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EXISTENCE RESULTS FOR NON-UNIFORMLY ELLIPTIC EQUATIONS WITH GENERAL GROWTH IN THE GRADIENT

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Abstract. We prove a priori estimates and existence results for a class of problems whose prototype is

$$-\mathrm{div}(b(|u|)\,|Du|^{p-2}\,Du) = k(|u|)\,|Du|^q + f, \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

where Ω is a bounded domain in $\mathbb{R}^n$, $p-1 < q \leq p$, and k and b are continuous functions.

1. Introduction

In this paper, we deal with Dirichlet problems whose prototype is

$$-\mathrm{div}(b(|u|)\,|Du|^{p-2}\,Du) = k(|u|)\,|Du|^q + f, \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad (1.1)$$

where Ω is a bounded open set in $\mathbb{R}^n$, $1 < p < +\infty$, $p-1 < q \leq p$, b and k are continuous functions, with $k \geq 0$ and $b > 0$, and $f \in L^r(\Omega)$, $r > \max\{n/p, 1\}$. Problems like (1.1) have been widely studied in the literature under various hypotheses.

In a series of paper of Boccardo, Murat and Puel (see [7, 8, 9, 10]), the existence of solutions of problems like (1.1), with $p = q$ and b constant, is proved. Moreover, similar problems with p–growth in the gradient have been considered, always with b constant, in [23] (for $p = 2$) and [17] (see also [3] for the case in which b and k are constant). The case where b is not necessarily constant is treated, for $k \equiv 0$, in [2], [1], [6], and for $k \not\equiv 0$ in [31], [12]. The existence of (possibly unbounded) solutions for equations similar to (1.1) in the case $f \in L^{n/p}(\Omega)$ is obtained in [14], with $p = 2$, and, with an additional term in the equation, in [15], always with further hypotheses on f. Moreover, existence results which do not depend on f when $q = p$ are given in [12] (for $p = 2$) and in [27].

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As concerns the case of q–growth in the gradient, $p-1 < q \le p$, existence results with b and k constant are given in [13], and in [11], [19].

The quoted results are of two kinds: the first one establishes existence of solutions without imposing any additional condition on f (see [12], [27]); the second one requires conditions on the smallness of some norm of f (see [23], [17], [31], [13], [14], [15], [19]). More precisely, when it is possible to remove the smallness hypotheses on f, what is needed are appropriate hypotheses on the structure of the equation, like sign conditions or particular hypotheses on the functions $k(s)$ and $b(s)$.

Our aim is to obtain an existence result for problems like (1.1). Our approach permits us to treat in a unified way both the cases in which a particular hypothesis on f is required and the cases in which such a hypothesis is not necessary.

For example, when $b \equiv 1$ in (1.1), our result reads as follows:

Theorem 1.1. *If*

$$c \,\|f\|_{L^r(\Omega)} < \sup_{s>0} W(s), \tag{1.2}$$

where

$$W(s) = \int_0^s e^{-C\int_r^s k(y)^{\frac{1}{1-p+q}}\,dy} dr,$$

and the constants c and C depend only on p, q, n and $|\Omega|$, then there exists a solution of (1.1).

Depending on $k(s)$, the function $W(s)$ can be bounded or not. In the first case, the condition (1.2) is a hypothesis on the norm of f. A simple example of the latter case occurs when $k(s)$ tends to zero at infinity, but obviously the condition $\sup_{s>0} W(s) = +\infty$ is more general.

The paper is organized as follows. In Section 2, we give the general assumptions on the Dirichlet problem we will consider, and state the main result. In Section 3, we prove an a priori estimate on the solutions of (1.1) in terms of the function W, namely

$$W(\|u\|_{L^\infty(\Omega)}) \le c\,\|f\|_{L^r(\Omega)}\,, \tag{1.3}$$

where c is the same constant which appears in (1.2). From the same proof we get a $W^{1,p}$ estimate of the gradient of u. In Section 4, we prove Theorem 1.1 using the estimate (1.3). If the condition (1.2) is not satisfied, it is clear that (1.3) cannot give any information on $\|u\|_{L^\infty(\Omega)}$, since it is trivially verified. On the other hand, if, for example, $W(s)$ is monotone, (1.2) and (1.3) immediately give us an L^∞−estimate for u. In general, under our assumptions $W(s)$ is not monotone, and then (1.2) and (1.3) do not imply directly an estimate for u. The main point in the proof of the existence result consists in showing that the only hypothesis (1.2) allows us to obtain a uniform estimate for the solutions to suitable problems which approximate (1.1). After passing to the limit of such approximate problems we get the result.

