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CONTROLLABILITY RESULTS OF LINEAR PARABOLIC INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this paper we analyze the controllability properties of a linear parabolic integrodifferential equation with homogeneous Dirichlet boundary conditions. Our method relies on the derivation of a Carleman estimate for the adjoint equation. As a consequence, we obtain the approximate and null controllability properties for the integrodifferential equation.

1. Introduction

In many areas of science and engineering, mathematical models consisting of boundary-value problems of parabolic partial differential equations, where the differential equations contain additional integral expressions including 'memory functions' describing the memory property of the material yielding a partial integrodifferential equation. Systems with memory have the property that the mathematical-physical description of their state at a given point of time includes states in which the materials have been at an earlier point of time [18].

Consider the linear parabolic integrodifferential equation of the form

$$\begin{cases} y_t(t,x) - \Delta y(t,x) - \displaystyle\int_0^t k(t,\tau)\Delta y(\tau,x)d\tau - \int_0^t l(t,\tau)y(\tau,x)d\tau \\ \qquad\qquad = \chi_\omega u(t,x) + f(t,x), \quad (t,x) \in (0,T)\times\Omega \\ y(0,x) \ = \ y_0(x), \quad x\in\Omega \\ y(t,x) \ = \ 0, \quad (t,x)\in(0,T)\times\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$, $y(t,x)$ is the temperature of the body at the point x and time t, $y_0(x)$ is the initial temperature distribution, the integral kernels $k(\cdot,\tau), l(\cdot,\tau)$ are called heat

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kernels, and have support in (t_1, t_2), where $0 < t_1 < t_2 < T$, $f(t,x)$ is a forcing term, such as a heat source, χ_ω is the characteristic function of the subset $\omega \subset \Omega$, and $u(t,x)$ is the control to be determined. The control is on the right-hand side of the parabolic equation and distributed over an arbitrary subdomain ω of the domain Ω. We assume that the kernels $k, l : (0,T) \times (0,T) \to \mathbb{R}$ are smooth. Such a type of equation as (1.1) occurs in viscoelasticity as well as in thermodynamics of materials with memory. The problem (1.1) is interesting because the integral term can be considered as a perturbation of Δy. There are also some other physical and engineering problems which can be described by an equation of the form (1.1) (see [12,19] for more details).

Barbu and Iannelli [5] obtained the controllability results of a similar type of equation by assuming the convolution kernel is completely monotone and also studied the exact controllability of the one-dimensional linear viscoelasticity equation

$$\begin{aligned} y_t(t,x) - \int_0^t a(t-s) y_{xx} ds &= m(x)u(t,x), \quad (t,x) \in Q \\ y(0,x) &= y_0(x), \quad x \in (0,1) \\ y(t,0) &= y(t,l) = 0, \quad t \in (0,T), \end{aligned}$$

for a sufficiently large interval of time $(0,T)$. The null controllability of linear parabolic equations without the memory kernels has been extensively studied by several authors (see Barbu [4], Fernandez-Cara *et al.* [9, 10], Fursikov and Imanuvilov [11] and references therein). Ainseba and Langlais [2] obtained the approximte controllability result for a population dynamics model with age dependence spatial structure. Fabre *et al.* [7] studied the approximate controllability of the semilinear heat equation and Fernandez and Zuazua [8] investigated the same problem for semilinear heat equations with gradient terms. Teresa [16] obtained the approximate controllability of the semilinear heat equation in unbounded domains and obtained some Carleman type estimates for a cascade system in [17]. This is done using a standard technique proved by Fursikov and Imanuvilov [11].

Yong and Zhang [20] considered the heat equation with hyperbolic memory kernel

$$\begin{aligned} y_t(t,x) - \nabla \cdot \int_0^t a(t-s,x) \nabla y(s,x) ds = u\chi_\omega, &\quad \text{in} \quad Q \\ y(0,x) = y_0(x), &\quad \text{in} \quad \Omega \\ y = 0, &\quad \text{on} \quad \Sigma. \end{aligned}$$

Due to the appearance of such kinds of memory, the speed of propagation for heat pulses is finite, and the corresponding controllability property has a certain nature similar to the wave equation. Recently, Sakthivel *et al.* [13, 14] studied the controllability results for linear and non-linear parabolic integrodifferential systems with the memory kernels and established a Carleman type estimate for the linear parabolic integrodifferential systems.

In this paper, we study the null and approximate controllability of the parabolic integrodifferential equation with general kernel (1.1). However, the controllability problem for (1.1) with convolution kernel is not yet fully discussed. The paper is organized as follows: In Section 2, we first give preliminary results and state our two main results on controllability. In Section 3, we obtain a Carleman-type estimate for the adjoint problem and we deduce an observability inequality. In Section 4, we prove Theorem 2.1 and Theorem 2.2.

2. Assumptions and the main results

Let $\Omega \subset \mathbb{R}^n$ be a connected bounded domain with boundary $\partial\Omega$ of class C^2. Let $\omega \subset \Omega$ be a non-empty open subset and let $T > 0$. We will use the notation $Q = (0,T)\times\Omega, Q_\omega = (0,T)\times\omega$ and $\Sigma = (0,T)\times\partial\Omega$ and we denote by $\nu(x)$ the outward unit normal to Ω at the point $x \in \partial\Omega$.

For convenience, we use the following standard notation for the Sobolev spaces $H^m(\Omega), H_0^m(\Omega)$ and the L^p spaces on Ω and Q, $1 \le p \le \infty$, with the norm denoted by $\|\cdot\|$ and we use $\langle\cdot\rangle$ for the inner product of $L^2(\Omega)$ and $|\cdot|$ for the usual norm in $\mathbb{R}^n$.

$$W_m^p(\Omega) = \Big\{ w(x) \;:\; \|w\|_{W_m^p} = \Big(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha w|^p dx \Big)^{\frac{1}{p}} < \infty \Big\},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\ \partial x_n^{\alpha_n}}$. When $p = 2$, instead of W_m^2 we shall write $H^m(\Omega)$. Further, we use

$$H^1([0,T]; L^2(\Omega)) = \{w \in L^2(0,T; L^2(\Omega)) : \frac{dw}{dt} \in L^2(0,T; L^2(\Omega))\},$$

where $\frac{dw}{dt}$ is taken in the sense of distributions. Besides, to formulate our results, we need the space $L^2(0,T; H^1(\Omega))$ of all equivalence classes of square integrable functions from $(0,T)$ to $H^1(\Omega)$. The space $L^2(0,T; L^2(\Omega))$ is analogously defined. For results on Sobolev spaces, one can refer to Adams [1] and Temam [15].

