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UNIFORM GRADIENT BOUNDS FOR THE PRIMITIVE EQUATIONS OF THE OCEAN

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Abstract. In this paper, we consider the 3D primitive equations of the ocean in the case of the Dirichlet boundary conditions on the side and bottom boundaries. We provide an explicit upper bound for the H^1 norm of the solution. We prove that, after a finite time, this norm is less than a constant which depends only on the viscosity ν, the force f, and the domain Ω. This improves our previous result from [7] where we established the global existence of strong solutions with an argument which does not give such explicit rates.

1. INTRODUCTION

In this paper, we consider the 3D primitive equations of the ocean

$$\frac{\partial u_k}{\partial t} - \nu\Delta u_k + \sum_{j=1}^{3}\partial_j(u_j u_k) + \partial_k p = f_k, \qquad k = 1,2 \tag{PE$_k$}$$

$$\sum_{k=1}^{3}\partial_k u_k = 0,$$

where $v(x_1,x_2,x_3) = (u_1(x_1,x_2,x_3), u_2(x_1,x_2,x_3))$ is the horizontal velocity and $p(x_1,x_2)$ is the pressure, with the Dirichlet boundary condition on the sides and on the bottom and Neumann boundary conditions on the top.

The mathematical framework for the primitive equations was established in [8, 9, 10] by Lions, Temam, and Wang, who in particular proved the global existence of weak solutions. Subsequently, the equations were extensively studied in [1, 2, 3, 5, 16, 18, 19]; see also [16] and the references therein. In the case of Navier boundary conditions on the side and Neumann conditions

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on the bottom and the top, global existence was established by Cao and Titi in [2] (see also [4, 5, 12, 13]). The case of the physical Dirichlet boundary conditions on the sides and the bottom does not follow from the methods from [2, 5] due to difficulties arising from the pressure (see [6] for more details). The global existence for this situation was established in [7]. Also, for the case of periodic boundary conditions on the side, we provided in [7] uniform H^1 bounds using pointwise estimates on the pressure. However, in the case of the Dirichlet boundary condition, our previous paper [7] only gives global existence but does not provide an explicit bound on the H^1 norm for a solution. This is due to the use of the uniform integrability of $\int_{t_1}^{t_2} \|\nabla u\|_{L^2}^2$, which can not be deduced with an explicit parameter. In this paper, we provide an explicit upper bound on the H^1 norm of any solution by improving the approach in [7]. The structure of the paper is as follows. In Section 2, we provide the uniform bound on the H^1 norm of the velocity under the condition of uniform square integrability on the force. In Section 3, we address the full primitive equations of the ocean including the Robin boundary condition on top and the Dirichlet boundary condition on the bottom sides and with the varying bottom topography.

2. Notation and the main result

Consider the primitive equations of the ocean (PE_k), where $u = (u_1, u_2, u_3)$ is the velocity, $v = (u_1, u_2)$ is the horizontal velocity, p is the pressure, and $f = (f_1, f_2)$ is the external force. The equations are set in a domain $\Omega = \Omega_2 \times (-h, 0) \subseteq \mathbb{R}^3$, where $\Omega_2 \subseteq \mathbb{R}^2$ is a smooth bounded domain. The case of varying bottom topography is treated in Section 3. The initial conditions read

$$v(\cdot, 0) = v_0,$$

where $v_0 \in H^1(\Omega)$ satisfies

$$\mathrm{div}_2 \int_{-h}^{0} v_0 \, dx_3 = 0.$$

The physical boundary conditions for the primitive equations are

$$\begin{aligned}
&\frac{\partial v}{\partial x_3} = 0 \text{ and } u_3 = 0 \text{ on } \Gamma_t = \Omega_2 \times \{0\} \\
&v = 0 \text{ and } u_3 = 0 \text{ on } \Gamma_b = \Omega_2 \times \{-h\} \\
&v = 0 \text{ on } \Gamma_s = \partial\Omega_2 \times [-h, 0].
\end{aligned}$$

