], page 234) (see the definition below).

Let us now introduce the Sobolev conjugate A_* of the N-function A. We shall always suppose that

$$\lim_{t\to 0} \int_t^1 \frac{A^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\mathrm{d}\tau < \infty, \tag{3.2}$$

replacing, if necessary, A by another N-function equivalent to A near infinity (which determines the same Orlicz space).

Suppose also that

$$\lim_{t\to\infty} \int_1^t \frac{A^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\mathrm{d}\tau = \infty. \tag{3.3}$$

With (3.3) satisfied, we define the *Sobolev conjugate* A_* of A by setting

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{\frac{N+1}{N}}} \mathrm{d}\tau,\ t \geq 0. \tag{3.4}$$

We say that the N-function A satisfies the Δ_2-*condition*, if there exist $k > 0$ and $t_0 > 0$ such that

$$A(2t) \leq kA(t), \text{ for all } t \geq t_0.$$

Next, we define the Orlicz class, the Orlicz space and the Orlicz-Sobolev space corresponding to an N-function A. The *Orlicz class* is defined by:

$$K_A(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ measurable}; \int_\Omega A(u(x))dx < \infty \Big\}.$$

The *Orlicz space* $L_A(\Omega)$ is defined as the linear hull of $K_A(\Omega)$ and is a Banach space with respect to the *Luxemburg norm*

$$\|u\|_{(A)} = \inf \Big\{ k > 0 : \int_\Omega A\Big(\frac{u(x)}{k}\Big) \mathrm{d}x \leq 1 \Big\}.$$

Generally, $K_A(\Omega) \subset L_A(\Omega)$. Moreover, $K_A(\Omega) = L_A(\Omega)$ if and only if A satisfies the Δ_2-condition.

One has a *Hölder's type inequality*: if $u \in L_A(\Omega)$ and $v \in L_{\overline{A}}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\Big| \int_\Omega u(x)v(x)\mathrm{d}x \Big| \leq 2 \|u\|_{(A)} \|v\|_{(\overline{A})}.$$

We shall denote the closure of $L^\infty(\Omega)$ in $L_A(\Omega)$ by $E_A(\Omega)$. One has $E_A(\Omega) \subset K_A(\Omega)$ and $E_A(\Omega) = K_A(\Omega)$ if and only if A satisfies the Δ_2-condition.

The *Orlicz-Sobolev space* $W^m L_A(\Omega)$ ($W^m E_A(\Omega)$) is the space of all $u \in L_A(\Omega)$ whose distributional derivatives $D^\alpha u$ are in $L_A(\Omega)$ ($E_A(\Omega)$) for any α, with $|\alpha| \leq m$. The spaces $W^m L_A(\Omega)$ and $W^m E_A(\Omega)$ are Banach spaces with respect to the norm

$$\|u\|_{W^m L_A(\Omega)} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|_{(A)}^2 \Big)^{1/2}.$$

The space $W^m L_A(\Omega)$ is reflexive if and only if the N-functions A and $\overline{A}$ satisfy the Δ_2-condition ([6]).

The space $W_0^m E_A(\Omega)$ is defined as the norm closure of $\mathcal{D}(\Omega)$ in $W^m E_A(\Omega)$. If $u \in W_0^m E_A(\Omega)$, then $D^\alpha u = 0$ on $\partial\Omega$, for $|\alpha| \leq m-1$, in the sense of the trace (see [11]).

The existence and multiplicity of weak solutions for the boundary-value problem

$$J_a u = \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha g_\alpha(x, D^\alpha u) \text{ in } \Omega, \tag{3.5}$$

$$D^\alpha\, u = 0 \;\text{ on } \partial\Omega,\; |\alpha| \le m-1, \tag{3.6}$$

is studied, in this section, in the following functional framework:

• $T[u,v]$ is a nonnegative symmetric bilinear form on the Orlicz-Sobolev space $W_0^m E_A(\Omega)$, involving the only generalized derivatives of order m of the functions $u, v \in W_0^m E_A(\Omega)$, satisfying

$$c_1 \sum_{|\alpha|=m} (D^\alpha u)^2 \le T[u,u] \le c_2 \sum_{|\alpha|=m} (D^\alpha u)^2,\, \forall u \in W_0^m L_A(\Omega), \tag{3.7}$$

with c_1, c_2 be positive constants;

• $\|u\|_{m,A} = \left\|\sqrt{T[u,u]}\right\|_{(A)}$ is a norm on $W_0^m E_A(\Omega)$, $\|\cdot\|_{(A)}$ designating the Luxemburg norm on the Orlicz space $L_A(\Omega)$;

• $J_a : \left(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\right) \to \left(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\right)^*$ is the duality mapping on $\left(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\right)$ subordinated to the gauge function a;

• $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, are Carathéodory functions satisfying hypotheses $(H)_1$ and $(H)_2$ below:

$(H)_1$ *there exist N-functions M_α, $|\alpha| < m$, which increase essentially more slowly than A_* near infinity and satisfy the Δ_2-condition, such that*

$$|g_\alpha(x,s)| \le c_\alpha(x) + d_\alpha \overline{M}_\alpha^{-1}\left(M_\alpha(s)\right),\, x \in \Omega,\, s \in \mathbb{R},\; |\alpha| < m, \tag{3.8}$$

where $\overline{M}_\alpha$ are the complementary N-functions to M_α, $c_\alpha \in K_{\overline{M}_\alpha}$ and d_α are positive constants;

$(H)_2$ *for any α with $|\alpha| < m$, there exist $s_\alpha > 0$ and $\theta_\alpha > p^* = \sup_{t>0} \frac{ta(t)}{A(t)}$ such that*

$$0 < \theta_\alpha G_\alpha(x,s) \le s g_\alpha(x,s), \tag{3.9}$$

for a.e. $x \in \Omega$ and all s with $|s| \ge s_\alpha$, where

$$G_\alpha(x,s) = \int_0^s g_\alpha(x,\tau)\mathrm{d}\tau. \tag{3.10}$$

Assume also that

$(H)_3$ *the function $\frac{a(t)}{t}$ is nondecreasing on* $(0,\infty)$, (3.2) *and* (3.3) *being fulfilled as well* (*see the beginning of this section*).

