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FULLY NONLINEAR GAUGE INVARIANT EVOLUTION OF THE PLANE WAVE

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Abstract. We consider fully nonlinear gauge invariant evolution of the plane wave. Although the plane wave does not decay at infinity, by an elementary and simple argument we find an extremely smooth solution which has an explicit expression. Additionally, we study the global behavior of the solution from its representation.

1. Introduction

We consider the following equation

$$\partial_t u + \mathcal{L}u = \lambda \mathcal{N}((\partial_x^\alpha u)_{|\alpha|\le l}, \partial_x^\nu u) \quad \text{in } [0,\infty)\times \mathbf{R}^n, \tag{1.1}$$

where $n \ge 1$, $\lambda \in \mathbf{C}$ (we denote $\lambda_1 = \mathrm{Re}\lambda$ and $\lambda_2 = \mathrm{Im}\lambda$), $l \in \mathbf{Z}_+$, $\nu \in \mathbf{Z}_+^n$, $\partial_t = \partial/\partial t$, and the unknown function $u = u(t,x)$ is complex-valued. Here, $\mathbf{Z}_+$ stands for the set of non-negative integers. The linear differential operator $\mathcal{L}\colon \mathcal{S}' \to \mathcal{S}'$ is defined by

$$\mathcal{L}u = \mathcal{F}^{-1}[P(\xi)\mathcal{F}[u](\xi)], \quad \text{for } u \in \mathcal{S}',$$

where $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^n)$ is the set of tempered distributions, $\mathcal{F}$ is the Fourier transform on $\mathcal{S}'$, and P is a slowly increasing function; that is, $P \in C^\infty(\mathbf{R}^n)$ and, for all $\alpha \in \mathbf{Z}_+^n$, there exists $N(\alpha) \in \mathbf{Z}_+$ such that

$$\sup_{\xi\in\mathbf{R}^n} (1+|\xi|)^{-N(\alpha)} |\partial_\xi^\alpha P(\xi)| < \infty.$$

The nonlinear function $\mathcal{N}\colon \mathbf{C}^{l^*}\times\mathbf{C}\to\mathbf{C}$ is given by

$$\mathcal{N}((v^\alpha)_{|\alpha|\le l}, w) = \prod_{|\alpha|\le l} |v^\alpha|^{k_\alpha} w \quad \text{for } ((v^\alpha)_{|\alpha|\le l}, w) \in \mathbf{C}^{l^*}\times\mathbf{C},$$

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where $k_\alpha \geq 0$ ($\alpha \in \mathbf{Z}_+^n$, $|\alpha| \leq l$), and

$$l^* = \sum_{j=0}^{l} \binom{n+j-1}{n-1} = \sum_{j=0}^{l} \frac{(n+j-1)!}{(n-1)!j!}.$$

This function satisfies the so-called gauge invariance; that is,

$$\mathcal{N}(e^{i\theta}((v^\alpha)_{|\alpha|\leq l}, w)) = e^{i\theta}\mathcal{N}((v^\alpha)_{|\alpha|\leq l}, w), \tag{1.2}$$

for $((v^\alpha)_{|\alpha|\leq l}, w) \in \mathbf{C}^{l^*} \times \mathbf{C}$ and $\theta \in \mathbf{R}$. Here, and in the following, we denote $i = \sqrt{-1}$.

We remark that (1.1) contains a large variety of equations. For example, when $P(\xi) = \gamma|\xi|^2 + \kappa$ (γ, $\kappa \in \mathbf{C}$), we have $\mathcal{L} = -\gamma\Delta + \kappa$. In this case, the equation (1.1) represents the complex Ginzburg-Landau equation. In particular, the cases $P(\xi) = |\xi|^2$ and $P(\xi) = i|\xi|^2$ correspond to, respectively, the nonlinear heat equation and the nonlinear Schrödinger equation (see Appendix for details). We note that we can also consider $P(\xi) = e^{ib\cdot\xi}$ ($b \in \mathbf{R}^n$), for instance. Then $\mathcal{L}$ is a differential operator of infinite order.