The sets Γ_t, Γ_b, and Γ_s represent the top, bottom, and the side boundaries respectively. On Γ_b and Γ_s, we assume the Dirichlet boundary conditions since this is where the ocean is in contact with the boundary, while on the top Γ_t of the ocean we assume the Neumann boundary conditions which represent a no-wind scenario. It is convenient to introduce the classical spaces

$$H = \Big\{ v \in (L^2(\Omega))^2 : \mathrm{div}_2 \int_{-h}^0 v\, dx_3 = 0 \text{ on } \Omega_2, \Big(\int_{-h}^0 v\, dx_3 \Big) \cdot n = 0 \text{ on } \Omega_s \Big\},$$

and

$$V = \Big\{ v \in H \cap H^1 : v = 0 \text{ on } \Gamma_b \cup \Gamma_s \Big\},$$

with the norms $\|v\|_H = \|v\|_{L^2}$ and $\|v\|_V = \|\nabla v\|_{L^2}$. By $A = -P\Delta$, where P is the orthogonal projector in $L^2(\Omega)$ with range H, we denote the Stokes-type operator. By [6], there exists a global strong solution $v \in L^\infty_{\mathrm{loc}}([0,\infty), V) \cap L^2_{\mathrm{loc}}([0,\infty), D(A))$. From [6] (except in the case of the periodic boundary conditions on the side), we cannot deduce the dissipativity property of the equation. This is due to the parameter δ in [6] whose value depends on v_0 in an implicit way. Moreover, since δ in [6] depends on the size of the interval, it also does not follow from [6] that $v \in L^\infty([0,\infty), V)$; i.e., it is not clear whether $\|v(\cdot,t)\|_V$ is a bounded function of $t \geq 0$. The following theorem shows that $v \in L^\infty([0,\infty), V)$ holds with an explicit estimate for the norm $\|u(\cdot,t)\|_V$. This bound is independent of the initial data when t is large.

Theorem 2.1. *Assume that*

$$\|f\|_{L^2_t L^2_x(\Omega \times [t, t+1/\nu])} \leq \overline{M}, \qquad t \geq 0.$$

Then there exists a constant K_0 depending on ν, $\overline{M}$, and Ω, such that the following holds. For every $v_0 \in V$ there exists $t_1 \geq 0$ depending on $\|v_0\|_H$, ν, $\overline{M}$, and Ω such that

$$\|v(\cdot,t)\|_V \leq K_0,$$

for all $t \geq t_1$.

The constant K_0 can be computed explicitly, cf. (2.17) below.

Proof. Let $v = (u_1, u_2) \in L^\infty_{\mathrm{loc}}([0,\infty), V) \cap L^2_{\mathrm{loc}}([0,\infty), D(A))$ be the strong solution of the primitive equations, which exists by [6]. Denote by u_3 the third component of the velocity determined from the divergence-free condition div $u = 0$ and the boundary condition $u_3 = 0$ on Γ_b, and let

$u=(u_1,u_2,u_3)$. The energy inequality reads

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2+\nu\|\nabla v\|_{L^2}^2=\int fv\le F\|v\|_{L^2},$$

where

$$F=F(t)=\|f(\cdot,t)\|_{L^2}.$$

Using the Poincaré inequality $\|\nabla v\|_{L^2}^2\ge C^{-1}\|v\|_{L^2}^2$ valid for $v\in V$ and the Cauchy-Schwarz inequality, we get

$$\frac{d}{dt}\|v\|_{L^2}^2+\nu\|\nabla v\|_{L^2}^2+\frac{\nu}{C}\|v\|_{L^2}^2\le\frac{C}{\nu}F^2. \tag{2.1}$$

Integrating this inequality between t and $t+\nu^{-1}$ gives

$$\left\|v\Big(\cdot,t+\frac{1}{\nu}\Big)\right\|_{L^2}^2+\frac{\nu}{C}\int_t^{t+1/\nu}\|v(\cdot,\tau)\|_{L^2}^2\,d\tau\le\frac{C}{\nu}\overline{M}^2+\|v(\cdot,t)\|_{L^2}^2,$$

which implies

$$\begin{aligned}\frac{d}{dt}\int_t^{t+1/\nu}\|v(\cdot,\tau)\|_{L^2}^2\,d\tau&=\|v(\cdot,t+1/\nu)\|_{L^2}^2-\|v(\cdot,t)\|_{L^2}^2\\&\le-\frac{\nu}{C}\int_t^{t+1/\nu}\|v(\cdot,\tau)\|_{L^2}^2\,d\tau+\frac{C}{\nu}\overline{M}^2.\end{aligned}$$

Therefore, there exists $\widetilde{t}_0\ge 0$ which depends on $\overline{M}$, ν, and $\|u_0\|_{L^2}$ such that

$$\int_t^{t+1/\nu}\|v(\cdot,\tau)\|_{L^2}^2\,d\tau\le\frac{C}{\nu^2}\overline{M}^2,\qquad t\ge\widetilde{t}_0. \tag{2.2}$$

Using (2.1) and (2.2), we obtain

$$\|v(\cdot,t)\|_{L^2}^2\le\frac{C}{\nu}\overline{M}^2,\qquad t\ge t_0=\widetilde{t}_0+\frac{1}{\nu}. \tag{2.3}$$