By (*weak*) *solution* of the problem (3.5) and (3.6), we understand a solution of the equation

$$J_a u = G'(u), \tag{3.11}$$

in the following functional framework:

(*i*) $X = W_0^m E_A(\Omega)$ normed with $\|\cdot\|_{m,A}$;

$Y = \bigcap_{|\beta|<m} W^{m-1} L_{M_\beta}(\Omega)$ normed with the norm

$$\|u\|_Y = \sum_{|\beta|<m} \|u\|_{W^{m-1}L_{M_\beta}(\Omega)};$$

(*ii*) $J_a =$ the duality mapping on $\left(W_0^m E_A(\Omega), \|\cdot\|_{m,A}\right)$ corresponding to the gauge function a;

(*iii*) $G' : Y \to Y^*$ is the differential of the functional $G : Y \to \mathbb{R}$,

$$G(u) = \sum_{|\alpha|<m} \int_\Omega G_\alpha\left(x, D^\alpha u(x)\right) \mathrm{d}x.$$

According to [10, Proposition 6.2], X is compactly imbedded in Y.

Proposition 4. *Let $A : \mathbb{R} \to \mathbb{R}_+$ be the N-function given by* (3.1). *Furthermore, we assume that A satisfies* (3.2) *and* (3.3), *the Δ_2-condition being also satisfied by A and $\overline{A}$.*

Let $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$ be Carathéodory functions satisfying condition $(H)_1$.

Then, the functional $H : W_0^m E_A(\Omega) \to \mathbb{R}$ defined by

$$H(u) = \Psi(u) - G(u), \tag{3.12}$$

with $\Psi(u) = A\left(\|u\|_{m,A}\right)$, and

$$G(u) = \sum_{|\alpha|<m} \int_\Omega G_\alpha\left(x, D^\alpha u(x)\right) dx,$$

for all $u \in W_0^m E_A(\Omega)$, is well defined and $\mathcal{C}^1$ on $W_0^m E_A(\Omega)$, with

$$H'(u) = J_a u - \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha g_\alpha(x, D^\alpha u).$$

Proof. Clearly, the well definedness of H on $W_0^m E_A(\Omega)$ reduces to that of G. In turn, the well definedness of G on $W_0^m E_A(\Omega)$ is proved in [10, Proposition 7.5].

We shall prove more: G is well defined on Y. Fix α with $|\alpha| \le m-1$. If $u \in Y$, then $u \in W^{m-1} L_{M_\beta}(\Omega)$, for all β with $|\beta| \le m-1$. In particular, $u \in W^{m-1} L_{M_\alpha}(\Omega)$, therefore $D^\alpha u \in L_{M_\alpha}(\Omega) = E_{M_\alpha}(\Omega)$.

Taking into account [10, Proposition 7.5, inequality (7.15)], one has

$$|G_\alpha(x,s)| \le c_\alpha |s| + 2 d_\alpha M_\alpha(|s|).$$

Therefore,

$$\int_\Omega G_\alpha\left(x, D^\alpha u(x)\right) dx \leq c_\alpha \int_\Omega \left|D^\alpha u(x)\right| \mathrm{d}x + 2d_\alpha \int_\Omega M_\alpha\left(\left|D^\alpha u(x)\right|\right) \mathrm{d}x.$$

Since $D^\alpha u \in E_{M_\alpha}(\Omega) \hookrightarrow L^1(\Omega)$, it follows that $\int_\Omega |D^\alpha u(x)| \mathrm{d}x$ makes sense. Also, $\int_\Omega M_\alpha\left(|D^\alpha u(x)|\right) \mathrm{d}x$ makes sense. Consequently,

$$\int_\Omega G_\alpha\left(x, D^\alpha u(x)\right) dx < \infty.$$

In order to prove that $H \in \mathcal{C}^1$, it is sufficient to prove that $\Psi \in \mathcal{C}^1$ and $G \in \mathcal{C}^1$. Indeed, one has ([10, Proposition 7.5]):

$$\Psi'(u) = J_a u, \ \forall u \in W_0^m E_A\left(\Omega\right),$$

where

$$J_a u = \begin{cases} 0, & \text{if } u = 0 \\ a(\|u\|_{m,A}) \left\|\cdot\right\|'_{m,A}(u), & \text{if } u \neq 0 \end{cases}$$

and

$$\left\langle \left\|\cdot\right\|'_{m,A}(u), h \right\rangle = \frac{\int_\Omega a\left(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\right) \frac{T[u,h](x)}{\sqrt{T[u,u](x)}} \mathrm{d}x}{\int_\Omega a\left(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\right) \frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}} \mathrm{d}x},$$

for all $u \in W_0^m E_A\left(\Omega\right)$, $u \neq 0$, for all $h \in W_0^m E_A\left(\Omega\right)$.

The continuity of the map $u \longmapsto \left\|\cdot\right\|'_{m,A}(u)$ at any $u \neq 0$ is proved in [10, Theorem 3.6] and for the continuity of Ψ' at $u = 0$, see the proof of Proposition 7.5 in [10]. Thus, $\Psi \in \mathcal{C}^1$.

