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WELL POSEDNESS FOR THE STOCHASTIC CAHN-HILLIARD EQUATION DRIVEN BY LÉVY SPACE-TIME WHITE NOISE

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Abstract. In this paper we study a stochastic Cahn-Hilliard equation driven by Lévy space-time white noise with Neumann boundary conditions. We establish the global existence and uniqueness of a mild solution under some regularity and boundedness on the coefficients.

1. Introduction

We consider the following stochastic version of the Cahn-Hilliard partial differential equation with initial and homogeneous Neumann boundary conditions:

$$(P) \qquad \begin{cases} \frac{\partial u}{\partial t} + \Delta^2 u = \Delta f(u) + \sigma(u)\dot{F}, & \text{in } [0,T]\times D, \\ u(0,\cdot) = u_0, \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 & \text{on } [0,T]\times \partial D, \end{cases}$$

where Δ denotes the Laplace operator, $D := [0,\pi]^d$ ($d \leq 3$ is the space dimension), u_0 is the initial function, $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying some assumptions given below, and $\dot{F}$ is a $d+1$-parameter Lévy space-time white noise which actually consists of a Brownian sheet, denoted by $\dot{W}$, on $D\times[0,\infty)$ and a Poisson space-time white noise on $D\times[0,\infty)$(the detail will be given below in this section).

The deterministic Cahn-Hilliard equation (i.e. $\sigma = 0$) was proposed in [3, 4] to describe the evolution of the phase separation of a binary alloy when temperature has been quenched from a value T_0, above the critical

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temperature T_c, to a temperature less than T_c. The function u represents a scaled concentration, and $-\Delta u+f(u)$ represents the chemical potential. The function f is the derivative of the homogeneous free energy E. In its original form, E contains a logarithmic term, which makes the study of this equation delicate. In some circumstances, E can be approximated by a polynomial of even degree with a strictly positive dominant coefficient. The Neumann boundary conditions reflect the conservation of mass and insulation from outside. This case has been extensively studied (see for instance [8, 9, 6] and references therein).

The stochastic Cahn-Hilliard equation, such as in the case $\dot{F} = \dot{W}$ (Gaussian space-time white noise), has been studied by several authors (see e.g. Da Prato and Debussche [13] and Cardon-Weber [5]). In the case when $\dot{F}$ is a Lévy space-time white noise, the local well posedness of the solution to problem (P) has already been proved by Bo and Wang [2] under some mild assumptions on the coefficients, where they failed to solve the global solution and left it as a challenging problem.

Enlightened by the approach used by Cardon-Weber [5], Gyöngy [10] and Da Prato and Gatarek [14], and noticing the Burkhölder-Davis-Gundy inequality which was proved in Knoche [11], we prove the global well posedness of the solution to the problem (P) in a weak sense, and our local well posedness is better than Bo and Wang's [2].

For this aim, we first introduce Lévy space-time white noise and stochastic integration (see [2, 16, 1]).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $(E_i, \mathcal{E}_i, \mu_i)(i = 1, 2)$ be two σ-finite measurable spaces. We call $N : (E_1, \mathcal{E}_1, \mu_1) \times (E_2, \mathcal{E}_2, \mu_2) \times (\Omega, \mathcal{F}, P) \to \mathbb{N} \cup \{0\} \cup \{\infty\}$ a Poisson noise on $(E_1, \mathcal{E}_1, \mu_1)$ if, for any $\mathbf{A} \in \mathcal{E}_1, B \in \mathcal{E}_2$ and $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$,

$$P(\omega \in \Omega : N(\mathbf{A}, B, \omega) = n) = \frac{e^{-\mu_1(\mathbf{A})\mu_2(B)}(\mu_1(\mathbf{A})\mu_2(B))^n}{n!}.$$

In particular, when $(E_1, \mathcal{E}_1, \mu_1) = ([0, \infty) \times D, \mathcal{B}([0, \infty) \times D), dt \times dx)$, we define the compensated random martingale measure

$$M(B, A, t) := N([0, t] \times A, B) - t\mu_1(A)\mu_2(B),$$

by assuming that $\mu_1([0, t] \times A)\mu_2(B) < \infty$ for all $(t, A, B) \in [0, \infty) \times \mathcal{B}(D) \times \mathcal{E}_2$. Furthermore, let $p : [0, \infty) \times D \times E_2 \times \Omega \to \mathbb{R}$ be an $\{\mathcal{F}_t\}_{t\geq 0}$-predictable function satisfying

$$\mathbb{E}\Big(\int_0^t \int_A \int_B |p(s, x, y)|^2 \mu_2(dy)dxds\Big) < \infty, \tag{1.1}$$

for all $t > 0$ and $(A, B) \in \mathcal{B}(D) \times \mathcal{E}_2$. We define a stochastic integral process

$$\Big\{R_t := \int_0^{t+} \int_A \int_B p(s,x,y) M(dy,dx,ds) : t \geq 0\Big\},$$

which is a square integrable $\{\mathcal{F}_t\}_{t\geq 0}$ martingale. Let $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ denote respectively the quadratic variation and the conditional quadratic variation of a process (refer to page 97 of Protter [15] for definitions). Then

$$\langle R, R\rangle_t = \int_0^t \int_A \int_B |p(s,x,y)|^2 \mu_2(dy) dx ds,$$
$$[R,R]_t = \int_0^{t+} \int_A \int_B |p(s,x,y)|^2 N(ds \times dx, dy).$$

It is well known that a Lévy space-time white noise has the following structure:

$$\dot{F}(x,t) = \dot{W}(x,t) + \int_{U_0} h_1(t,x,y)\dot{M}(dy,x,t) + \int_{E_2/U_0} h_2(t,x,y)\dot{N}(dy,x,t),$$

for some $U_0 \in \mathcal{E}_2$ such that $\mu_2(E_2/U_0) < \infty$, where $h_1, h_2 : [0,\infty) \times D \times E_2 \to \mathbb{R}$ are measurable functions, and $\dot{W}(t,x)$ is a Gaussian space-time white noise on $D \times [0,\infty)$; i.e., $\mathbb{E}[\dot{W}(x,t)\dot{W}(x',t')] = \delta(t-t')\delta(x-x')$, $\dot{M}$ and $\dot{N}$ are the Radon-Nikodym derivatives defined by

$$\dot{M}(dy,x,t) = \frac{M(dy,dx,dt)}{dt \times dx}, \qquad \dot{N}(dy,x,t) = \frac{N(dy,dx,dt)}{dt \times dx},$$

for $(t,x,y) \in [0,\infty) \times D \times E_2$.