(Indeed, let $t\ge\widetilde{t}_0+1/\nu$. Then, by (2.2), there exists $\tau_0\in(t-1/\nu,t)$ such that $\|v(\cdot,\tau_0)\|_{L^2}^2\le C\overline{M}^2/\nu$. Integrating (2.1) between τ_0 and t then leads to (2.3).) Now, integrating (2.1) between t and $t+1/\nu$, we obtain

$$\int_t^{t+1/\nu}\|\nabla v(\cdot,\tau)\|_{L^2}^2\,d\tau\le\frac{C}{\nu^2}\overline{M}^2,\qquad t\ge t_0.$$

We shall derive estimates for $t\ge t_1=t_0+1/\nu$. First, choose a sequence of times $t_0\le s_0\le s_1\le\cdots$ such that $s_0\le t_0+1/\nu$,

$$\|\nabla v(\cdot,s_j)\|_{L^2}\le\frac{C\overline{M}}{\nu^{1/2}},\qquad j=0,1,2,\ldots, \tag{2.4}$$

and

$$\frac{1}{2\nu} \le |s_{j+1} - s_j| \le \frac{1}{\nu}, \qquad j = 0, 1, 2, \ldots.$$

Note that

$$\|v(\cdot, t)\|_{L^2} \le \frac{C\overline{M}}{\nu^{1/2}}, \qquad t \ge s_0,$$

and

$$\int_t^{t+1/\nu} \|\nabla v(\cdot, \tau)\|_{L^2}^2 \, d\tau \le \frac{C\overline{M}^2}{\nu^2}, \qquad t \ge t_0. \tag{2.5}$$

Denote $\overline{E}(t) = \|\nabla v(\cdot, t)\|_{L^2}$. Now, fix $j \in \mathbb{N}_0$ and define $T_1 = s_j$ and $T_2 = s_{j+1}$. Also, fix a positive constant γ, which depends on the viscosity ν and is to be determined further below. Then form a sequence

$$T_1 = \tau_0 < \tau_1 < \cdots < \tau_m = T_2,$$

so that

$$\int_{\tau_j}^{\tau_{j+1}} \overline{E}(\tau)^2 \, d\tau \le \frac{1}{\gamma}, \qquad j = 0, 1, \ldots, m-1, \tag{2.6}$$

and

$$m \le \frac{C\gamma}{\nu^2} \overline{M}^2 + 1. \tag{2.7}$$

This may be shown by induction in the following way. Assume τ_j, where $j \in \mathbb{N}_0$, has already been constructed. If

$$\int_{\tau_j}^{T_2} \overline{E}(\tau)^2 \, d\tau \le \frac{1}{\gamma},$$

then set $m = j + 1$ and $\tau_m = T_2$, and terminate. Otherwise, let

$$\tau_{j+1} = \sup\Big\{ t \in (\tau_j, T_2) : \int_{\tau_j}^{t} \overline{E}(\tau)^2 \, d\tau < \frac{1}{\gamma} \Big\}.$$

By construction, we have

$$\int_{\tau_j}^{\tau_{j+1}} \overline{E}(\tau)^2 \, d\tau = \frac{1}{\gamma}, \tag{2.8}$$

for $j = 0, \ldots, m-2$ and

$$\int_{\tau_{m-1}}^{\tau_m} \overline{E}(\tau)^2 \, d\tau \le \frac{1}{\gamma}.$$

Since

$$\int_{\tau_0}^{\tau_m} \overline{E}(\tau)^2 \, d\tau = \int_{s_j}^{s_{j+1}} \overline{E}(\tau)^2 \, d\tau \le \int_{s_j}^{s_j+1/\nu} \overline{E}(\tau)^2 \, d\tau \le C\overline{M}^2/\nu^2,$$

we obtain (2.7). The estimates are performed on the quantities

$$J(t) = \Big(\sum_{k=1}^{2} \|u_k(\cdot,t)\|_{L^6}^6\Big)^{1/6}, \qquad K(t) = \Big(\sum_{k=1}^{2} \|\partial_3 u_k(\cdot,t)\|_{L^2}^2\Big)^{1/2},$$

$$\overline{K}(t) = \Big(\sum_{k=1}^{2} \|\nabla\partial_3 u_k(\cdot,t)\|_{L^2}^2\Big)^{1/2}.$$

Denote

$$M_j = \frac{1}{\nu^2} \sup_{t\in[\tau_{j-1},\tau_j]} J(t)^4 + \sup_{t\in[\tau_{j-1},\tau_j]} K(t)^2 + \nu\int_{\tau_{j-1}}^{\tau_j} \overline{K}(t)^2\,dt, \quad j = 1,2,\dots,m.$$