As far as the $\mathcal{C}^1$-regularity of G is concerned, for a later use, we shall prove more: G is $\mathcal{C}^1$ on Y and

$$\left\langle G'(u), h \right\rangle = \sum_{|\alpha|<m} \int_\Omega g_\alpha\left(x, D^\alpha u(x)\right) D^\alpha h(x) \mathrm{d}x, \ u, h \in Y. \tag{3.13}$$

Indeed, let $u, h \in Y$. One has

$$\begin{aligned} &\left|G(u+h) - G(u) - \left\langle G'(u), h \right\rangle\right| \\ &= \Big| \sum_{|\alpha|<m} \int_\Omega \Big[G_\alpha\left(x, D^\alpha u(x) + D^\alpha h(x)\right) - G_\alpha\left(x, D^\alpha u(x)\right) \\ &\qquad - g_\alpha(x, D^\alpha u(x)) D^\alpha h(x) \Big] dx \Big| \\ &= \Big| \sum_{|\alpha|<m} \int_\Omega \Big[g_\alpha\left(x, D^\alpha u(x) + \theta_{D^\alpha h}(x) \cdot D^\alpha h(x)\right) D^\alpha h(x) \end{aligned}$$

$$
\begin{aligned}
&\quad - g_\alpha(x, D^\alpha u(x)) D^\alpha h(x) \Big] dx \Big| \\
&\leq 2 \sum_{|\alpha|<m} \Big\| g_\alpha\left(x, D^\alpha u(x) + \theta_{D^\alpha h} \cdot D^\alpha h(x)\right) \\
&\quad - g_\alpha(x, D^\alpha u(x)) \Big\|_{(\overline{M}_\alpha)} \|D^\alpha h\|_{(M_\alpha)} \\
&\leq 2 \|h\|_Y \sum_{|\alpha|<m} \Big\| g_\alpha\left(x, D^\alpha u(x) + \theta_{D^\alpha h} \cdot D^\alpha h(x)\right) \\
&\quad - g_\alpha(x, D^\alpha u(x)) \Big\|_{(\overline{M}_\alpha)},
\end{aligned}
$$

where $0 \leq \theta_{D^\alpha h}(x) \leq 1$ ([13, Lemma 18.1]). Consequently,

$$
\begin{aligned}
&\frac{|G(u+h) - G(u) - \langle G'(u), h\rangle|}{\|h\|_Y} \\
&\leq 2 \sum_{|\alpha|<m} \left\| g_\alpha\left(x, D^\alpha u(x) + \theta_{D^\alpha h} \cdot D^\alpha h(x)\right) - g_\alpha(x, D^\alpha u(x)) \right\|_{(\overline{M}_\alpha)}.
\end{aligned}
$$

Suppose $\|h\|_Y \to 0$. It follows that $\|h\|_{W^{m-1}L_{M_\alpha}(\Omega)} \to 0$, therefore, $\|D^\alpha h\|_{(M_\alpha)} \to 0$, for any α with $|\alpha| < m$. Taking into account the continuity of Nemytskij operators (see [13, Theorem 17.6]), it follows that G is Fréchet differentiable on Y and G' is given by (3.13).

Moreover, the operator $G' : Y \to Y^*$ given by (3.13) is continuous (see [10, Proposition 6.3]). Now, since X is continuously imbedded in Y and G is $\mathcal{C}^1$ on Y, it follows that G is $\mathcal{C}^1$ on X. □

The main result is the following.

Theorem 3. *Let $A : \mathbb{R} \to \mathbb{R}_+$ be the N-function given by* (3.1), *fulfilling* (3.2), (3.3) *and hypothesis* $(H)_3$, *and let $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, be Carathéodory functions satisfying* $(H)_1$, $(H)_2$ *and odd in the second argument: $g_\alpha(x, -s) = -g_\alpha(x, s)$. Suppose that the N-functions A, $\overline{A}$ and $\overline{M}_\alpha$, $|\alpha| < m$, satisfy the Δ_2-condition. With*

$$
p_0 = \inf_{t>0} \frac{ta(t)}{A(t)},\ p^* = \sup_{t>0} \frac{ta(t)}{A(t)} < \infty, \tag{3.14}
$$

we further assume:

$(H)_4$ $p_0 < \gamma = \max_{|\alpha|<m} \gamma_\alpha,\ \gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)}$.

Then, the functional (3.12) *has a sequence of critical positive values which converges to $+\infty$.*

Proof. Theorem 1 applies.

Indeed, since $\frac{a(t)}{t}$ is nondecreasing on $(0,\infty)$, $W_0^m E_A(\Omega)$ is uniformly convex ([10, Theorem 3.14]). Consequently, $W_0^m E_A(\Omega)$ is reflexive and has the Kadeč-Klee property. The same space is smooth ([10, Theorem 3.6]), separable ([1, Theorem 8.28]), and compactly imbedded in $\bigcap_{|\beta|<m} W^{m-1} L_{M_\beta}(\Omega)$ ([10, Proposition 6.2]).

The functional $H \in \mathcal{C}^1(X,\mathbb{R})$ (Proposition 4), is even (since the g_αs are odd in the second argument) and satisfies the hypotheses $(i)-(iv)$ of Theorem 1.

Since (3.14) holds, the hypothesis (i) is obviously satisfied with $\varphi = a$. Since $G' : Y \to Y^*$ is continuous ([10, Proposition 6.3]), $(ii)_1$ is obviously satisfied.

Taking into account [10, Lemma 7.7]), we infer that there exists a positive constant C such that

$$\sum_{|\alpha|<m} \int_\Omega \Big[\frac{1}{\theta} g_\alpha\left(x, D^\alpha u_n(x)\right) D^\alpha u_n(x) - G_\alpha\left(x, D^\alpha u_n(x)\right) \Big] \mathrm{d}x \geq -C, \quad (3.15)$$

where $\theta = \min_{|\alpha|<m} \theta_\alpha$. We remark that (3.15) can be rewritten as

$$\frac{1}{\theta} \left\langle G'(u_n), u_n \right\rangle - G(u) \geq -C,$$

therefore, $(ii)_2$ in Theorem 1 is fulfilled.