Now we give a definition of the solution in a weak sense (as in Walsh [17]). We say that u is a weak solution of problem (P) if, for all $\phi \in C^4(D)$ with

$$\frac{\partial\phi}{\partial n} = \frac{\partial\Delta\phi}{\partial n} = 0 \quad \text{on } \partial D,$$

u satisfies

$$\begin{aligned}\int_D u(t,x)\phi(x)dx &= \int_D u_0(x)\phi(x)dx - \int_0^t \int_D \Delta^2\phi(x) u(s,x) dx ds \\ &+ \int_0^t \int_D \Delta\phi(x) f(u(s,x)) dx ds + \int_0^t \int_D \phi(x)\sigma(u(s,x)) W(dx,ds) \\ &+ \int_0^{t+} \int_D \int_{U_0} \phi(x)\sigma(u(s-,x)) h_1(s,x,z) M(dz,dx,ds) \\ &+ \int_0^{t+} \int_D \int_{E_2/U_0} \phi(x)\sigma(u(s-,x)) h_2(s,x,z) N(ds \times dx, dz). \qquad (1.2)\end{aligned}$$

The paper is organized as follows. Section 2 contains some preliminaries on some useful estimates. In Section 3, we give the local well posedness of the solution to the problem (P) under some mild assumptions. In the last section, the main results and their proofs are presented.

2. PRELIMINARIES

In this section, we first give some estimates about the Green's function G generated by the operator $\frac{\partial}{\partial t}+\Delta^2$ on the domain $[0,\infty)\times D$. The estimates are from Eidelman and Ivasisen [7] on Green's functions on smooth domains and were extended to the domain D by Cardon-Weber [5].

Let A denote the operator $-\Delta$ on the domain $\mathcal{D}(A)=\{u\in H^2(D):\frac{\partial u}{\partial n}=0 \text{ on } \partial D\}$. The following family $(\varepsilon_k)_{k\in\mathbb{N}^d}$ is a basis of eigenfunctions of A in $L^2(D)$. If $d=1$,

$$\varepsilon_k(x)=\cos(kx)\sqrt{\frac{2}{\pi}} \text{ if } k\neq 0, \quad \text{and} \quad \varepsilon_0=\frac{1}{\sqrt{\pi}};$$

and for $d\in\{1,2,3\}$,

$$\varepsilon_k(x)=\Pi_{i=1}^d\varepsilon_{k_i}(x_i),$$

associated with the eigenvalues $\lambda_k=\sum_{i=1}^d k_i^2=|k|^2$. The semigroup $S(t)$ generated by $-A^2$ is denoted by $S(t)=e^{-tA^2}$; that is, for $z\in L^2(D)$,

$$S(t)z=\sum_{k\in\mathbb{N}^d}e^{-\lambda_k^2 t}\langle z,\varepsilon_k\rangle\varepsilon_k,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L^2(D)$; this is a convolution semigroup with the Green's function G defined by

$$G(t,x,y)=\sum_{\mathbb{N}^d}e^{-\lambda_k^2 t}\varepsilon_k(x)\varepsilon_k(y).$$

We lists some useful properties about the Green's function (see Cardon-Weber [5] for details) to the following work.

Lemma 2.1. *There exist $C>0$ and $c>0$ such that, for $t\in(0,T]$, $x,y\in D$, α a dimensional exponent satisfying $|\alpha|\leq 3$,*

$$|G(t,x,y)|\leq\frac{C}{t^{d/4}}\exp(-c\frac{|x-y|^{4/3}}{|t|^{1/3}}); \tag{2.1}$$

$$|\partial_x^\alpha G(t,x,y)|\leq\frac{C}{t^{(d+|\alpha|)/4}}\exp(-c\frac{|x-y|^{4/3}}{|t|^{1/3}}); \tag{2.2}$$

$$|\partial_t G(t,x,y)|\leq\frac{C}{t^{(d+4)/4}}\exp(-c\frac{|x-y|^{4/3}}{|t|^{1/3}}). \tag{2.3}$$

The following result (Gyöngy [10]) gives precise estimates of the regularizing effect of convolution with $G, \Delta G$ and G^2.

Lemma 2.2. *Let J be defined, for all $v \in L^1([0,T], L^\rho(D))$, $0 \le t_0 \le t \le T$, and $x \in D$, by*

$$J(v)(t_0,t,x) = \int_{t_0}^t \int_D H(t-s,x,y)v(s,y)dyds.$$

Then, for any $\rho \in [1,\infty], q \in [\rho,\infty]$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{\rho} + 1 \in [0,1]$, J is a bounded operator from $L^1([0,T], L^\rho(D))$ to $L^\infty([0,T], L^q(D))$ such that:

1. *If $H(t,x,y) = G(s,x,y)$, there exists a constant $C > 0$ such that*

$$\|J(v)(t_0,t,\cdot)\|_q \le C \int_{t_0}^t (t-s)^{\frac{d}{4r}-\frac{d}{4}} \|v(s,\cdot)\|_\rho ds. \tag{2.4}$$

2. *If $H(t,x,y) = \Delta G(s,x,y)$(if $d = 3$, we also need $r < 3$; and if $d = 2$, $r \neq \infty$), there exists a constant $C > 0$ such that*

$$\|J(v)(t_0,t,\cdot)\|_q \le C \int_{t_0}^t (t-s)^{\frac{d}{4r}-\frac{d+2}{4}} \|v(s,\cdot)\|_\rho ds. \tag{2.5}$$

3. *If $H(t,x,y) = G^2(s,x,y)$ (if $d = 3$, we also need $r < \frac{3}{2}$; and if $d = 2$, $r \neq \infty$), there exists a constant $C > 0$ such that*

$$\|J(v)(t_0,t,\cdot)\|_q \le C \int_{t_0}^t (t-s)^{\frac{d}{4r}-\frac{d}{2}} \|v(s,\cdot)\|_\rho ds. \tag{2.6}$$

By the way, we give the Burkhölder-Davis-Gundy inequality from Knoche [11], which is very useful like the Burkhölder inequality.