We explicitly observe that our approach permits us to treat the case in which q assumes all the values between $p-1$ and p and the general case in which k and b are not necessarily monotone.

2. Assumptions, main result and comments

We deal with Dirichlet problem of the form

$$-\operatorname{div}(a(x,u,Du)) = H(x,u,Du) + f, \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{2.1}$$

where $1 < p < +\infty$, Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions satisfying the following assumptions:

$$(a(x,s,\xi) - a(x,s,\xi')) \cdot (\xi - \xi') > 0, \tag{2.2}$$

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$a(x,s,\xi) \geq b(|s|)|\xi|^p, \tag{2.3}$$

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^n$, and $b : [0,+\infty) \to (0,+\infty)$ is a continuous function,

$$|a(x,s,\xi)| \leq c_0(|\xi|^{p-1} + |s|^{p-1} + g(x)), \tag{2.4}$$

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^n$, with $g \in L^{p'}(\Omega)$,

$$|H(x,s,\xi)| \leq k(|s|)|\xi|^q, \tag{2.5}$$

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^n$, with $p-1 < q \leq p$, and the function $k : [0,+\infty) \to [0,+\infty)$ is continuous. Moreover, assume

$$f \in L^r(\Omega), \ r > \max\left\{\tfrac{n}{p}, 1\right\}. \tag{2.6}$$

We say that $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a *weak solution* of (2.1) if

$$\int_\Omega a(x,u,Du) \cdot Dv \, dx = \int_\Omega H(x,u,Du) v \, dx + \int_\Omega f v \, dx,$$

for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Now, we recall some basic definitions about decreasing rearrangements and Schwarz symmetrization. For an exhaustive treatment of the properties of rearrangements we refer to [4], [20], [21], [25].

Definition 2.1. *Let $u \in L^1(\Omega)$. The distribution function of u is the map $\mu : [0,\infty) \to [0,\infty)$ defined by*

$$\mu(t) \;=\; |\{x \in \Omega : |u(x)| > t\}|.$$

The function μ is decreasing and right continuous.

Definition 2.2. *The decreasing rearrangement of u is the map $u^* :\ [0,\infty) \to [0,\infty)$ defined by*

$$u^*(s) := \sup\{t \geq 0 : \mu_u(t) > s\}.$$

The function u^* is the generalized inverse of μ. We recall that, for every $1 \leq p < +\infty$,

$$\int_0^{|\Omega|} (u^*(s))^p ds = \int_\Omega |u(x)|^p \, dx,$$

and, if $u \in L^\infty(\Omega)$,

$$u^*(0) = \operatorname{ess\,sup}_\Omega |u| \,.$$

Definition 2.3. *The spherically symmetric decreasing rearrangement of u is defined by*

$$u^\#(x) = u^*(\omega_n |x|^n), \quad x \in \Omega^\#,$$

where $\Omega^\#$ is the ball centered at the origin having the same measure as Ω and ω_n is the measure of the unit ball in $\mathbb{R}^n$.

Now, we consider the Dirichlet problem:

$$\begin{cases} -\Delta_p V = f^\# & \text{in } \Omega^\#, \\ V = 0 & \text{on } \partial\Omega^\#. \end{cases} \tag{2.7}$$

We recall that the solution of the equation (2.7) can be explicitly written as

$$V(x) = \big(n\omega_n^{1/n}\big)^{-p'} \int_{\omega_n |x|^n}^{|\Omega|} t^{-(1-\frac{1}{n})p'} \Big(\int_0^t f^*(r) dr \Big)^{\frac{p'}{p}} dt. \tag{2.8}$$

Let us define the following functions:

$$Q : s \in [0,+\infty) \to C \int_0^s \Big[\frac{k(y)}{b(y)}\Big]^{\frac{1}{1-p+q}} dy; \tag{2.9}$$

$$F : s \in [0,+\infty) \to \int_0^s e^{Q(t)} b(t)^{\frac{1}{p-1}} dt; \tag{2.10}$$

$$W : s \in [0,+\infty) \to \frac{F(s)}{e^{Q(s)}}, \tag{2.11}$$

where

$$C = \frac{1-p+q}{p-1} \left(\frac{|\Omega|}{\omega_n} \right)^{\frac{p-q}{n}} .$$

We can state the following existence theorem:

Theorem 2.1. *Under assumptions* (2.2) – (2.6), *if*

$$C_0 V(0) < \sup_{s>0} W(s), \tag{2.12}$$

where C_0 *is a constant depending only on the data, namely*

$$C_0 = e^{\frac{p'}{p}(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}},$$

then there exists a weak solution of (2.1)*.*

Remark 2.1. If $p < n$, the condition (2.12) is a smallness assumption on the norm of f in the Lorentz space $L^{(n/p,\, p'/p)}(\Omega)$. In fact, in view of (2.8), (2.12) can be rewritten in the form

$$C_0 n^{-p'} \omega_n^{-p'/n} \|f\|_{L^{(n/p,\, p'/p)}(\Omega)} < \sup_{s>0} W(s);$$

it is well known that $L^{(n/p,\, p'/p)}(\Omega)$ contains $L^r(\Omega)$, for any $r > n/p$.

