Differential and Integral Equations Volume 21, Numbers 9-10 (2008), 859–890

A REMARK ON THE CAUCHY PROBLEM FOR THE GENERALIZED BENNEY-LUKE EQUATION

José Raúl Quintero
Departamento de Matemáticas, Universidad del Valle
A. A. 25360. Cali, Colombia

(Submitted by: Gustavo Ponce)

In memory of my professor and colleague Jurgen K. A. Tischer (1943–2007)

Abstract. In this article, we address the well posedness of the Cauchy problem associated with the generalized Benney–Luke equation in $\mathbb{R}^{1+2}$:

$$\Phi_{tt} - \Delta\Phi + a\Delta^2\Phi - b\Delta\Phi_{tt} + \theta\Big(\Phi_t\big[\partial_x\big[(\partial_x\Phi)^p\big] + \partial_y\big[(\partial_y\Phi)^p\big]\big] + 2\big[(\partial_x\Phi)^p\Phi_{xt} + (\partial_y\Phi)^p\Phi_{yt}\big]\Big) + \beta\nabla\cdot\big(|\nabla\Phi|^m\nabla\Phi\big) = 0,$$

under a reasonable "physical" initial condition, which is imposed from the formal derivation of the Benney-Luke water wave model.

1. Introduction

In this article, we study the well posedness of the Cauchy Problem in $\mathbb{R}^{1+2}$ associated with the nonlinear generalized Benney-Luke equation,

$$\Phi_{tt} - \Delta\Phi + a\Delta^2\Phi - b\Delta\Phi_{tt} + \theta\Big(\Phi_t\big[\partial_x\big[(\partial_x\Phi)^p\big] + \partial_y\big[(\partial_y\Phi)^p\big]\big] + 2\big[(\partial_x\Phi)^p\Phi_{xt} + (\partial_y\Phi)^p\Phi_{yt}\big]\Big) + \beta\nabla\cdot\big(|\nabla\Phi|^m\nabla\Phi\big) = 0, \quad (1.1)$$

where $a, b > 0$, $p = p_1/p_2$ with $(p_1, p_2) = 1$ (p_2 odd), $m \in \mathbb{R}^+$, and $\theta, \beta, \in \mathbb{R}$.

We begin the discussion noting that, in the case $\beta = 0$ and $p = 1$, equation (1.1) is the re-scaled version of the Benney-Luke equation

$$\Phi_{tt} - \Delta\Phi + \mu\left(a\Delta^2\Phi - b\Delta\Phi_{tt}\right) + \epsilon\left(\Phi_t\Delta\Phi + 2\nabla\Phi_t\cdot\nabla\Phi\right) = 0. \quad (1.2)$$

Regarding this water wave model, J. Quintero and R. Pego in ([5]) (also see [6], [7]) showed that the evolution of three-dimensional water waves with surface tension can be reduced to studying the solution $\Phi(x, y, t)$ of the isotropic equation (1.2), where ϵ represents the amplitude parameter, μ represents the long-wave or dispersion parameter and $a - b = \sigma - 1/3$ with

Accepted for publication: April 2008.
AMS Subject Classifications: 35L30, 35Q35, 76B15.

$a, b > 0$ (σ is called the Bond number). For instance, if $\overrightarrow{u}$ is the velocity of a particle in an irrotational, three-dimensional flow of an inviscid, incompressible fluid which at rest occupies the region $-\infty < x < \infty$, $-\infty < y < \infty$, $0 < z < h_0$, then, for some distribution ϕ, the velocity potential $\overrightarrow{u}$ takes the form $\overrightarrow{u} = (\nabla\phi, \partial_z\phi)$, where $\nabla = (\partial_x, \partial_y)$. Moreover, the study of the water waves with surface tension reduces to finding solutions of the linear equation

$$\Delta\phi + \phi_{zz} = 0 \quad \text{for } 0 < z < h_0 + \eta, \quad (\Delta = \partial_x^2 + \partial_y^2), \tag{1.3}$$

with the boundary and interface nonlinear conditions

$$\phi_z = 0 \quad \text{at } z = 0, \tag{1.4}$$

$$\eta_t + \nabla\eta \cdot \nabla\phi - \phi_z = 0 \quad \text{at } z = h_0 + \eta, \tag{1.5}$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\phi_z^2 + g\eta - \frac{2TH}{\rho} = 0 \quad \text{at } z = h_0 + \eta, \tag{1.6}$$

where $z = 0$ represents the solid boundary, $z = h_0 + \eta$ is the disturbed free surface, T is the coefficient of surface tension, ρ is density (assumed constant), g is the gravitational acceleration, and the mean curvature of the free surface $z = \eta(x, y, t)$ is given by

$$H = \frac{1}{2}\nabla \cdot \Big(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\Big).$$

In order to study long water waves with small amplitude, we introduce the amplitude parameter ϵ and the long wave parameter $\hat{\mu} = (h_0/L)^2$, where L stands for the horizontal length of motion. The long-wave regime corresponds to $\hat{\mu} \ll 1$. The system for long water waves with small amplitude is introduced through the following rescaling of the variables x, y, and z:

$$x = L\hat{x}, \quad y = L\hat{y}, \quad z = h_0\hat{z}, \quad t = L(gh_0)^{-\frac{1}{2}}\hat{t},$$

and the definition of the functions $\hat{\phi}$ and $\hat{\eta}$ as $\phi = \epsilon\frac{h_0}{\sqrt{\hat{\mu}}}(gh_0)^{\frac{1}{2}}\hat{\phi}$, and $\eta = \epsilon h_0\hat{\eta}$.

Note that a simple computation shows that

$$\partial_x\phi = \epsilon(gh_0)^{\frac{1}{2}}\partial_{\hat{x}}\hat{\phi}, \quad \partial_y\phi = \epsilon(gh_0)^{\frac{1}{2}}\partial_{\hat{y}}\hat{\phi},$$

implying that $\partial_x\phi$ and $\partial_y\phi$ are of order $O(\epsilon)$, as long as $\partial_{\hat{x}}\hat{\phi}$ and $\partial_{\hat{y}}\hat{\phi}$ are of order $O(1)$ with respect to ϵ. Taking $T = h_0^2\rho g\sigma$, and after dropping hats, we obtain that the couple (ϕ, η) satisfies the non-linear system

$$\mu\Delta\phi + \phi_{zz} = 0 \quad \text{for } 0 < z < 1 + \epsilon\eta, \tag{1.7}$$

$$\phi_z = 0 \quad \text{at } z = 0, \tag{1.8}$$

$$\eta_t + \epsilon \nabla\eta \cdot \nabla\phi - \frac{1}{\mu}\phi_z = 0 \quad \text{at } z = 1 + \epsilon\eta, \tag{1.9}$$

$$\phi_t + \frac{\epsilon}{2}|\nabla\phi|^2 + \frac{\epsilon}{2\mu}\phi_z^2 + \eta - \mu\sigma\nabla \cdot \Big(\frac{\nabla\eta}{\sqrt{1+\epsilon^2\mu|\nabla\eta|^2}}\Big) = 0 \quad \text{at} \quad z = 1 + \epsilon\eta, \tag{1.10}$$

where Δ and ∇ are the Laplacian and the gradient with respect to the variables x, y, respectively. Now, to derive the family of Benney-Luke equations, which describes long water waves with small amplitude, we have to assume that $\nabla\phi$, η, and its derivatives with respect to the variables x and y are $O(1)$ with respect to ϵ. Defining

$$\Phi(x, y, t) = \phi(x, y, z = 0, t),$$

and taking the Taylor expansion of the velocity potential at the bottom $z = 0$, one can see that

$$\phi = \Phi - \frac{\mu z^2}{2}\Delta\Phi + \frac{\mu^2 z^4}{4!}\Delta^2\Phi + O(\mu^3). \tag{1.11}$$

Plugging this in the previous equations, we have that $\Phi_t + \eta = O(\epsilon, \mu)$ (for details see [5]) and

$$\Phi_{tt} - \Delta\Phi + \mu\Big(\tfrac{1}{6}\Delta^2\Phi + \Big(\sigma - \tfrac{1}{2}\Big)\Delta\Phi_{tt}\Big) + \epsilon(\Phi_t\Delta\Phi + (\nabla\Phi)_t^2) = O(\mu^2, \epsilon^2). \tag{1.12}$$

In particular, note that $\Phi_{tt} = \Delta\Phi + O(\mu, \epsilon)$. Therefore, if we choose a, b such that

$$a = b + \sigma - \tfrac{1}{3}, \tag{1.13}$$

we find that

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi \cdot \nabla\Phi_t) = O(\mu^2, \epsilon^2).$$

Neglecting terms of order $O(\mu^2, \epsilon^2)$, we obtain a family of Benney-Luke equations. We want to point out that the relevant information from the physical viewpoint in the formal derivation of this water wave model is not in the distribution $\Phi(x, y)$, but in the velocity potential at the bottom $\nabla\Phi(x, y, t) = \nabla\phi(x, y, z = 0, t)$.

It is also important to mention that C. Christov in [2] derived some dispersive shallow-water equations (named dispersive shallow-water system) which are asymptotically correct to the first order of all small dispersion parameters, showing that it is Galilean invariant and possesses Hamiltonian structure. It can be seen directly that equation (1.1) is obtained from the system (2.12 - 2.13) in [2], when $p = 1$ and $m = 2$.

As is well known, one of the main features of water wave models is that they come equipped with a Hamiltonian structure. Moreover, as a principle, results of existence and uniqueness for the associated Cauchy problems follow from the existence of some conserved quantities and the use of energy estimates. It is also a fact that the natural space in which to consider the well posedness of such Cauchy problems is dictated by the definition of either the Hamiltonian or the energy. For our particular problem, the Hamiltonian and the energy are well defined when $\nabla\Phi$ and Φ_t are in $H^1(\mathbb{R}^2)$, being in a perfect concordance with the formal derivation of these models from the full water wave problem after some approximations, since the physical assumptions are not in the distribution Φ, but on the velocity potential $\nabla\Phi$ evaluated at some level, as we noted above.

We are aware of the existence of some recent works related to the Cauchy problem considered in this article. For $\beta = 0$, A. González in two spatial dimensions, with $\theta = 1$ and $p \geq 1$, established global well posedness in the energy space, but in three spatial dimensions and $p \geq 2$ proved the result in a smoother space than the energy space. In both cases, the author used Strichartz inequalities and properties of the commutators of Kato-Ponce type (see [3]). L. Paumond in two spatial dimensions, with $\theta = 1$ and $p = 1$, obtained global well posedness (see [8]) in a space with high regularity, which is smaller than the energy space. For $\beta \in \mathbb{R}$, S. Wang, G. Xu, G. Chen in spatial dimensions 2, 3, 4, $\theta \in R$ and $p = 1$ obtained global well posedness and a non-existence result (see [4]). The authors used a Sobolev multiplicative law. In works by L. Paumond in ([8]) and S. Wang, G. Xu, G. Chen in ([4]), the space for the well posedness of the Cauchy problem controls Φ in L^2, which is mathematically correct, but seems to be unnatural from the physical viewpoint.

Our goal is to present some results of local/global existence, and non-existence of solutions for the Cauchy problem associated with the generalized Benney-Luke equation (1.1), under the reasonable physical assumption on the initial condition (Φ_0, r_0):

$$\nabla\Phi_0 \in (H^1(\mathbb{R}^2))^2,\ r_0 \in H^1(\mathbb{R}^2),$$

which is associated with the formal derivation of the Benney–Luke model.

The article is organized as follows. In section 2, we establish the semigroup estimates, and the non-linear estimates to prove a local existence and uniqueness result for the Cauchy problem associated with the Benney-Luke equation, via a standard fixed point argument. Instead of using Strichartz inequalities, linear and non-linear estimates will be done by using essentially a Sobolev multiplicative law and some simple properties of the derivatives of the function $g(\Phi) = \Phi^p$, where Φ is a distribution with $\nabla\Phi \in H^1(\mathbb{R}^2)$. In section 3, we prove that the Hamiltonian energy is preserved in time in classical solutions of the generalized Benney-Luke equation. In section 4, we establish a global existence and uniqueness result for the Cauchy problem associated with equation (1.1) for $\beta \leq 0$ and $\theta \in \mathbb{R}$, and for $\beta > 0$ and $\theta \in \mathbb{R}$, imposing some restrictions on the initial conditions. We also prove a non-existence result of solutions of equation (1.1) for $\beta > 0$ and $\theta = 0$. In case $\beta > 0$, results of existence/non-existence are generalizations for $p \geq 1$ of the work by S. Wang, G. Xu, G. Chen in ([4]).

2. Local existence

We start by noting that the generalized Benney-Luke equation (1.1) can be written formally as

$$\Box\Phi + B^{-1}F(\Phi_x, \Phi_y, \Phi_t) = 0,$$

where the linear operator $\Box$ and the function F are defined as

$$\Box\Phi = \partial_{tt} - \Delta B^{-1}A,$$

with $A = I - a\Delta$ and $B = I - b\Delta$ defined on $H^2(\mathbb{R}^2)$ via the Fourier transform, and

$$\begin{aligned} F(\phi, \psi, r) = &-\Big[r\left[(\partial_x(\phi)^p) + (\partial_y(\psi)^p)\right] + 2[\phi^p r_x + \psi^p r_y] \\ &+ \beta\nabla\cdot\left(\|(\phi,\psi)\|^m \begin{pmatrix}\phi\\ \psi\end{pmatrix}\right)\Big] \\ = &-\left[G_1(\phi, r) + G_2(\psi, r) + H_1(\phi, r) + H_2(\psi, r) + \beta K(\phi, \psi)\right], \end{aligned}$$

where $G_i(\phi, r) = \partial_i\left(r\phi^p\right)$, $H_i(\phi, r) = \phi^p\partial_i r$, and

$$K(\phi, \psi) = \nabla\cdot\left(\|(\phi,\psi)\|^m \begin{pmatrix}\phi\\ \psi\end{pmatrix}\right).$$

In order to establish the existence and uniqueness of mild solutions for the Cauchy problem associated with equation (1.1), we consider the following space.

Definition 1. *Let $k \in \mathbb{R}^+$. $\mathcal{V}^k$ denotes the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm given by*

$$\|\psi\|^2_{\mathcal{V}^k} := \|\psi_x\|^2_{H^{k-1}} + \|\psi_y\|^2_{H^{k-1}}.$$

Note that $(\mathcal{V}^k, \|.\|)$ is a Hilbert space with inner product

$$(u,v)_{\mathcal{V}^k} = (\partial_x u, \partial_x v)_{H^{k-1}(\mathbb{R}^2)} + (\partial_y u, \partial_y v)_{H^{k-1}(\mathbb{R}^2)}.$$

The first observation is that the generalized Benney–Luke equation can be written as a first-order system in the variables Φ and $\Phi_t = r$:

$$\begin{pmatrix} \Phi \\ r \end{pmatrix}_t = M \begin{pmatrix} \Phi \\ r \end{pmatrix} + \mathcal{G}\begin{pmatrix} \Phi \\ r \end{pmatrix}, \tag{2.1}$$

where $M : \mathcal{V}^{k+1} \times H^k \to \mathcal{V}^k \times H^{k-1}$ is a linear operator, and $\mathcal{G}$ is a function defined on $\mathcal{V}^{k+1} \times H^k$ given by

$$M = \begin{pmatrix} 0 & I \\ \Delta B^{-1} A & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{G}\begin{pmatrix} \Phi \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ F(\Phi_x, \Phi_y, r) \end{pmatrix}.$$

It is not hard to verify that associated with the linear operator M there exists a semigroup $\mathcal{T}$ defined in $\mathcal{V}^k \times H^{k-1} := X^k$ as

$$\mathcal{T}(t) = \begin{pmatrix} \mathcal{F}^{-1} & 0 \\ 0 & \mathcal{F}^{-1} \end{pmatrix} \begin{pmatrix} \cos(|\zeta|\Lambda(\zeta)t) & \dfrac{\sin(|\zeta|\Lambda(\zeta)t)}{|\zeta|\Lambda(\zeta)} \\ -|\zeta|\Lambda(\zeta)\sin(|\zeta|\Lambda(\zeta)t) & \cos(|\zeta|\Lambda(\zeta)t) \end{pmatrix} \begin{pmatrix} \mathcal{F} & 0 \\ 0 & \mathcal{F} \end{pmatrix},$$

where $\mathcal{F}$ stands for the Fourier transform on $\mathbb{R}^2$ and $\Lambda(\zeta) = \sqrt{\frac{1+a|\zeta|^2}{1+b|\zeta|^2}}$.