Lemma 2.3. *Let $p : [0,\infty) \times D \times E_2 \times \Omega \to \mathbb{R}$ be $\{\mathcal{F}_t\}_{t\ge 0}$-predictable and satisfy* (1.1). *We denote by X the integral process*

$$\Big\{X_t := \int_0^{t+} \int_D \int_{E_2} p(s,y,z)M(dz,dy,ds) : t \ge 0\Big\}.$$

Then, for any $T > 0$ and $q \ge 2$, there exists a constant $C(q,T) > 0$ such that

$$\sup_{t\in[0,T]} (\mathbb{E}|X_t|^q) \le C(q,T)\Big(\int_0^T \int_D \int_{E_2} (\mathbb{E}[|p(s,y,z)|^q])^{\frac{2}{q}} \mu_2(dz)dyds\Big)^{\frac{q}{2}}. \tag{2.7}$$

Now we give an equivalent form of equation (1.2) represented by the Green function's G as follows, which will be used to prove the existence and uniqueness of the solution to the problem (P),

$$u(t,x) = \int_D G(t,x,y)u_0(y)dy + \int_0^t \int_D \Delta G(t-s,x,y)f(u(s,y))dyds \tag{2.8}$$

$$+\int_0^t\int_D\int_{E_2/U_0} G(t-s,x,y)\sigma(u(s,y))h_2(s,y,z)\mu_2(dz)dyds$$
$$+\int_0^t\int_D G(t-s,x,y)\sigma(u(s,y))W(dy,ds)$$
$$+\int_0^{t+}\int_D\int_{E_2} G(t-s,x,y)\sigma(u(s-,y))\big(h_1(s,y,z)I_{U_0}(z)$$
$$+h_2(s,y,z)I_{E_2/U_0}(z)\big)\mu_2(dz)dyds.$$

3. LOCAL SOLUTION

In this section, we apply the estimates of the Green's function to the existence and uniqueness of solutions for a similar equation with truncated coefficients. This yields the existence and uniqueness of the solution on the time interval $[0,\tau)$, where τ is a stopping time.

Let $n>0$ and denote by K_n, defined on $[0,\infty)$, a C^1 function such that $|K_n'|\le 2$ and

$$K_n(x)=\begin{cases} 1 & if\ x<n \\ 0 & if\ x\ge n+1. \end{cases}$$

Let

$$g(t,y)=\int_{E_2/U_0} h_2(t,y,z)\mu_2(dz)$$

and

$$h(t,y,z)=h_1(s,y,z)I_{U_0}(z)+h_2(s,y,z)I_{E_2/U_0}(z).$$

Then we prove the local well posedness for the equation obtained from the problem (P) by truncating the operators determined by f and σ,

$$u_n(t,x)=\int_D G(t,x,y)u_0(y)dy \tag{3.1}$$
$$+\int_0^t\int_D \Delta G(t-s,x,y)K_n(\|u_n(s,\cdot)\|_q)f(u_n(s,y))dyds$$
$$+\int_0^t\int_D G(t-s,x,y)K_n(\|u_n(s,\cdot)\|_q)\sigma(u_n(s,y))g(s,y)dyds$$
$$+\int_0^t\int_D G(t-s,x,y)K_n(\|u_n(s,\cdot)\|_q)\sigma(u_n(s,y))W(dy,ds)$$
$$+\int_0^{t+}\int_D\int_{E_2} G(t-s,x,y)K_n(\|u_n(s,\cdot)\sigma(u_n(s-,y))h(s,y,z)M(dz,dy,ds).$$

We quote the definition of the local solution to problem (P) coming from [2]. Define

$$\tau_n = \inf\{t \geq 0 : \|u_n(t,\cdot)\|_q \geq n\}.$$

Let $\tau = \lim_{n\to\infty} \tau_n$, and define $u(t,\cdot) := u_n(t,\cdot)$ on $\{t < \tau_n < \tau\}$. Then $u(t,\cdot)$ is a solution to (3.1) on $\{t < \tau\}$. This solution is called a local solution.

Before giving the local solution, we give some reasonable assumptions about the problem (P):

(a) f is a polynomial of degree 3 with positive dominant coefficients;

(b) σ is Lipschitzian and has linear growth; i.e., there exists a constant $C > 0$ such that

$$|\sigma(x)| \leq C(1 + |x|), \quad \text{for all } x \in \mathbb{R}.$$

For g, h and μ_2,

(c) $\sup_{t\in[0,T]} \|g(t,\cdot)\|_q < \infty$;

(d) $G(t-s,x,y)h(s,y,z)$ is $L^q([0,t] \times D^2 \times E_2)$, $ds \times dx \times dy \times \mu_2(E_2)$ integrable, for $0 \leq t \leq T$;

(e) $\mu_2(E_2) < \infty$.

Theorem 3.1. *Suppose the assumptions (a)-(e) are satisfied. Let* $d \in \{1,2,3\}$ *and* $q \in (4,\infty)$. *Then, for every* $\mathcal{F}_0$*-measurable* $u_0 : D \times \Omega \to \mathbb{R}$ *with* $\mathbb{E}(\|u_0(\cdot)\|_q^q) < \infty$, *there exists a unique local solution* $\{u(t,x) : (t,x) \in [0,T] \times D\}$ *to the problem* (P) *and there exists a stopping time* τ *such that* $\mathbb{E}(\|u(t\wedge\tau,\cdot)\|_q^q) < \infty$, *for any* $t \in [0,T]$.

We want to use a fixed-point argument to prove Theorem 3.1. So we first introduce our working space.

Let $\mathcal{H}$ be all $L^q(D)$-valued $\mathcal{F}_t$ adapted r.c.l.l. processes $u(t,x)$ such that the norm

$$\|u\|_{\mathcal{H}} = (\sup_{t\in[0,T]} (\mathbb{E}(\|u(t,\cdot)\|_q^q))^{1/q} \tag{3.2}$$

is finite. Then, under the norm (3.2), $\mathcal{H}$ is a Banach space.

For convenience, let us define operators A_i $(i = 0,1,2,3,4)$ such that

$$A_0(u_0)(t,x) = \int_D G(t,x,y)u_0(y)dy,$$

$$A_1(u)(t,x) = \int_0^t \int_D \Delta G(t-s,x,y)K_n(\|u(s,\cdot)\|_q)f(u(s,y))dyds,$$

$$A_2(u)(t,x) = \int_0^t \int_D G(t-s,x,y)K_n(\|u(s,\cdot)\|_q)\sigma(u(s,y))W(dy,ds),$$

$$A_3(u)(t,x) = \int_0^{t+} \int_D \int_{E_2} G(t-s,x,y)K_n(\|u(s,\cdot)\|_q)$$

$$\times\, \sigma(u(s^-,y))h(s,y,z)M(dz,dy,ds),$$

$$A_4(u)(t,x) = \int_0^t \int_D G(t-s,x,y)K_n(\|u(s,\cdot)\|_q)\sigma(u(s,y))g(s,y)dyds,$$

$$Y(u)(t,x) := A_0(u_0)(t,x) + \sum_{i=1}^4 A_i(u)(t,x).$$

In order to prove the local well posedness of the solution to the problem (P), i.e., Theorem 3.1, we first give the following propositions:

Proposition 1. *Under the conditions of Theorem* 3.1, *for any* $u \in \mathcal{H}$, $A_i(u) \in \mathcal{H}$.