Remark 2.2. We observe that if $p = q$ and b and k are respectively monotone decreasing and monotone increasing, the hypothesis (2.12) coincides with the existence condition given in [31].

Remark 2.3. We explicitly note that if

$$\sup_{s>0} W(s) = +\infty, \tag{2.13}$$

no smallness assumption on f is needed, therefore a solution exists for any $f \in L^r(\Omega)$, $r > \max\{n/p, 1\}$. In this case, the hypothesis (2.13) also weakens, when $q = p$, the existence condition given in [27] (see also [12]), in which it is required that $\lim_{s\to+\infty} W(s) = +\infty$.

As a matter of fact, if $b \equiv 1$, a simple condition which verifies (2.13) is, for example, the following:

$$\lim_{s\to+\infty} k(s) = 0.$$

In this case, $\sup_{s>0} W(s) = \lim_{s\to+\infty} W(s) = +\infty$. The condition (2.13) is obviously more general, as we explicitly see in the following example.

Let $b \equiv 1$, and $p = q = 2$. Let us define k in the following way:

$$k(s) = \begin{cases} \frac{\pi}{2}\tan\frac{\pi}{4}s, & \text{if } 0 \le s \le 1; \\ \frac{3\pi - 2[\sqrt{2}\sin\left(\log s + \frac{\pi}{4}\right) + 1 - \frac{2}{s^2}]}{2s[\sin(\log s) + 1 + \frac{2}{s^2}]}, & \text{if } s > 1. \end{cases}$$

We have that k is continuous in $[0,+\infty)$ and positive for any $s>0$, and

$$\limsup_{s\to+\infty} k(s)=+\infty, \qquad \liminf_{s\to+\infty} k(s)=0.$$

If we compute $W(s)$, we have that, for $s>1$,

$$W(s)=\frac{2}{3\pi}s\Big[\sin(\log s)+1+\frac{2}{s^2}\Big],$$

so we easily find that

$$\limsup_{s\to+\infty} W(s)=+\infty, \quad \text{and} \quad \liminf_{s\to+\infty} W(s)=0.$$

3. A priori estimates

The aim of this section is to prove *a priori* estimates of the solutions of (2.1).

Given $h>0$, we define the truncation function at level $\pm h$ as

$$T_h(s)=\begin{cases} s & \text{if } |s|\le h,\\ h\,\mathrm{sign} s & \text{if } |s|>h.\end{cases}$$

We prove an L^∞ estimate of u in terms of the function W.

Theorem 3.1. *Let V be the solution of* (2.7)*. Under assumptions* (2.2) – (2.6)*, if u is a solution of* (2.1)*, then*

$$W(u^*(0))\le C_0V(0), \tag{3.1}$$

where C_0 is the constant defined in Theorem 2.1*.*

Proof. Let u be a solution of (2.1). Using in (2.1) the test functions $T_t(u)$, $T_{t+h}(u)$, with $t,h>0$, and subtracting, we have:

$$\begin{aligned}&\int_{\{t<|u|\le t+h\}} a(x,u,Du)\cdot Du\,dx\\ &=\int_{\{t<|u|\le t+h\}} H(x,u,Du)(|u|-t)\mathrm{sign}u\,dx+\int_{\{t<|u|\le t+h\}} f(|u|-t)\mathrm{sign}u\,dx\\ &\qquad+h\int_{\{|u|>t+h\}} H(x,u,Du)\mathrm{sign}u\,dx+h\int_{\{|u|>t+h\}} f\mathrm{sign}u\,dx,\end{aligned}$$

so, dividing both sides by h and using (2.3) and (2.5), we get:

$$\frac{1}{h}\int_{\{t<|u|\le t+h\}} b(|u|)|Du|^p dx\le \int_{\{|u|>t\}} k(|u|)|Du|^q dx+\int_{\{|u|>t\}} |f|dx. \tag{3.2}$$

We note that, from (3.2) and from the fact that u is a solution of (2.1), it follows that the function

$$\Phi(t) = \int_{\{|u|>t\}} b(|u|)|Du|^p dx$$

is Lipschitz continuous. If $h \to 0$, since b is continuous, using the Hardy - Littlewood inequality, from (3.2) we get