We will prove that hypothesis (iii) of Theorem 1 is fulfilled. For the first term in (3.12), according to [10, Lemma 6.5 a)], we have

$$A(\|u\|_{m,A}) \geq A(1) \, \|u\|_{m,A}^{p_0}, \quad (3.16)$$

for all $u \in W_0^m E_A(\Omega)$ with $\|u\|_{m,A} > 1$.

We shall now handle the estimations for the second term in (3.12). As in [10, Proposition 7.5, inequality (7.15)], from (H_3) we deduce that for any α with $|\alpha| < m$ one has

$$|G_\alpha(x,s)| \leq |c_\alpha(x)| \, |s| + 2 d_\alpha M_\alpha(|s|), \; x \in \Omega, \; s \in \mathbb{R}. \quad (3.17)$$

Consequently, for all $u \in W_0^m E_A(\Omega)$,

$$\int_\Omega G_\alpha\left(x, D^\alpha u(x)\right) \mathrm{d}x \leq \int_\Omega |c_\alpha(x)| \, |D^\alpha u(x)| \, dx + 2 d_\alpha \int_\Omega M_\alpha(|D^\alpha u(x)|) dx. \quad (3.18)$$

From Hölder's inequality, we deduce

$$\left| \int_\Omega c_\alpha(x) \, |D^\alpha u(x)| \, dx \right| \leq 2 \, \|c_\alpha\|_{(\overline{M}_\alpha)} \, \|D^\alpha u\|_{(M_\alpha)}, \quad (3.19)$$

therefore,

$$\left|\int_\Omega c_\alpha(x)\,|D^\alpha u(x)|\,dx\right| \le 2\,\|c_\alpha\|_{(\overline{M}_\alpha)},$$

if $\|D^\alpha u\|_{(M_\alpha)} \le 1$ and

$$\left|\int_\Omega c_\alpha(x)\,|D^\alpha u(x)|\,dx\right| \le 2\,\|c_\alpha\|_{(\overline{M}_\alpha)}\,\|D^\alpha u\|^{\gamma}_{(M_\alpha)},$$

if $\|D^\alpha u\|_{(M_\alpha)} > 1$. Consequently,

$$\left|\int_\Omega c_\alpha(x)\,|D^\alpha u(x)|\,dx\right| \le k_\alpha(\|u\|_Y^\gamma + 1), \forall u \in W_0^m E_A(\Omega), \tag{3.20}$$

where $k_\alpha = 2\,\|c_\alpha\|_{(\overline{M}_\alpha)}$.

On the other hand, if $\|D^\alpha u\|_{(M_\alpha)} \le 1$, then

$$\int_\Omega M_\alpha(D^\alpha u(x))dx \le 1.$$

If $\|D^\alpha u\|_{(M_\alpha)} > 1$, then from [10, Lemma 6.5, b)]

$$\int_\Omega M_\alpha(D^\alpha(u(x)))dx \le \|D^\alpha(u)\|^{\gamma_\alpha}_{(M_\alpha)} \le \|u\|_Y^\gamma, \tag{3.21}$$

therefore

$$\int_\Omega M_\alpha(|D^\alpha u(x)|)dx \le \|u\|_Y^\gamma + 1,\ \forall u \in W_0^m E_A(\Omega). \tag{3.22}$$

Taking into account (3.18), (3.20) and (3.22), it follows that

$$\int_\Omega G_\alpha\left(x, D^\alpha u(x)\right)\mathrm{d}x \le (k_\alpha + 1)\,\|u\|_Y^\gamma + (k_\alpha + 1), \forall u \in W_0^m E_A(\Omega),\ |\alpha| < m.$$

Consequently, summing by α, we have

$$\sum_{|\alpha|<m}\int_\Omega G_\alpha\left(x, D^\alpha u(x)\right)\mathrm{d}x < c_2\,\|u\|_Y^\gamma + c_2, \tag{3.23}$$

where $c_2 = \sum_{|\alpha|<m}(k_\alpha + 1)$. Then, from (3.16) and (3.23), one obtains

$$F(u) \ge A(1)\,\|u\|^{p_0}_{m,A} - c_2\,\|u\|_Y^\gamma - c_2,$$

if $u \in W_0^m E_A(\Omega)$, $\|u\|_{m,A} > 1$, therefore, the hypothesis (iii) of Theorem 1 is fulfilled.

Now, we will prove that the hypothesis (iv) of Theorem 1 is fulfilled. Let Y_k be a finite dimensional subspace of $W_0^m E_A(\Omega)$. According to [10, Lemma 7.6, inequality (7.46)], it is shown that, for any α with $|\alpha| < m$, one has

$$G_\alpha(x,s) \geq \gamma_\alpha(x)\, |s|^{\theta_\alpha}\,, \text{ for a.e. } x \in \Omega \text{ and } |s| \geq s_\alpha,$$

where $\gamma_\alpha \in L^\infty(\Omega)$.