Hereafter, we say that a couple $(\Phi, r) \in C^0(\mathbb{R}_t; X^k)$ is a mild solution of the generalized Benney-Luke equation (1.1) with initial data $(U^0)^t = (\Phi_0, r_0)$, if (Φ, r) satisfies the integral equation

$$\begin{pmatrix} \Phi \\ r \end{pmatrix}(t) = \mathcal{T}(t)\begin{pmatrix} \Phi_0 \\ r_0 \end{pmatrix} + \int_0^t \mathcal{T}(t-y)\mathcal{G}\begin{pmatrix} \Phi \\ r \end{pmatrix}(y)\,dy := \mathcal{S}\begin{pmatrix} \Phi \\ r \end{pmatrix}(t). \tag{2.2}$$

The first result regarding this problem is to establish that $\mathcal{T}$ is a bounded semigroup. Hereafter, without any other indication, $k \in \mathbb{R}^+$.

Lemma 2.1. *$\mathcal{T}(t) \in \mathcal{L}_b(X^k)$ for $t \in \mathbb{R}$. Moreover, $\mathcal{T}$ is a bounded semigroup.*

Proof. Let $U^t = (\Phi, r) \in X^k$. Then

$$\mathcal{T}(t)\begin{pmatrix} \Phi \\ r \end{pmatrix} = \begin{pmatrix} \mathcal{F}^{-1}\left(\cos(|\zeta|\Lambda(\zeta)t)\hat{\Phi} + \dfrac{\sin(|\zeta|\Lambda(\zeta)t)}{|\zeta|\Lambda(\zeta)}\hat{r}\right) \\ \mathcal{F}^{-1}\left(-|\zeta|\Lambda(\zeta)\sin(|\zeta|\Lambda(\zeta)t)\hat{\Phi} + \cos(|\zeta|\Lambda(\zeta)t)\hat{r}\right) \end{pmatrix}.$$

Since $|\sin(|\zeta|\Lambda(\zeta)t)| \le 1$, $|\cos(|\zeta|\Lambda(\zeta)t)| \le 1$, and for some positive constant $C(a,b) > 1$, we have that $C^{-1} \le \Lambda(\zeta) \le C$, we conclude that

$$\left\|\mathcal{T}(t)\begin{pmatrix}\Phi\\ r\end{pmatrix}\right\|_{X^k}^2 = \left\|(1+|\zeta|^2)^{k/2}|\zeta|)\Big(\cos(|\zeta|\Lambda(\zeta)t)\hat{\Phi} + \frac{\sin(|\zeta|\Lambda(\zeta)t)}{|\zeta|\Lambda(\zeta)}\hat{r}\Big)\right\|_{L^2}^2$$
$$+ \left\|(1+|\zeta|^2)^{k/2}\Big(-|\zeta|\Lambda(\zeta)\sin(|\zeta|\Lambda(\zeta)t)\hat{\Phi} + \cos(|\zeta|\Lambda(\zeta)t)\hat{r}\Big)\right\|_{L^2}^2$$
$$\le C^2(a,b)\left\|\begin{pmatrix}\Phi\\ r\end{pmatrix}\right\|_{X^k}^2. \qquad \square$$

Proposition 2.1. *For any $T > 0$, the operator $\mathcal{S}$ maps $C^0([0,T];X^k)$ into itself.*

We will use the following Sobolev multiplication law (SML) to establish this result.

Lemma 2.2. ([10]) *Let s, s_1, s_2 be real numbers such that*

(1) $s < s_1, s_2$, $s_1 + s_2 > 0$, $s \le s_1 + s_2 - 1$,
(2) $s \le s_1, s_2$, $s_1 + s_2 \ge 0$, $s < s_1 + s_2 - 1$.

Then

$$\|uv\|_{H^s(\mathbb{R}^2)} \le C\|u\|_{H^{s_1}(\mathbb{R}^2)}\|v\|_{H^{s_2}(\mathbb{R}^2)}. \tag{2.3}$$

Before we proceed to estimate the nonlinear terms, we present some basic results.

Lemma 2.3. *Suppose that $\Phi \in \mathcal{V}^k$.*

(1) *If $k \ge 2$ and $p = 1$, then $\partial_i\Phi \in H^{k-1}$ and $\|\partial_i\Phi\|_{H^{k-1}} \le \|\Phi\|_{\mathcal{V}^k}$.*
(2) *If $k > 2$ and $p \ge 1$, $(\partial_i\Phi)^p \in H^1$ and $\|(\partial_i\Phi)^p\|_{H^1} \le C\|\Phi\|_{\mathcal{V}^2}^p$.*
(3) *If $p \ge l \ge 2$ and $k \ge l+1$, then $(\partial_i\Phi)^p \in H^l$ and for some constant $C = C(l,p)$*

$$\|(\partial_i\Phi)^p\|_{H^l} \le C\|\Phi\|_{\mathcal{V}^{l+1}}^p. \tag{2.4}$$

Proof. We first observe that, if $\Phi \in \mathcal{V}^k$, then

$$\partial_i\Phi \in H^{k-1}(\mathbb{R}^2) \hookrightarrow H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)\ (q \ge 2),\quad \partial_{ij}^2\Phi \in H^{k-2}(\mathbb{R}^2).$$

In particular, for $k \ge 2$ and $p = 1$ we conclude that

$$\|\partial_i\Phi\|_{H^{k-1}} \le \|\Phi\|_{\mathcal{V}^k}.$$

Moreover, $(\partial_i\Phi)^p \in L^q(\mathbb{R}^2)$ for $q \ge 2$ and $p \ge 1$. Now assume that $k > 2$. First note that $(\partial_i\Phi)^p \in H^1(\mathbb{R}^2)$. In fact, $(\partial_i\Phi)^p \in L^2(\mathbb{R}^2)$. On the other hand,

$$\partial_s(\partial_i\Phi)^p = p(\partial_i\Phi)^{p-1}\partial_{is}^2\Phi.$$

In this case, $\partial_i\Phi \in H^{k-1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and so it follows directly that $(\partial_i\Phi)^{p-1}\partial^2_{is}\Phi \in L^2(\mathbb{R}^2)$, since $\partial^2_{is}\Phi \in L^2(\mathbb{R}^2)$. It is easy to see that

$$\|(\partial_i\Phi)^p\|_{H^1} \leq C\,\|\partial_i\Phi\|^p_{H^1} \leq C\|\Phi\|^p_{\mathcal{V}^2}.$$

Now take $l \geq 2$ and suppose that $k \geq l+1$. Note that

$$\partial^2_{js}((\partial_i\Phi)^p) = \begin{cases} p(p-1)(\partial_i\Phi)^{p-2}\partial^2_{js}\Phi\partial^2_{is}\Phi + p(\partial_i\Phi)^{p-1}(\partial^3_{isj}\Phi) \in L^2(\mathbb{R}^2), \\ \qquad p \geq 2 \\ \partial^3_{irs}\Phi \in L^2(\mathbb{R}^2), \quad p = 1, \end{cases}$$

since $\partial_{isj}\Phi \in H^{k-3}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$, and $\partial_{is}\Phi \in H^{k-2}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$; thus it is easy to see that $(\partial_i\Phi)^p \in H^2(\mathbb{R}^2)$ for either $p = 1$ or $p \geq 2$. In general, for $p \geq l$ we have that $\partial^l_{j_1j_2\cdots j_l}((\partial_i\Phi)^p)$ has terms of the form

$$(\partial_i\Phi)^{p-t}D^{\rho_1}\Phi D^{\rho_2}\Phi\cdot\cdot D^{\rho_t}\Phi, \quad 1 \leq t \leq l,$$

where $2 \leq |\rho_n| \leq l+2-t$ and $|\rho_1| + |\rho_2| + \cdot\cdot + |\rho_t| = t + l$. In particular, for $t = 1$ we have the main term $(\partial_i\Phi)^{p-1}\partial^{l+1}_{j_1j_2\cdots j_l i}\Phi$. In order to guarantee that $(\partial_i\Phi)^p \in H^l$, we require that $\partial^{l+1}_{j_1j_2\cdots j_l i}\Phi \in L^2$, since $\partial_i\Phi \in H^{k-1}(\mathbb{R}^2) \hookrightarrow H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. But this is assured because we are taking $k \geq l+1$, and $\partial^{l+1}_{j_1j_2\cdots j_l i}\Phi \in H^{k-(l+1)}(\mathbb{R}^2)$, for $\Phi \in \mathcal{V}^k$. On the other hand, $D^\rho\Phi \in H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ $(q \geq 2)$, with $1 \leq |\rho| \leq l$. Moreover, using the Hölder inequality appropriately, we have that

$$\|(\partial_i\Phi)^{p-t}D^{\rho_1}\Phi D^{\rho_2}\Phi\cdot\cdot D^{\rho_t}\Phi\|_2 \leq \|\partial_i\Phi\|^{p-t}_{H^1}\|\Phi\|^t_{\mathcal{V}^{l+1}},$$

which implies that for some constant $C(l,p)$

$$\|(\partial_i\Phi)^p\|_{H^l} \leq C\|\Phi\|^p_{\mathcal{V}^{l+1}}. \qquad \square$$

Corollary 2.1. *Suppose that $\Phi, \Psi \in \mathcal{V}^k$.*

(1) *If $k \geq 2$ and $p = 1$, then*

$$\|\partial_i\Phi - \partial_i\Psi\|_{H^{k-1}} \leq C\|\Phi - \Psi\|_{\mathcal{V}^k}.$$

(2) *If $k > 2$ and $p \geq 2$, then*

$$\|(\partial_i\Phi)^p - (\partial_i\Psi)^p\|_{H^1} \leq C_1(p)\|\Phi - \Psi\|_{\mathcal{V}^2}\left(\|\Phi\|^{p-1}_{\mathcal{V}^2} + \|\Psi\|^{p-1}_{\mathcal{V}^2}\right).$$

(3) *If $p \geq l \geq 2$ and $k \geq l+1$, then for some constant $C_1 = C_1(l,p)$*

$$\|(\partial_i\Phi)^p - (\partial_i\Psi)^p\|_{H^{l-1}} \leq C_1(l,p)\|\Phi - \Psi\|_{\mathcal{V}^{l+1}}\left(\|\Phi\|^{p-1}_{\mathcal{V}^{l+1}} + \|\Psi\|^{p-1}_{\mathcal{V}^{l+1}}\right).$$

In particular, for $p \geq k-1 \geq 2$ we have that

$$\|(\partial_i\Phi)^p - (\partial_i\Psi)^p\|_{H^{k-2}} \leq C_1(k,p)\|\Phi - \Psi\|_{\mathcal{V}^k}\left(\|\Phi\|^{p-1}_{\mathcal{V}^k} + \|\Psi\|^{p-1}_{\mathcal{V}^k}\right).$$

Proof. For $k \geq 2$ and $p = 1$ the conclusion follows directly from the previous lemma. Now assume that $k > 2$ and $p \geq 2$. Then

$$\begin{aligned}
&|\partial_j [(\partial_i\Phi)^p - (\partial_i\Psi)^p]| \\
&\leq \left[|\partial_i\Phi|^{p-1} |\partial^2_{ij}\Phi - \partial^2_{ij}\Psi| + |\partial_{ij}\Psi| \left| (\partial_i\Phi)^{p-1} - (\partial_i\Psi)^{p-1} \right| \right] \\
&\leq C(p) \left[|\partial_i\Phi|^{p-1} |\partial^2_{ij}\Phi - \partial^2_{ij}\Psi| + |\partial_{ij}\Psi| \, |\partial_i\Phi - \partial_i\Psi| \, (|\partial_i\Phi| + |\partial_i\Psi|)^{p-2} \right].
\end{aligned}$$

Then we conclude that

$$\begin{aligned}
\|\partial_j [(\partial_i\Phi)^p - (\partial_i\Psi)^p] \|_{L^2} &\leq C(p) \Big[\|\partial_i\Phi\|^{p-1}_{L^\infty} \|\Phi - \Psi\|_{\mathcal{V}^2} \\
&\qquad + \|\Psi\|_{\mathcal{V}^2} \|\partial_i\Phi - \partial_i\Psi\|_{L^\infty} (\|\partial_i\Phi\|_{L^\infty} + \|\partial_i\Psi\|_{L^\infty})^{p-2} \Big] \\
&\leq C(p) \|\Phi - \Psi\|_{\mathcal{V}^2} \left[\|\Phi\|^{p-1}_{\mathcal{V}^2} + \|\Psi\|^{p-1}_{\mathcal{V}^2} \right],
\end{aligned}$$

where we have used the fact that $H^s(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, for $s > 1$. Moreover, we also have in a similar way that

$$\|(\partial_i\Phi)^p - (\partial_i\Psi)^p\|_{L^2} \leq C(p) \|\Phi - \Psi\|_{\mathcal{V}^2} \left[\|\Phi\|^{p-1}_{\mathcal{V}^2} + \|\Psi\|^{p-1}_{\mathcal{V}^2} \right].$$

Putting together the last two inequalities, we obtain for some constant $C = C(p)$ that

$$\|\partial_j [(\partial_i\Phi)^p - (\partial_i\Psi)^p]\|_{H^1} \leq C(p) \|\Phi - \Psi\|_{\mathcal{V}^2} \left(\|\Phi\|^{p-1}_{\mathcal{V}_2} + \|\Psi\|^{p-1}_{\mathcal{V}_2} \right).$$

Now, take $l \geq 2$ and suppose that $k \geq l + 1$. From the previous result, we have for $1 \leq q \leq l - 1$ that $\partial^q_{j_1 j_2 \cdots j_q} [(\partial_i\Phi)^p - (\partial_i\Psi)^p]$ has for $1 \leq s \leq q$ terms of the form $(\partial_i\Phi)^{p-s} D^{\rho_1}\Phi D^{\rho_2}\Phi \cdot\cdot D^{\rho_s}\Phi - (\partial_i\Psi)^{p-s} D^{\rho_1}\Psi D^{\rho_2}\Psi \cdot\cdot D^{\rho_s}\Psi$, where $2 \leq |\rho_n| \leq q + 2 - s$ and $|\rho_1| + |\rho_2| + \cdot\cdot + |\rho_s| = s + q$. For $s = 1$, we have that

$$\begin{aligned}
&\|(\partial_i\Phi)^{p-1} \partial^{q+1}_{j_1 j_2 \cdots j_q i}\Phi - (\partial_i\Psi)^{p-1} \partial^{q+1}_{j_1 j_2 \cdots j_q i}\Psi\|_2 \\
&\leq \|\partial_i\Phi\|^{p-1}_\infty \|\Phi - \Psi\|_{\mathcal{V}^{q+1}} + \|\partial^{q+1}_{j_1 j_2 \cdots j_q i}\Psi\|_{L^2} \left\| (\partial_i\Phi)^{p-1} - (\partial\Psi)^{p-1} \right\|_\infty \\
&\leq C(p) \|\Phi - \Psi\|_{\mathcal{V}^{q+1}} \left[\|\Phi\|^{p-1}_{\mathcal{V}^{q+1}} + \|\Phi\|^{p-1}_{\mathcal{V}^{q+1}} \right].
\end{aligned}$$

Note for $1 < s \leq q$ that $\partial_i\Phi$, $\partial_i\Psi$, $D^{\rho_n}\Phi$, and $D^{\rho_n}\Psi$ belong to $H^{k-|\rho_n|}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, since $k - |\rho_n| \geq k - l - 1 + s > 1$. Moreover,

$$\begin{aligned}
\|\partial_i\Phi\|_\infty \leq \|\partial_i\Phi\|_{H^1}, \quad \|D^{\rho_n}\Phi\|_\infty \leq \|\Phi\|_{\mathcal{V}^{q+2}}, \\
\|\partial_i\Psi\|_\infty \leq \|\partial_i\Psi\|_{H^1}, \quad \|D^{\rho_n}\Psi\|_\infty \leq \|\Psi\|_{\mathcal{V}^{q+2}}.
\end{aligned}$$