Consider the linear parabolic integrodifferential equation

$$\begin{cases} y_t - \Delta y - \displaystyle\int_0^t k(t,\tau)\Delta y(\tau,x)d\tau \\ \qquad\qquad - \displaystyle\int_0^t l(t,\tau)y(\tau,x)d\tau = \chi_\omega u(t,x), \quad \text{in} \quad Q \\ y(0,x) \ = \ y_0(x), \quad \text{in} \quad \Omega \\ y(t,x) \ = \ 0, \quad \text{on} \quad \Sigma. \end{cases} \tag{2.1}$$

In (2.1), $y = y(t,x)$ is the state, $u = u(t,x)$ is the control function, and χ_ω denotes the characteristic function of the subset $\omega \subset \Omega$; more precisely,

$$\chi_\omega = \begin{cases} 1 & \text{for} \quad x \in \omega \\ 0 & \text{for} \quad x \in \Omega\backslash\omega. \end{cases}$$

Therefore, the control u acts on the system through the subset ω and using this control we are able to achieve the controllability of the system (2.1).

The main goal of this paper is to analyze the controllability properties of (2.1).

System (2.1) is said to be *approximately controllable* in $L^2(\Omega)$ at time T if, for every $\epsilon > 0$, initial data $y_0 \in L^2(\Omega)$, final data $y_T(x) \in L^2(\Omega)$, there exists a control $u \in L^2(Q_\omega)$ such that the solution of (2.1) satisfies

$$\|y(T,x) - y_T(x)\|_{L^2(\Omega)} \leq \epsilon.$$

On the other hand, it will be said that system (2.1) is *null controllable* at time T if, for each $y_0 \in L^2(\Omega)$, there exists a control $u \in L^2(Q_\omega)$ such that the associated solution y satisfies

$$y(T,x) = 0 \text{ in } \Omega.$$

The following lemma is a fundamental tool in proving the Carleman estimate.

Lemma 2.1. *Suppose that Ω is a domain of class C^2. Let ω, ω_0 be arbitrary non-empty open subsets of Ω such that $\overline{\omega_0} \subset \omega \subset \Omega$. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that*

$$\psi(x) > 0 \ \ \forall\ x \in \Omega, \quad \psi(x)|_{\partial\Omega} = 0, \quad |\nabla\psi(x)| > 0\ \forall\ x \in \Omega\backslash\omega_0. \tag{2.2}$$

Proof. The proof is quite similar to that of Lemma 1.1 from Fursikov and Imanuvilov [11] and hence is omitted.

The first result of this paper concerns the approximate controllability; it can be stated as:

Theorem 2.1. *Let us fix $T > 0$. For each $y_T(x) \in L^2(\Omega)$ and for each $\epsilon > 0$, there exists a control $u \in L^2(Q_\omega)$ such that y, the corresponding* *solution to* (2.1), *satisfies*

$$||y(T,\cdot) - y_T(\cdot)||_{L^2(\Omega)} \le \epsilon.$$

We examine the controllability at a given time T and show that approximate controllability holds true for every fixed time T.

Our second result is stated as follows:

Theorem 2.2. *Let Ω be a bounded domain of class C^2. Let ω be a nonempty subset of Ω; then for each $y_0 \in L^2(\Omega)$ there exists $u \in L^2(Q)$ and associated solutions $y^u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$ which satisfy* (2.1) *and*

$$y^u(T,x) = 0 \quad a.e. \quad x \in \Omega.$$

Moreover, the control u can be chosen such that $\|u\|_{L^2(Q)} \le C(|y_0|_2^2)$.

3. Carleman Estimate

In this section, we derive the Carleman-type estimate for the adjoint linear parabolic integrodifferential system.

Let us define the functions ϕ, α (see [10, 11]) as follows:

$$\phi(x,t) = \frac{e^{\lambda\psi(x)}}{\beta(t)}, \quad \alpha(x,t) = \frac{e^{2\lambda\Psi} - e^{\lambda\psi(x)}}{\beta(t)}, \tag{3.1}$$

where $\beta(t) = t(T-t)$ and $\Psi = \|\psi(x)\|_{C(\overline{\Omega})}$. From (3.1), we have

$$\nabla\phi = \lambda\phi\nabla\psi, \quad \nabla\alpha = -\lambda\phi\nabla\psi. \tag{3.2}$$

Besides, throughout the proof of the estimate, we use C, the generic constant for all the space derivatives of ψ as well. The proof of the approximate controllability result is based on a unique continuation result for the adjoint system.

Let us denote by w the adjoint state variable. Then, w is a solution to

$$\begin{cases} w_t(t,x) + \Delta w(t,x) + \displaystyle\int_t^T k(\tau,t)\Delta w(\tau)d\tau \\ \qquad\qquad + \displaystyle\int_t^T l(\tau,t)w(\tau)d\tau = g(t,x), \quad \text{in} \quad Q \\ w(T,x) \;=\; w_T(x), \quad \text{in} \quad \Omega \\ w(t,x) \;=\; 0, \quad \text{in} \quad \Sigma, \end{cases} \tag{3.3}$$

where $g \in L^2(Q)$ and $w_T \in L^2(\Omega)$. However, our equation is involved with the general kernel so that a Carleman estimate for the equation (3.3) is not found in the existing papers. In order to treat the integral term, we have to assume the extra information $e^{2s\alpha}\phi \leq e^{2s\alpha\delta}$, for $\delta \in (0,1)$ as in Barbu [3, 4]. We have the following Carleman estimate for the equation (3.3):

Theorem 3.1. *Let the functions ϕ and α be defined as in* (3.1) *and the kernel have support in (t_1, t_2), where $0 < t_1 < t_2 < T$. Then there exist constants $\hat{\lambda} > 1, \hat{s}(\Omega, T)$ and $C_1(\Omega, T)$ such that the following estimate holds for every w solution of* (3.3) *and for every $\lambda > \hat{\lambda}$, $s > \hat{s}$:*

$$\int_Q e^{-2s\alpha}\Big((s\phi)^{-1}(|w_t|^2 + |\Delta w|^2) + s\phi|\nabla w|^2 + s^3\phi^3|w|^2\Big)dxdt$$
$$\leq C\Big(\int_Q e^{-2s\alpha}|g|^2 dxdt + \int_{Q_\omega} e^{-2s\alpha}s^3\phi^3|w|^2 dxdt\Big). \tag{3.4}$$

Furthermore, $\hat{s}$ can be chosen of the form $\eta_1 + \eta_2\|k\|_\infty^2 + \eta_3\|l\|_\infty^2$ where η_1, η_2 and η_3 are positive constants that depend only on Ω and T.

Proof. For simplicity, we will divide the proof into three steps.