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx \le \int_{\{|u|>t\}} k(|u|)|Du|^q dx + \int_0^{\mu(t)} f^*(s)ds. \qquad (3.3)$$

By the continuity of k we have

$$\int_{\{|u|>t\}} k(|u|)|Du|^q dx = \int_t^{+\infty} k(s)\Big(-\frac{d}{ds}\int_{\{|u|>s\}} |Du|^q dx\Big)ds,$$

moreover, by the Hölder inequality we obtain

$$-\frac{d}{ds}\int_{\{|u|>s\}} |Du|^q dx \le \Big(-\frac{d}{ds}\int_{\{|u|>s\}} |Du|^p dx\Big)^{\frac{q}{p}} (-\mu'(s))^{1-\frac{q}{p}},$$

and therefore

$$\int_{\{|u|>t\}} k(u)|Du|^q \le \int_t^{+\infty} k(s)\Big(-\frac{d}{ds}\int_{\{|u|>s\}} |Du|^p dx\Big)^{\frac{q}{p}} (-\mu'(s))^{1-\frac{q}{p}} ds. \qquad (3.4)$$

Now, using again the Hölder inequality, we get

$$\Big(-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx\Big)^{\frac{p-q}{p}} \ge \Big[\Big(-\frac{d}{dt}\int_{\{|u|>t\}} |Du| dx\Big)\,\big(-\mu'(t)\big)^{-1+\frac{1}{p}}\Big]^{p-q};$$

that is,

$$\Big(-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx\Big)^{\frac{q}{p}} (-\mu'(t))^{1-\frac{q}{p}} \le \Big(-\frac{d}{dt}\int_{\{|u|>t\}} |Du| dx\Big)^{q-p} \times \Big(-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx\Big)(-\mu'(t))^{(1-\frac{1}{p})(p-q)+\left(1-\frac{q}{p}\right)}.$$

Using the Coarea formula and isoperimetric inequality, and observing that $p - q = (1 - 1/p)(p - q) + (1 - q/p)$, we get

$$\Big(-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx\Big)^{\frac{q}{p}} (-\mu'(t))^{1-\frac{q}{p}}$$

$$\leq \Big(n\omega_n^{1/n}\mu(t)^{1-\frac{1}{n}}\Big)^{q-p}\Big(-\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx\Big)(-\mu'(t))^{p-q}. \quad (3.5)$$

So, from (3.4) and (3.5) we have

$$\int_{\{|u|>t\}} k(|u|)|Du|^q \leq \Big(n\omega_n^{1/n}\Big)^{q-p}\int_t^{+\infty} k(s)\Big(-\frac{d}{ds}\int_{\{|u|>s\}}|Du|^p dx\Big)\Big(\frac{-\mu'(s)}{\mu(s)^{1-\frac{1}{n}}}\Big)^{p-q} ds. \quad (3.6)$$

From (3.3) and (3.6) we obtain

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx \leq \big(n\omega_n^{1/n}\big)^{q-p}\int_t^{+\infty} k(s)\Big(-\frac{d}{ds}\int_{\{|u|>s\}}|Du|^p dx\Big)\Big(\frac{-\mu'(s)}{\mu(s)^{1-\frac{1}{n}}}\Big)^{p-q} ds + \int_0^{\mu(t)} f^*(s)ds.$$

The previous inequality and the Gronwall lemma[1] imply:

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx \leq \int_t^{+\infty}\exp\Big\{\big(n\omega_n^{1/n}\big)^{q-p}\int_t^s \frac{k(y)}{b(y)}\Big(\frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}}\Big)^{p-q}dy\Big\} f^*(\mu(s))[-d\mu(s)],$$

and then

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}}|Du|^p dx \leq \int_0^{\mu(t)}\exp\Big\{\big(n\omega_n^{1/n}\big)^{q-p}\int_t^{u^*(r)} \frac{k(y)}{b(y)}\Big(\frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}}\Big)^{p-q}dy\Big\} f^*(r)dr. \quad (3.7)$$

On the other hand, if $p-1<q<p$, using the Hölder inequality we have

$$\int_t^{u^*(r)} \frac{k(y)}{b(y)}\Big(\frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}}\Big)^{p-q}dy$$

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[1] Here we use the following generalization of the Gronwall lemma: if φ is a bounded function and $\varphi(t)\leq\int_t^{+\infty} g(s)\varphi(s)ds+\psi(t)$, for almost every $t>0$, where $g\geq 0$ is an integrable function and ψ is a BV function so that $\psi(+\infty)=0$, then, for almost every $t>0$, we have

$$\varphi(t)\leq\int_t^{+\infty} e^{\int_t^s g(r)dr}[-d\psi(s)].$$

$$\leq \Big[\int_t^{u^*(r)} \Big(\frac{k(y)}{b(y)}\Big)^{\frac{1}{1-p+q}} dy\Big]^{1-p+q} \Big[\int_t^{u^*(r)} \frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}} dy\Big]^{p-q}. \quad (3.8)$$