For α with $|\alpha| < m$ and $v \in W_0^m E_A(\Omega)$, we define

$$\Omega_{\geq}^{\alpha} = \{x \in \Omega : |D^\alpha v(x)| \geq s_\alpha\},\ \Omega_{<}^{\alpha} = \Omega \backslash \Omega_{\geq}^{\alpha}.$$

Then

$$\int_\Omega G_\alpha(x, D^\alpha v(x) dx \geq \int_{\Omega_{\geq}^{\alpha}} \gamma_\alpha(x)\, |D^\alpha v(x)|^{\theta_\alpha}\, dx + \int_{\Omega_{<}^{\alpha}} G_\alpha(x, D^\alpha v(x) dx.$$

But

$$\int_{\Omega_{\geq}^{\alpha}} \gamma_\alpha(x)\, |D^\alpha v(x)|^{\theta_\alpha}\, dx = \int_\Omega \gamma_\alpha(x)\, |D^\alpha v(x)|^{\theta_\alpha}\, dx - \int_{\Omega_{<}^{\alpha}} \gamma_\alpha(x)\, |D^\alpha v(x)|^{\theta_\alpha}\, dx.$$

Since

$$\int_{\Omega_{<}^{\alpha}} \gamma_\alpha(x)\, |D^\alpha v(x)|^{\theta_\alpha}\, dx \leq \|\gamma_\alpha\|_\infty\, s_\alpha^{\theta_\alpha} vol(\Omega),$$

we have

$$\int_\Omega G_\alpha(x, D^\alpha v(x) dx \geq \int_\Omega \gamma_\alpha(x)\, |D^\alpha v(x)|^{\theta_\alpha}\, dx + \int_{\Omega_{<}^{\alpha}} G_\alpha(x, D^\alpha v(x) dx - k_\alpha,$$

where $k_\alpha = \|\gamma_\alpha\|_\infty\, s_\alpha^{\theta_\alpha} vol(\Omega)$. On the other hand, it follows from (3.17) that

$$\int_{\Omega_{<}^{\alpha}} G_\alpha(x, D^\alpha v(x) dx \leq \|c_\alpha\|_{L^1(\Omega)}\, s_\alpha + 2 d_\alpha M_\alpha(s_\alpha) vol(\Omega),$$

therefore,

$$\int_\Omega G_\alpha(x, D^\alpha v(x) dx \geq \int_\Omega \gamma_\alpha(x)\, |D^\alpha v(x)|^{\theta_\alpha}\, dx - K_\alpha,$$

where $K_\alpha = k_\alpha + \|c_\alpha\|_{L^1(\Omega)}\, s_\alpha + 2 d_\alpha M_\alpha(s_\alpha) vol(\Omega)$. Consequently,

$$F(v) \leq A\left(\|v\|_{m,A}\right) - \sum_{|\alpha|<m} \int_\Omega \gamma_\alpha(x)\, |D^\alpha v(x)|^{\theta_\alpha}\, \mathrm{d}x + K,$$

where K is a positive constant and θ_α is given by $(H)_2$. Taking into account the definition of p^*, for $\|v\|_{m,A} > 1$, one obtains

$$F(v) \leq A(1)\, \|v\|_{m,A}^{p^*} - \sum_{|\alpha|<m} \int_\Omega \gamma_\alpha(x)\, |D^\alpha v(x)|^{\theta_\alpha}\, \mathrm{d}x + K. \tag{3.24}$$

Now, the functional $\|\cdot\|_\gamma : W_0^m E_A(\Omega) \to \mathbb{R}$ defined by

$$\|u\|_\gamma = \sum_{|\alpha|<m} \left(\int_\Omega \gamma_\alpha(x) \left| D^\alpha u(x) \right|^{\theta_\alpha} \mathrm{d}x \right)^{1/\theta_\alpha}$$

is a norm on $W_0^m E_A(\Omega)$. Denoting by

$$\|D^\alpha u\|_{\theta_\alpha} = \left(\int_\Omega \gamma_\alpha(x) \left| D^\alpha u(x) \right|^{\theta_\alpha} \mathrm{d}x \right)^{1/\theta_\alpha},$$

one has

$$\|u\|_\gamma = \sum_{|\alpha|<m} \|D^\alpha u\|_{\theta_\alpha}.$$

Let $\underline{\alpha}$ be a multiindex satisfying

$$\|D^{\underline{\alpha}} u\|_{\theta_{\underline{\alpha}}} = \max_{|\alpha|<m} \|D^\alpha u\|_{\theta_\alpha}.$$

Then

$$\|u\|_\gamma \le N_0 \|D^{\underline{\alpha}} u\|_{\theta_{\underline{\alpha}}},$$

where $N_0 = \sum_{|\alpha|<m} 1$. Therefore,

$$\begin{aligned} \sum_{|\alpha|<m} \int_\Omega \gamma_\alpha(x) \left| D^\alpha u(x) \right|^{\theta_\alpha} \mathrm{d}x &\ge \int_\Omega \gamma_{\underline{\alpha}}(x) \left| D^{\underline{\alpha}} u(x) \right|^{\theta_{\underline{\alpha}}} \mathrm{d}x \\ &= \|D^{\underline{\alpha}} u\|_{\theta_{\underline{\alpha}}}^{\theta_{\underline{\alpha}}} \ge \frac{1}{N_0} \|u\|_\gamma^{\theta_{\underline{\alpha}}}. \end{aligned} \tag{3.25}$$

Since the $\|\cdot\|_{m,A}$-norm and $\|\cdot\|_\gamma$-norm are equivalent on the finite-dimensional subspace Y_k, there is a constant $\delta = \delta(Y_k) > 0$ such that

$$\|u\|_{m,A} \le \delta \|u\|_\gamma. \tag{3.26}$$

Therefore,

$$F(v) \le A(1) \|v\|_{m,A}^{p^*} - \frac{1}{N_0 \delta^{\theta_{\underline{\alpha}}}} \|v\|_{m,A}^{\theta_{\underline{\alpha}}} + K,$$

if $v \in Y_k$, $\|v\|_{m,A} > 1$. Taking into account Theorem 1 , it follows that the functional F possesses a sequence of critical positive values. By Proposition 4, equation (3.11) possesses a sequence of solutions in $W_0^m E_A(\Omega)$ or, equivalently, the problem (3.5), (3.6) possesses a sequence of weak solutions in $W_0^m E_A(\Omega)$. □