Step 1: Change of variables and plan of the sequel. In this step, we set the parabolic integrodifferential equation satisfied by a new function v, which will be w up to a weight function. Thus, let us make the change of variables for the unknown function $w = e^{s\alpha}v$ in equation (3.3). Then, we easily obtain that

$$L_1 v + L_2 v = h_s, \tag{3.5}$$

where

$$L_1 v = \Delta v + s^2\lambda^2\phi^2|\nabla\psi|^2 v + s\alpha_t v, \tag{3.6}$$
$$L_2 v = v_t - 2s\lambda\phi\nabla\psi\cdot\nabla v, \tag{3.7}$$

and

$$h_s = ge^{-s\alpha} + s\lambda^2\phi|\nabla\psi|^2 v + s\lambda\phi\Delta\psi v \tag{3.8}$$
$$- e^{-s\alpha(t)}\Big(\int_t^T k(\tau,t)\Delta(e^{s\alpha(\tau)}v(\tau))d\tau\Big) - e^{-s\alpha(t)}\Big(\int_t^T l(\tau,t)e^{s\alpha(\tau)}v(\tau)d\tau\Big).$$

With the previous notation, we have from (3.5)

$$\|L_1 v\|^2_{L^2(Q)} + \|L_2 v\|^2_{L^2(Q)} + 2\langle L_1 v, L_2 v\rangle_{L^2(Q)} = \|h_s\|^2_{L^2(Q)}. \tag{3.9}$$

In the next steps, we will see that the definition we have made of α makes $2\langle L_1 v, L_2 v\rangle_{L^2(Q)}$ positive up to several terms that can be controlled. More precisely, in the second step we will make the computations of the double products $2\langle L_1 v, L_2 v\rangle_{L^2(Q)}$. In the third step, we will add two terms (involving v_t and Δv) to the left of (3.21). This will help us to eliminate the local term containing ∇v that appears in the right-hand side and will provide a Carleman estimate for the function v. Finally, we will go back to the original function w and eliminate the integral term containing Δv and v that appears in the right-hand side of (3.34) and will deduce the estimate (3.4).

Step 2: First estimates. In this step, we will develop the six terms appearing in $2\langle L_1 v, L_2 v\rangle_{L^2(Q)}$. For this, we will integrate by parts several times with respect to the space and time variables, so derivatives of the weight functions will be involved. We will use the estimates

$$|\phi_t| \le C\phi^2, \quad |\alpha_t| \le C\phi^2, \quad |\alpha_{tt}| \le C\phi^2, \tag{3.10}$$

where C does not depend on s, λ, t, x. The inner product in (3.9) yields

$$\begin{aligned}\langle L_1 v, L_2 v\rangle_{L^2(Q)} &= \int_Q \Delta v v_t dxdt - 2\int_Q s\lambda\phi\Delta v \nabla\psi\cdot\nabla v dxdt \\ &+ \int_Q s^2\lambda^2\phi^2|\nabla\psi|^2 v v_t dxdt - 2\int_Q s^3\lambda^3\phi^3|\nabla\psi|^2\nabla\psi\cdot\nabla v v dxdt \\ &+ \int_Q s\alpha_t v v_t dxdt - 2\int_Q s^2\lambda\phi\alpha_t\nabla\psi\cdot\nabla v v dxdt.\end{aligned} \tag{3.11}$$

Let us compute $\int_Q \Delta v v_t dxdt$. Integrating by parts over Q, one gets

$$\begin{aligned}\int_Q \Delta v v_t dxdt &= -\int_Q \nabla v\cdot\nabla v_t dxdt + \int_\Sigma \frac{\partial v}{\partial\nu} v_t d\Sigma \\ &= -\frac{1}{2}\int_Q \frac{d}{dt}|\nabla v|^2 dxdt + \int_\Sigma \frac{\partial v}{\partial\nu} v_t d\Sigma = 0,\end{aligned} \tag{3.12}$$

where ν is the outward normal to Ω at the point $x \in \partial\Omega$. Since $v = \psi = 0$ on $\partial\Omega$ and $\psi \ge 0$ in $\overline{\Omega}$, one has

$$\begin{aligned}\langle L_1 v, L_2 v\rangle_{L^2(Q)} &= -\int_\Sigma s\lambda\phi|\frac{\partial v}{\partial\nu}|^2\frac{\partial\psi}{\partial\nu}\, ds - \int_Q s\lambda^2\phi|\nabla\psi|^2|\nabla v|^2 dxdt \\ &- \int_Q s\lambda\phi\Delta\psi|\nabla v|^2 dxdt + 2\int_Q s\lambda^2\phi|\nabla\psi\cdot\nabla v|^2 dxdt \\ &+ 2\int_Q s\lambda\phi\Big(\sum_{i,j=1}^n \frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\Big)dxdt - 2\int_Q s^2\lambda^2\phi\phi_t|\nabla\psi|^2 v^2 dxdt \\ &+ 3\int_Q s^3\lambda^4\phi^3|\nabla\psi|^4 v^2 dxdt + \int_Q s^3\lambda^3\phi^3(\nabla\cdot(|\nabla\psi|^2\nabla\psi))v^2 dxdt\end{aligned} \tag{3.13}$$

$$-\frac{1}{2}\int_Q s\alpha_{tt}v^2dxdt-\int_Q s^2\lambda^2\phi\alpha_t|\nabla\psi|^2v^2dxdt-\int_Q s^2\lambda\phi\alpha_t\Delta\psi v^2dxdt.$$

Hence, for s and λ sufficiently large, one can find a constant (still denoted by) C such that

$$\langle L_1v,L_2v\rangle_{L^2(Q)}=Y_1+X_1+3\int_Q s^3\lambda^4\phi^3|\nabla\psi|^4v^2dxdt \tag{3.14}$$
$$+2\int_Q s\lambda^2\phi|\nabla\psi\cdot\nabla v|^2dxdt-\int_Q s\lambda^2\phi|\nabla\psi|^2|\nabla v|^2dxdt,$$

where

$$Y_1=-\int_\Sigma s\lambda\phi\Big|\frac{\partial v}{\partial\nu}\Big|^2\frac{\partial\psi}{\partial\nu}d\Sigma\geq 0 \quad \text{(by construction of } \psi) \tag{3.15}$$

$$|X_1|\leq C\Big\{\int_Q s\lambda\phi|\nabla v|^2dxdt+\int_Q s^2\lambda^3\phi^3v^2dxdt\Big\}. \tag{3.16}$$

