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BREZIS-MERLE TYPE INEQUALITY FOR A WEAK SOLUTION TO THE N-LAPLACE EQUATION IN LORENTZ-ZYGMUND SPACES

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Abstract. We consider a regularity estimate for a solution of the homogeneous Dirichlet problem for N-Laplace equations in a bounded domain $\Omega \subset \mathbb{R}^N$ with external force $f \in L^1(\Omega)$. Introducing the generalized Lorentz-Zygmund space, we show the multiple exponential integrability of the Brezis-Merle type for an entropy solution of the Dirichlet problem of the N-Laplace equation. We also discuss conditions on f that guarantee the solutions are bounded.

1. Introduction and Main Results

This paper deals with the Dirichlet problem of the N-Laplace equation. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary; the N-Laplace equation is as follows:

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{N-2}\nabla u\big) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

We consider the regularity of a weak solution u for a given external force f, in particular, boundedness of the solution u.

Let us initially recall the case $N = 2$ in (1.1), namely the Dirichlet problem of the Poisson equation in 2-dimension,

$$\begin{cases} -\Delta u = f, & x \in \Omega \subset \mathbb{R}^2, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

It is a well-known result that if $f \in L^q(\Omega)$ $(q > 1)$ then the solution u of (1.2) is bounded (cf. [13]). This result can be proved from boundedness of the singular integral in L^q and the Sobolev embedding theorem $W^{2,q}(\Omega) \subset$

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$L^\infty(\Omega)$. When $f \in L^1(\Omega)$, Brezis-Merle proved in [3] that u has exponential integrability: For every $0 \le \alpha < 4\pi$, there exists $C > 0$ such that

$$\int_\Omega \exp\Big[\frac{\alpha|u(x)|}{\|f\|_{L^1}}\Big]dx \le C|\Omega|. \tag{1.3}$$

Moreover, they proved that if $f \in L^1(\Omega)$ then u is not bounded in general. On the other hand, a solution u is bounded for $f \in L\log L(\Omega)$, where $L\log L(\Omega)$ is one of the interpolation spaces between $L^1(\Omega)$ and $L^q(\Omega)$ $(q > 1)$.

Trombetti [22] developed the integrability estimate (1.3) for a quasi-linear version of (1.1) when f is in the Lorentz space $L^{1,\frac{1}{p}}(\Omega)$. We show the corresponding integurability estimate when f is in the Lorentz-Zygmund space $L(\log L)^p(\Omega)$ $(p \ge 0)$. The Lorentz-Zygmund space is introduced as follows. Let f^* be the non-increasing rearrangement of f defined by

$$f^*(r) := \inf\{\lambda > 0 : \mu_f(\lambda) \le r\},$$

where $\mu_f(\lambda)$ is a distribution function of f. For $0 \le p < \infty$, the Lorentz-Zygmund space $L(\log L)^p(\Omega)$ is a set of measurable functions on Ω defined by

$$L(\log L)^p(\Omega) := \Big\{f \in L^1(\Omega) : \|f\|_{L(\log L)^p} := \int_0^{|\Omega|}\Big(\log\frac{e|\Omega|}{r}\Big)^p f^*(r)dr < \infty\Big\}.$$

Then one may obtain the following estimate.

Proposition 1.1 (Brezis-Merle type inequality for the Poisson equation [22])**.** *For $f \in L(\log L)^p(\Omega)$, $0 \le p < 1$, let u be a weak solution of* (1.2) *in the sense of distributions. Then, for every $0 \le \alpha < 4\pi$ there exists $C = C(p, \alpha) > 0$ such that*

$$\int_\Omega \exp\Big[\Big\{\frac{\alpha|u(x)|}{\|f\|_{L(\log L)^p}}\Big\}^{\frac{1}{1-p}}\Big]dx \le C|\Omega|.$$

Trombetti [22] showed a result similar to Proposition 1.1 for f in Lorentz spaces. Our result involves f in Lorentz-Zygmund spaces.

The main purpose of this paper is to generalize the above result for (1.2) to the higher-dimensional case, the N-Laplace equation with the Dirichlet boundary condition,

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{N-2}\nabla u\big) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and the external force f belongs to $L^1(\Omega)$.