From [7], we recall the differential inequalities

$$\frac{d}{dt}J^2 \le \frac{C}{\nu}\|\nabla_2 p\|_{L^{3/2}(\Omega_2)}^2 + \frac{C}{\nu}F^2, \tag{2.9}$$

and

$$\frac{d}{dt}K^2 + \nu\overline{K}^2 \le \frac{C}{\nu^3}J^4K^2 + \frac{C}{\nu}\|\nabla_2 p\|_{L^{3/2}(\Omega_2)}^2 + \frac{C}{\nu}F^2. \tag{2.10}$$

The first inequality is obtained by multiplying (PE$_k$) by u_k^5, where $k = 1,2$, integrating over Ω, and adding; the second inequality follows by multiplying (PE$_k$) by $-\partial_{33}u_k$, integrating over Ω and adding for $k = 1,2$. Also, recall from [7]:

$$\begin{aligned}\|\nabla_2 p\|_{L_t^2 L_x^{3/2}(\Omega_2\times(T_1,t))}^2 \le C\nu^2\|K^{1/2}\overline{K}^{1/2}\|_{L_t^2(T_1,t)}^2 + C\|J\overline{E}\|_{L_t^2(T_1,t)}^2 \\ + C\|F\|_{L_t^2(T_1,t)}^2 + C\nu\|\nabla v(\cdot,T_1)\|_{L^2}^2,\end{aligned} \tag{2.11}$$

which holds for all $t \ge T_1$; this inequality is obtained by averaging the primitive equations in the x_3 variable and using the pressure estimate due to Sohr and Van Wahl [15]. Fix $j \in \{0,\dots,m-1\}$. Integrating (2.9) between τ_j and $t \in [\tau_j,\tau_{j+1}]$, we get

$$\begin{aligned}J(t)^2 &\le J(\tau_j)^2 + \frac{C}{\nu}\|\nabla_2 p\|_{L_t^2L_x^{3/2}(\Omega_2\times[\tau_j,t])}^2 + \frac{C}{\nu}\overline{M}^2 \\ &\le J(\tau_j)^2 + \frac{C}{\nu}\|\nabla_2 p\|_{L_t^2L_x^{3/2}(\Omega_2\times[T_1,t])}^2 + \frac{C}{\nu}\overline{M}^2.\end{aligned}$$

Using the pressure estimate (2.11) and $\|K\|_{L^2} \le \|\overline{E}\|_{L^2}$, we obtain

$$\sup_{\tau_j\le t\le\tau_{j+1}} J(t)^2 \le \big(M_j\nu^2\big)^{1/2} + C\nu\|\overline{E}\|_{L_t^2(T_1,\tau_j)}\|\overline{K}\|_{L_t^2(T_1,\tau_j)}$$

$$+ C\nu\|\overline{E}\|_{L^2_t(\tau_j,\tau_{j+1})}\|\overline{K}\|_{L^2_t(\tau_j,\tau_{j+1})} + \frac{C}{\nu}\|\overline{E}\|^2_{L^2_t(T_1,\tau_j)} \sup_{T_1\le t\le \tau_j} J(t)^2$$
$$+ \frac{C}{\nu}\|\overline{E}\|^2_{L^2_t(\tau_j,\tau_{j+1})} \sup_{\tau_j\le t\le \tau_{j+1}} J(t)^2 + \frac{C}{\nu}\overline{M}^2 + C\|\nabla v(\cdot,T_1)\|^2_{L^2},$$

where we also used $\tau_{j+1}-\tau_j \le T_2 - T_1 \le 1/\nu$. Squaring this inequality, we conclude

$$\sup_{\tau_j\le t\le\tau_{j+1}} J(t)^4 \le CM_j\nu^2 + \frac{C}{\nu}\overline{M}^2\Big(\sum_{i=1}^{j} M_i\Big) + \frac{C\nu}{\gamma}M_{j+1}$$
$$+ \frac{C}{\nu^4}\overline{M}^4 \sup_{1\le i\le j} M_i + \frac{C}{\gamma^2}M_{j+1} + \frac{C}{\nu^2}\overline{M}^4. \qquad (2.12)$$