Now, multiply (3.5) by $s\lambda^2\phi v|\nabla\psi|^2$ to get

$$\int_Q s\lambda^2h_s\phi v|\nabla\psi|^2dxdt=\int_Q s\lambda^2\phi v|\nabla\psi|^2L_2vdxdt+\int_Q s\lambda^2\phi v\Delta v|\nabla\psi|^2dxdt$$
$$+\int_Q s^3\lambda^4\phi^3|\nabla\psi|^4v^2dxdt+\int_Q s^2\lambda^2\phi\alpha_t|\nabla\psi|^2v^2dxdt. \tag{3.17}$$

After integration by parts and some manipulation, we get

$$\int_Q s^3\lambda^4\phi^3|\nabla\psi|^4v^2dxdt=\int_Q s\lambda^2\phi|\nabla v|^2|\nabla\psi|^2dxdt+X_2, \tag{3.18}$$

where

$$|X_2|\leq\frac{1}{16}\|L_2v\|^2+C\int_Q s^2\lambda^4\phi^3v^2dxdt+\frac{1}{16}\|h_s\|^2. \tag{3.19}$$

Replacing $2\int_Q s^3\lambda^4\phi^3|\nabla\psi|^4v^2dxdt$ in (3.14) with the expression (3.18) we get:

$$\langle L_1v,L_2v\rangle_{L^2(Q)}=X_1+2X_2+\int_Q s^3\lambda^4\phi^3|\nabla\psi|^4v^2dxdt$$
$$+2\int_Q s\lambda^2\phi|\nabla\psi\cdot\nabla v|^2dxdt+\int_Q s\lambda^2\phi|\nabla\psi|^2|\nabla v|^2dxdt. \tag{3.20}$$

Substituting into (3.9) and recalling that $|\nabla\psi(x)| > 0$ for all $x \in \Omega\backslash\omega_0$, it is clear that there exists $\hat{\lambda}$ and $\hat{s}$ large enough, such that, for every $\lambda > \hat{\lambda}, s > \hat{s}$,

$$\frac{1}{4}\|L_2 v\|^2_{L^2(Q)} + \frac{1}{3}\|L_1 v\|^2_{L^2(Q)} + \big(\frac{2}{3} - \hat{\epsilon}\big)\int_0^T\int_{\omega_0} s^3\lambda^4\phi^3|v|^2 dxdt$$
$$+\big(\frac{2}{3} - \hat{\epsilon}\big)\int_0^T\int_{\omega_0} s\lambda^2\phi|\nabla v|^2 dxdt \le \|h_s\|^2. \tag{3.21}$$

The constant $\hat{\epsilon}$ can be chosen arbitrarily small but the selection of $\hat{\lambda}$ and $\hat{s}$ depends on $\hat{\epsilon}$.

Step 3: Indirect estimates and conclusion. The final step will be to add integrals of $|\Delta v|^2$ and $|v_t|^2$ to the left-hand side of (3.21). This can be done using the expressions for $L_1 v$ and $L_2 v$. Multiplying (3.6) and (3.7) by $(s\phi)^{-\frac{1}{2}}$ and squaring and doing estimates with the help of (3.10), we have

$$\int_Q (s\phi)^{-1}|\Delta v|^2 dxdt \le C\Big(\int_Q (s\phi)^{-1}|L_1 v|^2 + s^3\lambda^4\phi^3|v|^2\Big)dxdt, \tag{3.22}$$
$$\int_Q (s\phi)^{-1}|v_t|^2 dxdt \le C\Big(\int_Q (s\phi)^{-1}|L_2 v|^2 + s\lambda^2\phi|\nabla v|^2\Big)dxdt. \tag{3.23}$$

From (3.21),(3.22) and (3.23), we get

$$\begin{aligned}
&\frac{1}{8}\int_Q (s\phi)^{-1}|\Delta v|^2 dxdt + \frac{1}{6}\int_Q (s\phi)^{-1}|v_t|^2 dxdt \\
&+ \big(\frac{5}{6} - \hat{\epsilon}\big)\int_Q s^3\lambda^4\phi^3|v|^2 dxdt + \big(\frac{1}{6} - \hat{\epsilon}\big)\int_Q s\lambda^2\phi|\nabla v|^2 dxdt \\
&\le C\Big(\int_Q e^{-2s\alpha}|g|^2 dxdt + \int_{Q_{\omega_0}} s^3\lambda^4\phi^3|v|^2 dxdt + \int_{Q_{\omega_0}} s\lambda^2\phi|\nabla v|^2 dxdt \\
&+ \int_Q e^{-2s\alpha}\Big|\int_t^T k(\tau,t)\Delta(e^{s\alpha(\tau)}v(\tau))d\tau\Big|^2 dxdt \\
&+ \int_Q e^{-2s\alpha}\Big|\int_t^T l(\tau,t)e^{s\alpha(\tau)}v(\tau)d\tau\Big|^2 dxdt\Big).
\end{aligned} \tag{3.24}$$

In the estimate (3.24), one wants to eliminate the gradients of v on ω_0. Now, we shall express the term $|\nabla v|^2$ on the right-hand side of (3.24) in terms of $|v|^2$ in the larger domain ω (since $\bar{\omega_0} \subset \omega \subset \Omega$). Let us define the truncating function $\rho \in C_0^\infty(\Omega)$ with $\rho(x) = 1$ in $\overline{\omega_0}$ and $\rho(x) = 0$ in $\Omega\backslash\omega$. We multiply (3.5) by $\rho v s\lambda\phi$ in $L^2(Q)$and integrate it on Q and use the Cauchy inequality (with $\epsilon = 1$) to have

$$\begin{aligned}
&\int_Q \rho s\lambda^2\phi|\nabla v|^2 dxdt \leq \frac{1}{4}\int_Q e^{-2s\alpha}|g|^2 dxdt\\
&+\int_{Q_\omega}\Big(s^2\lambda^2(\phi^2+\phi^3)+s^2\lambda^3\phi+s^3\lambda^3\phi^3\Big)|v|^2 dxdt\\
&-\int_{Q_\omega} s\lambda\nabla(\rho\phi)v\nabla v dxdt-\frac{1}{2}\int_{Q_\omega} s\lambda\phi_t|v|^2 dxdt\\
&+\int_{Q_\omega}\Big(s^2\lambda^2(\phi^2+\phi^3)+s^2\lambda^3\phi+s^3\lambda^3\phi^3\Big)|v|^2 dxdt\\
&+\frac{1}{4}\int_Q e^{-2s\alpha}\Big|\int_t^T k(\tau,t)\Delta(e^{s\alpha(\tau)}v(\tau))d\tau\Big|^2 dxdt\\
&+\frac{1}{4}\int_Q e^{-2s\alpha}\Big|\int_t^T l(\tau,t)e^{s\alpha(\tau)}v(\tau)d\tau\Big|^2 dxdt;
\end{aligned}$$

again using integration by parts and some manipulations, we obtain

$$\begin{aligned}
&\int_{Q_{\omega_0}} s\lambda^2\phi|\nabla v|^2 dxdt \leq C\Big(\int_Q e^{-2s\alpha}|g|^2 dxdt+\int_{Q_\omega} s^3\lambda^4\phi^3|v|^2 dxdt\\
&+\int_Q e^{-2s\alpha}\Big|\int_t^T k(\tau,t)\Delta(e^{s\alpha(\tau)}v(\tau))d\tau\Big|^2 dxdt\\
&+\int_Q e^{-2s\alpha}\Big|\int_t^T l(\tau,t)e^{s\alpha(\tau)}v(\tau)d\tau\Big|^2 dxdt\Big).
\end{aligned} \tag{3.25}$$