Then using the mean value inequality applied to the function $g(x, y_1, \cdot\cdot, y_s) = x^{p-s} y_1 \cdot \cdot y_s$, we conclude that

$$\|(\partial_i \Phi)^{p-s} D^{\rho_1}\Phi D^{\rho_2}\Phi \cdot\cdot D^{\rho_s}\Phi - (\partial_i \Psi)^{p-s} D^{\rho_1}\Psi D^{\rho_2}\Psi \cdot\cdot D^{\rho_s}\Psi\|_2$$

$$\leq C(p,s)\left(\|\Phi\|_{\mathcal{V}^{q+2}} + \|\Psi\|_{\mathcal{V}^{q+2}}\right)^{p-1}\left(\|\partial_i(\Phi-\Psi)\|_{L^2} + \sum_{j=1}^{s}\|D^{\rho_j}(\Phi-\Psi)\|_{L^2}\right)$$

$$\leq C_1(p,s)\|\Phi-\Psi\|_{\mathcal{V}^{q+2}}\left(\|\Phi\|_{\mathcal{V}^{q+2}} + \|\Psi\|_{\mathcal{V}^{q+2}}\right)^{p-1},$$

and so,

$$\|\partial^q_{j_1 j_2 \cdots j_q}\left[(\partial_i\Phi)^p - (\partial_i\Psi)^p\right]\|_2 \leq C(p,q)\|\Phi-\Psi\|_{\mathcal{V}^{q+2}}\left(\|\Phi\|_{\mathcal{V}^{q+2}} + \|\Psi\|_{\mathcal{V}^{q+2}}\right)^{p-1}.$$

In particular, we have shown for $p \geq l$ and $k \geq l+1$ that

$$\|(\partial_i\Phi)^p - (\partial_i\Psi)^p\|_{H^{l-1}} \leq C(p,l)\|\Phi-\Psi\|_{\mathcal{V}^{l+1}}\left(\|\Phi\|_{\mathcal{V}^{l+1}} + \|\Psi\|_{\mathcal{V}^{l+1}}\right)^{p-1}. \qquad \square$$

Lemma 2.4. *Suppose that k and p are such that*

(1) $k \geq 2$ *and* $p = 1$, *or*
(2) $k = 2$ *and* $p > 1$, *or*
(3) $2 < k \leq 4$ *and* $p \geq 2$, *or*
(4) $p \geq k - 1 > 2$.

If $r \in H^{k-1}$ and $\Phi, \Psi \in \mathcal{V}^k$, then for some constant $C = C(p)$

$$\|r\left((\partial_i\Phi)^p - (\partial_i\Psi)^p\right)\|_{H^{k-2}} \leq C\|r\|_{H^{k-1}}\|\Phi-\Psi\|_{\mathcal{V}^k}\left(\|\Phi\|^{p-1}_{\mathcal{V}^k} + \|\Psi\|^{p-1}_{\mathcal{V}^k}\right). \tag{2.5}$$

In particular, we have for $r_1, r_2 \in H^{k-1}(\mathbb{R}^2)$ and $\Phi \in \mathcal{V}^k$ that

$$\|\left(\partial_i\Phi\right)^p (r_1 - r_2)\|_{H^{k-2}} \leq C\|\Phi\|^p_{\mathcal{V}^k}\|r_1 - r_2\|_{H^{k-1}}.$$

Proof. Suppose $p = 1$ and $k \geq 2$. Then we apply the Sobolev multiplication law with $s = k-2$, $s_1 = k-1$ and $s_2 = k-1$. In this case,

$$\|r\left(\partial_i\Phi - \partial_i\Psi\right)\|_{H^{k-2}} \leq C\|r\|_{H^{k-1}}\|\Phi-\Psi\|_{\mathcal{V}^k}.$$

Suppose now that $p > 1$ and $k = 2$. For $r \in H^1$, and $\Phi, \Psi \in \mathcal{V}^2$, we choose $\rho > 1$ with $\rho(p-1) > 1$ and $\gamma > 2$ such that $2/\gamma + 1/\rho = 1$. Then we have that

$$\|r\left((\partial_i\Phi)^p - (\partial_i\Psi)^p\right)\|_{L^2} \leq C_1(p)\|\,|r|\,|\partial_i\Phi - \partial_i\Psi|\,[|\partial_i\Phi| + |\partial_i\Psi|]^{p-1}\,\|_{L^2}$$

$$\leq C_2(p)\|r\|_{L^{2\gamma}}\|\partial_i\Phi - \partial_i\Psi\|_{L^{2\gamma}}\left(\|\partial_i\Phi\|^{p-1}_{L^{2\rho(p-1)}} + \|\partial_i\Psi\|^{p-1}_{L^{2\rho(p-1)}}\right)$$

$$\leq C_3(p)\|r\|_{H^1}\|\Phi-\Psi\|_{\mathcal{V}^2}\left(\|\Phi\|^{p-1}_{\mathcal{V}^2} + \|\Psi\|^{p-1}_{\mathcal{V}^2}\right).$$

Now for $k > 2$ we know that $(\partial_i \Phi)^p \in H^2(\mathbb{R}^2)$ for $p \geq 2$. Thus, using the Sobolev multiplication law with $s = k-2$, $s_1 = k-1$ and $s_2 = 2$, for $k \leq 4$, we have $s \leq s_i$ and $k-2 = s < s_1 + s_2 - 1 = k$. As a consequence of this,

$$\|r\,((\partial_i \Phi)^p - (\partial_i \Psi)^p)\|_{H^{k-2}} \leq C\|r\|_{H^{k-1}} \|\,(\partial_i \Phi)^p - (\partial_i \Psi)^p\,\|_{H^2}. \tag{2.6}$$

But from Corollary 2.1 we have that

$$\|\,(\partial_i \Phi)^p - (\partial_i \Psi)^p\,\|_{H^1} \leq C_1(p)\|\Phi - \Psi\|_{\mathcal{V}^k} \left(\|\Phi\|^{p-1}_{\mathcal{V}^k} + \|\Psi\|^{p-1}_{\mathcal{V}^k}\right),$$

and from (2.6) that

$$\|r\,((\partial_i \Phi)^p - (\partial_i \Psi)^p)\|_{H^{k-2}} \leq C\|r\|_{H^{k-1}} \|\partial_i \Phi - \partial_i \Psi\|_{\mathcal{V}^k} \left[\|\Phi\|^{p-1}_{\mathcal{V}^k} + \|\Psi\|^{p-1}_{\mathcal{V}^k}\right].$$

Now suppose that $p \geq k-1 > 2$. In this case, $(\partial_i \Phi)^p \in H^{k-1}$. Then applying the Sobolev multiplication law with $s = k-2$, $s_1 = k-1$ and $s_2 = k-2$, we have that

$$\|r\,((\partial_i \Phi)^p - (\partial_i \Psi)^p)\|_{H^{k-2}} \leq C\|r\|_{H^{k-1}} \|\,(\partial_i \Phi)^p - (\partial_i \Psi)^p\,\|_{H^{k-2}}.$$

But we obtained in a previous result that

$$\|\,(\partial_i \Phi)^p - (\partial_i \Psi)^p\,\|_{H^{k-2}} \leq C_2(p)\|\Phi - \Psi\|_{\mathcal{V}^k} \left(\|\Phi\|^{p-1}_{\mathcal{V}^k} + \|\Psi\|^{p-1}_{\mathcal{V}^k}\right),$$

and so,

$$\|r\,((\partial_i \Phi)^p - (\partial_i \Psi)^p)\|_{H^{k-2}} \leq C\|r\|_{H^{k-1}} \|\Phi - \Psi\|_{\mathcal{V}^k} \left[\|\Phi\|^{p-1}_{\mathcal{V}^k} + \|\Psi\|^{p-1}_{\mathcal{V}^k}\right]. \quad \square$$

Using the same type of estimates and similar arguments, we are able to establish the following:

Lemma 2.5. *Suppose that k and p are as in Lemma 2.4. If $r \in H^{k-1}$ and $\Phi, \Psi \in \mathcal{V}^k$, then for some constant $C = C(p)$,*

$$\|\partial_i r\,[(\partial_i \Phi)^p - (\partial_i \Psi)^p]\,\|_{H^{k-3}} \leq C\|r\|_{H^{k-1}} \|\Phi - \Psi\|_{\mathcal{V}^k} \left(\|\Phi\|^{p-1}_{\mathcal{V}^k} + \|\Psi\|^{p-1}_{\mathcal{V}^k}\right). \tag{2.7}$$

In particular, we have for $r_1, r_2 \in H^{k-1}(\mathbb{R}^2)$ and $\Phi \in \mathcal{V}^k$ that

$$\|\,(\partial_i \Phi)^p\,(\partial_i r_1 - \partial_i r_2)\,\|_{H^{k-3}} \leq C(p)\|\Phi\|^p_{\mathcal{V}^k} \|r_1 - r_2\|_{H^{k-1}}.$$

Lemma 2.6. *Suppose that k and m satisfy*

(1) *$k = 2$ and $m > 0$, or*
(2) *$2 < k \leq 4$ and either $m = 1$ or $m \geq 2$, or*
(3) *$m \geq k-1 > 2$.*

If $\Phi, \Psi \in \mathcal{V}^k$*, there exists* $C = C(m)$ *such that*

$$\||\nabla\Phi|^m\nabla\Phi - |\nabla\Psi|^m\nabla\Psi\|_{H^{k-2}} \leq C\|\Phi - \Psi\|_{\mathcal{V}^k}\left(\|\Phi\|^m_{\mathcal{V}^k} + \|\Psi\|^m_{\mathcal{V}^k}\right), \tag{2.8}$$

and so, for m *and* k *as in* (1), (2), *and* (3),

$$\||\nabla\Phi|^m\nabla\Phi\|_{H^{k-2}} \leq C\|\Phi\|^{m+1}_{\mathcal{V}^k}. \tag{2.9}$$

Proof. Let $k = 2$. First note that

$$|\nabla\Phi|^m\nabla\Phi - |\nabla\Psi|^m\nabla\Psi| \leq C(m)|\nabla\Phi - \nabla\Psi|\left(|\nabla\Phi|^m + |\nabla\Psi|^m\right).$$

Since we know that $\nabla\Phi, \nabla\Psi \in H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$, for $q \geq 2$, then after applying the Hölder inequality with appropriate exponents, we obtain that

$$\||\nabla\Phi|^m\nabla\Phi - |\nabla\Psi|^m\nabla\Psi\|_{L^2} \leq C(m)\|\Phi - \Psi\|_{\mathcal{V}^2}\left(\|\Phi\|^m_{\mathcal{V}^2} + \|\Psi\|^m_{\mathcal{V}^2}\right).$$

Now assume that $k > 2$. In this case we have that $\nabla\Phi \in (H^{k-1}(\mathbb{R}^2))^2 \hookrightarrow (L^\infty(\mathbb{R}^2))^2$, and that $|\nabla\Phi|^m \in H^1(\mathbb{R}^2)$ for $m \geq 1$. Moreover, for $2 < k \leq 3$, we can use the Sobolev multiplication law with $s = k-2$, $s_1 = k-1$ and $s_2 = 1$. Then, $s \leq s_i$ and $k - 2 = s < s_1 + s_2 - 1 = k - 1$. Thus,

$$\begin{aligned}\|\nabla\Phi\left(|\nabla\Phi|^m - |\nabla\Psi|^m\right)\|_{H^{k-2}} &\leq C\|\nabla\Phi\|_{H^{k-1}}\|\,|\nabla\Phi|^m - |\nabla\Psi|^m\,\|_{H^1} &(2.10)\\ &\leq C\|\Phi\|_{\mathcal{V}^k}\|\,|\nabla\Phi|^m - |\nabla\Psi|^m\,\|_{H^1}. &(2.11)\end{aligned}$$

If $m = 1$, then we conclude directly that

$$\|\nabla\Phi\left(|\nabla\Phi| - |\nabla\Psi|\right)\|_{H^{k-2}} \leq \|\Phi\|_{\mathcal{V}^k}\|\Phi - \Psi\|_{\mathcal{V}^k}.$$

Now suppose that $m \geq 2$. As in Lemma 2.4, it is easy to check that

$$\|\,|\nabla\Phi|^m - |\nabla\Psi|^m\,\|_{H^1} \leq C_1(m)\|\Phi - \Psi\|_{\mathcal{V}^k}\left(\|\Phi\|^{m-1}_{\mathcal{V}^k} + \|\Psi\|^{m-1}_{\mathcal{V}^k}\right). \tag{2.12}$$

Then from (2.10) we have that

$$\|\nabla\Phi\left(|\nabla\Phi|^m - |\nabla\Psi|^m\right)\|_{H^{k-2}} \leq C\|\Phi\|_{H^{k-1}}\|\Phi - \Psi\|_{\mathcal{V}^k}\left[\|\Phi\|^{m-1}_{\mathcal{V}^k} + \|\Psi\|^{m-1}_{\mathcal{V}^k}\right].$$

Now observe that

$$\begin{aligned}&\||\nabla\Phi|^m\nabla\Phi - |\nabla\Psi|^m\nabla\Psi\|_{H^{k-2}}\\ &\leq \|\nabla\Phi\left(|\nabla\Phi|^m - |\nabla\Psi|^m\right)\|_{H^{k-2}} + \||\nabla\Psi|^m\left(\nabla\Phi - \nabla\Psi\right)\|_{H^{k-2}}.\end{aligned}$$

Then, combining previous inequalities, we conclude, as desired, that

$$\begin{aligned}\||\nabla\Phi|^m\nabla\Phi - |\nabla\Psi|^m\nabla\Psi\|_{H^{k-2}} &\leq C\|\Phi\|_{\mathcal{V}^k}\|\Phi - \Psi\|_{\mathcal{V}^k}\left(\|\Phi\|^{m-1}_{\mathcal{V}^k} + \|\Psi\|^{m-1}_{\mathcal{V}^k}\right)\\ &\leq C\|\Phi - \Psi\|_{\mathcal{V}^k}\left(\|\Phi\|^m_{\mathcal{V}^k} + \|\Psi\|^m_{\mathcal{V}^k}\right).\end{aligned}$$

Again as in Lemma 2.4, it follows directly for $3 < k \leq 4$ and either $m = 1$ or $m \geq 2$ that, $|\nabla\Phi|^m \in H^2(\mathbb{R}^2)$ and

$$\|\,|\nabla\Phi|^m - |\nabla\Psi|^m\,\|_{H^2} \leq C_2(p)\|\Psi - \Psi\|_{\mathcal{V}^k}\left(\|\Phi\|^{m-1}_{\mathcal{V}^k} + \|\Psi\|^{m-1}_{\mathcal{V}^k}\right).$$

Moreover, using again the Sobolev multiplication law with $s = k-2$, $s_1 = k-1$ and $s_2 = 2$, we have that

$$\begin{aligned}&\|\nabla\Phi\left(|\nabla\Phi|^m - |\nabla\Psi|^m\right)\|_{H^{k-2}} \le C\|\nabla\Phi\|_{H^{k-1}}\|\,|\nabla\Phi|^m - |\nabla\Psi|^m\,\|_{H^2}\\&\le C\|\Phi\|_{\mathcal{V}^k}\|\Phi-\Psi\|_{\mathcal{V}^k}\left(\|\Phi\|_{\mathcal{V}^k}^{m-1} + \|\Psi\|_{\mathcal{V}^k}^{m-1}\right)\\&\le C\|\Phi-\Psi\|_{\mathcal{V}^k}\left(\|\Phi\|_{\mathcal{V}^k}^{m} + \|\Psi\|_{\mathcal{V}^k}^{m}\right).\end{aligned}$$

As above we obtain the desired inequality. Finally, for $m \ge k-1 > 2$, we know that $|\nabla\Phi|^m, |\nabla\Psi|^m \in H^{k-1}$ (see Corollary 2.1). Using the fact that H^{k-2} is an algebra, we conclude that

$$\begin{aligned}\|\nabla\Phi\left(|\nabla\Phi|^m - |\nabla\Psi|^m\right)\|_{H^{k-2}} &\le C\|\nabla\Phi\|_{H^{k-2}}\|\,|\nabla\Phi|^m - |\nabla\Psi|^m\,\|_{H^{k-2}}\\&\le C\|\Phi\|_{\mathcal{V}k}\|\Phi-\Psi\|_{\mathcal{V}^k}\left(\|\Phi\|_{\mathcal{V}^k}^{m-1} + \|\Psi\|_{\mathcal{V}^k}^{m-1}\right)\\&\le C\|\Phi-\Psi\|_{\mathcal{V}^k}\left(\|\Phi\|_{\mathcal{V}^k}^{m} + \|\Psi\|_{\mathcal{V}^k}^{m}\right).\end{aligned}$$

Then we have that

$$\begin{aligned}&\||\nabla\Phi|^m\nabla\Phi - |\nabla\Psi|^m\nabla\Psi\|_{H^{k-2}}\\&\le \|\nabla\Phi\left(|\nabla\Phi|^m - |\nabla\Psi|^m\right)\|_{H^{k-2}} + \||\nabla\Psi|^m\left(\nabla\Phi-\nabla\Psi\right)\|_{H^{k-2}}\\&\le C\|\Phi-\Psi\|_{\mathcal{V}^k}\left(\|\Phi\|_{\mathcal{V}k}^{m} + \|\Psi\|_{\mathcal{V}k}^{m}\right).\end{aligned}$$
□

Now we are able to handle the non-linear terms.