In general, the problem (1.1) does not have a classical solution. Therefore we need to employ the notion of weak solutions. We introduce here two kinds of weak solutions.

Definition 1.2 (Variational solutions). *We say that u is a weak solution in $W_0^{1,N}(\Omega)$ of* (1.1) *if $u \in W_0^{1,N}(\Omega)$ and for any $\phi \in C_0^\infty(\Omega)$*

$$\int_\Omega |\nabla u|^{N-2}\nabla u \cdot \nabla \phi dx = \int_\Omega f\phi dx.$$

When we consider $f \in L^1(\Omega)$, then even the existence of a variational solution to (1.1) is not obvious. Therefore we need to introduce a weaker weak solution. Bénilan-Boccardo-Galloüet-Gariepy-Pierre-Vazquez [4] introduced a weak solution called an entropy solution.

Definition 1.3 (Entropy solutions [4], [6]). *We say that $u \in \mathcal{T}_0^{1,N}(\Omega)$ if $T_\lambda(u) \in W_0^{1,N}(\Omega)$ for any $\lambda > 0$, where T_λ is the truncating operator defined by*

$$T_\lambda(s) := \begin{cases} s, & |s| \le \lambda, \\ \lambda \dfrac{s}{|s|}, & |s| > \lambda. \end{cases} \tag{1.4}$$

If $u \in \mathcal{T}_0^{1,N}(\Omega)$ satisfies the entropy condition

$$\int_\Omega |\nabla u|^{N-2}\nabla u \cdot \nabla\big(T_\lambda(u-\phi)\big)dx \le \int_\Omega T_\lambda(u-\phi) f dx,$$

for every $\lambda > 0$, $\phi \in C_0^\infty(\Omega)$, then u is an entropy solution to problem (1.1)*.*

Aguilar-Peral proved in [2] that the entropy solution u of (1.1) for $f \in L^1(\Omega)$ has the exponential integrability such as Brezis-Merle's inequality (1.3). Here we consider the unknown case $0 < p < N-1$.

Theorem 1.4 (Brezis-Merle type inequality). *Let $f \in L(\log L)^p(\Omega)$* $(0 \le p < N-1)$*. Then there exists an entropy solution of the Dirichlet problem* (1.1)*. Moreover, let*

$$\beta := \begin{cases} N\omega_N^{\frac{1}{N-1}}, & 0 \le p \le 1, \\ \left(p\Big(1 - \dfrac{p-1}{N-2}\Big)^{N-2}\right)^{\frac{1}{N-1}} N\omega_N^{\frac{1}{N-1}}, & 1 < p < N-1; \end{cases}$$

then for every $0 \le \alpha < \beta$, there exists $C = C(\alpha, p, N) > 0$ such that

$$\int_\Omega \exp\Big[\Big\{\frac{\alpha|u(x)|}{\|f\|^{\frac{1}{N-1}}_{L(\log L)^p}}\Big\}^{\frac{N-1}{N-1-p}}\Big]dx \le C|\Omega|,$$

where ω_N is the surface measure of the unit sphere in $\mathbb{R}^N$.

Boccardo-Peral-Vazquez proved in [6] that the boundedness of a solution u of (1.1) in the case $N \ge 3$, more precisely, if $p > N-1$, then the solution u is bounded and if $p = N-1$ then the solution u is not necessarily bounded. To specify more detailed integrability of the solution u, we choose to have f in an interpolation space between $L(\log L)^{N-1}(\Omega)$ and $L(\log L)^p(\Omega)$ $(p > N-1)$. To this end, we use a slightly generalized version of the Lorentz-Zygmund space. For $0 \le p_1, p_2 < \infty$, we define

$$L(\log L)^{p_1}(\log \log L)^{p_2}(\Omega) := \left\{ f \in L^1(\Omega) : \|f\|_{L(\log L)^{p_1}(\log\log L)^{p_2}} < \infty \right\},$$

$$\|f\|_{L(\log L)^{p_1}(\log\log L)^{p_2}} := \int_0^{|\Omega|} \Big(\log\frac{|\Omega|}{r}\Big)^{p_1}\Big(\log\log\frac{e|\Omega|}{r}\Big)^{p_2} f^*(r)dr.$$

We obtain an explicit regularity estimate of a solution u for

$$f \in L(\log L)^{N-1}(\log\log L)^p(\Omega).$$

Proposition 1.5. *Let $p \ge 0$, $N \in \mathbb{N}$, $N \ge 3$, and*

$$f \in L(\log L)^{N-1}(\log\log L)^p(\Omega).$$

Then there exists a variational solution of (1.1) such that the solution u satisfies the following.