Using (3.25) in (3.24) and finally going back to our original function, which was given by $w = e^{s\alpha}v$, we have

$$\begin{aligned}
&\int_Q s^{-1}\phi^{-1}(|\Delta v|^2+|v_t|^2)dxdt+\int_Q e^{-2s\alpha}s^3\lambda^4\phi^3|w|^2 dxdt+\int_Q s\lambda^2\phi|\nabla v|^2 dxdt\\
&\leq C\Big(\int_Q e^{-2s\alpha}|g|^2 dxdt+\int_{Q_\omega} e^{-2s\alpha}s^3\lambda^4\phi^3|w|^2 dxdt \qquad (3.26)\\
&+\int_Q e^{-2s\alpha}\Big|\int_t^T k(\tau,t)\Delta w(\tau)d\tau\Big|^2 dxdt+\int_Q e^{-2s\alpha}\Big|\int_t^T l(\tau,t)w(\tau)d\tau\Big|^2 dxdt\Big).
\end{aligned}$$

For ∇w, we use the identity

$$\nabla w = e^{s\alpha}(\nabla v - s\lambda\nabla\psi\phi v), \tag{3.27}$$

to obtain

$$\int_Q e^{-2s\alpha} s\lambda^2\phi|\nabla w|^2 dxdt$$
$$\leq C\int_Q s\lambda^2\phi|\nabla v|^2 dxdt + C\int_Q e^{-2s\alpha}s^3\lambda^4\phi^3|w|^2 dxdt. \tag{3.28}$$

Consequently, we can add the previous integral of $|\nabla w|^2$ to the left-hand side of (3.26). We obtain

$$\begin{aligned}\int_Q s^{-1}\phi^{-1}(|\Delta v|^2 + |v_t|^2)dxdt + \int_Q s^3\lambda^4 e^{-2s\alpha}\phi^3|w|^2 dxdt + \int_Q s\lambda^2\phi|\nabla w|^2 dxdt\\ \leq \quad C\Big(\int_Q e^{-2s\alpha}|g|^2 dxdt + \int_{Q_\omega} e^{-2s\alpha}s^3\lambda^4\phi^3|w|^2 dxdt\\ + \int_Q e^{-2s\alpha}\Big|\int_t^T k(\tau,t)\Delta w(\tau)d\tau\Big|^2 dxdt\\ + \int_Q e^{-2s\alpha}\Big|\int_t^T l(\tau,t)w(\tau)d\tau\Big|^2 dxdt\Big).\end{aligned} \tag{3.29}$$

For Δw, we use the identity

$$\Delta v = e^{-s\alpha}(\Delta w + s\lambda\Delta\psi\phi w + s\lambda^2|\nabla\psi|^2\phi w + 2s\lambda\phi\nabla\psi\cdot\nabla w + s^2\lambda^2|\nabla\psi|^2\phi^2 w), \tag{3.30}$$

to obtain

$$\begin{aligned}\int_Q e^{-2s\alpha}s^{-1}\phi^{-1}|\Delta w|^2 dxdt \leq C\Big(\int_Q s^{-1}\phi^{-1}|\Delta v|^2 dxdt\\ + \int_Q e^{-2s\alpha}s\lambda^2\phi|w|^2 dxdt + \int_Q e^{-2s\alpha}s\lambda^4\phi|w|^2 dxdt\\ + \int_Q e^{-2s\alpha}s\lambda^2\phi|\nabla w|^2 dxdt + \int_Q e^{-2s\alpha}s^3\lambda^4\phi^3|w|^2 dxdt\Big).\end{aligned} \tag{3.31}$$

For w_t, we use the identity

$$w_t = e^{s\alpha}(v_t + s\alpha_t v), \tag{3.32}$$

to obtain

$$\begin{aligned}\int_Q e^{-2s\alpha}s^{-1}\phi^{-1}|w_t|^2 dxdt\\ \leq C\Big(\int_Q s^{-1}\phi^{-1}|v_t|^2 dxdt + \int_Q e^{-2s\alpha}s\phi^3|w|^2 dxdt\Big).\end{aligned} \tag{3.33}$$

Thus, taking s and λ sufficiently large, we are able to introduce all the terms involving $|\Delta w|^2$ and $|w_t|^2$ in the left-hand side of (3.29) to get

$$\begin{aligned}
&\int_Q s^{-1}\phi^{-1}(|\Delta w|^2+|w_t|^2)dxdt+\int_Q e^{-2s\alpha}s^3\lambda^4\phi^3|w|^2dxdt+\int_Q s\lambda^2\phi|\nabla w|^2dxdt\\
&\le C\Big(\int_Q e^{-2s\alpha}|g|^2dxdt+\int_{Q_\omega}e^{-2s\alpha}s^3\lambda^4\phi^3|w|^2dxdt \qquad (3.34)\\
&+\int_Q e^{-2s\alpha}\Big|\int_t^T k(\tau,t)\Delta w(\tau)d\tau\Big|^2dxdt+\int_Q e^{-2s\alpha}\Big|\int_t^T l(\tau,t)w(\tau)d\tau\Big|^2dxdt\Big).
\end{aligned}$$