Theorem 2.1. *Let k, p, m be as in Lemmas 2.4, 2.5, and 2.6. If $\Phi \in C([0,T],\mathcal{V}^k)$ and $r \in C([0,T], H^{k-1}(\mathbb{R}^2))$, we have for $q, s \in [0,T]$ the following estimates*

$$\begin{aligned}&\|B^{-1}\left[G_i(\partial_i\Phi(q), r(q)) - G_i(\partial_i\Phi(s), r(s))\right]\|_{H^{k-1}}\\&\qquad\le \Lambda_1\|\left(\Phi(q)-\Phi(s), r(q)-r(s)\right)\|_{X^k},\\&\|B^{-1}\left[H_i(\partial_i\Phi(q), r(q)) - H_i(\partial_i\Phi(s), r(s))\right]\|_{H^{k-1}}\\&\qquad\le \Lambda_2\|\left(\Phi(q)-\Phi(s), r(q)-r(s)\right)\|_{X^k},\\&\|B^{-1}\nabla\cdot\left[\left(|\nabla\Phi|^m\nabla\Phi\right)(q) - \left(|\nabla\Phi|^m\nabla\Phi\right)(s)\right]\|_{H^{k-1}} \le \Lambda_3\|\Phi(q)-\Phi(s)\|_{\mathcal{V}^k},\end{aligned}$$

where

$$\begin{aligned}\Lambda_i &= \Lambda_i(\|\Phi\|_{L^\infty([0,T],\mathcal{V}^k)}, \|r\|_{L^\infty([0,T],H^1)}, b, p) \quad (i=1,2),\\\Lambda_3 &= \Lambda_3(\|\Phi\|_{L^\infty([0,T],\mathcal{V}^k)}, b, m).\end{aligned}$$

Proof. First, we observe that

$$\begin{aligned}&G_i(\partial_i\Phi(q), r(q)) - G_i(\partial_i\Phi(s), r(s))\\&\qquad= \partial_i\left[(\partial_i\Phi(q))^p\left(r(q)-r(s)\right) + r(s)\left((\partial_i\Phi(q))^p - (\partial_i\Phi(s))^p\right)\right].\end{aligned}\tag{2.13}$$

Then we have for a generic constant C depending only on p that

$$
\begin{aligned}
&\|G_i(\partial_i\Phi(q), r(q)) - G_i(\partial_i\Phi(s), r(s))\|_{H^{k-3}}\\
&\le \| (\partial_i\Phi(q))^p (r(q)-r(s)) + r(s) ((\partial_i\Phi(q))^p - (\partial_i\Phi(s))^p) \|_{H^{k-2}}\\
&\le \| (\partial_i\Phi(q))^p (r(q)-r(s))\|_{H^{k-2}} + \|r(s) ((\partial_i\Phi(q))^p - (\partial_i\Phi(s))^p) \|_{H^{k-2}}\\
&\le C\Big[\|\Phi(s)\|^p_{\mathcal{V}^k}\|r(q)-r(s)\|_{H^{k-1}} + \|r(s)\|_{H^{k-1}}\|\Phi(q)\\
&\qquad - \Phi(s)\|_{\mathcal{V}^k}\Big[\|\partial_i\Phi(q)\|^{p-1}_{\mathcal{V}^k} + \|\Phi(s)\|^{p-1}_{\mathcal{V}^k}\Big]\Big]\\
&\le C\|\Phi\|^{p-1}_{L^\infty([0,T],\mathcal{V}^k)}\Big[\|\Phi\|_{L^\infty([0,T],\mathcal{V}^k)}\|r(q)-r(s)\|_{H^{k-1}}\\
&\qquad + \|r\|_{L^\infty([0,T],H^{k-1})}\|\Phi(q)-\Phi)(s)\|_{\mathcal{V}^k}\Big].
\end{aligned}
$$

Since B^{-1} is an operator of order -2, we may conclude from the previous inequality that

$$
\begin{aligned}
&\left\|B^{-1}\left[G_i(\partial_i\Phi(q), r(q)) - G_i(\partial_i\Phi(s), r(s))\right]\right\|_{H^{k-1}}\\
&\le \Lambda_1\| (\Phi(q)-\Phi(s), r(q)-r(s)) \|_{X^k},
\end{aligned}
$$

where $\Lambda_1 = \Lambda_1(\|\Phi\|_{L^\infty([0,T],\mathcal{V}^k)}, \|r\|_{L^\infty([0,T],\mathcal{V}^k)}, b, p)$. On the other hand, in a similar fashion we are able to obtain that

$$
\begin{aligned}
&\|B^{-1}\left[H_i(\partial_i\Phi(q), r(q)) - H_i(\partial_i\Phi(s), r(s))\right]\|_{H^{k-1}}\\
&\le C\| (\partial_i r (\partial_i\Phi)^p)(q) - (\partial_i r (\partial_i\Phi)^p)(s)\|_{H^{k-3}}\\
&\le C\|\Phi\|^{p-1}_{L^\infty([0,T],\mathcal{V}^k)}\Big[p\|r\|_{L^\infty([0,T],H^{k-1})}\|\Phi(q)-\Phi)(s)\|_{\mathcal{V}^k}\\
&\qquad + \|\Phi\|_{L^\infty([0,T],\mathcal{V}^k)}\|r(q)-r(s)\|_{H^{k-1}}\Big]\\
&\le \Lambda_2\| (\Phi(q)-\Phi(s), r(q)-r(s)) \|_{X^k},
\end{aligned}
$$

where $\Lambda_2 = \Lambda_2(\|\Phi\|_{L^\infty([0,T],\mathcal{V}^k)}, \|r\|_{L^\infty([0,T],\mathcal{V}^k)}, b, p)$. Finally, we also have that

$$
\begin{aligned}
&\nabla\cdot(|\nabla\Phi|^m\nabla\Phi)(q) - \nabla\cdot(|\nabla\Phi|^m\nabla\Phi)(s)\\
&\quad = \nabla\cdot\{|\nabla\Phi|^m(q)\left[\nabla\Phi(q)-\nabla\Phi(s)\right]) + \nabla\Phi(s)(|\nabla\Phi(q)|^m - |\nabla\Phi(s)|^m\}.
\end{aligned}
$$

Since ∇ has order one, we also have for a generic constant $C = C(b,m)$ independent of T that

$$
\begin{aligned}
&\|B^{-1}(\nabla\cdot(|\nabla\Phi|^m\nabla\Phi)(q) - \nabla\cdot(|\nabla\Phi|^m\nabla\Phi)(s))\|_{H^{k-1}}\\
&\qquad \le C\|(|\nabla\Phi|^m\nabla\Phi)(q) - (|\nabla\Phi|^m\nabla\Phi)(s)\|_{H^{k-2}}\\
&\qquad \le C\|(|\nabla\Phi|^m(q)\left[\nabla\Phi(q)-\nabla\Phi(s)\right])\|_{H^{k-2}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\nabla\Phi(s)\left(|\nabla\Phi(q)|^m - |\nabla\Phi(s)|^m\right)\|_{H^{k-2}} \\
&\leq C\left(\|\Phi(q)\|_{\mathcal{V}^k}^m + \|\Phi(s)\|_{\mathcal{V}^k}^m\right)\|\Phi(q)-\Phi(s)\|_{\mathcal{V}^k} \\
&\leq \Lambda_3(\|\Phi\|_{L^\infty([0,T],\mathcal{V}^k)}, b, m)\|\Phi(q)-\Phi(s)\|_{\mathcal{V}^k}. \qquad \square
\end{aligned}
$$

Proof of Proposition 2.1. From Lemma 2.1, we only need to study the continuity of the operator

$$\mathcal{K}(t) = \int_0^t \mathcal{T}(t-y)\mathcal{G}(U)(y)\,dy.$$

Let t_0 be fixed and $t \in \mathbb{R}$ near t_0. To prove the continuity of $\mathcal{S}$ at t_0, we need to estimate $\mathcal{K}(t) - \mathcal{K}(t_0)$ in X^k. Note that, after a change of variables in both integrals,

$$
\begin{aligned}
\mathcal{K}(t) - \mathcal{K}(t_0) &= \int_0^t \mathcal{T}(y)\mathcal{G}(U)(t-y)\,dy - \int_0^{t_0} \mathcal{T}(y)\mathcal{G}(U)(t_0-y)\,dy \\
&= \int_0^{t_0} \mathcal{T}(y)\left(\mathcal{G}(U)(t-y) - \mathcal{G}_0(U)(t_0-y)\right)dy + \int_{t_0}^t \mathcal{T}(y)\mathcal{G}(U)(t-y)\,dy.
\end{aligned}
$$

Thus, we obtain that

$$
\begin{aligned}
\|\mathcal{K}(t) - \mathcal{K}(t_0)\|_{X^k} \leq& \int_0^{t_0} \|\mathcal{T}(y)\left(\mathcal{G}(U)(t-y) - \mathcal{G}(U)(t_0-y)\right)\|_{X^k}\,dy \\
&+ \int_{t_0}^t \|\mathcal{T}(y)\mathcal{G}(U)(t-y)\|_{X^k}\,dy. \qquad (2.14)
\end{aligned}
$$

As we showed above,

$$
\begin{aligned}
&\|\mathcal{T}(y)\left[\mathcal{G}(U)(t-y) - \mathcal{G}(U)(t_0-y)\right]\|_{X^k} \\
&\quad\leq C(a,b)\|\mathcal{G}(U)(t-y) - \mathcal{G}(U)(t_0-y)\|_{X^k} \\
&\quad\leq C(a,b)\|F(\Phi_x,\Phi_y,r)(t-y) - F(\Phi_x,\Phi_y,r)(t_0-y)\|_{H^{k-3}}.
\end{aligned}
$$

Using this fact and estimates in Theorem 2.1, we have that

$$
\begin{aligned}
&\|\mathcal{T}(y)\left[\mathcal{G}(U)(t-y) - \mathcal{G}(U)(t_0-y)\right]\|_{X^k} \\
&\quad\leq (\Lambda_1+\Lambda_2+\Lambda_3)\|(\Phi(t-y) - \Phi(t_0-y), r(t-y) - r(t_0-y))\|_{X^k}.
\end{aligned}
$$

Moreover, we also have that

$$\|\mathcal{T}(y)\left(\mathcal{G}(U)\right)(t-y)\|_{X^k} \leq (\Lambda_1+\Lambda_2+\Lambda_3)\|(\Phi(t-y), r(t-y))\|_{X^k}.$$

Recall that we are assuming that the following functions are continuous:

$$t \to \|\Phi(t-y) - \Phi(t_0-y)\|_{\mathcal{V}^k}, \quad \text{and} \quad t \to \|(r(t-y) - r(t_0-y))\|_{H^{k-1}}.$$

Then the dominated convergence theorem implies that

$$\lim_{t\to t_0}\int_0^{t_0}\|\Phi(t-y)-\Phi(t_0-y)\|_{\mathcal{V}}\,dy=0,$$

and

$$\lim_{t\to t_0}\int_0^{t_0}\|(r(t-y)-r(t_0-y))\|_{H^1}=0.$$

Moreover,

$$\int_{t_0}^{t}\|\mathcal{T}(y)\,(\mathcal{G}(U))\,(t-y)\|_{X^k}\le(t-t_0)(\Lambda_1+\Lambda_2+\Lambda_3).$$

Using previous estimates in (2.14), we conclude that

$$\lim_{t\to t_0}\|\mathcal{K}(t)-\mathcal{K}(t_0)\|_{X^k}=0. \qquad \square$$

Now we are in position to establish the local existence and uniqueness result for the Cauchy problem associated with the generalized Benney-Luke equation.

Theorem 2.2. *Let k,p,m be as in Lemmas 2.4, 2.5, and 2.6. If $\Phi_0\in\mathcal{V}^k$ and $r_0\in H^{k-1}(\mathbb{R}^2)$, then there exists $T=T(\Phi_0,r_0)>0$ such that the integral equation (2.2) has a unique solution (Φ,r) such that*

$$\Phi\in C^0([0,T],\mathcal{V}^k),\quad r\in C^0([0,T],H^{k-1}(\mathbb{R}^2))\cap C^1([0,T],H^{k-2}(\mathbb{R}^2)).$$

Moreover, equation (1.1) has a unique classical solution $\Phi\in C^0([0,T],\mathcal{V}^k)$, with

$$\Phi_t\in C^0([0,T],H^{k-1}(\mathbb{R}^2))\cap C^1([0,T],H^{k-2}(\mathbb{R}^2)),$$

that satisfies the initial conditions

$$\nabla\Phi(0,\cdot)=\nabla\Phi_0,\quad \Phi_t(0,\cdot)=r_0.$$

Proof. The strategy of the proof will be to show that for some $R>0$, $\mathcal{S}$ is a contraction on $\mathcal{B}_R\subset C^0\left([0,T],X^k\right)$. From the estimate in Lemma 2.1, if $(U^0)^t=(\Phi_0,r_0))\in\mathcal{V}^k\times H^{k-1}(\mathbb{R}^2)$, then

$$\left\|\mathcal{T}(t)U^0\right\|_{X^k}\le C(a,b)\left\|U^0\right\|_{X^k}.$$

Moreover, if $U^t=(\Phi_1,r_1)$ and $V^t=(\Phi_2,r_2)$, then following the same computations as in the proof of Proposition 2.1 we have for some constant $C=C(p,m,\beta,b)$ (independent of T) that

$$\begin{aligned}\|\mathcal{S}(U)-\mathcal{S}(V)\|_{X^k}&\le TC\|U-V\|_{L^\infty([0,T],X^k)}\\&\quad\times\Big[\Big(\|U\|_{L^\infty([0,T],X^k)}+\|V\|_{L^\infty([0,T],X^k)}\Big)^p\end{aligned}$$

$$+ \left(\|U\|_{L^\infty([0,T],X^k)} + \|V\|_{L^\infty([0,T],X^k)} \right)^m \Big],$$

and

$$\|\mathcal{S}(U)\|_{X^k} \le C(a,b)\|U^0\|_{L^\infty([0,T],X^k)} + TC\Big[\|U\|^{p+1}_{L^\infty([0,T],X^k)} + \|U\|^{m+1}_{L^\infty([0,T],X^k)}\Big].$$