(1) *If $0 \le p < N-2$, then there exists $\beta_1 > 0$ such that, for every $0 < \alpha < \beta_1$, u has the following double exponential integurability:*

$$\int_\Omega \exp\Big(\exp\Big(\Big(\frac{\alpha|u(x)|}{\|f\|_{L(\log L)^{N-1}(\log\log L)^p}}\Big)^{\frac{N-1}{N-2-p}}\Big)\Big)dx < \infty.$$

(2) *If $p = N-2$, then there exists $\beta_2 > 0$ such that for every $0 < \alpha < \beta_2$, u has the following triple exponential integurability:*

$$\int_\Omega \exp\Big(\exp\Big(\exp\Big(\Big(\frac{\alpha|u(x)|}{\|f\|_{L(\log L)^{N-1}(\log\log L)^{N-2}}}\Big)^{\frac{N-1}{N-2}}\Big)\Big)\Big)dx < \infty.$$

(3) *If $p > N-2$, then the solution u is bounded.*

From Proposition 1.5, we obtain that $L(\log L)^{N-1}(\log\log L)^{N-2}$ is the border whether the solution is bounded or not. We remark that, for any $\varepsilon > 0$,

$$L(\log L)^{N-1+\varepsilon}(\Omega) \subset L(\log L)^{N-1}(\log\log L)^{N-2}(\Omega) \subset L(\log L)^{N-1}(\Omega).$$

One may obtain more generalized results. For example, we may specify the condition on f in much more detail. For $\lambda > 0$, $k \in \mathbb{N}$, let $e_k[\lambda]$ and $\log^{[k]}\lambda$ be defined such that

$$e_1[\lambda] := e^{\lambda}, \quad e_{k+1}[\lambda] := \exp[e_k[\lambda] - 1],$$
$$\log^{[1]}\lambda := \log\lambda, \quad \log^{[k+1]}\lambda := \log\bigl(1 + \log^{[k]}\lambda\bigr).$$

The generalized Lorentz-Zygmund space

$$L(\log L)^{p_1}(\log^{[2]} L)^{p_2}\cdots(\log^{[k]} L)^{p_k}(\Omega)$$

for $0 \le p_1, p_2, \cdots, p_k < \infty$ is defined in the following way:

$$L(\log L)^{p_1}\cdots(\log^{[k]}L)^{p_k}(\Omega) := \Bigl\{f \in L^1(\Omega)\,;\, \|f\|_{L(\log L)^{p_1}\cdots(\log^{[k]}L)^{p_k}} < \infty\Bigr\},$$
$$\|f\|_{L(\log L)^{p_1}\cdots(\log^{[k]}L)^{p_k}} := \int_0^{|\Omega|}\Bigl(\log\frac{|\Omega|}{r}\Bigr)^{p_1}\cdots\Bigl(\log^{[k]}\frac{|\Omega|}{r}\Bigr)^{p_k} f^*(r)dr.$$

Here we show the generalization of Proposition 1.5.

Theorem 1.6 (Generalized Brezis-Merle type inequality)**.** *Let $p \ge 0$, $k, N \in \mathbb{N}$, $k \ge 2$, $N \ge 3$, and*

$$f \in L(\log L)^{N-1}(\log^{[2]}L)^{N-2}\cdots(\log^{[k-1]}L)^{N-2}(\log^{[k]}L)^{p}(\Omega).$$

Then there exists a variational solution of (1.1) *such that the solution u satisfies the following properties:*