Similarly, we integrate (2.10) between τ_j and $t\in[\tau_j,\tau_{j+1}]$ and use (2.11) in order to get

$$K(t)^2 + \nu\int_{\tau_j}^{t}\overline{K}(\tau)^2\,d\tau \le K(\tau_j)^2 + \frac{C}{\nu^3}\|J^4K^2\|_{L^1_t(\tau_j,\tau_{j+1})}$$
$$+ C\nu\left(\|K^{1/2}\overline{K}^{1/2}\|^2_{L^2(T_1,\tau_j)} + \|K^{1/2}\overline{K}^{1/2}\|^2_{L^2(\tau_j,\tau_{j+1})}\right)$$
$$+ \frac{C}{\nu}\left(\|J\overline{E}\|^2_{L^2_t(T_1,\tau_j)} + \|J\overline{E}\|^2_{L^2_t(\tau_j,\tau_{j+1})}\right) + \frac{C}{\nu}\overline{M}^2 + C\|\nabla v(\cdot,T_1)\|^2_{L^2},$$

whence, observing that $K(t)\le\overline{E}(t)$,

$$\sup_{\tau_j\le t\le\tau_{j+1}} K(t)^2 + \nu\int_{\tau_j}^{\tau_{j+1}}\overline{K}(\tau)^2\,d\tau$$
$$\le CM_j + \frac{C}{\nu^3\gamma}\sup_{\tau_j\le t\le\tau_{j+1}} J(t)^4 + C\overline{M}\|\overline{K}\|_{L^2_t(T_1,\tau_j)} + \frac{C\nu}{\gamma^{1/2}}\|\overline{K}\|_{L^2_t(\tau_j,\tau_{j+1})}$$
$$+ \frac{C}{\nu}\|\overline{E}\|^2_{L^2_t(T_1,\tau_j)}\sup_{T_1\le t\le\tau_j} J(t)^2 + \frac{C}{\nu\gamma}\sup_{\tau_j\le t\le\tau_{j+1}} J(t)^2 + \frac{C}{\nu}\overline{M}^2.$$

Therefore,

$$\sup_{\tau_j\le t\le\tau_{j+1}} K(t)^2 + \nu\int_{\tau_j}^{\tau_{j+1}}\overline{K}(\tau)^2\,d\tau \le CM_j + \frac{C}{\nu\gamma}M_{j+1} + \frac{C\overline{M}}{\nu^{1/2}}\Big(\sum_{i=1}^{j}M_i\Big)^{1/2}$$
$$+ \frac{C\nu^{1/2}}{\gamma^{1/2}}M_{j+1}^{1/2} + \frac{C}{\nu^2}\overline{M}^2\sup_{1\le i\le j}M_i^{1/2} + \frac{C}{\gamma}M_{j+1}^{1/2} + \frac{C}{\nu}\overline{M}^2. \qquad (2.13)$$

Dividing (2.12) by ν^2 and adding the resulting inequality to (2.13) gives

$$M_{j+1} \le CM_j + \frac{C}{\nu^3}\overline{M}^2 \sum_{i=1}^{j} M_i + \frac{C}{\nu\gamma} M_{j+1} + \frac{C}{\nu^6}\overline{M}^4 \sup_{1\le i\le j} M_i + \frac{C}{\nu^2\gamma^2} M_{j+1} + \frac{C\overline{M}^4}{\nu^4} + \frac{C\overline{M}}{\nu^{1/2}}\Big(\sum_{i=1}^{j} M_i\Big)^{1/2} + \frac{C\nu^{1/2}}{\gamma^{1/2}} M_{j+1}^{1/2} + \frac{C}{\nu^2}\overline{M}^2 \sup_{1\le i\le j} M_i^{1/2} + \frac{C}{\gamma} M_{j+1}^{1/2} + \frac{C}{\nu}\overline{M}^2. \tag{2.14}$$

Now, note that

$$\sum_{i=1}^{j} M_i \le m \sup_{1\le i\le j} M_i \le (C\overline{M}^2\nu^{-2}\gamma + 1) \sup_{1\le i\le j} M_i,$$

by (2.7). Therefore,

$$M_{j+1} \le CM_j + \frac{C\gamma}{\nu^5}\overline{M}^4 \sup_{1\le i\le j} M_i + \frac{C}{\nu^3}\overline{M}^2 \sup_{1\le i\le j} M_i + \frac{C}{\nu\gamma} M_{j+1} + \frac{C}{\nu^6}\overline{M}^4 \sup_{1\le i\le j} M_i + \frac{C}{\nu^2\gamma^2} M_{j+1} + \frac{C\overline{M}^4}{\nu^4} + \frac{C\overline{M}^2\gamma^{1/2}}{\nu^{3/2}} \sup_{1\le i\le j} M_i^{1/2} + \frac{C\overline{M}}{\nu^{1/2}} \sup_{1\le i\le j} M_i^{1/2} + \frac{C\nu^{1/2}}{\gamma^{1/2}} M_{j+1}^{1/2} + \frac{C}{\nu^2}\overline{M}^2 \sup_{1\le i\le j} M_i^{1/2} + \frac{C}{\gamma} M_{j+1}^{1/2} + \frac{C}{\nu}\overline{M}^2. \tag{2.15}$$