Let $R = 2C(a,b)\|U^0\|_{L^\infty([0,T],X^k)}$ and choose $T > 0$ satisfying $2(R^p + R^m)TC_6 < 1$. Under those conditions we have that $\mathcal{S}$ maps $\mathcal{B}$ into $\mathcal{B}$. Then, the contraction mapping theorem guarantees the existence of a fixed point. In other words, there exists a local mild solution for the integral equation (2.2). In order to establish that a mild solution is already a classical solution, we have to use the regularizing effect due to the good behavior of the nonlinear part, since $\mathcal{G}$ already maps X^k into X^k. □

3. Energy estimates

As it is well known, a principle for establishing global existence in time for problems with a Hamiltonian structure is that this follows by local existence and the use of energy estimates. For this particular problem, we will see that a solution Φ of (1.1) conserves the Hamiltonian energy, which has the form

$$\begin{aligned}\mathcal{E}(\Phi,\Phi_t) &= \frac{1}{2}\int_{\mathbb{R}^2}\left[\Phi_t^2 + b|\nabla\Phi_t|^2 + |\nabla\Phi|^2 + a|\Delta\Phi|^2 - \frac{\beta}{m+2}|\nabla\Phi|^{m+2}\right]dx\,dy \\ &= \frac{1}{2}\mathcal{H}(\Phi,\Phi_t) - \frac{\beta}{m+2}\int_{\mathbb{R}^2}|\nabla\Phi|^{m+2}\,dx\,dy. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let k,p,m be as in Lemmas 2.4, 2.5, and 2.6. Then if $\Phi \in C^0([0,T_0),\mathcal{V}^k)$ is a maximal solution of equation (1.1) with*

$$\Phi_t \in C^0([0,T_0),H^{k-1}(\mathbb{R}^2)) \cap C^1([0,T_0),H^{k-2}(\mathbb{R}^2)),$$

with $\nabla\Phi(0,\cdot) = \nabla\Phi_0$, and $\Phi_t(0,\cdot) = r_0$, it follows that

$$\mathcal{E}(\Phi,\Phi_t) = \mathcal{E}(\Phi_0,r_0), \qquad \textit{for all } t \in [0,T_0).$$

Before we proceed, we note that $H^1(\mathbb{R}^2) \hookrightarrow L^l(\mathbb{R}^2)$, for $l \ge 2$ and $L^q(\mathbb{R}^2) \hookrightarrow H^{-1}(\mathbb{R}^2)$ for $1 < q \le 2$. Moreover, for $w \in L^2(\mathbb{R}^2)$ and $r \in H^1(\mathbb{R}^2)$ we have, via a duality argument, that $wr \in L^q(\mathbb{R}^2)$ for $1 < q < 2$, and that

$$\|wr\|_{H^{-1}} \le \|wr\|_{L^q} \le \|w\|_{L^2}\|r\|_{L^{2q/(2-q)}} \le \|w\|_{L^2}\|r\|_{H^1}.$$

In particular, for $p = 1$,

$$\|r\partial^2_{ii}\Phi\|_{H^{-1}} \le \|r\partial^2_{ii}\Phi\|_{L^q} \le \|\partial^2_{ii}\Phi\|_{L^2}\|r\|_{L^{2q/(2-q)}} \le \|\Phi\|_{\mathcal{V}^k}\|r\|_{H^1}. \tag{3.2}$$

On the other hand, for $p > 1$ we have the following result.

Lemma 3.1. *Let $p > 1$ and $1 < \alpha < 2$ such that $p > 2/\alpha$. If $r \in H^{k-1}$, $\Phi \in \mathcal{V}^k$ and $\psi \in L^2(\mathbb{R}^2)$, then we have that $(\partial_i \Phi)^{p-1}\,\psi \in L^\alpha(\mathbb{R}^2)$ with*

$$\|\,(\partial_i\Phi)^{p-1}\,\psi\|_{L^\alpha} \le \|\partial_i\Phi\|^{p-1}_{L^{2\alpha/(2-\alpha)}}\|\psi\|_{L^2} \le \|\Phi\|^{p-1}_{\mathcal{V}^k}\|\psi\|_{L^2},$$

and that $r\,(\partial_i\Phi)^{p-1}\,\psi \in L^\alpha(\mathbb{R}^2)$ with

$$\|r\,(\partial_i\Phi)^{p-1}\,\psi\|_{H^{-1}} \le \|r\|_{H^{k-1}}\|\Phi\|^{p-1}_{\mathcal{V}^k}\|\psi\|_{L^2}.$$

Proof. Let $\gamma = \alpha/(\alpha-1)$ and take $w \in L^\gamma(\mathbb{R}^2)$. Then, since $p > 2/\alpha$ if and only if $\frac{2\alpha(p-1)}{2-\alpha} > 2$, we have from the Hölder inequality that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2}(\partial_i\Phi)^{p-1}\,\psi w\,dx\Big|\\
&\le \Big(\int_{\mathbb{R}^2}|\partial_i\Phi|^{\frac{2\alpha(p-1)}{2-\alpha}}\,dx\Big)^{\frac{2-\alpha}{2\alpha}}\Big(\int_{\mathbb{R}^2}|\psi|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^2}|w|^\gamma\,dx\Big)^{1/\gamma}\\
&\le \|\partial_i\Phi\|^{p-1}_{L^{\frac{2\alpha(p-1)}{2-\alpha}}}\|\psi\|_{L^2}\|w\|_{L^\gamma} \le \|\Phi\|^{p-1}_{\mathcal{V}^k}\|\psi\|_{L^2}\|w\|_{L^\gamma}.
\end{aligned}$$

To prove the second inequality, we proceed as before. Let $1 < q < 2$ be such that $q < \alpha$. From the Hölder inequality,

$$\begin{aligned}
&\|r\,(\partial_i\Phi)^{p-1}\,\psi\|_{H^{-1}} \le C\|r\,(\partial_i\Phi)^{p-1}\,\psi\|_{L^q}\\
&\le C\Big(\int_{\mathbb{R}^2}|r|^{\frac{q\alpha}{\alpha-q}}\,dx\Big)^{\frac{\alpha-q}{\alpha q}}\|\,(\partial_i\Phi)^{p-1}\,\psi\|_{L^\alpha} \le C\|r\|_{H^{k-1}}\|\Phi\|^{p-1}_{\mathcal{V}^k}\|\psi\|_{L^2},
\end{aligned}$$

since $q\alpha/(\alpha-q) > 2$ if and only if $q > 2\alpha/(2+\alpha)$, and $q > 1 > 2\alpha/(2+\alpha)$. □

As a consequence of the previous result, we have the following.

Corollary 3.1. *Let either $p = 1$ and $\alpha = 2$ or $1 < \alpha < 2$ and $p > 2/\alpha$. If $r \in H^{k-1}$ and $\Phi \in \mathcal{V}^k$, we then have that*

(1) *$(\partial_i\Phi)^{p-1}\,\partial_{ii}\Phi \in L^\alpha(\mathbb{R}^2)$, and*

$$\begin{aligned}
\|\,(\partial_i\Phi)^{p-1}\,\partial_{ii}\Phi\|_{L^\alpha} &\le \|\partial_i\Phi\|^{p-1}_{L^{\frac{2\alpha(p-1)}{2-\alpha}}}\|\partial_{ii}\Phi\|_{L^2} \quad (\textit{only for } p>1)\\
&\le \|\Phi\|^p_{\mathcal{V}^k}.
\end{aligned}$$

(2) *$r\,(\partial_i\Phi)^{p-1}\,\partial_{ii}\Phi \in L^\alpha(\mathbb{R}^2)$, and*

$$\begin{aligned}
\|r\,(\partial_i\Phi)^{p-1}\,\partial_{ii}\Phi\|_{H^{-1}} &\le C\|r\,(\partial_i\Phi)^{p-1}\,\partial_{ii}\Phi\|_{L^\alpha}\\
&\le \|r\|_{H^{k-1}}\|\partial_i\Phi\|^{p-1}_{L^{\frac{2\alpha(p-1)}{2-\alpha}}}\|\partial_{ii}\Phi\|_{L^2} \quad (\textit{only for } p>1)\\
&\le \|r\|_{H^{k-1}}\|\Phi\|^p_{\mathcal{V}^k},
\end{aligned}$$

(3) $r\,(\partial_i\Phi)^p \in L^\alpha(\mathbb{R}^2)$ *and*

$$\|r\,(\partial_i\Phi)^p\|_{H^{-1}} \le \|r\,(\partial_i\Phi)^p\|_{L^\alpha} \le \|r\|_{H^{k-1}}\|\Phi\|^p_{\mathcal{V}^k}.$$

On the other hand, we have the following.

Lemma 3.2. *Let either $l > 1$ and $\gamma < 2$, or $l = 1$ and $\gamma = 2$ be such that $l\gamma \ge 2$. Then the function $L : \mathcal{V}^k \to L^\gamma$ defined as $L(\Phi) = (\partial_i\Phi)^l$ is continuous. Moreover,*

$$\begin{aligned}&\|L(\Phi) - L(\Psi)\|_\gamma \\ &\le C(l)\left(\|\partial_i\Phi\|^{(l-1)}_{L^{2\gamma(l-1)/(2-\gamma)}} + \|\partial_i\Psi\|^{(l-1)}_{L^{2\gamma(l-1)/(2-\gamma)}}\right)\|\partial_i\Phi - \partial_i\Psi\|_{L^2} \\ &\le C(l,\gamma)\left(\|\Phi\|^{l-1}_{\mathcal{V}^k} + \|\Psi\|^{l-1}_{\mathcal{V}^k}\right)\|\Phi - \Psi\|_{\mathcal{V}^k}.\end{aligned} \tag{3.3}$$

Proof. The proof for the case $l = 1$ and $\gamma = 2$ follows directly since $H^1(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$. Now assume that $l > 1$ and $\gamma < 2$. Then, from the mean value inequality,

$$|(\partial_i\Phi)^l - (\partial_i\Psi)^l| \le C(l)(|\partial_i\Phi|^{l-1} + |\partial_i\Psi|^{l-1})|\partial_i\Phi - \partial_i\Psi|.$$

Then, by applying the Hölder inequality and using the fact that $2\gamma(l-1)/(2-\gamma) \ge 2$ if and only if $l\gamma \ge 2$, we get the desired result. □

Proof of Theorem 3.1. Hereafter, for $l \in \mathbb{R}$, we set $\langle\cdot,\cdot\rangle_{-l,l}$ as the dual pairing between $H^l(\mathbb{R}^2)$ and its dual space H^{-l}. As we established in previous results, we have for $\Phi \in \mathcal{V}^k$ that $\partial_i\Phi_t\,(\partial_i\Phi)^p = \partial_i\left[\Phi_t\,(\partial_i\Phi)^p\right] - \Phi_t\partial_i\,(\partial_i\Phi)^p \in H^{-1}$, and also $K(\partial_x\Phi, \partial_y\Phi) = \nabla\cdot(|\nabla\Phi|^m\nabla\Phi) \in H^{-1}$ (see Lemma 2.6). Moreover, for either $p = 1$ and $\alpha = 2$ or $1 < \alpha < 2$ and $p > 2/\alpha$,

$$F(\Phi_x, \Phi_y, \Phi_t) \in C^0([0,T_0), L^\alpha(\mathbb{R}^2)) \subset C^0([0,T_0), H^{-1}(\mathbb{R}^2)). \tag{3.4}$$

So, we have that

$$\begin{aligned}&\langle\Phi_t\left[\partial_x\left[(\partial_x\Phi)^p\right] + \partial_y\left[(\partial_y\Phi)^p\right]\right] + 2\left[(\partial_x\Phi)^p\,\Phi_{xt} + (\partial_y\Phi)^p\,\Phi_{yt}\right], \Phi_t\rangle_{-1,1} \\ &\qquad\qquad = -\int_{\mathbb{R}^2}\nabla\left(\Phi_t^2\cdot\begin{pmatrix}(\Phi_x)^p\\(\Phi_y)^p\end{pmatrix}\right)dx\,dy = 0,\end{aligned}$$

and

$$\langle\nabla\cdot(|\nabla\Phi|^m\nabla\Phi), \Phi_t\rangle_{-1,1} = -\frac{1}{m+2}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\Phi|^{m+2}\,dx\,dy.$$

In other words, we have already shown that

$$\langle F(\partial_x\Phi, \partial_y\Phi, \partial_t\Phi), \partial_t\Phi\rangle_{-1,1} = -\frac{1}{m+2}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\Phi|^{m+2}\,dx\,dy.$$

Since $\Phi_{tt} - \Delta\Phi \in H^{k-2}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$, we also have that

$$\langle \Phi_{tt} - \Delta\Phi, \Phi_t \rangle_{-1,1} = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \left[\Phi_t^2 + |\nabla\Phi|^2\right]\, dx\, dy.$$

Now, we observe that $\Delta^2\Phi$ and $\Delta\Phi_{tt}$ are elements of H^{-2}. Then by the Banach theorem there exist $v, w \in H^{-1}$ such that $v|_{H^2} = \Delta^2\Phi$ and $w|_{H^2} = \Delta\Phi_{tt}$. Moreover, if $g \in H^1(\mathbb{R}^2)$ and $(g_n) \subset H^2(\mathbb{R}^2)$ is such that $g_n \to g$ in $H^1(\mathbb{R}^2)$, then

$$v(g) = \lim_{n\to\infty} \left\langle \Delta^2\Phi, g_n \right\rangle_{-2,2}, \quad w(g) = lim_{n\to\infty} \left\langle \Delta\Phi_{tt}, g_n \right\rangle_{-2,2},$$

and so for $g \in H^1(\mathbb{R}^2)$ and $(g_n) \subset H^2(\mathbb{R}^2)$ such that $g_n \to g$ in $H^1(\mathbb{R}^2)$,

$$(av - bw)(g) = lim_{n\to\infty} \left\langle a\Delta^2\Phi - b\Delta\Phi_t, g_n \right\rangle_{-2,2}.$$

Finally, to evaluate $av - bw$ in $\Phi_t \in H^1(\mathbb{R}^2)$, we will use a regularization procedure. Take a positive test function $\phi \in C_0^\infty(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \phi = 1$, and define the approximation of unity

$$\phi^{(j)}(x, y) = j^2\phi(jx, jy).$$

The first observation is that, for $l \geq 1$, we have $\phi^{(j)} * \partial_i\Phi \in C^1([0, T_0), H^l(\mathbb{R}^2))$, and $\phi^{(j)} * \partial_t\Phi \in C^0([0, T_0), H^l(\mathbb{R}^2))$, where $*$ denotes the convolution in space. As a consequence of this, for $\Psi^{(j)} = \phi^{(j)} * \Phi$,

$$\begin{aligned} v(\Phi_t) &= \lim_{j\to\infty} \left\langle \Delta^2(\Phi - \Psi^{(j)}), \Psi_t^{(j)} \right\rangle_{-2,2} + \lim_{j\to\infty} \left\langle \Delta^2\Psi^{(j)}, \Psi_t^{(j)} \right\rangle_{-2,2} \\ &= \lim_{j\to\infty} \left[\left\langle \Delta^2(\Phi - \Psi^{(j)}), \Psi_t^{(j)} \right\rangle_{-2,2} + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\Delta\Psi^{(j)}|^2\, dx\, dy \right] \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\Delta\Phi|^2\, dx\, dy, \end{aligned}$$

since we have that

$$\begin{aligned} \lim_{j\to\infty} \left| \left\langle \Delta^2(\Phi - \Psi^{(j)}), \Psi_t^{(j)} \right\rangle_{-2,2} \right| &\leq \lim_{j\to\infty} \|\Delta^2(\Phi - \Psi^{(j)})\|_{H^{-2}} \|\Psi_t^{(j)}\|_{H^2} \\ &\leq \lim_{j\to\infty} \|\nabla\Phi - \nabla\Psi^{(j)}\|_{H^1} \|\Psi_t^{(j)}\|_{H^2} = 0. \end{aligned}$$

In the same way,

$$w(\Phi_t) = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla\Phi_t|^2\, dx\, dy.$$

In other words,

$$\left(\Phi_{tt} - \Delta\Phi + a\Delta^2\Phi - b\Delta\Phi_{tt} + F(\Phi_x, \Phi_y, \Phi_t)\right)(\Phi_t)$$

$$= \frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^2}\left[\Phi_t^2 + b|\nabla\Phi_t|^2 + |\nabla\Phi|^2 + a|\Delta\Phi|^2 - \frac{\beta}{m+2}|\nabla\Phi|^{m+2}\right]dx\,dy = 0.$$

So, we have shown that $\mathcal{E}(\Phi, \Phi_t)$ is constant, and so,

$$\mathcal{E}(\Phi, \Phi_t) = \mathcal{E}(\Phi_0, r_0), \text{ for } 0 \le t < T_0. \qquad \square$$

4. Global existence

Now, we are ready to establish the existence of a global solution. For $\beta \le 0$, we use the standard energy method. For $\beta > 0$ and $p \ge 1$, we extend the results obtained by S. Wang, G. Xu, G. Chen in ([4]) when $p = 1$.