Proof. We always assume that $u \in \mathcal{H}$ in the following proof. Using (2.1) and Young's inequality for $1/q = 1 + 1/q - 1$, we have

$$\begin{aligned}\|A_0(u_0)\|_q &= \|\int_D G(t,\cdot,y)u_0(y)dy\|_q \\ &\le Kt^{-\frac{d}{4}}\|\int_D \exp(-C\frac{|\cdot - y|^{\frac{4}{3}}}{t^{\frac{1}{3}}})u_0(y)dy\|_q \\ &\le Kt^{-\frac{d}{4}}\|\exp(-C\frac{|\cdot - y|^{\frac{4}{3}}}{t^{\frac{1}{3}}})\|_1\|u_0(\cdot)dy\|_q \le K\|u_0(\cdot)dy\|_q.\end{aligned} \tag{3.3}$$

Thus, $A_0(u_0) \in \mathcal{H}$, under the condition $\mathbb{E}(\|u_0\|_q^q) < \infty$.

As for $A_1(u)$, we apply (2.5) with $\rho = q/3$; we have $1/r = 1 - 2/q$, which satisfies $r_1 \neq \infty$ if $d = 3$ and $r_1 < 3$ if $d = 3$. Then

$$\begin{aligned}\|A_1(u)\|_q &\le \int_0^t |t-s|^{\frac{d}{4r} - \frac{d+2}{4}}\|f(u(s,\cdot))K_n(\|u(s,\cdot)\|_q)\|_\rho ds \\ &\le C(n+1)^3 T^{1+d/(4r_1)-(d+2)/4},\end{aligned}$$

so that $A_1(u) \in \mathcal{H}$.

As for $A_2(u)$, applying Burkhölder's inequality, assumption (b),and (2.6) with $1/r = 1/(q/2) - 1/\rho + 1$, we have

$$\|A_2(u)\|_{\mathcal{H}}^q = \sup_{t\in[0,T]} \mathbb{E}(\|\int_0^t\int_D G(t-s,x,y)\sigma(u(s,y))K_n(\|u(s,\cdot)\|_q)W(dy,ds)\|_q^q)$$

$$\le C\sup_{t\in[0,T]} \mathbb{E}\|\int_0^t\int_D G^2(t-s,x,y)\sigma^2(u(s,y))K_n^2(\|u(s,\cdot)\|_q)dyds\|_{\frac{q}{2}}^{\frac{q}{2}} \tag{3.4}$$

$$\le C\sup_{t\in[0,T]} \big(\int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{2}}\|\sigma^2(u(s,y))K_n^2(\|u(s,\cdot)\|_q)\|_\rho ds\big)^{\frac{q}{2}}$$

$$\leq C \sup_{t\in[0,T]} \Big(\int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{2}} \|(1+u^2(s,\cdot))K_n^2(\|u(s,\cdot)\|_q)\|_\rho ds\Big)^{\frac{q}{2}}$$

$$\leq C \sup_{t\in[0,T]} \Big(\int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{2}} (1+(n+1)^2) ds\Big)^{\frac{q}{2}} \leq C(T)(n+1)^q.$$

In the last inequality, one can set $\rho = q/2$, and $r = 1$. Thus we have $A_2(u) \in \mathcal{H}$.

As for $A_3(u)$, let $q > 4$. By assumptions $(b), (d)$ and Lemma 2.3, using the Hölder inequality, we have

$$\|A_3(u)\|_{\mathcal{H}}^q = \sup_{t\in[0,T]} \mathbb{E}(\| \int_0^{t+} \int_D \int_{E_2} G(t-s,x,y)\sigma(u(s^-,y)) \tag{3.5}$$

$$\times K_n(\|u(s,\cdot)\|_q) h(s,y,z) M(dz,dy,ds)\|_q^q)$$

$$\leq C \sup_{t\in[0,T]} \int_D \mathbb{E}(| \int_0^{t+} \int_D \int_{E_2} G(t-s,x,y)\sigma(u(s,y))K_n(\|u(s,\cdot)\|_q)h(s,y,z)$$

$$\times M(dz,dy,ds)|^q) dx$$

$$\leq C \sup_{t\in[0,T]} \int_D (| \int_0^{t} \int_D \int_{E_2} (\mathbb{E}|G(t-s,x,y)\sigma(u(s,y)K_n(\|u(s,\cdot)\|_q)$$

$$\times h(s,y,z)|^q)^{\frac{2}{q}}) \mu_2(dz)dyds|^{\frac{q}{2}} dx$$

$$\leq C \sup_{t\in[0,T]} \int_D | \int_0^{t} \int_D \int_{E_2} |G(t-s,x,y)h(s,y,z)|^2$$

$$\times K_n(\|u(s,\cdot)\|_q)(\mathbb{E}(1+|u(s,y)|)^q)^{2/q} \mu_2(dz)dyds|^{\frac{q}{2}} dx$$

$$\leq C(T)\|n+1\|^q \mathbb{C} \sup_{t\in[0,T]} \int_D | \int_0^{t} \int_D \int_{E_2} |G(t-s,x,y))h(s,y,z)|^q)$$

$$\times \mu_2(dz)dyds|dx.$$

Thus, we have $A_3(u) \in \mathcal{H}$.

Applying (2.4) with $\rho = q/2$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{\rho} + 1$, we have

$$\|A_4(u)\|_q \leq C \int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{4}} \|g(s,\cdot)\sigma(u(s,\cdot))K_n(\|u(s,\cdot)\|_q)\|_\rho ds \tag{3.6}$$

$$\leq C \int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{4}} (\|g(s,\cdot)\|_{q/2} + \|g(s,\cdot)u(s,\cdot)\|_{q/2}) K_n(\|u(s,\cdot)\|_q) ds$$

$$\leq C \int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{4}} (C(q)\|g(s,\cdot)\|_q + (n+1)\|g(s,\cdot)\|_q) ds$$

$$\leq C \sup_{t\in[0,T]} \|g(t,\cdot)\|_q \int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{4}} ds.$$

Since $q > 4$ and $\frac{d}{4r} - \frac{d}{4} > -1$, it follows that $\|A_4(u)\|_{\mathcal{H}} < \infty$.