Choosing $\gamma = C_0/\nu$ with a sufficiently large constant C_0, we get, after a short calculation,

$$M_{j+1} \le K_1 \sup_{1\le i\le j} M_i + K_2, \qquad j = 0, \dots, m-1, \tag{2.16}$$

where $K_1 = \max\{C\overline{M}^4/\nu^6 + C, 1\}$ and $K_2 = C\nu^2 + C\overline{M}^2/\nu$. The inequality (2.16) is valid for $j = 0$, which gives $M_1 \le K_2$. By induction, we get $M_j \le jK_2K_1^{j-1}$ for $j = 1, \dots, m$, whence by (2.7)

$$M_j \le mK_2K_1^m \le \frac{C}{\nu^3}\overline{M}^2 K_2 \exp\Big(\frac{C}{\nu^3}\overline{M}^2 \log K_1\Big) = K_3, \qquad j = 1, \dots, m.$$

We conclude

$$\frac{1}{\nu^2}\sup_{t\ge t_1} J(t)^4 + \sup_{t\ge t_1} K(t)^2 \le K_3,$$

for all t, and

$$\nu \int_{s_j}^{s_{j+1}} \overline{K}(t)^2\, dt \le K_3,$$

for $j = 0, 1, \ldots$, which implies

$$\nu \int_{t}^{t+1/\nu} \overline{K}(t)^2\, dt \le CK_3,$$

for all $t \ge t_1$. Next, from [7], we have

$$\frac{d}{dt}\|A^{1/2}v\|_{L^2}^2 + \nu\|Av\|_{L^2}^2 \le \frac{C}{\nu^3}(J^4 + K^2\overline{K}^2)\|A^{1/2}v\|_{L^2}^2 + \frac{CF^2}{\nu},$$

from which, using the Gronwall lemma,

$$\|A^{1/2}v(\cdot,t)\|_{L^2}^2 + \nu \int_{t}^{t+1/\nu} \|Av(\cdot,\tau)\|_{L^2}^2\, d\tau \le \frac{C\overline{M}^2}{\nu} e^{CK_3/\nu^2}, \tag{2.17}$$

for all $t \ge t_1$, and the theorem is proven. □

3. THE UNIFORM GRADIENT BOUNDS FOR THE FULL PRIMITIVE EQUATIONS

In this section, we show how to extend the analysis of Section 2 to the case of Robin boundary conditions with varying bottom topography, as well as to the full primitive equations which include the temperature and the salinity.

3.1. **Robin boundary condition.** The Robin boundary condition, which is used classically for the primitive equations, reads

$$\frac{\partial v}{\partial x_3} + \alpha v = 0, \qquad (x_1, x_2, x_3) \in \Gamma_t = \Omega_2 \times \{0\},$$

where $\alpha \ge 0$ is given. As pointed out in [7], the equation (2.9) reads

$$\frac{d}{dt}J^2 + \frac{\nu\alpha}{CJ^4}\|v\|_{L^6(\Gamma_t)}^6 \le \frac{C}{\nu}\|\nabla_2 p\|_{L^{3/2}(\Omega_2)}^2 + \frac{C}{\nu}F^2.$$

Thus, in the inequality (2.12) we get an extra term

$$\Big(\frac{\nu\alpha}{C}\int_{\tau_j}^{\tau_{j+1}} \frac{1}{J(\tau)^4}\|v(\cdot,\tau)\|_{L^6(\Gamma_t)}^6\, d\tau\Big)^2,$$

on the left side. Also, by [7], we get an additional term $(C\alpha/\nu)\|v\|_{L^4(\Gamma_t)}^4$ on the right side of (2.10), and thus we get an extra term

$$\frac{C\alpha}{\nu}\int_{\tau_j}^{\tau_{j+1}} \|v(\cdot,\tau)\|_{L^4(\Gamma_t)}^4\, d\tau,$$

on the right side of (2.13). Therefore, the equation (2.14) is the same except for an additional expression

$$\frac{C\alpha}{\nu}\int_{\tau_j}^{\tau_{j+1}}\|v(\cdot,\tau)\|_{L^4(\Gamma_t)}^4\,d\tau-\Big(\frac{\alpha}{C}\int_{\tau_j}^{\tau_{j+1}}\frac{1}{J(\tau)^4}\|v(\cdot,\tau)\|_{L^6(\Gamma_t)}^6\,d\tau\Big)^2. \tag{3.1}$$