Case 1: $\beta \le 0$.

Theorem 4.1. *Let k be an integer, p, m real numbers such that*

(1) $k = 2$, $p \ge 1$ *and* $m > 0$, *or*
(2) $k = 3, 4$ *and* $m, p \ge 2$
(3) $m, p \ge k - 1 \ge 3$.

Let $T_0 > 0$, $\Phi_0 \in \mathcal{V}^k$, $r_0 \in H^{k-1}(\mathbb{R}^2)$ and $\beta \le 0$. Then equation (1.1) *has a unique global solution $\Phi \in C^0([0,\infty), \mathcal{V}^k)$ such that*

$$\Phi_t \in C^0([0,\infty), H^{k-1}(\mathbb{R}^2)) \cap C^1([0,\infty), H^{k-2}(\mathbb{R}^2)),$$

with $\nabla\Phi(x,y,0) = \nabla\Phi_0(x,y)$, and $\Phi_t(x,y,0) = r_0(x,y)$. Moreover, $\mathcal{H}(\Phi,\Phi_t)$ is bounded independent of T_0.

Proof. The first observation is that there exists a positive constant $M = M(a,b) > 1$ such that

$$M^{-1}\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{X^2}^2 \le \mathcal{H}(\phi,\psi) \le M\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{X^2}^2. \tag{4.1}$$

In other words, $\sqrt{\mathcal{H}(\cdot,\cdot)}$ is an equivalent norm to the norm in $\|\cdot\|_{X^2}$.

Let us assume that $k = 2$ and suppose that there exist $0 < T_{\max} < \infty$ such that the initial-value problem associated with equation (1.1) has a unique local classical solution Φ on the interval $[0, T_{\max})$. In this case, we have that

$$\lim_{t\uparrow T_{\max}}\left\|\begin{pmatrix}\Phi\\ \Phi_t\end{pmatrix}\right\|_{X^2} = \infty, \tag{4.2}$$

But from the previous lemma, we have that $\mathcal{E}(\cdot,\cdot)$ is conserved in time along classical solutions, and

$$\mathcal{H}(q_0, r_0) - \frac{2\beta}{m+2}\|\nabla\Phi_0\|_{m+2}^{m+2} = \mathcal{H}\left(q(t,\cdot), r(t,\cdot)\right) - \frac{2\beta}{m+2}\|\nabla\Phi\|_{m+2}^{m+2},$$

for all $t \in [0, T_{\max})$. Since $\beta \leq 0$, then we have that

$$\mathcal{H}\left(q(t,\cdot), r(t,\cdot)\right) \leq \mathcal{H}(q_0, r_0) + \frac{2|\beta|}{m+2}\|\nabla\Phi_0|_{m+2}^{m+2}, \text{ for all } t \in [0, T_{\max}).$$

As a consequence of this and inequality (4.1)

$$\begin{aligned}\left\|\begin{pmatrix}\Phi\\ \Phi_t\end{pmatrix}\right\|_{\mathcal{V}\times H^1}^2 &\leq M\left(\mathcal{H}(q_0, r_0) + \frac{2|\beta|}{m+2}\|\nabla\Phi_0\|_{m+2}^{m+2}\right)\\ &\leq C(m,\beta)\left\|\begin{pmatrix}\Phi_0\\ r_0\end{pmatrix}\right\|_{\mathcal{V}\times H^1}^2\left(1 + \left\|\begin{pmatrix}\Phi_0\\ r_0\end{pmatrix}\right\|_{\mathcal{V}\times H^1}^m\right),\end{aligned}$$

which contradicts the limit (4.2). In other words, Φ is a global classical solution of the initial-value problem associated with equation (1.1).

Now, assume that $k \geq 3$. From the local result Proposition 2.2, there exists a maximal solution $\Phi \in C^0([0,T_1), \mathcal{V}^k)$ such that

$$\Phi_t \in C^0([0,T_1), H^{k-1}(\mathbb{R}^2)) \cap C^1([0,T_1), H^{k-2}(\mathbb{R}^2)),$$

with $\nabla\Phi(x,y,0) = \nabla\Phi_0(x,y)$, and $\Phi_t(x,y,0) = r_0(x,y)$. Suppose that $T_1 < T_0$, and that $\|\nabla\Phi\|_{H^{k-2}}$ and $\|\Phi_t\|_{H^{k-2}}$ are bounded on $[0,T_1)$.

Let $\alpha \in \mathbb{N}^2$ be any index with $|\alpha| = k-2$. Then, by assumption, we have that $D^\alpha\Phi_t \in H^1(\mathbb{R}^2)$ and that $D^\alpha(\Phi_{tt})$, $D^\alpha(\Delta\Phi) \in L^2$. Moreover, we also have that

$$\begin{aligned}\langle D^\alpha(\Phi_{tt} - \Delta\Phi), D^\alpha\Phi_t\rangle_{-1,1} &= \int_{\mathbb{R}^2} D^\alpha(\Phi_{tt} - \Delta\Phi)D^\alpha\Phi_t\, dxdy\\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\left[|D^\alpha\Phi_t|^2 + |D^\alpha\nabla\Phi|^2\right]\, dx\, dy.\end{aligned}$$

On the other hand, $D^\alpha\Delta^2\Phi$ and $D^\alpha\Delta\Phi_{tt}$ are elements of H^{-2}. Then by the Hahn-Banach theorem there exist $v_\alpha, w_\alpha \in H^{-1}$ such that $v_\alpha|_{H^2} = \Delta^2\Phi$ and $w_\alpha|_{H^2} = \Delta\Phi_{tt}$. Moreover, if $g \in H^1(\mathbb{R}^2)$ and $(g_n) \subset H^2(\mathbb{R}^2)$ are such that $g_n \to g$ in $H^1(\mathbb{R}^2)$,

$$(av_\alpha - bw_\alpha)(g) = lim_{n\to\infty}\left\langle D^\alpha(a\Delta^2\Phi - b\Delta\Phi_t), g_n\right\rangle_{-2,2}.$$

Now to evaluate $av_\alpha - bw_\alpha$ in $D^\alpha\Phi_t \in H^1(\mathbb{R}^2)$, we proceed as in Theorem 2.2. If we use the same notation $\Psi^{(j)} = \Phi^{(j)} * \Phi$ and the same type of arguments, we have that

$$\begin{aligned}(av_\alpha - bw_\alpha)(D^\alpha\Phi_t) &= \frac{1}{2}\lim_{j\to\infty}\frac{d}{dt}\left[\int_{\mathbb{R}^2} a|D^\alpha\Delta\Psi^{(j)}|^2 + b|D^\alpha\nabla\Psi_t^{(j)}|^2\, dx\, dy\right]\\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|aD^\alpha\Delta\Phi|^2 + b|D^\alpha\nabla\Phi_t|^2\, dx\, dy.\end{aligned}$$

In other words,

$$\begin{aligned}&\left[D^\alpha\left(\Phi_{tt}-\Delta\Phi+a\Delta^2\Phi-b\Delta\Phi_{tt}\right)\right](D^\alpha\Phi_t)\\&\quad=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\left[|D^\alpha\Phi_t|^2+b|D^\alpha\nabla\Phi_t|^2+|D^\alpha\nabla\Phi|^2+a|D^\alpha\Delta\Phi|^2\right]\,dx\,dy.\end{aligned}$$

We also have that $D^\alpha F(\Phi_x,\Phi_y,\Phi_t)\in H^{-(k-1)}$. Thus we assure the existence of $F_\alpha\in H^{-1}$ such that $F_\alpha|_{H^{k-1}}=D^\alpha F(\Phi_x,\Phi_y,\Phi_t)$. Moreover,

$$\begin{aligned}&\left|\left\langle D^\alpha F(\Phi_x,\Phi_y,\Phi_t)-D^\alpha F(\Psi_x^{(j)},\Psi_y^{(j)},\Psi_t^{(j)}),D^\alpha\Psi_t^{(j)}\right\rangle_{-(k-1),k-1}\right|\\&\qquad\le\|F(\Phi_x,\Phi_y,\Phi_t)-F(\Psi_x^{(j)},\Psi_y^{(j)},\Psi_t^{(j)})\|_{H^{-1}}\|D^\alpha\Psi_t^{(j)}\|_{H^{k-1}}.\end{aligned}$$

If we assume that $p=1$, then using the fact that $\partial_i\Phi,\partial_s\Phi\in H^{k-1}(\mathbb{R}^2)\hookrightarrow H^2(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$, we conclude directly that

$$\begin{aligned}&\|\partial_{ii}^2\Phi\Phi_s-\partial_{ii}^2\Psi^{(j)}\Psi_s^{(j)}\|_{L^2}\\&\le\|\partial_{ii}^2\Phi\|_{L^2}\|\Phi_s-\Psi_s\|_{H^1}+\|\Psi_s\|_{H^1}\|\partial_{ii}^2\Phi_s-\partial_{ii}^2\Psi_s\|_{L^2},\\&\|\partial_i\Phi\partial_i\Phi_s-\partial_i\Psi^{(j)}\partial_i\Psi_s^{(j)}\|_{L^2}\\&\le\|\partial_i\Phi\|_{H^1}\|\partial_i\Phi_s-\partial_i\Psi_s\|_{L^2}+\|\partial_i\Psi_s\|_{H^1}\|\partial_i\Phi_s-\partial_i\Psi_s\|_{L^2}.\end{aligned}$$

Now, for $p>1$, we choose $1<\alpha<2$ such that $p>2/\alpha$. Then since $\Phi_s,\Psi_s\in H^{k-1}(\mathbb{R}^2)\hookrightarrow H^2(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$, we have that

$$\begin{aligned}&\|(\partial_i\Phi)^{p-1}\partial_{ii}^2\Phi\Phi_s-(\partial_i\Psi^{(j)})^{p-1}(\partial_{ii}^2\Psi^{(j)})(\Psi_s^{(j)})\|_{L^\alpha}\\&\le\|(\partial_i\Phi)^{p-1}\partial_{ii}^2\Phi(\Phi_s-\Psi_s^{(j)})\|_{L^\alpha}\\&\qquad+\|\Psi_s^{(j)}\left[(\partial_i\Phi)^{p-1}\partial_{ii}^2\Phi-(\partial_i\Psi^{(j)})^{p-1}\partial_{ii}^2\Psi^{(j)}\right]\|_{L^\alpha}\\&\le\|(\partial_i\Phi)^{p-1}\partial_{ii}^2\Phi\|_{L^\alpha}\|\Phi_s-\Psi_s^{(j)}\|_{H^1}\\&\qquad+\|\Psi_s^{(j)}\|_{H^1}\|(\partial_i\Phi)^{p-1}\partial_{ii}^2\Phi-(\partial_i\Psi^{(j)})^{p-1}\partial_{ii}^2\Psi^{(j)}\|_{L^\alpha}.\end{aligned}$$

Since we have that $p>2/\alpha$ is equivalent to $\frac{2(p-1)\alpha}{2-\alpha}\ge 2$, we have the following estimates:

$$\begin{aligned}&\|(\partial_i\Phi)^{p-1}\partial_{ii}^2\Phi-(\partial_i\Psi^{(j)})^{p-1}\partial_{ii}^2\Psi^{(j)}\|_{L^\alpha}\\&\le\|(\partial_i\Phi)^{p-1}(\partial_{ii}^2\Phi-\partial_{ii}^2\Psi^{(j)})\|_{L^\alpha}+\|\partial_{ii}^2\Psi^{(j)}((\partial_i\Phi)^{p-1}-(\partial_i\Psi^{(j)})^{p-1})\|_{L^\alpha}\\&\le\|\partial_i\Phi\|_{L^{\frac{2\alpha(p-1)}{2-\alpha}}}\|\partial_{ii}^2\Phi-\partial_{ii}^2\Psi^{(j)}\|_{L^2}+\|\partial_{ii}^2\Psi^{(j)}\|_{L^2}\|(\partial_i\Phi)^{p-1}-(\partial_i\Psi^{(j)})^{p-1}\|_{L^{\frac{2\alpha}{2-\alpha}}}\\&\le\|\Phi\|_{\mathcal{V}^k}^{p-1}\|\partial_{ii}^2\Phi-\partial_{ii}^2\Psi^{(j)}\|_{L^2}+\|\partial_{ii}^2\Psi^{(j)}\|_{L^2}\|(\partial_i\Phi)^{p-1}-(\partial_i\Psi^{(j)})^{p-1}\|_{L^{\frac{2\alpha}{2-\alpha}}}.\end{aligned}$$

But we have that $\Psi_s^{(j)} \to \Phi_s$ in $L^{\frac{q\alpha}{q-\alpha}}$, and $\partial_{ii}^2\Psi_s^{(j)} \to \partial_{ii}^2\Phi_s$ in L^2. Then inequality (3.3) in Lemma 3.2 implies that

$$\|(\partial_i\Phi)^{p-1}\partial_{ii}^2\Phi\Phi_s - (\partial_i\Psi^{(j)})^{p-1}(\partial_{ii}^2\Psi^{(j)})(\Psi_s^{(j)})\|_{L^\alpha} \to 0, \quad \text{as} \quad j \to \infty.$$

On the other hand, for $p > 1$ and $p > 2/\alpha$,

$$\begin{aligned}
&\|(\partial_i\Phi)^p\partial_i\Phi_s - (\partial_i\Psi^{(j)})^p(\partial_i\Psi_s^{(j)})\|_{L^\alpha} \\
&\le \|(\partial_i\Phi)^p(\partial_i\Phi_s - \partial_i\Psi_s^{(j)})\|_{L^\alpha} + \|\partial_i\Psi_s^{(j)}\left[(\partial_i\Phi)^p - (\partial_i\Psi^{(j)})^p\right]\|_{L^\alpha} \\
&\le \|\partial_i\Phi\|^p_{L^{\frac{2p\alpha}{2-\alpha}}}\|\partial_i\Phi_s - \partial_i\Psi_s^{(j)}\|_{L^2} + \|\partial_i\Psi_s^{(j)}\|_{L^2}\|(\partial_i\Phi)^p - (\partial_i\Psi^{(j)})^p\|_{L^{\frac{2p\alpha}{2-\alpha}}} \\
&\le \|\Phi\|^p_{\mathcal{V}^k}\|\partial_i\Phi_s - \partial_i\Psi_s^{(j)}\|_{L^2} + \|\partial_i\Psi_s^{(j)}\|_{L^2}\|(\partial_i\Phi)^p - (\partial_i\Psi^{(j)})^p\|_{L^{\frac{2p\alpha}{2-\alpha}}}.
\end{aligned}$$

Again, using that $\partial_i\Psi_s^{(j)} \to \partial_i\Phi_s$ in L^2, inequality (3.3) in Lemma 3.2 implies that

$$\|(\partial_i\Phi)^p\partial_i\Phi_s - (\partial_i\Psi^{(j)})^p(\partial_i\Psi_s^{(j)})\|_{L^\alpha} \to 0, \quad \text{as} \quad j \to \infty.$$

In other words, for either $p = 1$ and $\alpha = 2$ or $p > 1$ and $p > 2/\alpha$,

$$\|G_i(\partial_i\Phi, \Phi_s) - G_i(\phi^{(j)} * \partial_i\Phi, \phi^{(j)} * \Phi_s)\|_{L^\alpha} \to 0, \quad \text{as} \quad j \to \infty.$$