We shall study the equation (1.1) with the plane wave as an initial data

$$u(0,x) = \mu E_a(x) \quad \text{for } x \in \mathbf{R}^n, \tag{1.3}$$

where $\mu \in \mathbf{C}$, $a \in \mathbf{R}^n$, and $E_a(x) = e^{ia\cdot x}$ for $x \in \mathbf{R}^n$. Here, we note that the plane wave E_a does not belong to $H^s(\mathbf{R}^n)$ ($s \in \mathbf{R}$), where $H^s(\mathbf{R}^n)$ is the Sobolev space. To the author's knowledge, there are only a few pieces of literature on nonlinear evolution equations involving initial data which are not in $H^s(\mathbf{R}^n)$. This kind of problem is treated in [1]–[4], [6]–[8], [11], for instance. As for the case that the initial data is a bounded measure, Brezis-Friedman [2] studied the nonlinear parabolic equation with the Dirichlet condition, and Tsutsumi [11] considered the Korteweg-de Vries equation. Kenig-Ponce-Vega [6] proved ill posedness for the nonlinear Schrödinger equation with supercritical nonlinearity involving the delta function as initial data, which is an example of bounded measures. On the other hand, Kita [7], [8] constructed a solution for the nonlinear Schrödinger equation with subcritical nonlinearity involving superposed delta functions as initial data (see also [3]), and Banica-Vega [1] studied the same problem for perturbed initial data from a delta function. We notice that E_a is unbounded as a measure. Another feature of the plane wave is that it is likely to be an *eigenfunction*; namely, we have $\mathcal{L}E_a = P(a)E_a$ in $\mathcal{S}'$.

The aim of this paper is to find an extremely smooth solution to the problem (1.1)-(1.3) which has an explicit expression, although the nonlinearity can be chosen quite freely except for the gauge condition (1.2). More precisely, we will construct a function $u(t,x)$ satisfying the following:

$$\begin{cases} \dfrac{d}{dt}\langle u(t),\varphi\rangle + \langle u(t),\mathcal{L}^*\varphi\rangle = \lambda\langle \mathcal{N}((\partial_x^\alpha u)_{|\alpha|\le l},\partial_x^\nu u),\varphi\rangle, \\ \lim\limits_{t\to 0}\langle u(t),\varphi\rangle = \mu\langle E_a,\varphi\rangle, \end{cases}$$

for each $\varphi \in \mathcal{S}$. Here, $\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$ is the set of rapidly decreasing functions, $\langle\cdot,\cdot\rangle$ is the pairing between $\mathcal{S}'$ and $\mathcal{S}$, and $\mathcal{L}^*$ is the dual operator of $\mathcal{L}$. Moreover, we will study the global behavior of the solution by using its explicit expression. When one considers the linear case (i.e., $\lambda = 0$), such an approach is widely used in order to guess behavior of solutions for general data.

In order to state the main results, we denote by $U(t,x) \equiv U_a^P(t,x)$ the solution (in the distributional sense) of the following linear equation

$$\begin{cases} \partial_t u + \mathcal{L}u = 0 & \text{in } [0,\infty)\times\mathbf{R}^n, \\ u(0,x) = E_a(x) & \text{for } x\in\mathbf{R}^n. \end{cases}$$

Using the identity $\mathcal{L}E_a = P(a)E_a$ in $\mathcal{S}'$, we have

$$U(t,x) = e^{-P(a)t}E_a(x) = e^{-\mathrm{Re}P(a)t}e^{i(a\cdot x-\mathrm{Im}P(a)t)},$$

which belongs to $C^\infty([0,\infty)\times\mathbf{R}^n)$. It is remarkable that $\mathrm{Re}P(a)$ determines the representation of the solution to (1.1)-(1.3) and characterizes the dissipative nature and global behavior for the problem, as we will see in Theorems 2.1 and 2.2 below.

We conclude this section with the following remark. Quite recently, the author [4] constructed a solution to the problem (1.1) with $\sum_{j\in\mathbf{Z}}\mu_j E_{ja}(x)$ ($\mu_j \in \mathbf{C}$) as initial data under the condition: $\mathrm{Re}P(ja) = 0$ for every $j \in \mathbf{Z}$ (e.g., $P(\xi) = i|\xi|^2$ (Schrödinger type), $P(\xi) = -i\xi^3$ ($n = 1$) (Airy type), $P(\xi) = ie^{i\pi a\cdot\xi/|a|^2}$ (infinite order type), etc.).

2. Main results

In this section, we consider a simple case where the nonlinearity is given as follows. Let $l = 0$, $k_{(0,\ldots,0)} = p-1$ ($p > 1$) and $\nu = (0,\ldots,0)$; that is,

$$\mathcal{N}(u,u) = |u|^{p-1}u. \tag{2.1}$$

The treatment for the general case is outlined in the next section. Now, we state main theorems.