(Observe that the last inequality is trivial if $q = p$). Furthermore, by the properties of the distribution function μ of u, we have

$$\int_t^{u^*(r)} \frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}} dy \leq \int_0^{+\infty} \frac{-\mu'(y)}{\mu(y)^{1-\frac{1}{n}}} dy \leq n|\Omega|^{\frac{1}{n}}. \quad (3.9)$$

Using (3.8) and (3.9) in (3.7), we get

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx$$
$$\leq \int_0^{\mu(t)} \exp\Big\{\Big(\frac{|\Omega|}{\omega_n}\Big)^{\frac{p-q}{n}} \Big[\int_t^{u^*(r)} \Big(\frac{k(y)}{b(y)}\Big)^{\frac{1}{1-p+q}} dy\Big]^{1-p+q}\Big\} f^*(r)dr. \quad (3.10)$$

Now, we recall that if $x \geq 0$, and $0 \leq \alpha \leq 1$, we have

$$x^\alpha \leq \alpha x + (1-\alpha), \quad (3.11)$$

so, applying (3.11) to (3.10), we have

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx$$
$$\leq \int_0^{\mu(t)} e^{\left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-q}{n}} \left[(p-q)+(1-p+q)\int_t^{u^*(r)} \left(\frac{k(y)}{b(y)}\right)^{\frac{1}{1-p+q}} dy\right]} f^*(r)dr;$$

that is,

$$b(t)\Big(-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx\Big)$$
$$\leq e^{(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}} \int_0^{\mu(t)} e^{(p-1)[Q(u^*(r))-Q(t)]} f^*(r)dr, \quad (3.12)$$

with

$$Q(t) = \Big(\frac{|\Omega|}{\omega_n}\Big)^{\frac{p-q}{n}} \frac{1-p+q}{p-1} \int_0^t \Big(\frac{k(y)}{b(y)}\Big)^{\frac{1}{1-p+q}} dy.$$

On the other hand, we have by the Hölder inequality that

$$-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx \geq \Big(-\frac{d}{dt}\int_{\{|u|>t\}} |Du| dx\Big)^p (-\mu'(t))^{-\frac{p}{p'}}.$$

Applying again the coarea formula and the isoperimetric inequality, we have

$$-\frac{d}{dt}\int_{\{|u|>t\}}|Du|dx \geq n\omega_n^{1/n}\mu(t)^{1-\frac{1}{n}},$$

and thus

$$b(t)\left(n\omega_n^{1/n}\right)^p \mu(t)^{p-\frac{p}{n}}(-\mu'(t))^{-\frac{p}{p'}} \leq e^{(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}}\int_0^{\mu(t)} e^{(p-1)[Q(u^*(r))-Q(t)]}f^*(r)dr;$$

that is,

$$b(t)e^{(p-1)Q(t)} \leq e^{(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}}\left(n\omega_n^{1/n}\right)^{-p}\left(\frac{-\mu'(t)^{\frac{1}{p'}}}{\mu(t)^{1-\frac{1}{n}}}\right)^p\int_0^{\mu(t)} e^{(p-1)Q(u^*(r))}f^*(r)dr;$$

recalling that u^* is a decreasing function, it follows that

$$\frac{b(t)^{1/(p-1)}e^{Q(t)}}{e^{Q(u^*(0))}} \leq e^{\frac{p'}{p}(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}}\left(n\omega_n^{1/n}\right)^{-p'}\frac{-\mu'(t)}{\mu(t)^{p'-\frac{p'}{n}}}\left(\int_0^{\mu(t)} f^*(r)dr\right)^{\frac{p'}{p}}.$$

Integrating between 0 and σ,

$$\frac{F(\sigma)}{e^{Q(u^*(0))}} \leq e^{\frac{p'}{p}(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}}\left(n\omega_n^{1/n}\right)^{-p'}\int_0^{\sigma}\frac{-\mu'(t)}{\mu(t)^{p'-\frac{p'}{n}}}\left(\int_0^{\mu(t)} f^*(r)dr\right)^{\frac{p'}{p}}dt,$$

where $F(s)$ and $Q(s)$ are the functions defined in (2.9), (2.10). If $\sigma = u^*(s)$, using the properties of the rearrangements, we have

$$\frac{F(u^*(s))}{e^{Q(u^*(0))}} \leq e^{\frac{p'}{p}(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}}\left(n\omega_n^{1/n}\right)^{-p'}\int_s^{|\Omega|} t^{-\left(1-\frac{1}{n}\right)p'}\left(\int_0^t f^*(r)dr\right)^{\frac{p'}{p}}dt; \quad (3.13)$$

that is,

$$W(u^*(0)) \leq e^{\frac{p'}{p}(p-q)(|\Omega|/\omega_n)^{\frac{p-q}{n}}}V(0). \qquad \square$$