In the inequality (3.34), one wants to eliminate the memory kernel that appears on the right-hand side. Making use of the assumptions on the kernels, and using Hölder's inequality and changing the order of integration, we have

$$\begin{aligned}
&\int_Q e^{-2s\alpha}\Big|\int_t^T k(\tau,t)\Delta w(\tau)d\tau\Big|^2dxdt\\
&\le\int_Q e^{-2s\alpha}\Big(\int_t^T|k(\tau,t)|^2s\phi(\tau)e^{2s\alpha(\tau)}d\tau\Big)\\
&\qquad\times\Big(\int_t^T s^{-1}\phi^{-1}e^{-2s\alpha(\tau)}|\Delta w(\tau)|^2d\tau\Big)dxdt\\
&\le\int_Q e^{-2s\alpha}\Big(\int_{t_1}^{t_2}|k(t,\tau)|^2e^{2s\alpha(\tau)}s\phi(\tau)d\tau\Big)\\
&\qquad\times\Big(\int_{t_1}^{t_2}e^{-2s\alpha(\tau)}s^{-1}\phi^{-1}(\tau)|\Delta w(\tau)|^2d\tau\Big)dxdt\\
&\le\int_Q e^{-2s\alpha}\Big(\int_{t_1}^{t_2}|k(t,\tau)|^2e^{2s\alpha(\tau)\delta}sd\tau\Big)\\
&\qquad\times\Big(\int_{t_1}^{t_2}e^{-2s\alpha(\tau)}s^{-1}\phi^{-1}(\tau)|\Delta w(\tau)|^2d\tau\Big)dxdt\\
&\le C\|k\|_{L^\infty}^2\int_Q e^{-2s\alpha}\Big(\int_{t_1}^{t_2}e^{2s\alpha(\tau)\delta}sd\tau\Big)\\
&\qquad\times\Big(\int_{t_1}^{t_2}e^{-2s\alpha(\tau)}s^{-1}\phi^{-1}(\tau)|\Delta w(\tau)|^2d\tau\Big)dxdt.
\end{aligned}$$

From the above, we obtain the estimate

$$\int_Q e^{-2s\alpha}\Big|\int_t^T k(\tau,t)\Delta w(\tau)d\tau\Big|^2dxdt$$

$$\leq \bar{C}\|k\|^2_{L^\infty} \int_0^T \int_\Omega e^{-2s\alpha} s^{-1}\phi^{-1}|\Delta w|^2 \Big(\int_{t_1}^{t_2} e^{-2s\alpha(\tau)} d\tau \Big) dxdt$$
$$\leq \bar{C}\|k\|^2_{L^\infty} \int_Q e^{-2s\alpha} s^{-1}\phi^{-1}|\Delta w(\tau)|^2 dxdt, \tag{3.35}$$

where $\bar{C} = C\max\{se^{2s\alpha\delta}\}$ is independent of s, but $\bar{C}$ depends on Ω, ω, t_1, t_2 and T. Using the fact that $e^{2s\alpha}\phi^{-3} \leq C < \infty$ (see [6]) from below, we evaluate the second integral term

$$\int_Q e^{-2s\alpha} \Big| \int_t^T l(\tau,t)v(\tau)d\tau \Big|^2 dxdt$$
$$\leq \int_Q e^{-2s\alpha(t)} \Big(\int_{t_1}^{t_2} e^{2s\alpha(\tau)} |l(t,\tau)|^2 s^{-3}\phi^{-3}(\tau) d\tau \Big) \tag{3.36}$$
$$\times \Big(\int_{t_1}^{t_2} e^{-2s\alpha(\tau)} s^3\phi^3(\tau)|v(\tau)|^2 d\tau \Big) dxdt$$
$$\leq C\|l\|^2_{L^\infty} \int_0^T \int_\Omega e^{-2s\alpha} s^3\phi^3|v|^2 \Big(\int_{t_1}^{t_2} e^{-2s\alpha} d\tau \Big) dxdt$$
$$\leq C\|l\|^2_{L^\infty} \int_Q e^{-2s\alpha} s^3\phi^3|v|^2 dxdt,$$

where C depends on Ω, ω, t_1, t_2 and T. The last two integrals (3.35) and (3.36) can be absorbed in the left-hand side and this together with (3.34) implies that one can obtain the Carleman estimate defined in (3.4). Thus the proof is completed.

Remark 3.1 In particular, (3.4) implies the unique continuation property of solutions to equation (3.3); that is, if $w = 0$ in $(0,T) \times \omega$ then $w \equiv 0$.

We prove the observability inequality for the adjoint system (3.3). The proof of the null controllability result is based on the observability inequality.

Lemma 3.1. *Under the assumptions of Theorem* 3.1, *there exists a positive constant C and μ, independent of s and w, such that the following inequality holds:*

$$\int_\Omega |w(0,x)|^2 dx \leq Ce^{\mu s} \Big(\int_Q |g|^2 dxdt + \int_{Q_\omega} e^{-2s\alpha}\phi^3|w|^2 dxdt \Big). \tag{3.37}$$

Proof. We multiply (3.3) by w and integrate on Ω and use the Cauchy inequality to obtain

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |w|^2 dx + \int_\Omega |\nabla w|^2 dx \leq \frac{1}{2}\int_\Omega (|g|^2 + 3|w|^2) dx \tag{3.38}$$

$$+\frac{1}{2}\int_\Omega\Big|\int_t^T k(\tau,t)\Delta w(\tau)d\tau\Big|^2dx+\frac{1}{2}\int_\Omega\Big|\int_t^T l(\tau,t)w(\tau)d\tau\Big|^2dx.$$

Note that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w|^2dx-\int_\Omega|\nabla w|^2dx\geq -C\Big(\int_\Omega(|g|^2+|w|^2)dx$$
$$+\int_\Omega\Big|\int_t^T k(\tau,t)\Delta w(\tau)d\tau\Big|^2dx+\int_\Omega\Big|\int_t^T l(\tau,t)w(\tau)d\tau\Big|^2dx\Big).$$

We define

$$\gamma(t)=\sup\{e^{2s\alpha}\phi^{-3}(x,t):x\in\Omega\}\leq Ce^{\frac{\mu s}{\beta(t)}},\mu=2e^{2\lambda m||\eta^0||_\infty}.$$

Integrating (3.38) on $(0,t)$, we have

$$\int_\Omega|w(0,x)|^2dx\leq C\Big(\gamma(t)\int_\Omega e^{-2s\alpha}\phi^3|w|^2dx+\int_0^t\int_\Omega|g|^2dxdt$$
$$+\int_0^t\int_\Omega\Big|\int_t^T k(\tau,t)\Delta w(\tau)d\tau\Big|^2dxdt$$
$$+\int_0^t\int_\Omega\Big|\int_t^T l(\tau,t)w(\tau)d\tau\Big|^2dxdt\Big)\quad\text{for}\quad t\in(0,T).\tag{3.39}$$