Theorem 2.1 (local result)**.** *Assume* (2.1)*. Then there exist a time* $T > 0$ *and a unique solution of the problem* (1.1)-(1.3) *described as*

$$u(t,x) = A(t)U(t,x) \in C^{\infty}([0,T] \times \mathbf{R}^n), \tag{2.2}$$

such that

$$A(t) = \begin{cases} \mu\Big(1 - \dfrac{\lambda_1|\mu|^{p-1}}{\mathrm{Re}P(a)}\Big(1 - e^{-(p-1)\mathrm{Re}P(a)t}\Big)\Big)^{-\frac{\lambda}{(p-1)\lambda_1}} & \text{if } \mathrm{Re}P(a) \neq 0, \lambda_1 \neq 0, \\ \mu \exp\Big(\dfrac{i\lambda_2|\mu|^{p-1}}{(p-1)\mathrm{Re}P(a)}\Big(1 - e^{-(p-1)\mathrm{Re}P(a)t}\Big)\Big) & \text{if } \mathrm{Re}P(a) \neq 0, \lambda_1 = 0, \\ \mu\Big(1 - (p-1)\lambda_1|\mu|^{p-1}t\Big)^{-\frac{\lambda}{(p-1)\lambda_1}} & \text{if } \mathrm{Re}P(a) = 0, \lambda_1 \neq 0, \\ \mu \exp(i\lambda_2|\mu|^{p-1}t) & \text{if } \mathrm{Re}P(a) = 0, \lambda_1 = 0. \end{cases} \tag{2.3}$$

This theorem is shown by reducing the problem to an ordinary differential equation. This kind of approach can be found in Ozawa-Yamazaki [9], for instance.

Our next interest is to study the global behavior of the solution to (1.1)-(1.3). We can immediately show the following theorem by calculating (2.3) precisely (see also (2.5) below).

Theorem 2.2 (global and blowup results)**.** *Let* $A(t)$ *be defined by* (2.3)*. If* $\mu \neq 0$*, then we have the following.*

(1) *Suppose that* $\lambda_1 \leq 0$*. Then* $A(t)$ *exists globally in time. Furthermore if* $\lambda_1 < 0$*, then*

$$\lim_{t\to\infty} A(t) = \begin{cases} \mu\Big(1 - \dfrac{\lambda_1|\mu|^{p-1}}{\mathrm{Re}P(a)}\Big)^{-\frac{\lambda}{(p-1)\lambda_1}} & \text{if } \mathrm{Re}P(a) > 0, \\ 0 & \text{if } \mathrm{Re}P(a) \leq 0. \end{cases}$$

(2) *Suppose that* $\lambda_1 > 0$*. Then the following hold.*

(a) *If* $\mathrm{Re}P(a) \leq 0$*, then* $A(t)$ *blows up at a time* T^* *which is given by*

$$T^* = \begin{cases} -\dfrac{1}{(p-1)\mathrm{Re}P(a)} \log\Big(1 - \dfrac{\mathrm{Re}P(a)}{\lambda_1|\mu|^{p-1}}\Big) & \text{if } \mathrm{Re}P(a) < 0, \\ \dfrac{1}{(p-1)\lambda_1|\mu|^{p-1}} & \text{if } \mathrm{Re}P(a) = 0. \end{cases}$$

(b) *Suppose that* $\mathrm{Re}P(a) > 0$*. If* $|\mu|^{p-1} \leq \lambda_1^{-1}\mathrm{Re}P(a)$*, then* $A(t)$ *exists globally in time. Conversely if* $|\mu|^{p-1} > \lambda_1^{-1}\mathrm{Re}P(a)$*, then*

$A(t)$ blows up at a time T^* which is given by

$$T^* = -\frac{1}{(p-1)\mathrm{Re}P(a)} \log\Big(1 - \frac{\mathrm{Re}P(a)}{\lambda_1|\mu|^{p-1}}\Big).$$