This completes the proof of this proposition. ♯

Proposition 2. *Under the conditions of Theorem* 3.1, *the operator* Y *is a contraction on* $\mathcal{H}$*; i.e., there exists a constant* $\kappa \in (0,1)$ *such that*

$$\|Y(u) - Y(v)\|_{\mathcal{H}} \leq \kappa \|u - v\|_{\mathcal{H}}$$

for any $u, v \in \mathcal{H}$*.*

Proof. Let $u, v \in \mathcal{H}$; we have the following estimates. One can see ([2, 5]) that

$$\|K_n(\|u(s,\cdot)\|_q) f(u(s,\cdot)) - K_n(\|v(s,\cdot)\|) f(v(s,\cdot))\|_\rho \leq C_n \|u(s,\cdot) - v(s,\cdot)\|_q,$$
$$\|K_n(\|u(s,\cdot)\|_q) \sigma(u(s,\cdot)) - K_n(\|v(s,\cdot)\|) \sigma(v(s,\cdot))\|_\rho \leq C_n \|u(s,\cdot) - v(s,\cdot)\|_q.$$

Let $u \in \mathcal{H}$. If $\rho \in [1, q]$ and $r \neq \infty$ if $d = 3$ and $r < 3$ if $d = 3$ such that $1/q = 1/r + 1/\rho - 1$, then, for A_1, we have

$$\|A_1(u) - A_1(v)\|_{\mathcal{H}}^q \leq C_n \sup_{t\in[0,T]} \mathbb{E} \int_0^t |t-s|^{\frac{d}{4r}-\frac{d+2}{4}} \|u(s,\cdot) - v(s,\cdot)\|_q ds$$
$$\leq C_n \sup_{t\in[0,T]} [(\int_0^t |t-s|^{\frac{d}{4r}-\frac{d+2}{4}} ds)^{q-1}$$
$$\times \int_0^t |t-s|^{\frac{d}{4r}-\frac{d+2}{4}} \mathbb{E}\|u(s,\cdot) - v(s,\cdot)\|_q^q ds] \leq C_n T^{q(\frac{2-d}{4}+\frac{d}{4r})} \|u - v\|_{\mathcal{H}}^q.$$

Choose T_1 such that $C_n T^{q(\frac{2-d}{4}+\frac{d}{4r})} < 1$. Then the map A_1 is a contraction on $\mathcal{H}$.

Let $u \in \mathcal{H}$; Burkhölder's inequality, (2.1), (2.6) and assumption (b) imply,

$$\|A_2(u) - A_2(v)\|_{\mathcal{H}}^q$$
$$\leq C(q) \sup_{t\in[0,T]} \int_D \mathbb{E}(|\int_0^t \int_D G^2(t-s,x,y)(K_n(\|u(s,\cdot)\|_q)\sigma(u(s,\cdot))$$
$$- K_n(\|v(s,\cdot)\|)\sigma(v(s,\cdot)))^2) dy ds|^{q/2}) dx$$
$$\leq C(n,q) \sup_{t\in[0,T]} \mathbb{E}(|\int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{2}} \|u(s,\cdot) - v(s,\cdot)\|_q^2 ds|^{q/2})$$
$$\leq C(n,q) T^{q(-d/2+d/(4r))} \|u - v\|_{\mathcal{H}}^q.$$

Choose T_2 such that $C(n,q)T^{q(-d/2+d/(4r))} < 1$. Then the map A_2 is a contraction on $\mathcal{H}$.

Let $q > 4$. By assumptions $(b),(d)$ and Lemma 2.3, using the Hölder inequality, we have

$$\|A_3(u) - A_3(v)\|_{\mathcal{H}}^q = \sup_{t\in[0,T]} \mathbb{E}(\|A_3(u(t,\cdot)) - A_3(v(t,\cdot))\|_q^q)$$
$$\leq C \sup_{t\in[0,T]} \int_D \mathbb{E}(|\int_0^{t+} \int_D \int_{E_2} G(t-s,x,y)h(s,y,z)(K_n(\|u(s,\cdot)\|_q)\sigma(u(s^-,y))$$
$$- K_n(\|v(s,\cdot)\|_q)\sigma(v(s^-,y)))M(dz,dy,ds)|^q)dx$$
$$\leq C(q) \sup_{t\in[0,T]} \int_D (|\int_0^t \int_D \int_{E_2} (\mathbb{E}|G(t-s,x,y)h(s,y,z)(K_n(\|u(s,\cdot)\|_q)\sigma(u(s^-,y))$$
$$- K_n(\|v(s,\cdot)\|_q)\sigma(v(s^-,y)))|^q)^{\frac{2}{q}})\mu_2(dz)dyds|^{\frac{q}{2}}dx$$
$$\leq C(q) \sup_{t\in[0,T]} \int_D |\int_0^t \int_D \int_{E_2} |G(t-s,x,y)h(s,y,z)|^2$$
$$\times (\mathbb{E}(|(K_n(\|u(s,\cdot)\|_q)\sigma(u(s^-,y))$$
$$- K_n(\|v(s,\cdot)\|_q)\sigma(v(s^-,y)))|)^q)^{2/q}\mu_2(dz)dyds|^{\frac{q}{2}}dx$$
$$\leq C(q) \sup_{t\in[0,T]} \int_D |\int_0^t \int_D \int_{E_2} \mathbb{E}(|K_n(\|u(s,\cdot)\|_q)u(s,y)) - K_n(\|v(s,\cdot)\|_q)v(s,y))|)^q$$
$$\times \mu_2(dz)dyds(\int_0^t \int_D \int_{E_2} |G(t-s,x,y)h(s,y,z)|^{\frac{2q}{q-2}})\mu_2(dz)dyds|^{\frac{q}{2}})^{\frac{q-2}{2}}dx$$
$$\leq C(n,q)T\mu_2(E_2)\|u-v\|_{\mathcal{H}}^q \int_0^t \int_D \int_{E_2} |G(t-s,x,y)h(s,y,z)|^q \mu_2(dz)dyds.$$

Choose T_3 such that

$$C(n,q)T\mu_2(E_2) \int_0^t \int_D \int_{E_2} |G(t-s,x,y)h(s,y,z)|^q \mu_2(dz)dyds < 1.$$

Then the map A_3 is a contraction on $\mathcal{H}$.

Applying (2.4), with $\rho = q/2$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{\rho} + 1$, we have

$$\|A_4(u) - A_4(v)\|_{\mathcal{H}}^q \leq C \sup_{t\in[0,T]} \int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{4}} \|g(s,\cdot)(\sigma(u(s,\cdot))K_n(\|u(s,\cdot)$$
$$- \sigma(v(s,\cdot))K_n(\|v(s,\cdot))\|_q)\|_\rho ds$$
$$\leq C \int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{4}} C(n,q)\|g(s,\cdot)\|_q \|u(s,y) - v(s,y))\|_q^q ds$$
$$\leq C \sup_{t\in[0,T]} \|g(t,\cdot)\|_q T^{\frac{d}{4r}-\frac{d}{4}+1} \|u-v\|_{\mathcal{H}}^q.$$

Since $q > 4$ and $\frac{d}{4r} - \frac{d}{4} > -1$, choose T such that

$$C \sup_{t\in[0,T]} \|g(t,\cdot)\|_q T^{\frac{d}{4r}-\frac{d}{4}+1} < 1.$$

Then the map A_4 is a contraction on $\mathcal{H}$.