From the proof of Theorem 3.1 we get easily a $W^{1,p}$ estimate of u:

Proposition 3.1. *Let V be the solution of* (2.7). *Under assumptions* (2.3)–(2.6), *if u is a solution of* (2.1), *then*

$$\int_\Omega |Du|^p dx \le \frac{u^*(0)}{m} \exp\Big\{ \Big(\frac{|\Omega|}{\omega_n}\Big)^{\frac{p-q}{n}} \Big[\int_0^{u^*(0)} \Big(\frac{k(y)}{b(y)}\Big)^{\frac{1}{1-p+q}} dy \Big]^{1-p+q} \Big\} \int_0^{|\Omega|} f^*(r)dr, \tag{3.14}$$

where $m = \min_{t\in[0,u^*(0)]} b(t)$.

Proof. By (3.10) it follows that

$$-b(t)\frac{d}{dt}\int_{\{|u|>t\}} |Du|^p dx \le \exp\Big\{ \Big(\frac{|\Omega|}{\omega_n}\Big)^{\frac{p-q}{n}} \Big[\int_0^{u^*(0)} \Big(\frac{k(y)}{b(y)}\Big)^{\frac{1}{1-p+q}} dy \Big]^{1-p+q} \Big\} \int_0^{|\Omega|} f^*(r)dr,$$

so integrating between 0 and $u^*(0)$ we obtain the estimate (3.14). □

4. Proof of Theorem 2.1

In order to prove Theorem 2.1, we need to define some auxiliary functions and study their behavior. For any $\lambda > 0$, we put

$$k_\lambda(s) = k(T_\lambda(s)), \qquad b_\lambda(s) = b(T_\lambda(s)).$$

Furthermore, we set

$$Q_\lambda : s \in [0,+\infty) \to C\int_0^s \Big[\frac{k_\lambda(y)}{b_\lambda(y)}\Big]^{\frac{1}{1-p+q}} dy, \tag{4.1}$$

$$F_\lambda : s \in [0,+\infty) \to \int_0^s e^{Q_\lambda(t)} b_\lambda(t)^{\frac{1}{p-1}} dt, \tag{4.2}$$

$$W_\lambda : s \in [0,+\infty) \to \frac{F_\lambda(s)}{e^{Q_\lambda(s)}}, \tag{4.3}$$

where

$$C = \frac{1-p+q}{p-1}\Big(\frac{|\Omega|}{\omega_n}\Big)^{\frac{p-q}{n}}.$$

We observe that W is continuously differentiable, $W(0) = 0$ and

$$W'(s) = b(s)^{\frac{1}{p-1}} - CW(s)\Big(\frac{k(s)}{b(s)}\Big)^{\frac{1}{1-p+q}}.$$

It follows that $W'(0) > 0$. When $s > 0$, we have

$$W'(s) > 0 \text{ if } k(s) = 0, \tag{4.4}$$

while if $k(s) \neq 0$ then

$$W'(s) \geq 0 \iff W(s) \leq C^{-1} \frac{b(s)^{\frac{1}{p-1}} \left(\frac{q}{p-1}\right)'}{k(s)^{\frac{1}{1-p+q}}}. \tag{4.5}$$

Clearly, we have

$$W_\lambda(s) = W(s) \qquad \text{if } 0 \leq s \leq \lambda,$$

while for $s > \lambda$, we have:

$$\begin{aligned} W_\lambda(s) &= \int_0^s e^{C\int_s^r \left[\frac{k_\lambda(y)}{b_\lambda(y)}\right]^{\frac{1}{1-p+q}} dy}\, b_\lambda(r)^{\frac{1}{p-1}} dr \\ &= W(\lambda) e^{C\int_s^\lambda \left[\frac{k_\lambda(y)}{b_\lambda(y)}\right]^{\frac{1}{1-p+q}} dy} + \int_\lambda^s e^{C\int_s^r \left[\frac{k_\lambda(y)}{b_\lambda(y)}\right]^{\frac{1}{1-p+q}} dy} b_\lambda(r)^{\frac{1}{p-1}} dr \\ &= W(\lambda) e^{C(\lambda-s)\left(\frac{k(\lambda)}{b(\lambda)}\right)^{\frac{1}{1-p+q}}} + b(\lambda)^{\frac{1}{p-1}} \int_\lambda^s e^{(r-s)\left(\frac{k(\lambda)}{b(\lambda)}\right)^{\frac{1}{1-p+q}}} dr. \end{aligned}$$