Remark 2.1. For the global behavior of the solutions, there is no restriction on the power $p > 1$. We particularly take notice of the part (1) of Theorem 2.2. When we consider the case that $P(\xi) = i|\xi|^2$ for $\xi \in \mathbf{R}^n$ (the nonlinear Schrödinger equation), $\lambda_1 = 0$, $\lambda_2 > 0$, and $\mathcal{N}$ is defined by (2.1), we observe a different consequence from the results of [5] and [10]: if the initial data $u(0, x)$ belongs to $H^1(\mathbf{R}^n)$, $|x|u(0, x) \in L^2(\mathbf{R}^n)$,

$$\frac{1}{2}\|\nabla u(0)\|^2_{L^2(\mathbf{R}^n)} + \frac{1}{p+1}\|u(0)\|^{p+1}_{L^{p+1}(\mathbf{R}^n)} < 0,$$

and

$$1 + \frac{4}{n} \le p < p^*(n) := \begin{cases} \infty & \text{if } n = 1, 2, \\ \dfrac{n+2}{n-2} & \text{if } n \ge 3, \end{cases}$$

then the solution of (1.1) cannot exist beyond a certain time. We obtain, however, the global solution even if $1 + 4/n \le p$, when the initial data is the plane wave.

Proof of Theorem 2.1. Substitution of (2.2) into (1.1)-(1.3) yields the ODE of $A(t)$:

$$\begin{cases} \dfrac{dA}{dt} = \lambda e^{-(p-1)\mathrm{Re}P(a)t}|A|^{p-1}A, \quad t \in [0, T], \\ A(0) = \mu. \end{cases} \tag{2.4}$$

To solve (2.4), we multiply both sides of the equation by $\overline{A(t)}$, where $\overline{A(t)}$ denotes the complex conjugate of $A(t)$. If we take the real part of the resulting equation, then we have

$$\frac{d}{dt}\big(|A(t)|^{-(p-1)}\big) = -(p-1)\lambda_1 e^{-(p-1)\mathrm{Re}P(a)t},$$

and hence, we obtain

$$|A(t)|^{p-1} = \begin{cases} \dfrac{|\mu|^{p-1}}{1 - \dfrac{\lambda_1|\mu|^{p-1}}{\mathrm{Re}P(a)}(1 - e^{-(p-1)\mathrm{Re}P(a)t})} & \text{if } \mathrm{Re}P(a) \ne 0, \\ \dfrac{|\mu|^{p-1}}{1 - (p-1)\lambda_1|\mu|^{p-1}t} & \text{if } \mathrm{Re}P(a) = 0. \end{cases} \tag{2.5}$$

Thus, we find (2.3) by substituting (2.5) into (2.4). This completes the proof. □

3. Complex generalized KdV equations

The argument described in the previous section permits us to treat more general nonlinearity. However, we discuss (1.1)-(1.3) with a specific nonlinearity, because we can easily generalize the argument. We, for instance, consider the case $n = 1$, $\mathcal{N}(u, \partial_x u) = |u|^{p-1}\partial_x u$ $(p > 1)$, and $P(\xi) = -i\xi^3$ for $\xi \in \mathbf{R}$ (i.e., $\mathcal{L} = \partial_x^3$). In this situation, the equation (1.1) represents the complex generalized KdV equation (or simply gKdV)

$$\partial_t u + \partial_x^3 u = \lambda |u|^{p-1}\partial_x u \quad \text{in } [0,\infty) \times \mathbf{R}. \tag{3.1}$$

Let us show the existence of the solution of (3.1)-(1.3) displayed as (2.2), with $U(t,x) = e^{iax+ia^3t}$. Analogously to the proof of Theorem 2.1, substituting $u(t,x) = A(t)U(t,x)$ into (3.1), we arrive at the following ODE:

$$\begin{cases} \dfrac{dA}{dt} = ia\lambda |A|^{p-1}A, \quad t \in [0,T], \\ A(0) = \mu. \end{cases}$$

We can solve this equation in the same fashion, and we have

$$|A(t)|^{p-1} = \frac{|\mu|^{p-1}}{1 + (p-1)a\lambda_2 |\mu|^{p-1}t}.$$

We sum up the above argument in the following proposition.