4. Examples

Example 1. Consider the problem (3.5), (3.6), under the following hypotheses:

(*i*) the function $a : \mathbb{R} \to \mathbb{R}$ is defined by $a(t) = \sum_{i=1}^{n} a_i |t|^{p_i - 2} t$, where $a_i > 0$, $1 \le i \le n$, $p_{i+1} > p_i \ge 2$, $1 \le i \le n-1$, $p_n < N$;

(*ii*) the Carathéodory functions $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, are odd in the second argument: $g_\alpha(x, -s) = -g_\alpha(x, s)$;

(*iii*) there exist q_α, $p_1 < q_\alpha < \frac{Np_n}{N - p_n}$, $|\alpha| < m$, such that

$$|g_\alpha(x,s)| \le a_\alpha + b_\alpha |s|^{q_\alpha - 1},\ x \in \Omega,\ s \in \mathbb{R},\ a_\alpha,\ b_\alpha \text{ positive constants;} \quad (4.1)$$

(*iv*) if G_α, $|\alpha| < m$, are given by (3.10), then there exist $s_\alpha > 0$ and $\theta_\alpha > p_n$ such that

$$0 < \theta_\alpha G_\alpha(x,s) \le s g_\alpha(x,s), \text{ for a.e. } x \in \Omega \text{ and all } s \text{ with } |s| \ge s_\alpha. \quad (4.2)$$

Under these conditions, the problem (3.5), (3.6) has a sequence of weak solutions.

Proof. The idea of the proof is as follows: the preceding assumptions imply that the hypotheses of Theorem 3 are fulfilled.

First, we prove that hypothesis $(H)_3$ is satisfied. Since $\frac{a(t)}{t} = \sum_{i=1}^{n} a_i t^{p_i - 2}$ for all $t > 0$, it follows that $\frac{a(t)}{t}$ is nondecreasing on $(0, \infty)$. In order to prove that (3.2) and (3.3) are satisfied, the following result is needed (see [10, Lemma 8.1, *(ii)*]).

Lemma 1. *Let* $A : \mathbb{R} \to \mathbb{R}_+$, $A(t) = \int_0^{|t|} a(s)ds$, *be an* N*-function. Assume that*

$$p^* = \sup_{t>0} \frac{ta(t)}{A(t)} < N$$

and there are constants $0 < \gamma < N$ *and* $\delta > 0$ *such that*

$$A(t) \ge Ct^\gamma,\ \forall t \in (0, A^{-1}(\delta)). \quad (4.3)$$

Then, (3.2) *and* (3.3) *are satisfied* (*consequently, the Sobolev conjugate,* A_* *of* A*, can be defined*).

In our case, $p^* = p_n$ and $p_n < N$ (by *(i)*). Since

$$A(t) = \sum_{i=1}^{n} \frac{a_i}{p_i} t^{p_i} \ge \frac{a_1}{p_1} t^{p_1},\ \forall t > 0,$$

it follows that (4.3) is satisfied with $C = \frac{a_1}{p_1}$, $\gamma = p_1$ and any $\delta > 0$.

Secondly, we prove that hypothesis $(H)_1$ is satisfied. By setting

$$M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha},\ |\alpha| < m,\ s \in \mathbb{R},$$

we may rewrite as follows:

$$|g_\alpha(x,s)| \le a_\alpha + b_\alpha \left(q_\alpha - 1\right)^{\frac{1}{q_\alpha}} \overline{M}_\alpha^{-1}\left(M_\alpha(s)\right),\ x \in \Omega,\ s \in \mathbb{R},\ |\alpha| < m,$$

showing that (3.8) is satisfied.

What remains to be proved is that M_α, $|\alpha| < m$, satisfies the Δ_2-condition and increases essentially more slowly than A_* near infinity. It is easy to check (by definition) that M_α, $|\alpha| < m$, satisfies the Δ_2-condition.

By using l'Hôspital's rule, we also have

$$\lim_{t\to\infty} \frac{A_*^{-1}(t)}{M_\alpha^{-1}(t)} = \lim_{t\to\infty} \underline{c}_\alpha \frac{A^{-1}(t)}{t^{\frac{1}{q_\alpha}+\frac{1}{N}}} = \lim_{s\to\infty} \underline{c}_\alpha \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}} = 0, \tag{4.4}$$

$$\underline{c}_\alpha = q_\alpha^{(q_\alpha-1)/q_\alpha},$$

since, from *(iii)*, the degree of denominator is $p_n(\frac{1}{q_\alpha} + \frac{1}{N}) > 1$. Thus, M_α, $|\alpha| < m$, increases essentially more slowly than A_*.

The hypothesis $(H)_2$ is covered by (iv) (with the g_αs odd functions in the second argument, according to (ii)).

In order to prove that A and $\overline{A}$ satisfy the Δ_2-condition, the following result is needed (see [10, Lemma 8.1, *(i)*]):

Lemma 2. *Let $A : \mathbb{R} \to \mathbb{R}_+$, $A(t) = \int_0^{|t|} a(s)ds$, be an N-function and $\overline{A}$ be the complementary N-function to A. Assume that*

$$p^* = \sup_{t>0} \frac{ta(t)}{A(t)} < \infty \text{ and } p_0 = \inf_{t>0} \frac{ta(t)}{A(t)} > 1.$$

Then, both A and $\overline{A}$ satisfy the Δ_2-condition.

In our case, as already one has seen, $p^* = p_n < N$ and $p_0 = p_1 > 1$ (according to (i)). Since $\overline{M}_\alpha(s) = \frac{|s|^{q'_\alpha}}{q'_\alpha}$, $\frac{1}{q_\alpha} + \frac{1}{q'_\alpha} = 1$, $|\alpha| < m$, $s \in \mathbb{R}$, it is easy to check (by definition) that $\overline{M}_\alpha$, $|\alpha| < m$, satisfies the Δ_2-condition.