Finally, let $T = \frac{1}{4}\min\{T_1, T_2, T_3, T_4\} < 1$. The map Y is a contraction on $\mathcal{H}$, thus we complete the proof of this proposition. □

Using Proposition 1, Proposition 2, and the fixed-point argument on the space $\mathcal{H}$ endowed with the norm 3.2, one can get Theorem 3.1.

Remark 1. Bo and Wang [2] have proved that Theorem 3.1 holds for $q \in (8,\infty)$ under the same assumptions in Theorem 3.1. We can improve the result if we use the norm (3.2), but Bo and Wang define a norm $\|\cdot\|_{\mathcal{H}}$ (depending on (λ, q)) on $\mathcal{H}$ by $\|u\|_{\mathcal{H}} = (\sup_{t\in[0,T]} e^{-\lambda t}(\mathbb{E}(\|u(t,\cdot)\|_q^q))^{1/q}$.

4. Global solution

By uniqueness of the solution to (3.1), the local property of stochastic integrals yields, for $m > n$, $u_m(t,\cdot) = u_n(t,\cdot)$ if $t \le \tau_n$, so that we can define a process u by $u(t,\cdot) = u_n(t,\cdot)$ on $t \le \tau_n$. Set $\tau_\infty = \lim_{n\to\infty}\tau_n$; clearly u is a solution to (3.1) on the interval $[0,\tau_\infty)$, and is unique. We just need to prove that $\tau_\infty = +\infty$ almost surely, and use an argument similar to that of reference [13, 5].

In order to obtain the global solution, we need stronger conditions than the assumptions given in Section 3:

$(b)'$ $\sigma : \mathbb{R} \to \mathbb{R}$ is a bounded and Lipschitzian function;

$(d)'$ $G(t-s,x,y)h(s,y,z)$ is $L^{2q}([0,t]\times D^2 \times E_2)$, $ds \times dx \times dy \times \mu_2(E_2)$ integrable, for $0 \le t \le T$.

Now we can give our main result, the global well posedness of the problem (P), in the present paper.

Theorem 4.1. *Suppose the assumptions* $(a), (b)', (c), (d)'$ *and* (e) *are satisfied. Let* $d \in \{1,2,3\}$ *and* $q \in (4,\infty)$. *Then for every* $\mathcal{F}_0$*-measurable* $u_0 : D\times\Omega \to \mathbb{R}$ *with* $\mathbb{E}(\|u_0(\cdot)\|_q^q) < \infty$, *for any* $T > 0$, *there exists a unique solution* $\{u(t,x) : (t,x) \in [0,T]\times D\}$ *to the problem* (P) *such that* $\mathbb{E}(\|u(t,\cdot)\|_q^q) < \infty$, *for any* $t \in [0,T]$.

Before proving this theorem, we first prove two a priori estimates in the following two propositions.

Let H, L be defined respectively by

$$H(u)(t,x) = A_1(u)(t,x),$$
$$L(u)(t,x) = A_2(u)(t,x) + A_3(u)(t,x) + A_4(u)(t,x).$$

It is obvious that, for all $L^q(D)$-valued $\mathcal{F}_t$ adapted r.c.l.l. processes $u(t,x)$, the following norm

$$\|u\|_{\mathcal{H}_1} = (\sup_{t\in[0,T]} (\mathbb{E}(\|u(t,\cdot)\|_q^{\beta}))^{1/\beta} \tag{4.1}$$

is finite. Thus $\mathcal{H}_1$ is a Banach space.

Proposition 3. *Let the assumptions* $(a), (b)', (c), (d)'$ *and* (e) *hold; then we have*

$$\sup_n \sup_{t\in[0,T]} \sup_{x\in D} \mathbb{E}(|L(u_n)(t,x)|^{\delta}) < +\infty,$$

for $\delta \in [1, 2q]$.

Proof. Assume the assumptions $(a), (b)', (c), (d)'$ and (e) hold. We have the following facts, for $u \in \mathcal{H}_1$.

As for $A_2(u)(t,x)$, similar to (3.4), we have

$$\begin{aligned}
&\|A_2(u)\|_{\mathcal{H}_1}^{\beta} \\
&\leq C \sup_{t\in[0,T]} \int_D \mathbb{E}(|\int_0^t \int_D G(t-s,x,y)\sigma(u(s,y))K_n(\|u(s,\cdot)\|_q)W(dy,ds)|^{\beta})dx \\
&\leq C \sup_{t\in[0,T]} \int_D \mathbb{E}(|\int_0^t \int_D G^2(t-s,x,y)\sigma^2(u(s,y))K_n^2(\|u(s,\cdot)\|_q)dyds|)^{\frac{\beta}{2}} dx \\
&\leq C(T)\|\sigma\|_{\infty}^{\beta} \leq \infty.
\end{aligned}$$

As for $A_3(u)(t,x)$, similar to (3.5), for $\beta \leq 2q$, we have

$$\begin{aligned}
&\|A_3(u)\|_{\mathcal{H}_1}^{\beta} = \sup_{t\in[0,T]} \mathbb{E}(\|A_3(u(t,\cdot))\|_q^{\beta}) \\
&\leq C \sup_{t\in[0,T]} \int_D \mathbb{E}(|\int_0^{t+} \int_D \int_{E_2} G(t-s,x,y)\sigma(u(s,y))h(s,y,z)M(dz,dy,ds)|^{\beta})dx \\
&\leq C \sup_{t\in[0,T]} \int_D |\int_0^t \int_D \int_{E_2} (\mathbb{E}|G(t-s,x,y)\sigma(u(s,y))h(s,y,z)|^{\beta})^{\frac{2}{\beta}} \mu_2(dz)dyds|^{\frac{\beta}{2}} dx \\
&\leq C(T)\|\sigma\|_{\infty}^{\beta} \sup_{t\in[0,T]} \int_D |\int_0^t \int_D \int_{E_2} |G(t-s,x,y)h(s,y,z)|^2 \mu_2(dz)dyds|^{\frac{\beta}{2}} dx \\
&\leq C(T)\|\sigma\|_{\infty}^{\beta} \sup_{t\in[0,T]} \int_D \int_0^t \int_D \int_{E_2} |G(t-s,x,y)h(s,y,z)|^{2q} \mu_2(dz)dydsdx.
\end{aligned}$$

For $A_4(u)(t,x)$, similar to (3.6), we have

$$\|A_4(u)\|_{\mathcal{H}_1}^{\beta} \leq C \sup_{t\in[0,T]} \|g(s,\cdot)\|_q^{\beta} \|\sigma\|_{\infty}^{\beta} (\int_0^t (t-s)^{\frac{d}{4r}-\frac{d}{4}} ds)^{\beta}.$$