Proposition 3.1 (gKdV)**.**

(1) *There exist a time $T > 0$ and a unique solution of the problem* (3.1)-(1.3) *described as*

$$u(t,x) = A(t)U(t,x) \in C^\infty([0,T] \times \mathbf{R}),$$

such that

$$A(t) = \begin{cases} \mu\big(1 + (p-1)a\lambda_2|\mu|^{p-1}t\big)^{\frac{i\lambda}{(p-1)\lambda_2}}, & \text{if } a\lambda_2 \neq 0, \\ \mu \exp(ia\lambda_1 |\mu|^{p-1}t), & \text{if } a\lambda_2 = 0. \end{cases}$$

(2) *Suppose that $a\lambda_2 \geq 0$. Then $A(t)$ exists globally in time. Furthermore if $a\lambda_2 > 0$, then we have $A(t) \to 0$ as $t \to \infty$. Conversely suppose that $a\lambda_2 < 0$. Then $A(t)$ blows up at a time T^* which is given by*

$$T^* = -\frac{1}{(p-1)a\lambda_2|\mu|^{p-1}}.$$

APPENDIX

We give two typical examples of Theorems 2.1 and 2.2, assuming (2.1).

The complex nonlinear heat equation. First, we consider the case $P(\xi) = |\xi|^2$ for $\xi \in \mathbf{R}^n$. In this situation, the equation (1.1) represents the complex nonlinear heat equation. Then we have $U(t,x) = e^{-|a|^2 t} e^{ia\cdot x}$ and

$$A(t) = \begin{cases} \mu\Big(1 - \dfrac{\lambda_1|\mu|^{p-1}}{|a|^2}\big(1 - e^{-(p-1)|a|^2 t}\big)\Big)^{-\frac{\lambda}{(p-1)\lambda_1}} & \text{if } a \neq 0, \lambda_1 \neq 0, \\ \mu \exp\Big(\dfrac{i\lambda_2|\mu|^{p-1}}{(p-1)|a|^2}\big(1 - e^{-(p-1)|a|^2 t}\big)\Big) & \text{if } a \neq 0, \lambda_1 = 0, \\ \mu\big(1 - (p-1)\lambda_1|\mu|^{p-1} t\big)^{-\frac{\lambda}{(p-1)\lambda_1}} & \text{if } a = 0, \lambda_1 \neq 0, \\ \mu \exp(i\lambda_2|\mu|^{p-1} t) & \text{if } a = 0, \lambda_1 = 0. \end{cases}$$

We proceed to discuss the global behavior of $A(t)$ under the condition $\mu \neq 0$. We suppose that $\lambda_1 \leq 0$. Then $A(t)$ exists globally in time. Furthermore, if $\lambda_1 < 0$, then

$$\lim_{t\to\infty} A(t) = \begin{cases} \mu\Big(1 - \dfrac{\lambda_1|\mu|^{p-1}}{|a|^2}\Big)^{-\frac{\lambda}{(p-1)\lambda_1}} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Conversely, we suppose that $\lambda_1 > 0$. If $a \neq 0$ and $|\mu|^{p-1} \leq \lambda_1^{-1}|a|^2$, then $A(t)$ exists globally in time. If either $a \neq 0$ and $|\mu|^{p-1} > \lambda_1^{-1}|a|^2$ or $a = 0$, then $A(t)$ blows up at a time T^* which is given by

$$T^* = \begin{cases} -\dfrac{1}{(p-1)|a|^2} \log\Big(1 - \dfrac{|a|^2}{\lambda_1|\mu|^{p-1}}\Big) & \text{if } a \neq 0 \text{ and } |\mu|^{p-1} > \dfrac{|a|^2}{\lambda_1}, \\ \dfrac{1}{(p-1)\lambda_1|\mu|^{p-1}} & \text{if } a = 0. \end{cases}$$

The nonlinear Schrödinger equation. Next, we consider the case $P(\xi) = i|\xi|^2$ for $\xi \in \mathbf{R}^n$. The equation (1.1) represents the nonlinear Schrödinger equation in this situation. Then, we have $U(t,x) = e^{ia\cdot x - i|a|^2 t}$, and

$$A(t) = \begin{cases} \mu\big(1 - (p-1)\lambda_1|\mu|^{p-1} t\big)^{-\frac{\lambda}{(p-1)\lambda_1}} & \text{if } \lambda_1 \neq 0, \\ \mu \exp(i\lambda_2|\mu|^{p-1} t) & \text{if } \lambda_1 = 0. \end{cases}$$

Let $\mu \neq 0$. We suppose that $\lambda_1 \leq 0$. Then $A(t)$ exists globally in time. Furthermore, if $\lambda_1 < 0$, then we have $A(t) \to 0$ as $t \to \infty$. Conversely, we suppose that $\lambda_1 > 0$. Then $A(t)$ blows up at a time T^* which is given by

$$T^* = \frac{1}{(p-1)\lambda_1|\mu|^{p-1}}.$$

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