In a similar fashion, it can be established that

$$\|H_i(\partial_i\Phi, \partial_i\Phi_s) - H_i(\phi^{(j)} * \partial_i\Phi, \phi^{(j)} * \partial_i\Phi_s)\|_{L^\alpha} \to 0, \quad \text{as} \quad j \to \infty.$$

On the other hand, in Lemma 2.6 we obtain for $m > 0$ and $k = 2$ that

$$\|\nabla\Phi|^m\Delta\Phi - |\nabla\Psi^{(j)}|^m\Delta\Psi^{(j)}\|_{L^2} \le \left(\|\Phi\|^m_{\mathcal{V}^2} + \|\Psi^{(j)}\|^m_{\mathcal{V}^2}\right)\|\Phi - \Psi^{(j)}\|_{\mathcal{V}^2}.$$

In other words, we have established that

$$\|K(\partial_x\Phi, \partial_y\Phi) - K(\phi^{(j)} * \partial_x\Phi, \phi^{(j)} * \partial_y\Phi)\|_{L^2} \to 0, \quad \text{as} \quad j \to \infty.$$

Since $L^\alpha(\mathbb{R}^2) \hookrightarrow H^{-1}$ for either $p = 1$ and $\alpha = 2$ or $p > 1$ and $1 < \alpha < 2$, then

$$\begin{aligned}
\|G_i(\partial_i\Phi, \partial_s\Phi) - G_i(\partial_i\Psi^{(j)}, \partial_s\Psi^{(j)})\|_{H^{-1}} \to 0, \quad \text{as} \quad j \to \infty. \\
\|H_i(\partial_i\Phi, \partial_i\partial_s\Phi) - H_i(\partial_x\Psi^{(j)}, \partial_i\partial_s\Psi^{(j)})\|_{H^{-1}} \to 0, \quad \text{as} \quad j \to \infty, \\
\|K(\partial_x\Phi, \partial_y\Phi) - K(\partial_x\Psi^{(j)}, \partial_y\Psi^{(j)})\|_{H^{-1}} \to 0, \quad \text{as} \quad j \to \infty.
\end{aligned}$$

As a direct consequence of the previous estimates, we are able to conclude that

$$\begin{aligned}
F_\alpha(D^\alpha\Phi_t) &= \lim_{j\to\infty}\left\langle D^\alpha F(\Phi_x, \Phi_y, \Phi_t), D^\alpha\Psi_t^{(j)}\right\rangle_{-(k-1),k-1} \\
&= \lim_{j\to\infty}\left\langle D^\alpha F(\Psi_x^{(j)}, \Psi_y^{(j)}, \Psi_t^{(j)}), D^\alpha\Psi_t^{(j)}\right\rangle_{-(k-1),k-1}
\end{aligned}$$

$$= \lim_{j\to\infty} \int_{\mathbb{R}^2} D^\alpha F(\Psi_x^{(j)}, \Psi_y^{(j)}, \Psi_t^{(j)}) D^\alpha \Psi_t^{(j)}\, dx\, dy.$$

Thus, we have that

$$\frac{1}{2}\frac{d\mathcal{H}_\alpha(\Phi,\Phi_t)}{dt} = -\lim_{j\to\infty} \int_{\mathbb{R}^2} D^\alpha F(\Psi_x^{(j)}, \Psi_y^{(j)}, \Psi_t^{(j)}) D^\alpha \Psi_t^{(j)}\, dx\, dy, \qquad (4.3)$$

where

$$\mathcal{H}_\alpha(\Phi,\Phi_t) = \|D^\alpha \Phi_t\|_{L^2}^2 + b\|D^\alpha \nabla \Phi_t\|_{L^2}^2 + \|D^\alpha \nabla\Phi\|_{L^2}^2 + a\|D^\alpha \Delta\Phi\|_{L^2}^2.$$

Now assume that $p \geq k-1 \geq 3$. In this case, we have that $H^{|\alpha|}(\mathbb{R}^2)$ is an algebra, since $|\alpha| \geq 2$. Recall that we are assuming that $\|\Phi_t\|_{|\alpha|}$ and $\|\nabla\Phi\|_{|\alpha|}$ are bounded in $[0,T_1)$. First, we observe that

$$\|F(\Psi_x^{(j)}, \Psi_y^{(j)}, \Psi_t^{(j)})\|_{H^{|\alpha|}} \leq \||\nabla\Psi^{(j)}|^m \nabla\Psi^{(j)}\|_{H^{|\alpha|}} +$$
$$\sum_{i=1}^{2}\left(p\|\Psi_t^{(j)}\|_{H^{|\alpha|}}\|(\partial_i\Psi^{(j)})^{p-1}\partial_{ii}^2\Psi^{(j)}\|_{H^{|\alpha|}} + 2\|\partial_i\Psi_t^{(j)}\|_{H^{|\alpha|}}\|(\partial_i\Psi^{(j)})^p\|_{H^{|\alpha|}}\right).$$

Using the fact that $H^{|\alpha|}(\mathbb{R}^2)$ is an algebra we conclude that

$$\begin{aligned}
\|(\partial_i\Psi^{(j)})^{p-1}\partial_{ii}^2\Psi^{(j)}\|_{H^{|\alpha|}} &\leq \|(\partial_i\Psi^{(j)})^{p-1}\|_{H^{|\alpha|}}\|\Delta\Psi^{(j)}\|_{H^{|\alpha|}},\\
\|(\partial_i\Psi^{(j)})^{p}\|_{H^{|\alpha|}} &\leq \|(\partial_i\Psi^{(j)})^{p-1}\|_{H^{|\alpha|}}\|\nabla\Psi^{(j)}\|_{H^{|\alpha|}},\\
\||\nabla\Psi^{(j)}|^m\nabla\Psi^{(j)}\|_{H^{|\alpha|}} &\leq \||\nabla\Psi^{(j)}|^m\|_{H^{|\alpha|}}\|\nabla\Psi^{(j)}\|_{H^{|\alpha|}}.
\end{aligned}$$

But for some constant independent of k we have that

$$\|(\partial_i\Psi^{(j)})^m\|_{H^{|\alpha|}},\ \|(\partial_i\Psi^{(j)})^{p-1}\|_{H^{|\alpha|}} \leq C,$$

since, for $|\rho| \leq |\alpha|$, we have that $D^\rho\left((\partial_i\Psi^{(j)})^{p-1}\right)$ and $D^\rho\left(|\nabla\Psi^{(j)}|^m\right)$ are controlled by $\|\nabla\Psi^{(j)}\|_{H^{|\alpha|}}$. These facts show that, for some constant $C = C(p)$,

$$\begin{aligned}
&\|F(\Psi_x^{(j)}, \Psi_y^{(j)}, \Psi_t^{(j)})\|_{H^{|\alpha|}}\\
&\leq C\left(\|\Psi_t^{(j)}\|_{H^{|\alpha|}}\|\Delta\Psi^{(j)}\|_{H^{|\alpha|}} + \left(\|\nabla\Psi_t^{(j)}\|_{H^{|\alpha|}} + C\right)\|\nabla\Psi^{(j)}\|_{H^{|\alpha|}}\right).
\end{aligned}$$

Then we have for $t \in [0,T_1)$ that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left|D^\alpha F(\Psi_x^{(j)}, \Psi_y^{(j)}, \Psi_t^{(j)}) D^\alpha\Psi_t^{(j)}\right| dx\, dy\\
&\leq \|F(\Psi_x^{(j)}, \Psi_y^{(j)}, \Psi_t^{(j)})\|_{H^{|\alpha|}}\|\Psi_t^{(j)}\|_{H^{|\alpha|}}\\
&\leq C_1\Big(1 + \|D^\alpha\Psi_t^{(j)}\|_{L^2}^2 + b\|D^\alpha\nabla\Psi_t^{(j)}\|_{L^2}^2 + \|D^\alpha\nabla\Psi^{(j)}\|_{L^2}^2 + a\|D^\alpha\Delta\Psi^{(j)}\|_{L^2}^2\Big).
\end{aligned}$$

So, for $t \in [0, T_1)$, and after taking the limit as $j \to \infty$ in inequality (4.3),

$$\frac{d\mathcal{H}_\alpha(\Phi, \Phi_t)}{dt} \le C_2(1 + \mathcal{H}_\alpha(\Phi, \Phi_t)),$$

and so integrating from 0 to $t \in (0, T_1)$,

$$\mathcal{H}_\alpha(\Phi, \Phi_t) \le (\mathcal{H}_\alpha(\Phi_0, r_0) + C_2 t) + C_2 \int_0^t \mathcal{H}_\alpha(\Phi, \Phi_s)\, ds.$$

From Gronwall's inequality we conclude that $\mathcal{H}_\alpha(\Phi, \Phi_t)$ is bounded in $[0, T_1)$ by a constant depending only on $\mathcal{H}_\alpha(\Phi_0, r_0)$ and T_0. The case $k = 3$ follows by a regularization process as in the discussion in Proposition 3.1.

Case 2: $\beta > 0$**.** In this section, we will adapt to the general Benney-Luke model the results obtained in [4], which correspond to $p = 1$. For the sake of completeness we will include the proofs of some results. Before we go further, we adapt our notation to that in [4]: For $u \in \mathcal{V}^2$, we define

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^2} a|\Delta u|^2 + |\nabla u|^2 - \Big(\frac{2\beta}{m+2}\Big)|u|^{m+2}\, dx\, dy,$$
$$I(u) = \int_{\mathbb{R}^2} a|\Delta u|^2 + |\nabla u|^2 - \beta |u|^{m+2}\, dx\, dy.$$

Note in particular that

$$J(u) = \frac{m}{2(m+2)} \left(a\|\Delta u\|_2^2 + \|\nabla u\|_2^2\right) + \frac{1}{m+2} I(u). \tag{4.4}$$

We also observe for $m > 0$ that the Sobolev embedding from $H^1(\mathbb{R}^2)$ to $L^{m+2}(\mathbb{R}^2)$ implies, for any $u \in \mathcal{V}^2$, that,

$$\|\nabla u\|_{L^{m+2}(\mathbb{R}^2)} \le C\|\nabla u\|_{H^1(\mathbb{R}^2)} \le C(a) \left[a\|\Delta u\|_{L^2(\mathbb{R}^2)} + \|\nabla u\|_{L^2(\mathbb{R}^2)}\right].$$

The right side of the previous inequality is obtained easily by recalling that the H^s-norm is defined through the Fourier transform. Moreover, $C(a)$ is characterized in terms of

$$C(a, m) = \sup_{u \in \mathcal{V}^2, \nabla u \ne 0} \frac{\|\nabla u\|_{m+2}}{\left(a\|\Delta u\|^2 + \|\nabla u\|\right)^{1/2}}.$$

In other words, we have, for $u \in \mathcal{V}^2$,

$$\|\nabla u\|_{L^{m+2}(\mathbb{R}^2)} \le C(a, m) \left[a\|\Delta u\|_{L^2(\mathbb{R}^2)} + \|\nabla u\|_{L^2(\mathbb{R}^2)}\right]. \tag{4.5}$$

As shown in [4], we have the following variational characterizations of $C(a, m)$: Let d be defined as

$$d = \inf\{\sup_{\lambda \ge 0} J(\lambda u) : u \in \mathcal{V}^2, \nabla u \ne 0\},$$

then we have that

$$
\begin{aligned}
d &= \inf\{J(u) : u \in \mathcal{V}^2, \nabla u \neq 0, I(u) = 0\} \\
&= \Big(\frac{m}{2(m+2)}\Big)\beta^{-2/m} C(a,m)^{-2(m+2)/m} > 0. \qquad (4.6)
\end{aligned}
$$

Using this characterization of $C(a,m)$ and the Sobolev embedding from $H^1(\mathbb{R}^2)$ to $L^{m+2}(\mathbb{R}^2)$, we obtain some properties of the stable and unstable set given, respectively, by

$$
\begin{aligned}
\mathcal{W} &= \left\{u \in \mathcal{V}^2 : J(u) < d,\ I(u) > 0\right\} \cup \left\{u \in \mathcal{V}^2 : \nabla u = 0\right\}, \\
\mathcal{V} &= \left\{u \in \mathcal{V}^2 : J(u) < d,\ I(u) < 0\right\}.
\end{aligned}
$$

Lemma 4.1. *Let $\beta > 0$, $\Phi(x,t)$ be a local solution of* (1.1) *with initial condition $\Phi_0 \in \mathcal{V}^2$ and $r_0 \in H^1(\mathbb{R}^2)$ on $[0,T_0)$, with $\mathcal{E}(\Phi_0, r_0) < d$. If $I(\Phi(\cdot,0)) > 0$ or $\nabla\Phi_0(\cdot,0) = 0$, then $\Phi(\cdot,t) \in \mathcal{W}$ and $\mathcal{E}(\Phi(\cdot,t), r(\cdot,t)) < d$ for all $t \in [0,T_0)$. Moreover, for $t \in [0,T_0)$,*

$$
\beta\|\nabla\Phi(\cdot,t)\|^{m+2}_{L^{m+2}(\mathbb{R}^2)} \leq \left(\frac{\mathcal{E}(\Phi_0,r_0)}{d}\right)^{\frac{m}{2}} \left[a\|\Delta\Phi(\cdot,t)\|_{L^2(\mathbb{R}^2)} + \|\nabla\Phi(\cdot,t)\|_{L^2(\mathbb{R}^2)}\right]. \qquad (4.7)
$$

Proof. Note that, for $t \in [0,T_0)$,

$$
J(\Phi(\cdot,t)) \leq \mathcal{E}(\Phi(\cdot,t), r(\cdot,t)) = \mathcal{E}(\Phi_0, r_0) < d.
$$

Now, suppose that there exists $t_0 \in [0,T_0)$ such that, for some $t_1 \in (t_0,T_0)$, we have that $I(\Phi(\cdot,t_1)) = 0$ and $\nabla\Phi(\cdot,t_1) \neq 0$. In other words, $\Phi(\cdot,t) \notin \mathcal{W}$, for $t \in [t_0,T_0)$. As a consequence of this and from (4.4),

$$
\begin{aligned}
0 < a\|\Delta\Phi(\cdot,t_1)\|_2^2 + \|\nabla\Phi(\cdot,t_1)\|_2^2 &= \frac{2(m+2)}{m} J(\Phi(\cdot,t_1)) - \frac{2}{m} I(\Phi(\cdot,t_1)) \\
&\leq \frac{2(m+2)}{m}\mathcal{E}(\Phi(\cdot,t_1)) < \frac{2(m+2)}{m} d. \qquad (4.8)
\end{aligned}
$$

But, this is a contradiction because this inequality implies that $I(\Phi(\cdot,t_1)) > 0$. In fact, from (4.5), and (4.6),

$$
\begin{aligned}
&\beta\|\nabla\Phi(\cdot,t_1))\|_{m+2}^{m+2} \leq \beta(C(a,m))^{m+2}(a\|\Delta\Phi(\cdot,t_1)\|_2^2 + \|\nabla\Phi(\cdot,t_1)\|_2^2)^{m+2/2} \\
&< \beta(C(a,m))^{m+2}\left(\frac{2(m+2)d}{m}\right)^{m/2}\left[a\|\Delta\Phi(\cdot,t_1)\|_2^2 + \|\nabla\Phi(\cdot,t_1)\|_2^2\right] \\
&< a\|\Delta\Phi(\cdot,t_1)\|_2^2 + \|\nabla\Phi(\cdot,t_1)\|_2^2. \qquad (4.9)
\end{aligned}
$$

Thus, we have shown that $I(\Phi(\cdot,t)) > 0$ for $t \in [t_0, T_0)$, and so, $I(\Phi(\cdot,t)) > 0$ for $t \in [0, T_0)$, since $I(\Phi(\cdot,0)) > 0$. Note that the condition $I(\Phi(\cdot,0)) > 0$ implies from (4.8) that

$$0 < a\|\Delta\Phi(\cdot,0)\|_2^2 + \|\nabla\Phi(\cdot,0)\|_2^2 < \frac{2(m+2)}{2}d.$$

We now assume that $\nabla\Phi(\cdot,0) = 0$. If $\nabla\Phi(\cdot,t) = 0$ for $t \in [0,T_0)$, then we already have $\Phi(\cdot,t) \in \mathcal{W}$, for $t \in [0,T_0)$. By continuity, we have the existence of $0 < t_1 < T_0$ such that for $t \in [0,t_1)$, we have that $a\|\Delta\Phi(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2 \not\equiv 0$, and that

$$0 \le a\|\Delta\Phi(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2 < \frac{2(m+2)}{m}d.$$

Finally, suppose that $I(\Phi(\cdot,t)) > 0$ and $J(\Phi(\cdot,t)) < d$. Then we have that

$$\begin{aligned} a\|\Delta\Phi(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2 &= \frac{2(m+2)}{m}J(\Phi(\cdot,t)) - \frac{2}{m}I(\Phi(\cdot,t)) \\ &\le \frac{2(m+2)}{m}J(\Phi(\cdot,t)). \end{aligned}$$

Then we obtain the desired inequality, since

$$\begin{aligned} &\beta\|\nabla\Phi(\cdot,t))\|_{m+2}^{m+2} \le \beta(C(a,m))^{m+2}(a\|\Delta\Phi(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2)^{m+2/2} \\ &\le \beta(C(a,m))^{m+2}\left(\frac{2(m+2)J(\Phi(\cdot,t))}{m}\right)^{m/2}\left(a\|\Delta\Phi(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2\right) \\ &\le \left(\frac{\mathcal{E}(\Phi(\cdot,t), r(\cdot,t))}{d}\right)^{m/2}\left(a\|\Delta\Phi(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2\right) \\ &\le \left(\frac{\mathcal{E}(\Phi_0,r_0)}{d}\right)^{m/2}\left(a\|\Delta\Phi(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2\right). \qquad \square \end{aligned}$$

An existence result. The proof of the existence of global solutions is based on the invariance of a set under the flow for the generalized Benney-Luke equation: the stable set $\mathcal{W}$.