Since $q > 4$ and $\frac{d}{4r} - \frac{d}{4} > -1$, it follows that $\|A_4(u)\|^{\beta}_{\mathcal{H}_1} < \infty$. The above estimates tell us that

$$\sup_n \sup_{t\in[0,T]} \sup_{x\in D} \mathbb{E}(|L(u_n)(t,x)|^{\delta}) < \infty,$$

where $\delta \in [1, 2q]$. Thus we complete the proof of this proposition. □

We just need to prove a uniform upper estimate for $H(u_n)$. To this end we note first that $v_n = u_n - L(u_n)$ is a solution of the following system:

$$\begin{aligned}\frac{\partial v_n}{\partial t}(t,x) &+ \Delta^2 v_n(t,x) \\ &- \Delta[K_n(\|v_n(t,\cdot) + L(u_n)(t,\cdot)\|_q) f(v_n(t,x) + L(u_n)(t,x))] = 0,\end{aligned} \tag{4.2}$$

with the initial boundary conditions

$$v_n(0,\cdot) = u_0(\cdot), \quad \frac{\partial v_n}{\partial n} = \frac{\partial \delta v_n}{\partial n} = 0.$$

Again the above problem can be made rigorous as problem (P). For any $\phi \in C^4(D)$ such that ϕ satisfies $\frac{\partial \phi}{\partial n} = \frac{\partial \Delta \phi}{\partial n} = 0$ on ∂D, u satisfies

$$\begin{aligned}\int_D [v_n(t,x) - u_0(x)]\phi(x)dx &= -\int_0^t \int_D v_n(s,x)\Delta^2\phi(x)dxds \\ + \int_0^t \int_D K_n(\|v_n(s,\cdot) &+ L(u_n)(s,\cdot)\|_q) f(v_n(s,x) + L(u_n)(s,x))\Delta\phi(x)dxds.\end{aligned}$$

Using the Galerkin method, we can prove that, for every $T > 0$, v_n is the weak solution on $[0,T]$ to the above system, and $v_n \in L^2([0,T], W^{2,2}(D))$, where $W^{2,2}(D)$ denotes the Sobolev space.

Proposition 4. *Let the assumptions* $(a), (b)', (c), (d)'$ *and* (e) *hold; then we have, for* $\kappa \in [1, q/3]$,

$$\sup_n \mathbb{E}(\sup_{t\in[0,T]} \|H(u_n)(t,\cdot)\|^{\kappa}_q) < \infty.$$

Proof. Recall that, for $A = -\Delta, \alpha \neq 0$, and $u \in Dom(A^{\alpha})$, by convention, denote by $\mathbb{N}^{d,*}$ the set $\mathbb{N}^d/\{0\}$, then

$$A^{\alpha} = \sum_{k\in\mathbb{N}^{d,*}} \lambda_k^{\alpha} \langle \varepsilon_k, u\rangle \varepsilon_k.$$

$A^{\alpha}u$ exists for every u such that $\sum_{k\in\mathbb{N}^{d,*}} \lambda_k^{2\alpha}\langle \varepsilon_k, u\rangle^2 < \infty$. In what follows, for a function $u : [0,T] \times D \to \mathbb{R}$, we set

$$m(u)(t) = \langle \varepsilon_0, u(t,\cdot)\rangle = \pi^{-\frac{d}{2}} \int_D u(t,x)dx \quad \text{and} \quad \tilde{u}(t,y) = u(t,y) - m(u)(t).$$

Notice that $A^\alpha \tilde{u}(t,y) = A^\alpha u(t,y)$ for $\alpha \neq 0$. Apply A^{-1} to equation (4.2) and take its scalar product in $L^2(D)$ with $\tilde{v}_n(t,\cdot)$; this leads to

$$\|A^{-\frac{1}{2}}\tilde{v}_n(t,\cdot)\|_2^2 - \|A^{-\frac{1}{2}}\tilde{v}_n(0,\cdot)\|_2^2 + \int_0^t \|A^{\frac{1}{2}}\tilde{v}_n(s,\cdot)\|_2^2 ds$$
$$+\int_0^t K_n(\|v_n(s,\cdot)+L(u_n)(s,\cdot)\|_q)\int_D f(v_n(s,x)+L(u_n)(s,x))\tilde{v}_n(t,x)dxds = 0.$$

Let us deal with the last term in the above inequality.

$$\begin{aligned}&\int_D f(v_n(s,x)+L(u_n)(s,x))\tilde{v}_n(t,x)dx\\ &= \int_D f(v_n(s,x)+L(u_n)(s,x))(v_n(s,x)+L(u_n)(s,x))dx\\ &- \int_D f(v_n(s,x)+L(u_n)(s,x))(L(u_n)(s,x)+m(v_n)(s))dx = I + II.\end{aligned}$$

The polynomial f is of degree 3 with positive dominant coefficient, hence, $\lim_{|x|\to+\infty} f(x) = +\infty$, and there exists a, C and $c > 0$ such that

$$I \geq \frac{7}{8}a\|v_n(s,\cdot)+L(u_n)(s,\cdot)\|_4^4 - c.$$

In addition,

$$\begin{aligned}II &\leq \frac{5}{4}\int_D |v_n(s,x)+L(u_n)(s,x)|^3(|L(u_n)(s,x)|+|m(v_n)(s)|)dx\\ &+ C\int_D(|L(u_n)(s,x)|+|m(v_n)(s)|)dx\\ &\leq \frac{5}{8}a\|v_n(s,\cdot)+L(u_n)(s,\cdot)\|_4^4 + C(|m(v_n)(s)|^4 + \|L(u_n)(s,\cdot)\|_4^4).\end{aligned}$$

Since K_n is a positive bounded function,

$$\begin{aligned}&\|A^{-\frac{1}{2}}\tilde{v}_n(t,\cdot)\|_2^2 - \|A^{-\frac{1}{2}}\tilde{v}_n(0,\cdot)\|_2^2 + \int_0^t \|A^{\frac{1}{2}}\tilde{v}_n(s,\cdot)\|_2^2 ds\\ &+ \frac{a}{4}\int_0^t K_n(\|v_n(s,\cdot)+L(u_n)(s,\cdot)\|_q)\|v_n(s,\cdot)+L(u_n)(s,\cdot)\|_4^4 ds\\ &\leq \int_0^t C(1+|m(v_n)(s)|^4 + \|L(u_n)(s,\cdot)\|_4^4)ds.\end{aligned}$$