Now, by Hölder's inequality,

$$\begin{aligned}&\frac{C\alpha}{\nu}\int_{\tau_j}^{\tau_{j+1}}\|v(\cdot,\tau)\|_{L^4(\Gamma_t)}^4\,d\tau\le\frac{C\alpha}{\nu}\int_{\tau_j}^{\tau_{j+1}}\|v(\cdot,\tau)\|_{L^6(\Gamma_t)}^4\,d\tau \\ &\le\frac{C\alpha}{\nu}\Big(\int_{\tau_j}^{\tau_{j+1}}\frac{1}{J(\tau)^4}\|v(\cdot,\tau)\|_{L^6(\Gamma_t)}^6\,d\tau\Big)^{2/3}\Big(\int_{\tau_j}^{\tau_{j+1}}J(\tau)^8\,d\tau\Big)^{1/3} \\ &\le\Big(\frac{\alpha}{\beta}\int_{\tau_j}^{\tau_{j+1}}\frac{1}{J(\tau)^4}\|v(\cdot,\tau)\|_{L^6(\Gamma_t)}^6\,d\tau\Big)^2+\frac{C\alpha^{1/2}\beta}{\nu^{3/2}}(\tau_{j+1}-\tau_j)^{\frac12}\sup_{\tau\in[\tau_j,\tau_{j+1}]}J(\tau)^4,\end{aligned} \tag{3.2}$$

for any constant $\beta>0$. By choosing $\beta>0$ sufficiently large, the first term on the far right of (3.2) is dominated by the second term in (3.1). Therefore, the inequality (2.15) is the same except for an extra term

$$C\nu^{1/2}\alpha^{1/2}(\tau_{j+1}-\tau_j)^{1/2}M_{j+1},$$

on the right. This term can be absorbed if we have

$$\tau_{j+1}-\tau_j\le\frac{1}{C_0\nu\alpha}, \tag{3.3}$$

where C_0 is a sufficiently large constant. We may achieve this by altering the construction of τ_j in the proof of Theorem 2.1 so that we have (2.6),

$$m\le\frac{C\gamma\overline{M}^2}{\nu^2}+C_0\alpha+1, \tag{3.4}$$

and (3.3). The construction is by induction starting with $\tau_0=T_1$. If

$$\int_{\tau_j}^{T_2}\overline{E}(\tau)^2\,d\tau\le\frac{1}{\gamma},$$

and $T_2-\tau_j\le 1/C_0\nu\alpha$, we terminate by setting $m=j+1$ and $\tau_m=T_2$. Otherwise, let

$$\tau_{j+1}=\sup\Big\{t\in(\tau_j,T_2):\int_{\tau_j}^{t}\overline{E}(\tau)^2\,d\tau<\frac{1}{\gamma}\text{ and }t-\tau_j\le\frac{1}{C_0\nu\alpha}\Big\}.$$

With this construction, on every interval except the last one we have either (2.8) or $\tau_{j+1}-\tau_j=1/C_0\nu\alpha$, and (3.4) follows. The rest of the proof is the same, except that now K_1 and K_2 depend on α in an explicit way determined by (3.4).

3.2. **The varying bottom topography.** The case of the varying bottom topography is handled as above. The only difference is in the pressure estimate (2.11). By [7], we have

$$\begin{aligned} \|\nabla_2 p\|^2_{L^2_t L^{3/2}_x(\Omega_2\times(T_1,t))} &\le C\nu^2\|K^{1/2}\overline{K}^{1/2}\|^2_{L^2_t(T_1,t)} + C\|J\overline{E}\|^2_{L^2_t(T_1,t)} \\ &\quad + C\overline{M}^2 + C\nu\|\nabla v(\cdot,T_1)\|^2_{L^2} \qquad (3.5) \\ &+ \nu^2\Big\|h^{-1}|\nabla_2 h|^2\partial_3 u_k\Big|_{\Gamma_t} + h\Delta_2(h^{-1})Mu_k + 2h\sum_{j=1}^{2} M(\partial_j u_k)\partial_j(h^{-1})\Big\|^2_{L^2}, \end{aligned}$$

instead of (2.11). The last term in (3.5) is bounded from above by

$$C\nu^2\|K^{1/2}\overline{K}^{1/2}\|^2_{L^2_t(T_1,t)} + C\overline{M}^2,$$

where C now also depends on h. This expression can be absorbed by the first and the third term on the right side of (3.5), and thus the rest of the proof is unchanged, except that the constants depend on h (in an explicit way).