Finally, hypothesis $(H)_4$ is satisfied. Indeed, since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha,\ |\alpha| < m,$$

it follows that $p_0 = p_1 < q_\alpha,\ |\alpha| < m$. The result follows by Theorem 3. □

Example 2. Consider the problem (3.5), (3.6), under the following hypotheses:

(*i*) the function $a : \mathbb{R} \to \mathbb{R}$ is defined by $a(t) = |t|^{p-2}\, t\sqrt{t^2+1}$, $2 \leq p < N-1$;

(*ii*) the Carathéodory functions $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, are odd in the second argument: $g_\alpha(x,-s) = -g_\alpha(x,s)$;

(*iii*) there exist q_α, $p < q_\alpha < \frac{N(p+1)}{N-p-1}$, $|\alpha| < m$, such that the growth conditions (4.1) hold;

(*iv*) there exist $s_\alpha > 0$ and $\theta_\alpha > p+1$ such that the conditions (4.2) hold.

Under these conditions, the problem (3.5), (3.6) has a sequence of weak solutions.

Proof. The idea of the proof is the same as that used for Example 1; namely, we shall show that the preceding assumptions entail the fulfillment of those of Theorem 3.

First, we prove that hypothesis $(H)_3$ is satisfied. Since $\frac{a(t)}{t} = t^{p-2}\sqrt{t^2+1}$ for all $t > 0$, it follows that $\frac{a(t)}{t}$ is nondecreasing on $(0,\infty)$. In order to prove that (3.2) and (3.3) are satisfied, we shall use Lemma 1. In our case, $p^* = p+1$ ([10, Example 8.6]) and $p+1 < N$ (by *(i)*). Since $a(t) \geq t^{p-1}$, $t > 0$, one has $A(t) \geq \frac{1}{p}t^p$, for all $t > 0$, therefore (4.3) is satisfied with $C = \frac{1}{p}$, $\gamma = p < N$ and any $\delta > 0$.

Secondly, the hypothesis $(H)_1$ in Theorem 3 is satisfied with $M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}$, $|\alpha| < m$, which, obviously, satisfies the Δ_2-condition. Also, M_α, $|\alpha| < m$, increases essentially more slowly than A_* near infinity. Indeed, as in (4.4),

$$\lim_{t\to\infty} \frac{A_*^{-1}(t)}{M_\alpha^{-1}(t)} = \lim_{s\to\infty} c_\alpha \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}}.$$

It suffices to show that

$$\lim_{s\to\infty} \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}} = 0.$$

Since $a(t) \geq t^p$, for all $t \geq 0$, it follows that $A(t) \geq \frac{t^{p+1}}{p+1}$, for all $t \geq 0$. Consequently,

$$\lim_{s\to\infty} \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}} \leq \lim_{s\to\infty} \frac{s}{(p+1)^{\frac{1}{q_\alpha}+\frac{1}{N}} \cdot s^{\left(\frac{1}{q_\alpha}+\frac{1}{N}\right)(p+1)}} = 0,$$

since, from *(iii)*, the degree of the denominator is $(p+1)\left(\frac{1}{q_\alpha}+\frac{1}{N}\right) > 1$.

The hypothesis $(H)_2$ is covered by (iv) (with the g_αs odd functions in the second argument, according to (ii)).

In order to prove that A and $\overline{A}$ satisfy the Δ_2-condition, we shall use Lemma 2.

In our case, as one has already seen, $p^* = p+1 < N$ and $p_0 = p > 1$ (according to (i)). Also, the functions $\overline{M}_\alpha(s) = \frac{|s|^{q'_\alpha}}{q'_\alpha}$, $\frac{1}{q_\alpha} + \frac{1}{q'_\alpha} = 1$, $|\alpha| < m$, $s \in \mathbb{R}$, satisfy the Δ_2-condition.

Finally, the hypothesis $(H)_4$ is satisfied. Indeed, since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha, \ |\alpha| < m,$$

it follows that $p_0 = p < q_\alpha, \ |\alpha| < m$. The result follows by Theorem 3. □

Example 3. Consider the problem (3.5), (3.6), under the following hypotheses:

(i) the function $a : \mathbb{R} \to \mathbb{R}$ is defined by $a(t) = |t|^{p-2}\, t \ln(1+|t|)$, $2 \le p < N-1$;

(ii) the Carathéodory functions $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, are odd in the second argument: $g_\alpha(x,-s) = -g_\alpha(x,s)$;

(iii) there exist q_α, $p < q_\alpha < \frac{Np}{N-p}$, $|\alpha| < m$, such that the growth conditions (4.1) hold;

(iv) there exist $s_\alpha > 0$ and $\theta_\alpha > p+1$ such that the conditions (4.2) hold.

Under these conditions, the problem (3.5), (3.6) has a sequence of weak solutions.

Proof. The idea of the proof is the same as that used for Example 1, namely, we shall show that the preceding assumptions entail the fulfillment of those of Theorem 3.

First, we prove that hypothesis $(H)_3$ is satisfied. Since $\frac{a(t)}{t} = t^{p-2}\ln(1+t)$ for all $t > 0$, it follows that $\frac{a(t)}{t}$ is nondecreasing on $(0,\infty)$. In order to prove that (3.2) and (3.3) are satisfied, we shall use Lemma 1. In our case, $p^* = p+1$ ([10, Example 8.8]) and $p+1 < N$ (by *(i)*). Since (see [10, Example 8.8, inequality (8.17)])

$$A(t) \ge \frac{2}{p+1} t^{p+1}, \ \forall t \in (0, \underline{\delta} = A^{-1}(\delta)),$$

it follows that (4.3) is satisfied with $C = \frac{2}{p+1}$, $\gamma = p+1 < N$ and any $\delta > 0$.