Taking the scalar product of the solution to the equation with the function ε_0, we obtain

$$\frac{\partial}{\partial t}\langle v_n(t,\cdot),\varepsilon_0\rangle = 0, \langle v_n(0,\cdot),\varepsilon_0\rangle = \langle u_0(\cdot),\varepsilon_0\rangle,$$

hence, $m(u_0) = \langle \varepsilon, u_0 \rangle$. Since $A^{-\frac{1}{2}}\tilde{u} = A^{-\frac{1}{2}}u_0$, it follows that, for every $T > 0$, we have

$$\begin{aligned}
&\|A^{-\frac{1}{2}}\tilde{v}_n(T,\cdot)\|_2^2 - \|A^{-\frac{1}{2}}u_0(\cdot)\|_2^2 + \int_0^T \|A^{\frac{1}{2}}v_n(s,\cdot)\|_2^2 ds \\
&+ \frac{a}{4}\int_0^T K_n(\|v_n(s,\cdot) + L(u_n)(s,\cdot)\|_q)\|v_n(s,\cdot) + L(u_n)(s,\cdot)\|_4^4 ds \\
&\le \int_0^T C(1 + |m(u_0)(s)|^4 + \|L(u_n)(s,\cdot)\|_4^4) ds.
\end{aligned}$$

This yields

$$\begin{aligned}
&\int_0^t K_n(\|v_n(s,\cdot) + L(u_n)(s,\cdot)\|_q)\|v_n(s,\cdot) + L(u_n)(s,\cdot)\|_4^4 ds \\
&\le C_T(1 + |m(u_0)|^4 + \|L(u_n)(s,\cdot)\|_\infty^4) ds + \|A^{-\frac{1}{2}}\tilde{v}_n(0,\cdot)\|_2^2.
\end{aligned} \tag{4.3}$$

We now take the scalar product in $L^2(D)$ of equation (4.2) with v_n; using Green's formula, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|v_n\|_2^2 + \|\Delta v_n(t,\cdot)\|_2^2 \\
&= K_n(\|v_n(t,\cdot) + L(u_n(t,\cdot))\|_q)\int_D f(v_n(t,x) + L(u_n)(t,x))\Delta v_n(t,x)dx.
\end{aligned}$$

Noticing (4.3), we have

$$\begin{aligned}
&\|v_n\|_2^2 + \int_0^t [\|\Delta[v_n(t,\cdot)\|_2^2 + m(v_n(s,\cdot))^2] ds \\
&\le \|u_0\|_2^2 + C_T(1 + \|L(u_n)\|_\infty^6) + C_T(1 + \|L(u_n)\|_\infty^6)[\|A^{-\frac{1}{2}}u_0\|_2^2 + m(u_0)^4].
\end{aligned}$$

Proposition 3 tells us that, for $\kappa \in [1, q/3]$,

$$\begin{aligned}
&\sup_n \mathbb{E}(\sup_{t\in[0,T]} \|v_n(t,\cdot)\|_2^{2\kappa}) < \infty, \\
&\sup_n \mathbb{E}(\int_0^T [\|\Delta[v_n(t,\cdot)\|_2^2 + m(v_n(s,\cdot))^2] ds]^\kappa) < \infty.
\end{aligned}$$

Furthermore, by Sobolev's embedding theorem there exists $C > 0$ such that, for $r \ge 2, d < 4$, if $u \in W^{2,2}(D)$,

$$\|u\|_{L^r(D)} \le C\|u\|_{W^{2,2}(D)}.$$

Thus, for $2 \le r < \infty$,

$$\sup_n \mathbb{E}([\int_0^T \|v_n(t,\cdot)\|_r^2 dt]^\kappa) < \infty.$$

For $2 \leq r < \infty$,

$$\sup_n \mathbb{E}(\sup_{t\in[0,T]} \|u_n(t,\cdot)\|_2^{2\kappa}) < \infty, \quad \sup_n \mathbb{E}([\int_0^T \|\Delta u_n(t,\cdot)\|_r^2 ds]^\kappa) < \infty.$$

Let us use the interpolation method to prove that u_n belongs almost surely to $L^a([0,T],L^q(D))$, with $2 \leq q \leq r, a \geq 1 \vee \frac{2q}{r}$. The Hölder inequality implies that, if $q = 2(1-\lambda) + r\lambda$, for $\lambda \in [0,1]$, that

$$\int_0^T \|u_n(t,\cdot)\|_q^a dt \leq \int_0^T \|u_n(t,\cdot)\|_2^{2a(1-\lambda)/q} \|u_n(t,\cdot)\|_r^{ar\lambda/q} dt.$$

Taking $\lambda = 2q/(ar)$, we obtain

$$\int_0^T \|u_n(t,\cdot)\|_q^a dt \leq \sup_{t\in[0,T]} \|u_n(t,\cdot)\|_2^{2a(1-\lambda)/q} \int_0^T \|u_n(t,\cdot)\|_r^2 dt$$

for $q \in [2,\infty)$ and $a \in [q,\infty)$ and

$$\sup_n \mathbb{E}([\int_0^T \|u_n(t,\cdot)\|_q^a dt]^\kappa) < \infty,$$

$$\|H(u_n)(t,\cdot)\|_q \leq C \int_0^t (t-s)^{\frac{d}{4r}-\frac{d+2}{4}} (\|u_n(s,\cdot)\|_q^3 + 1) ds.$$

Let $\gamma, \gamma' \in (1,\infty)$ be conjugate exponents, with γ close enough to unity to ensure $(\frac{d}{4r} - \frac{d+2}{4})\gamma > -1$; the Hölder inequality implies that

$$\|H(u_n)(t,\cdot)\|_q \leq C[\int_0^t (t-s)^{(\frac{d}{4r}-\frac{d+2}{4})\gamma} ds]^{\frac{1}{\gamma}} [\int_0^t (\|u_n(s,\cdot)\|_q^3 + 1)^{\gamma'} ds]^{\frac{1}{\gamma'}}.$$

We obtain

$$\sup_n \mathbb{E}(\sup_{t\in[0,T]} \|H(u_n)(t,\cdot)\|_q^\kappa) < \infty.$$

Thus we end the proof of this proposition. □

The proof of Theorem (4.1). Noticing Proposition 3, Proposition 4, and the estimates (3.3), we have

$$\sup_n \mathbb{E}(\sup_{t\in[0,T]} \|u_n(t,\cdot)\|_q^\kappa) < \infty.$$

We can now conclude that $\tau_\infty = +\infty$ almost surely; indeed, for every $T > 0$,

$$P(\tau_n \leq T) = P(\sup_{t\leq T} \|u_n(t,\cdot)\|_q \geq n) \leq \mathbb{E}(\sup_{t\leq T} \|u_n(t,\cdot)\|_q^{2\kappa}) n^{-2\kappa},$$

so that $\lim_{n\to\infty} P(\tau_n \leq T) = 0$. Therefore, we can construct the solution to the problem (P) on every interval $[0,T]$. Then we obtain the main theorem in this paper. □

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