It follows that, for $s > \lambda$,

$$W_\lambda(s) = \\ = \begin{cases} W(\lambda) + (s-\lambda) b(\lambda)^{\frac{1}{p-1}} & \text{if } k(\lambda) = 0; \\ \left[W(\lambda) - C^{-1} \frac{b(\lambda)^{\frac{1}{p-1}}\left(\frac{q}{p-1}\right)'}{k(\lambda)^{\frac{1}{1-p+q}}}\right] e^{C(\lambda-s)\left(\frac{k(\lambda)}{b(\lambda)}\right)^{\frac{1}{1-p+q}}} & \\ \qquad\qquad + C^{-1} \frac{b(\lambda)^{\frac{1}{p-1}}\left(\frac{q}{p-1}\right)'}{k(\lambda)^{\frac{1}{1-p+q}}} & \text{if } k(\lambda) > 0. \end{cases} \tag{4.6}$$

Thus, from (4.4), (4.5) and (4.6) it follows that $W'(\lambda) \geq 0$ implies

$$W_\lambda(s) \geq W(\lambda), \quad \forall\, s > \lambda. \tag{4.7}$$

Inequality (4.7) is immediate for $k(\lambda) = 0$, while for $k(\lambda) > 0$ we have to observe that, in view of (4.5), $W_\lambda(s)$ is increasing with respect to s.

For any fixed $\lambda > 0$, we consider the truncated problem:

$$-\operatorname{div}(a_\lambda(x, u_\lambda, Du_\lambda)) = H_\lambda(x, u_\lambda, Du_\lambda) + f, \quad u_\lambda \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{4.8}$$

where $a_\lambda(x,\eta,\xi)$ and $H_\lambda(x,\eta,\xi)$ are defined in the following way:

$$a_\lambda(x,s,\xi) = \begin{cases} a(x,s,\xi) & \text{if } |s| \le \lambda, \\ a(x,\lambda,\xi) & \text{if } s > \lambda, \\ a(x,-\lambda,\xi) & \text{if } s < -\lambda, \end{cases}$$

and

$$H_\lambda(x,s,\xi) = \begin{cases} H(x,s,\xi) & \text{if } |s| \le \lambda, \\ H(x,\lambda,\xi) & \text{if } s > \lambda, \\ H(x,-\lambda,\xi) & \text{if } s < -\lambda. \end{cases}$$

The functions $a_\lambda(x,s,\xi)$ and $H_\lambda(x,s,\xi)$ satisfy the assumptions (2.2)−(2.5) with b_λ and k_λ instead of b and k. So by Theorem 3.1 it follows that, if u_λ is a solution of (4.8), then

$$W_\lambda(u_\lambda^*(0)) \le C_0 V(0), \tag{4.9}$$

where W_λ is the function defined in (4.3), $V(0)$ is given by (2.8) and C_0 is the constant of Theorem 3.1.

Remark 4.1. We observe that, if there exists $\lambda > 0$ such that $u_\lambda^*(0) \le \lambda$, then u_λ also solves the problem (2.1).

Now, we obtain a uniform estimate for the solutions of the approximate problems (4.8):

Proposition 4.1. *Under assumptions* (2.2) − (2.6)*, if*

$$C_0 V(0) < \sup_{s>0} W(s), \tag{4.10}$$

where $C_0 = C_0(p,q,n,|\Omega|)$ *is given in Theorem* 3.1*, then there exists* $\lambda > 0$ *such that, if* u_λ *is a solution of* (4.8)*, we have*

$$u_\lambda^*(0) \le \lambda. \tag{4.11}$$

Proof. We define

$$\tau = \sup\{s \in [0,+\infty) : W(\sigma) \le C_0 V(0),\ \forall\, \sigma \in [0,s]\}; \tag{4.12}$$

because of assumption (4.10), we have $0 \le \tau < +\infty$, and obviously $W(\tau) = C_0 V(0)$; moreover, there exists $\lambda > \tau$ such that $W(\tau) < W(\lambda)$ and $W'(\lambda) \ge 0$. By (4.7), we get

$$C_0 V(0) = W(\tau) < W(\lambda) \le W_\lambda(s), \quad \forall\, s > \lambda.$$

On the other hand, if u_λ is a solution of the approximate equation (4.8) at the truncation value λ, by (4.9) we have $W_\lambda(u_\lambda^*(0)) \le C_0 V(0)$, so it follows that $u_\lambda^*(0) \le \lambda$. □