Corollary 4.1. *Let $p \ge 1$, $\beta > 0$, and let $\Phi_0 \in \mathcal{V}^2$ and $r_0 \in H^1$ be such that $\mathcal{E}(\Phi_0, r_0) < d$ and $I(\Phi_0) > 0$ or $\nabla\Phi_0 = 0$. Then, there exists a unique weak global solution $\Phi \in C([0,\infty), \mathcal{V}^2)$ of equation* (1.1) *such that*

$$\Phi_t \in C([0,\infty), H^1(\mathbb{R}^2)) \cap C^1([0,\infty), L^2(\mathbb{R}^2)),$$

satisfying the initial conditions

$$\nabla\Phi(\cdot,0) = \nabla\Phi_0, \quad \Phi_t(\cdot,0) = r_0.$$

Proof. Note that

$$\mathcal{E}(\Phi_0, r_0) = \mathcal{E}(\Phi, \Phi_t) = \frac{\|\Phi_t\|_2^2 + b\|\nabla\Phi_t\|_2^2}{2} + \frac{m\left(a\|\Delta\Phi\|_2^2 + \|\nabla\Phi\|_2^2\right)}{2(m+2)} + \frac{I(\Phi)}{m+2}.$$

From the previous result, we have that $I(\Phi(\cdot,t)) \geq 0$. Then we have that

$$\|\Phi_t\|_2^2 + b\|\nabla\Phi_t\|_2^2 + \frac{m}{(m+2)}\left(a\|\Delta\Phi\|_2^2 + \|\nabla\Phi\|_2^2\right) \leq 2\mathcal{E}(\Phi_0, r_0).$$

But for some positive constant $C = C(a,b,m)$ we have that

$$\begin{aligned}\left\| \begin{pmatrix}\Phi \\ \Phi_t\end{pmatrix} \right\|_X &\leq C\Big[\|\Phi_t\|_2^2 + b\|\nabla\Phi_t\|_2^2 + \frac{m}{(m+2)}\left(a\|\Delta\Phi\|_2^2 + \|\nabla\Phi\|_2^2\right)\Big] \\ &\leq 2C\mathcal{E}(\Phi_0, r_0).\end{aligned}$$

This fact implies that, for any finite time $T_0 < \infty$,

$$\lim_{t\to T_0} \left\| \begin{pmatrix}\Phi(\cdot,t) \\ \Phi_t(\cdot,t)\end{pmatrix} \right\|_{X^2} < \infty.$$

So, any local solution can be extended in time. □

A non-existence result on global solutions. As in the proof of the existence of global solutions, the proof of non-existence is based on the invariance of a set under the flow for the generalized Benney-Luke equation: the unstable set $\mathcal{V}$.

Lemma 4.2. *Let $\beta > 0$, $\Phi(x,t)$ be a local solution of* (1.1) *with initial condition $\Phi_0 \in \mathcal{V}^2$ and $r_0 \in H^1(\mathbb{R}^2)$ on $[0,T_0)$. If $\mathcal{E}(\Phi_0, r_0) < d$ and $I(\Phi_0) < 0$, then $\Phi(\cdot,t) \in \mathcal{V}$ and $\mathcal{E}(\Phi(\cdot,t), r(\cdot,t)) < d$ for all $t \in [0,T_0)$. Moreover, for $t \in [0,T_0)$,*

$$a\|\Delta\Phi(\cdot,0)\|_2^2 + \|\nabla\Phi(\cdot,0)\|_2^2 > \frac{2(m+2)}{m}d. \tag{4.10}$$

Proof. For $t \in [0,T_0)$,

$$J(\Phi(\cdot,t)) \leq \mathcal{E}(\Phi(\cdot,t), r(\cdot,t)) = \mathcal{E}(\Phi_0, r_0) < d.$$

By hypothesis, $I(\Phi(\cdot,0)) < 0$. Then if we assume that $\Phi(\cdot,t) \notin \mathcal{V}$, using the continuity of $I(\Phi(\cdot,t))$, there exists $t_0 > 0$ such that, for $t \in [0,t_0)$,

$$I(\Phi(\cdot,t)) < 0, \quad \text{and} \quad I(\Phi(\cdot,t_0)) = 0.$$

Then we have that

$$\begin{aligned} a\|\Delta\Phi(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2 &< \beta\|\nabla\Phi(\cdot,t))\|_{m+2}^{m+2} \\ &\leq \beta(C(a,m))^{m+2}(a\|\Delta\Phi(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2)^{m+2/2}.\end{aligned}$$

In other words, for $t \in [0, t_0)$,

$$a\|\Delta\Phi(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2 > \beta^{-2/m}(C(a,m))^{-2(m+2)/m} = \frac{2(m+2)}{m}d.$$

In particular,

$$J(\Phi(\cdot,t_0)) = \frac{m}{2(m+2)}\left[a\|\Delta\Phi(\cdot,t_0)\|_2^2 + \|\nabla\Phi(\cdot,t_0)\|_2^2\right] \geq d.$$

This contradiction shows that $I(\Phi(\cdot,t)) < 0$, for $t \in [0, T_0)$. □

Using this result, we are able to show that any local solution cannot be a global solution.

Corollary 4.2. *Let $p \geq 1$, $\beta > 0$, and $\theta = 0$. Let $\Phi_0 \in \mathcal{V}^2$ and $r_0 \in H^1$ be such that $\mathcal{E}(\Phi_0, r_0) < d$ and $I(\Phi_0) < 0$. If $\Phi \in C([0,T_0), \mathcal{V}^2)$ is a solution of equation* (1.1) *with*

$$\Phi_t \in C([0,T_0), H^1(\mathbb{R}^2)) \cap C^1([0,T_0), L^2(\mathbb{R}^2))$$

satisfying the initial conditions $\nabla\Phi(\cdot,0) = \nabla\Phi_0$ and $\Phi_t(\cdot,0) = r_0$, then $T_0 < \infty$.

The proof of this result requires establishing the following concavity result:

Lemma 4.3. ([9]) *Let $P \in C^2([0,T_0))$ be a non-negative function such that, for some $0 < a < T_0$, P is positive in $[a, T_0)$ and $P'(a) > 0$. If for some $\alpha > 0$ we have that*

$$PP'' - (\alpha+1)(P') > 0 \quad \text{in } (a, T_0),$$

then $T_0 < \infty$.

Proof. Assume that $T_0 = \infty$. We will see that for some $a < t_0 < T_0$,

$$\lim_{t \to t_0^-} P(t) = \infty.$$

Define the function $g = P^{-\alpha}$. Then a simple computation implies that, in $[a, \infty)$,

$$g'' = -\alpha P^{-(\alpha+2)}\left[PP'' - (\alpha+1)(P')\right] \leq 0.$$

In other words, g is a concave function in $[a, \infty)$. Thus, for $t \in (a, \infty)$,

$$g(t) \leq g(a) + g'(a)t,$$

which is equivalent to

$$0 < P^{-\alpha}(t) \leq P^{-\alpha}(a) - \alpha P^{-(\alpha+1)}P'(a)t = P^{-(\alpha+1)}(a)\left[P(a) - \alpha P'(a)t\right].$$

Thus, we have for $t \in (a, \infty)$ that $P(a) - \alpha P'(a)t > 0$ and

$$P^{\alpha}(t) \geq P^{(\alpha+1)}(a)\left[P(a) - \alpha P'(a)t\right]^{-1}.$$

Since $P'(a) > 0$ we have for $t_0 = P(a)/\alpha P'(a)$ that

$$\lim_{t \to t_0^-} P(t) = \infty,$$

as claimed above. As a consequence of this, we conclude that $T_0 < \infty$. □

Proof Corollary 4.2. Suppose that $T_0 = \infty$. Let P be the function defined in $[0, \infty)$ as

$$P(t) = \|\Phi(\cdot, t)\|_{L^2}^2 + b\|\nabla\Phi(\cdot, t)\|_{L^2}^2.$$

It is not hard to see that we have the identity

$$\frac{d^2P(t)}{dt^2} = \|\Phi_t(\cdot, t)\|_2^2 + b\|\nabla\Phi_t(\cdot, t)\|_2^2 + \langle\Phi_{tt}(\cdot, t) - a\Delta\Phi_{tt}(\cdot, t), \Phi(\cdot, t)\rangle_{-2,2}\,.$$

On the other hand, multiplying equation (1.1) by $\Phi(\cdot, t)$ we obtain that

$$\begin{aligned}&\langle\Phi_{tt}(\cdot, t) - a\Delta\Phi_{tt}(\cdot, t), \Phi(\cdot, t)\rangle_{-2,2}\\ &= \beta\|\nabla\Phi(\cdot, t)\|_{L^{m+2}}^{m+2} - a\|\Delta\Phi(\cdot, t)\|_2^2 - \|\nabla\Phi(\cdot, t)\|_2^2.\end{aligned}$$

But, from $\mathcal{E}(\Phi(\cdot, t), \Phi_t(\cdot, t)) = \mathcal{E}(\Phi_0, r_0)$, we conclude that (see Theorem 3.1)

$$\begin{aligned}&\frac{2\beta}{m+2}\|\nabla\Phi(\cdot, t)\|_{L^{m+2}}^{m+2} - a\|\Delta\Phi(\cdot, t)\|_2^2 - \|\nabla\Phi(\cdot, t)\|_2^2 + 2\mathcal{E}(\Phi_0, r_0)\\ &= \|\Phi_t(\cdot, t)\|_2^2 + b\|\nabla\Phi_t(\cdot, t)\|_2^2,\end{aligned}$$

or equivalently,

$$\begin{aligned}&\frac{m+2}{2}\left(\|\Phi_t(\cdot, t)\|_2^2 + b\|\nabla\Phi_t(\cdot, t)\|_2^2\right)\\ &\qquad + \frac{m}{2}\left(a\|\Delta\Phi(\cdot, t)\|_2^2 + \|\nabla\Phi(\cdot, t)\|_2^2\right) + I(\Phi(\cdot, t))\\ &= (m+2)\mathcal{E}(\Phi_0, r_0).\end{aligned}$$

Using those previous facts,

$$\begin{aligned}\frac{d^2P(t)}{dt^2} &= 2\|\Phi_t(\cdot, t)\|_2^2 + 2b\|\nabla\Phi_t(\cdot, t)\|_2^2 - 2I(\Phi(\cdot, t))\\ &= \frac{m+2}{2}\left(\|\Phi_t(\cdot, t)\|_2^2 + b\|\nabla\Phi_t(\cdot, t)\|_2^2\right)\\ &\quad + m\left(a\|\Delta\Phi(\cdot, t)\|_2^2 + \|\nabla\Phi(\cdot, t)\|_2^2\right) - 2(m+2)\mathcal{E}(\Phi_0, r_0)\\ &\geq 2(m+2)(d - \mathcal{E}(\Phi_0, r_0)) > 0. \end{aligned} \tag{4.11}$$

Integrating from 0 to t, we obtain

$$\begin{aligned}\frac{dP(t)}{dt} &\geq \frac{dP(t)}{dt}|_{t=0} + 2(m+2)(d - \mathcal{E}(\Phi_0, r_0))t \\ &\geq 2\langle \Phi_0, r_0\rangle + 2\langle \nabla\Phi_0, \nabla r_0\rangle + 2(m+2)(d - \mathcal{E}(\Phi_0, r_0))t.\end{aligned}$$

This fact implies that, for some $a > 0$, $\frac{dP(t)}{dt} > 0$, and $P(t) > 0$ in $[a, \infty)$. Note from Young's inequality that

$$\begin{aligned}|P^{'}(t)|^2 &\leq 4\left(\|\Phi(\cdot,t)\|\|\Phi_t(\cdot,t) + b\|\nabla\Phi(\cdot,t)\|\|\nabla\Phi_t(\cdot,t)\right)^2 \\ &\leq 4P(t)\left(\|\Phi_t(\cdot,t)\|^2 + b\|\nabla\Phi_t(\cdot,t)\|^2\right).\end{aligned}$$

Using this and inequality (4.11), we are able to show for $t > a$ that

$$P(t)P^{''}(t) - \frac{m+4}{4}|P^{'}(t)|^2 \geq 2(m+2)(d - \mathcal{E}(\Phi_0, r_0))P(t) > 0.$$

From Lemma 4.3 with $\alpha = \frac{m+4}{4}$, we conclude that T_0 must be finite. In other words, any local solution can not be extended in time. $\square$

Acknowledgments. This work was supported by the Universidad del Valle, Cali-Colombia.

References

[1] J.R. Quintero, *Existence and analyticity of lumps for a generalized Benney-Luke equation,* Revista Colombiana de Matematicas, 36 (2002), 71–95.

[2] C.I. Christov, *An energy-consistent dispersive shallow-water-wave model,* Wave motions, 734 (2002), 161–174.

[3] A. González, *The Cauchy problem for Benney-Luke equation and generalized Benney-Luke equations,* Differential and Integral Equations, 20 (2007), 1341–1362.

[4] S. Wang, G. Xu, and G. Chen, *The Cauchy Problem for the Generalized Benney-Luke Equation,* J. Math. Phys., to appear.

[5] R.L. Pego and J.R. Quintero, *Two–dimensional solitary Waves for a Benney-Luke equation,* Physica D, **1**32 (1999), 476–496.

[6] D.J. Benney and J.C. Luke, *Interactions of permanent waves of finite amplitude,* J. Math. Phys., **4**3 (1964), 309–313.

[7] P.A. Milewski and J. B. Keller, *Three dimensional water waves,* Studies Appl. Math., **3**7 (1996), 149–166.

[8] L. Paumond, *A rigorous link between Kp and a Benney-Luke equation,* Differential and Integral Equations, **1**6 (2003), 1039–1064.

[9] H. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form* $\mathcal{P}u_{tt} = -\mathcal{A}u + \mathcal{F}(u)$*,* Trans. Amer. Math. Soc., **1**92 (1974), 1–21.

[10] C. Tao, *Multilinear weighted convolution of* L^2 *functions and applications to nonlinear dispersive equations,* Amer J. Math., **1**22 (2001), 839–908.

[11] A. Pazy, "Semigroups of Linear Operator and Applications to Partial Differential Equations," Springer Verlag, New York, 1983.