By the assumption on the kernels and using Hölder's inequality, we have

$$\int_0^t\int_\Omega\Big|\int_t^T k(\tau,t)\Delta w(\tau)d\tau\Big|^2dxdt\leq\int_0^T\int_\Omega\Big(\int_{t_1}^{t_2}|k(\tau,t)|^2e^{2s\alpha(\tau)}s\phi(\tau)d\tau\Big)$$
$$\times\Big(\int_{t_1}^{t_2}e^{-2s\alpha(\tau)}s^{-1}\phi^{-1}(\tau)|\Delta w(\tau)|^2d\tau\Big)dxdt\tag{3.40}$$
$$\leq C\|k\|^2_{L^\infty}\int_0^T\int_\Omega\Big(\int_{t_1}^{t_2}se^{2s\alpha(\tau)\delta}d\tau\Big)\Big(\int_{t_1}^{t_2}e^{-2s\alpha(\tau)}s^{-1}\phi^{-1}(\tau)|\Delta w|^2d\tau\Big)dxdt$$
$$\leq\bar{C}\|k\|^2_{L^\infty}\int_0^T\int_\Omega e^{-2s\alpha}s^{-1}\phi^{-1}|\Delta w|^2\Big(\int_{t_1}^{t_2}d\tau\Big)dxdt$$
$$\leq\bar{C}\|k\|^2_{L^\infty}\int_Q e^{-2s\alpha}s^{-1}\phi^{-1}|\Delta w|^2dxdt,$$

where $\bar{C}=C\max\{e^{2s\alpha\delta}s\}$, is independent of s. Similarly we estimate the integral and taking into account $e^{2s\alpha}\phi^{-3}\leq C$, we obtain

$$\int_0^t\int_\Omega\Big|\int_t^T l(\tau,t)w(\tau)d\tau\Big|^2dxdt\leq\int_0^T\int_\Omega\Big(\int_{t_1}^{t_2}|l(\tau,t)|^2e^{2s\alpha(\tau)}\phi^{-3}(\tau)d\tau\Big)$$

$$\times \Big(\int_{t_1}^{t_2} e^{-2s\alpha(\tau)} \phi^3(\tau) |w(\tau)|^2 d\tau \Big) dxdt$$
$$\leq C \|l\|^2_{L^\infty} \int_{t_1}^{t_2} \int_\Omega e^{-2s\alpha} s^3 \phi^3 |w|^2 dxdt, \tag{3.41}$$

where C depends on Ω, ω, t_1, t_2 and T. Now, we fix t_0 and t_1, such that $0 < t_0 < t_1 < T$. Substituting (3.40) and (3.41) in (3.39) and integrating the latter on the interval $(t_0, t_1) \subset (0, T)$, we obtain

$$\int_{t_0}^{t_1} \int_\Omega |w(0,x)|^2 e^{-\frac{\mu s}{t(t-T)}} dxdt \leq C \Big(\int_Q |g|^2 dxdt + \int_{t_0}^{t_1} \int_\Omega e^{-2s\alpha} \phi^3 |w|^2 dxdt$$
$$+ \int_Q e^{-2s\alpha} \big(s^{-1} \phi^{-1} |\Delta w|^2 + s^3 \phi^3 |w|^2 \big) dxdt \Big). \tag{3.42}$$

As a result, combining the estimate (3.42) with Theorem 3.1 we arrive at the proof of Lemma 3.1.

4. Controllability results

In this section, we study the approximate and null controllability of the linear integrodifferential system

$$\begin{cases} y_t - \Delta y - \displaystyle\int_0^t k(t,\tau) \Delta y(\tau, x) d\tau \\ \qquad\qquad - \displaystyle\int_0^t l(t,\tau) y(\tau,x) d\tau = \chi_\omega u(t,x), \quad \text{in } \ Q \\ y(0,x) \ = \ y_0(x), \quad \text{in} \quad \Omega \\ y(t,x) \ = \ 0, \quad \text{on} \quad \Sigma. \end{cases} \tag{4.1}$$

Now, we are ready to give the proof of the approximate controllability which forms one of the main parts of our work.

Proof of Theorem 2.1. The proof of the approximate controllability result is based on the unique continuation result for the adjoint system (4.2). Observe that due to the linearity of system (4.1), it is enough to prove the approximate controllability of (4.1) when $y_0(x) = 0$.

In fact, suppose that the result is valid for $y_0 = 0$ given $y_0 \in L^2(\Omega)$ take $\hat{y}$ the solution to (4.1) with $u = 0$. Take the control u that drives (4.1) with $y(0) = 0$ close to $y_T(x) - \hat{y}(T, x)$. This control solves the problem for $y(0) = y_0$

To prove the approximate controllability, let us define a reachable set $\overline{B_d^0} = \{y(T,x)$, for u in $L^2(Q_\omega)$ and y solution of (4.1)$\}$. We want to show that $\overline{B_d^0} = L^2(\Omega)$, where $\overline{B_d^0}$ denotes the closure of B_d^0 in $L^2(\Omega)$. Let us prove this by contradiction. Assume that the above result is false. This means that there exists a function $w_T(x)$, not identically zero, belonging to the orthogonal subspace of B_d^0 in $L^2(\Omega)$, and consider the corresponding solution to the dual problem

$$\begin{cases} w_t(t,x) + \Delta w(t,x) + \displaystyle\int_t^T k(\tau,t)\Delta w(\tau)d\tau \\ \qquad\qquad + \displaystyle\int_t^T l(\tau,t)w(\tau)d\tau = 0, \quad (t,x) \in Q \\ w(T,x) \;=\; w_T(x), \quad x \in \Omega \\ w(t,x) \;=\; 0, \quad (t,x) \in \Sigma. \end{cases} \tag{4.2}$$

Multiply (4.2) by the solution y to the original problem (4.1). Integrating by parts, we get

$$-\int_{Q_\omega} u(t,x)w(t,x)dxdt + \int_\Omega w_T(x)y(T,x)dx = 0. \tag{4.3}$$

But $w_T(x)$ is orthogonal to B_d^0 and then

$$\int_\Omega w_T(x)y(T,x)dx = 0 \;\; \forall \;\; u \in L^2(\omega \times (0,T)).$$

That means that

$$\int_{Q_\omega} u(t,x)w(t,x)dxdt = 0,$$

and then

$$w(t,x) = 0 \;\; \text{on} \;\; (0,T) \times \omega. \tag{4.4}$$

Using the Carleman estimate (3.4), one can easily verify that $w \equiv 0$ and so $w_T(x) \equiv 0$. This is a contradiction to our assumption. Hence, $w_T(x) \neq 0$. Hence, Theorem 2.1 is proved.