Secondly, hypothesis $(H)_1$ in Theorem 3 is satisfied with $M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}$, $|\alpha| < m$, which, obviously, satisfies the Δ_2-condition. Also, M_α, $|\alpha| < m$, increases essentially more slowly than A_* near infinity. As in the preceding two examples, this turns out to show that

$$\lim_{s\to\infty} \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}} = 0.$$

This last equality is true since $A(t) \geq A(1)t^p$, for all $t > 1$ ([10, Lemma 6.5 a)]), therefore,

$$\lim_{s\to\infty} \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}} \leq \lim_{s\to\infty} \frac{s}{(A(1))^{\frac{1}{q_\alpha}+\frac{1}{N}} \cdot s^{\left(\frac{1}{q_\alpha}+\frac{1}{N}\right)p}} = 0,$$

since, from *(iii)*, the degree of the denominator is $p(\frac{1}{q_\alpha} + \frac{1}{N}) > 1$.

The arguments needed for proving that hypothesis $(H)_2$ of Theorem 3 is satisfied are those used in the preceding two examples.

In order to prove that A and $\overline{A}$ satisfy the Δ_2-condition, we shall use Lemma 2. In our case, as one has already seen, $p^* = p + 1 < N$ and $p_0 = p > 1$ (according to (i)). Also, the functions $\overline{M}_\alpha(s) = \frac{|s|^{q'_\alpha}}{q'_\alpha}$, $\frac{1}{q_\alpha} + \frac{1}{q'_\alpha} = 1$, $|\alpha| < m$, $s \in \mathbb{R}$, satisfy the Δ_2-condition.

Finally, hypothesis $(H)_4$ is satisfied, since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha, \ |\alpha| < m,$$

and, by (iii), $p_0 = p < q_\alpha$, $|\alpha| < m$. The result follows by Theorem 3. □

Example 4. Consider the problem (3.5), (3.6), under the following hypotheses:

(i) the function $a : \mathbb{R} \to \mathbb{R}$ is defined by $a(t) = |t|^{p-2}\, t \ln(1 + \alpha + |t|)$, $2 \leq p \leq N - 1$, $\alpha > 0$;

(ii) the Carathéodory functions $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, are odd in the second argument:

$$g_\alpha(x, -s) = -g_\alpha(x, s);$$

(iii) there exist q_α, $p < q_\alpha < \frac{Np}{N-p}$, $|\alpha| < m$, such that the growth conditions (4.1) hold;

(iv) there exist $s_\alpha > 0$ and $\theta_\alpha > p+1$ such that the conditions (4.2) hold.

Under these conditions, the problem (3.5), (3.6) has a sequence of weak solutions.

Proof. The idea of the proof is that used for Example 1; namely, we shall show that the preceding assumptions entail the fulfillment of those of Theorem 3.

First, we prove that hypothesis $(H)_3$ is satisfied. Since $\frac{a(t)}{t} = t^{p-2}\ln(1+\alpha+t)$ for all $t>0$, it follows that $\frac{a(t)}{t}$ is nondecreasing on $(0,\infty)$. In order to prove that (3.2) and (3.3) are satisfied, we shall use Lemma 1. In our case, $p^* = p + C_0 < p+1$ ([10, Example 8.10]) and $p+1 \le N$ (by *(i)*). Since (see [10, Example 8.10, inequality (8.23)])

$$A(t) \geq \frac{\ln(1+\alpha)}{p} t^p,\ \forall t \geq 0, \tag{4.5}$$

it follows that (4.3) is satisfied with $C = \frac{\ln(1+\alpha)}{p}$, $\gamma = p < N$ and any $\delta > 0$.

Secondly, the hypothesis $(H)_1$ in Theorem 3 is satisfied with $M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}$, $|\alpha| < m$, which, obviously, satisfies the Δ_2-condition. Also, M_α, $|\alpha| < m$, increases essentially more slowly than A_* near infinity. As in the preceding three examples, this reduces to showing that

$$\lim_{s\to\infty} \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}} = 0.$$

This last equality is true since (4.5) holds, therefore,

$$\lim_{s\to\infty} \frac{s}{(A(s))^{\frac{1}{q_\alpha}+\frac{1}{N}}} \leq \lim_{s\to\infty} \frac{s}{(\frac{\ln(1+\alpha)}{p})^{\frac{1}{q_\alpha}+\frac{1}{N}} \cdot s^{(\frac{1}{q_\alpha}+\frac{1}{N})p}} = 0,$$

since, from *(iii)*, the degree of the denominator is $p(\frac{1}{q_\alpha}+\frac{1}{N}) > 1$.

The necessary arguments to prove that hypothesis $(H)_2$ of Theorem 3 is satisfied are those used in the preceding three examples.

In order to prove that A and $\overline{A}$ satisfy the Δ_2-condition, we shall use Lemma 2. In our case, as one has already seen, $p^* = p + C_0 < N$ and $p_0 = p > 1$ (according to (i)). Also, the functions $\overline{M}_\alpha(s) = \frac{|s|^{q'_\alpha}}{q'_\alpha}$, $\frac{1}{q_\alpha}+\frac{1}{q'_\alpha} = 1$, $|\alpha| < m$, $s \in \mathbb{R}$, satisfy the Δ_2-condition.

Finally, hypothesis $(H)_4$ is satisfied, since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha,\ |\alpha| < m,$$

and, by (iii), $p_0 = p < q_\alpha$, $|\alpha| < m$. The result follows by Theorem 3. □

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