Remark 4.2. As a matter of fact, we can choose λ in order to get a better estimate of the solutions. The following two facts can happen:

a) There exists $\bar\lambda > \tau$ such that $W(s) > W(\tau)$ for any $s \in]\tau, \bar\lambda[$; then if $u_{\bar\lambda}$ is a solution of (4.8) at the truncation value $\bar\lambda$, by Proposition 4.1 we get $u^*_{\bar\lambda}(0) \leq \bar\lambda$. On the other hand, $W_{\bar\lambda}(s) = W(s)$ for any $s \leq \bar\lambda$, so since $W_{\bar\lambda}(u^*_{\bar\lambda}(0)) \leq C_0 V(0)$ we obtain $u^*_{\bar\lambda}(0) \leq \tau$.

b) Such a value $\bar\lambda$ does not exist; in this case, for any $\varepsilon > 0$ small, we can choose λ in the proof of Proposition 4.1 such that $\tau < \lambda < \tau + \varepsilon$ and $W(\lambda) > W(\tau)$, obtaining (4.11).

Remark 4.3. We observe that in the proof of Proposition 4.1 only the functions W and W_λ are involved. Thus, we deduce that if the condition (4.10) is satisfied, there exists $\lambda > 0$ such that every solution v of a problem like (2.1) satisfying, for almost every $x \in \Omega$, for every $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$, instead of (2.3) and (2.5) the assumptions:

$$a(x, s, \xi)\xi \geq b_\lambda(|s|)|\xi|^p, \quad |H(x, s, \xi)| \leq k_\lambda(|s|)|\xi|^q,$$

satisfies $v^*(0) \leq \lambda$.

It is clear that, in order to prove Theorem 2.1, we need only to prove existence for the problem (4.8), and then use Remark 4.1.

Proof of Theorem 2.1. Given $\lambda > 0$ and $\varepsilon > 0$, we consider the approximate problem

$$-\operatorname{div}(a_\lambda(x, u_{\lambda,\varepsilon}, Du_{\lambda,\varepsilon})) = H_{\lambda,\varepsilon}(x, u_{\lambda,\varepsilon}, Du_{\lambda,\varepsilon}) + f, \qquad u_{\lambda,\varepsilon} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{4.13}$$

where

$$H_{\lambda,\varepsilon}(x, s, \xi) = \frac{H_\lambda(x, s, \xi)}{1 + \varepsilon |H_\lambda(x, s, \xi)|}.$$

We have

$$|H_{\lambda,\varepsilon}(x, s, \xi)| \leq |H_\lambda(x, s, \xi)|$$

and

$$|H_{\lambda,\varepsilon}(x, s, \xi)| \leq \frac{1}{\varepsilon}.$$

It is well known (see [22]) that problem (4.13) admits a solution $u_{\lambda,\varepsilon}$ for every $\varepsilon > 0$. Let us observe that

$$a_\lambda(x, s, \xi)\xi \geq b_\lambda(|s|)|\xi|^p, \quad |H_{\lambda,\varepsilon}(x, s, \xi)| \leq k_\lambda(|s|)|\xi|^q.$$

By Proposition 4.1 and Remark 4.3 it follows that there exists $\lambda > 0$ such that

$$u^*_{\lambda,\varepsilon}(0) \leq \lambda, \tag{4.14}$$

for every $\varepsilon > 0$. Thus $\{u_{\lambda,\varepsilon}\}_{\varepsilon>0}$ is bounded in $L^\infty(\Omega)$. By (3.14) and (4.14) it follows that

$$\int_\Omega |Du_{\lambda,\varepsilon}|^p dx \leq \frac{\lambda}{m_\lambda} \exp\left\{ \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-q}{n}} \left[\int_0^\lambda \left(\frac{k_\lambda(y)}{b_\lambda(y)}\right)^{\frac{1}{1-p+q}} dy \right]^{1-p+q} \right\} \int_0^{|\Omega|} f^*(r)dr,$$

where $m_\lambda = \min_{t\in[0,\lambda]} b_\lambda(t)$. So $\{u_{\lambda,\varepsilon}\}_{\varepsilon>0}$ is bounded in $W^{1,p}(\Omega)$, and weakly converges to a function $u_\lambda \in W^{1,p}(\Omega)$. Using standard techniques (see [7, 8, 9, 10]), it is possible to extract a subsequence which strongly converges in $W^{1,p}(\Omega)$ to a function $u_\lambda \in W^{1,p}(\Omega)$ which solves (4.8). Since $u^*_\lambda(0) \leq \lambda$, this is a solution of (2.1). □

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