Now let us give the detailed proof of the null controllability result:

Proof of Theorem 2.2. For $\epsilon > 0$, we consider the following optimal control problem:

$$\mathit{Minimize} \;\; \{H_\epsilon(u) : u \in L^2(Q)\},$$

where the functional H_ϵ is defined by

$$H_\epsilon(u) = \int_Q |u|^2 dxdt + \frac{1}{\epsilon}\int_\Omega |y_u(T,x)|^2 dx, \tag{4.5}$$

where y_u is a solution of (4.1) associated with the control u. Since H_ϵ is a continuous, strictly convex functional in $L^2(Q)$ and is coercive, H_ϵ has a unique solution u_ϵ for any $\epsilon > 0$. The limit of (u_ϵ, y_ϵ) as $\epsilon \to 0$ is a solution of the null controllability for the system (4.2) due to the penalization term $\frac{1}{\epsilon}\int_\Omega |y(T,x)|^2 dx$, provided the limit exists in an appropriate norm. By the Pontriagin minimum principle, the minimum is characterized by

$$u_\epsilon = \chi_\omega w_\epsilon \quad a.e \ \text{ in } \ Q,$$

where $w_\epsilon \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega))$ is the solution to the dual problem

$$\begin{cases} (w_\epsilon)_t + \Delta w_\epsilon + \displaystyle\int_t^T k(\tau,t)\Delta w_\epsilon(\tau)d\tau \\ \qquad\qquad + \displaystyle\int_t^T l(\tau,t)w_\epsilon(\tau)d\tau = 0 \quad \text{in} \quad Q \\ w_\epsilon(T) = -\frac{1}{\epsilon}y_\epsilon(T,x), \quad \text{in} \quad \Omega \\ w_\epsilon = 0, \quad \text{in} \quad \Sigma. \end{cases} \tag{4.6}$$

If we multiply (4.6) by y_ϵ, (4.1) (where $y = y_\epsilon$) by w_ϵ and integrate on Q, we get the relation

$$\int_{Q_\omega} |w_\epsilon|^2 dxdt + \frac{1}{\epsilon}\int_\Omega |y_\epsilon(T,x)|^2 dx - \int_\Omega y_0(x)w_\epsilon(x,0)dx = 0. \tag{4.7}$$

Applying the observability inequality (3.37), we have

$$\Big|\int_\Omega y_0(x)w_\epsilon(x,0)dx\Big| \le C\Big(\int_{Q_\omega} |w_\epsilon(t,x)|^2 dxdt\Big)^{\frac{1}{2}}|y_0|_2. \tag{4.8}$$

By virtue of (4.7) and (4.8), we obtain

$$\int_{Q_\omega} |w_\epsilon|^2 dxdt + \frac{1}{\epsilon}\int_\Omega |y_\epsilon(T,x)|^2 dx \le C(|y_0|_2^2), \quad \forall \epsilon > 0. \tag{4.9}$$

By the condition (4.5), one can have

$$\int_Q |u_\epsilon|^2 dxdt + \frac{1}{\epsilon}\int_\Omega |y_\epsilon(T,x)|^2 dx \le C(|y_0|_2^2). \tag{4.10}$$

Using the inequalities (4.9) and (4.10), we have

$$\int_\Omega |y_\epsilon(T,x)|^2 dx \le C\epsilon, \tag{4.11}$$

where C is a positive constant independent of ϵ. Since u_ϵ is bounded in $L^2(Q)$, there exists a subsequence denoted by ϵ, such that

$$\begin{aligned} u_\epsilon &\rightharpoonup u^* \text{ weakly in } L^2(Q), \\ y_\epsilon &\rightharpoonup y^* \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ as } \epsilon \to 0. \end{aligned}$$

Clearly, $y^* = y^{u^*}$. Letting $\epsilon \to 0$ in (4.11), we have $y^*(T,x) = 0$ for almost all $x \in \Omega$. The estimate for the control u^* follows by letting $\epsilon \to 0$ in (4.10). This completes the proof of Theorem 2.2.

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REFERENCES

[1] R.A. Adams and J.F. Fournier, "Sobolev Spaces," Second Edition, New York, Academic Press, 2003.

[2] B.E. Ainseba and M. Langlais, *On a population dynamics control problem with age dependence and spatial structure*, J. Math. Anal. Appl., 248 (2000), 455–474.

[3] V. Barbu, *Local internal controllability of the Navier-Stokes equations* , Advances in Differential Equations, 12 (2001), 1443-1462.

[4] V. Barbu, *Controllability of parabolic and Navier-Stokes equations*, Scientiae Mathematicae Japonicae, 56 (2002), 143–211.

[5] V. Barbu and M. Iannelli, *Controllability of the heat equation with memory*, Differential and Integral Equations, 13 (2000), 1393–1412.

[6] O. Bodart, M. Gonzalez-Burgos, and R. Perez-Garcia, *Insensitizing controls for a heat equation with a nonlinear term involving the state and the gradient*, Nonlinear Analysis, 57 (2004), 687–711.

[7] C. Fabre, J. P. Puel, and E. Zuazua, *Approximate controllability of the semilinear heat equation*, Proceedings of the Royal Society of Edinburgh, 125A (1995), 31–61.

[8] L. A. Fernandez and E. Zuazua, *Approximate controllability of the semilinear heat equation involving gradient terms*, Journal of Optimization Theory and Aapplications, 101 (1999), 307–328.

[9] E. Fernandez-Cara, M. Gonzalez-Burgos, S. Guerrero, and J.P. Puel, *Null controllability of the heat equation with boundary Fourier conditions: The linear case*, ESAIM:Control, Optimization and Calculus of Variations, 12 (2006), 442–465.

[10] E. Fernandez-Cara and S. Guerrero, *Global Carleman inequalities for parabolic systems and applications to controllability*, SIAM Journal on Control and Optimization, 45 (2006), 1395–1446.

[11] A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Series 34, Seoul National University, RIM, Seoul, 1996.

[12] D. Sforza, *Parabolic integrodifferential equations with singular kernels*, Journal of Integral Equations and Applications, 3 (1991), 601–623.

[13] K. Sakthivel, K. Balachandran, and S. S. Sritharan, *Controllability and observability theory of certain parabolic integro-differential equations*, Computers & Mathematics with Applications, 52 (2006), 1299–1316.

[14] K. Sakthivel, K. Balachandran, and B. R. Nagaraj, *On a class of nonlinear parabolic control systems with memory effects*, International Journal of Control, 81 (2008), 764–777.

[15] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1983.

[16] L. de Teresa, *Approximate controllability of the semilinear heat equation in unbounded domains*, Nonlinear Analysis, 37 (1999), 1059–1090.

[17] L. de Teresa, *Insensitizing controls for a semilinear heat equation*, Communication in Partial Differential Equations, 25 (2000), 39–72.

[18] F. Ungerand L.V. Wolfersdorf, *Identification of memory kernels for materials with memory*, Journal of Materials Processing Technology, 67 (1997), 173–176.

[19] H.M. Yin, *Weak and classical solutions of some nonlinear Volterra integrodifferential equations*, Communication in Partial Differential Equations, 17 (1992), 1369–1385.

[20] J. Yong and X. Zhang, *Exact controllability of the heat erquation with hyperbolic memory kernel*, Control theory of Partial Differential Equations, Champan and Hall/CRC, Boca Raton, (2005), 387–401.