3.3. **The full primitive equations of the ocean.** Here, we indicate the modifications in the case of the full primitive equations including the temperature T and the salinity S. The full primitive equations of the ocean read

$$\begin{aligned} &\frac{\partial v}{\partial t} + (v\cdot\nabla_2)v + u_3\frac{\partial v}{\partial x_3} + \frac{1}{\rho_0}\nabla_2 p + 2\omega\sin\theta e_3\times v - \mu_v\Delta_2 v - \nu_v\frac{\partial^2 v}{\partial x_3^2} \\ &\qquad = F(T,S) + F_v \\ &\operatorname{div}_2 v + \frac{\partial u_3}{\partial x_3} = 0 \\ &\frac{\partial T}{\partial t} + v\cdot\nabla_2 T + u_3\frac{\partial T}{\partial x_3} - \mu_T\Delta_2 T - \nu_T\frac{\partial^2 T}{\partial x_3^2} = F_T \\ &\frac{\partial S}{\partial t} + v\cdot\nabla_2 S + u_3\frac{\partial S}{\partial x_3} - \mu_S\Delta_2 S - \nu_S\frac{\partial^2 S}{\partial x_3^2} = F_S \\ &\operatorname{div}_2\int_{-h}^{0} v\,dx_3 = 0, \end{aligned}$$

where

$$F(T,S) = g\int_{x_3}^{0}\bigl(\beta_T\nabla_2(T-T_r)+\beta_S\nabla_2(S-S_r)\bigr)dx_3,$$

(cf. [7] for the physical meaning of various parameters arising in the above equations). We assume, for $t\geq 0$,

$$\|F_v\|_{L_t^2L_x^2(\Omega\times(t,t+\nu^{-1}))}, \|F_T\|_{L_t^2L_x^2(\Omega\times(t,t+\nu^{-1}))}, \|F_S\|_{L_t^2L_x^2(\Omega\times(t,t+\nu^{-1}))} \leq \overline{M}.$$

The energy inequality (2.1) in this context reads

$$\frac{d}{dt}(\|v\|_{L^2}^2+\|T\|_{L^2}^2+\|S\|_{L^2}^2)+\nu(\|\nabla v\|_{L^2}^2+\|\nabla T\|_{L^2}^2+\|\nabla S\|_{L^2}^2) + \frac{\nu}{C}(\|v\|_{L^2}^2+\|T\|_{L^2}^2+\|S\|_{L^2}^2) \leq \frac{C}{\nu}F^2,$$

with $\nu = \min\{\mu_v,\mu_T,\mu_S,\nu_v,\nu_T,\nu_S\}$. Therefore, we obtain as before

$$\|F(T,S)\|_{L_t^2L_x^2(\Omega\times(t,t+\nu^{-1}))} \leq C\overline{M}, \qquad t\geq 0,$$

where the constant depends also on β_T and β_S. Thus, the estimates in the proof of Theorem 2.1 for the velocity are unchanged except for the additional dependence of the constants on β_T and β_S. Next, we need H^1 estimates on T and S. Since the derivations are completely the same, we only show details for T. We multiply by $-\Delta T$ and integrate in order to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla T\|_{L^2}^2+\nu\|\Delta T\|_{L^2}^2 \leq \int_\Omega (v\cdot\nabla_2 T)\cdot\Delta T + \int_\Omega u_3\frac{\partial T}{\partial x_3}\Delta T$$
$$\leq C\|v\|_{L^6}\|\nabla_2 T\|_{L^2}^{\frac12}\|\Delta T\|_{L^2}^{\frac32} + C\|\nabla v\|_{L^2}^{\frac12}\|\Delta v\|_{L^2}^{\frac12}\Bigl\|\frac{\partial T}{\partial x_3}\Bigr\|_{L^2}^{\frac12}\Bigl\|\nabla\frac{\partial T}{\partial x_3}\Bigr\|_{L^2}^{\frac12}\|\Delta T\|_{L^2},$$

where we used the anisotropic estimates from [16], whence

$$\frac{d}{dt}\|\nabla T\|_{L^2}^2+\nu\|\Delta T\|_{L^2}^2 \leq \frac{C}{\nu^2}\|v\|_{L^6}^2\|\nabla_2 T\|_{L^2}^2 + \frac{C}{\nu^2}\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}^2\,\|\nabla T\|_{L^2}^2\,.$$

Using (2.17) and the Gronwall lemma, the uniform estimate for $\|\nabla T\|_{L^2}^2$ follows.

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