(1) *If $0 \le p < N-2$, then there exists $\beta_1 > 0$ such that, for every $0 < \alpha < \beta_1$,*

$$\int_\Omega e_k\Bigl[\Bigl(\frac{\alpha|u(x)|}{\|f\|_{L(\log L)^{N-1}(\log^{[2]}L)^{N-2}\cdots(\log^{[k-1]}L)^{N-2}(\log^{[k]}L)^{p}}}\Bigr)^{\frac{N-1}{N-2-p}}\Bigr]dx < \infty.$$

(2) *If $p = N-2$, then there exists $\beta_2 > 0$ such that, for every $0 < \alpha < \beta_2$,*

$$\int_\Omega e_{k+1}\Bigl[\Bigl(\frac{\alpha|u(x)|}{\|f\|_{L(\log L)^{N-1}(\log^{[2]}L)^{N-2}\cdots(\log^{[k-1]}L)^{N-2}(\log^{[k]}L)^{N-2}}}\Bigr)^{\frac{N-1}{N-2}}\Bigr]dx < \infty.$$

(3) *If $p > N-2$, then the solution u is bounded.*

From Theorem 1.6, we obtain that, for every $k \geq 2$,

$$L(\log L)^{N-1}(\log^{[2]} L)^{N-2} \cdots (\log^{[k]} L)^{N-2}$$

is the border whether the solution is bounded or not.

It is worthwhile to notice that there exists a well-known result of the exponential integrability of critical Sobolev functions due to Trudinger [23] and Moser [16]. As an extension of the Trudinger-Moser inequality, the multiple exponential integrability of Sobolev functions in the Lorentz-Zygmund space can be seen in [9], [10] and [19]. Those results showed that, if $\nabla u \in L^p(\log L)^q(\Omega)$, then u has a multiple exponential integrability. For the definition of $L^p(\log L)^q(\Omega)$, see [5], [9] and [10]. Our problem for the Poisson equation (1.2) can be interpreted as the embedding problem of the Sobolev space if the solution is represented by the fundamental solution. However the main difference between those results and our result is that the fundamental solution corresponding to the problem (1.2) is not the Riesz potential but the logarithmic function. Besides, for the higher-dimensional case, there is no representation of the solution by the potential.

We summarize the notation that is used in this paper. Let α be the multi-index $\alpha = (\alpha_1, \cdots, \alpha_N)$, $\alpha_j \in \mathbb{N} \cup \{0\}$, $j = 1, \cdots, N$, $|\alpha| := \sum_{j=1}^N \alpha_j$, and ∂_x^α be a differential operator $\partial_x^\alpha := \Pi_{k=1}^N \left(\frac{\partial}{\partial x_k}\right)^{\alpha_k}$. The Lebesgue space is denoted by $L^p(\Omega)$ with the norm $\|f\|_{L^p} := \left(\int_\Omega |f|^p dx\right)^{1/p}$ for $1 \leq p < \infty$, and $\|f\|_{L^\infty} := \text{ess.sup}_{x \in \Omega} |f(x)|$. For $p \geq 1, k \in \mathbb{N} \cup \{0\}$, the Sobolev space denoted by $W^{k,p}$ is equipped with the norm

$$\|f\|_{W^{k,p}} := \Big(\sum_{|\alpha| \leq k} \|\partial_x^\alpha f\|_{L^p}^p \Big)^{1/p}.$$

The measure of Ω on $\mathbb{R}^N$ is denoted by $|\Omega|$. The distribution function of f is denoted by $\mu_f(\lambda) := \big|\{x : |f(x)| > \lambda\}\big|$ and the rearrangement of f, f^* is defined by $f^*(r) := \inf\{\lambda : \mu_f(\lambda) \leq r\}$. The set of all smooth functions with compact support is denoted by $C_0^\infty(\Omega)$. Let $W_0^{k,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. The dual space of $W_0^{k,p}(\Omega)$ is denoted by $W^{-k,p'}(\Omega)$, where p' is Hölder conjugate for $p \geq 1$, $p' = \frac{p}{p-1}$.

2. PRELIMINARIES

2.1. **The rearrangement.** For the proof of the main results, some properties of the rearrangement play an important role. We summarize them as follows.

Proposition 2.1 ([14, 20, 24])**.** *Let f, g be measurable functions and f^*, g^* be the non-increasing rearrangement of f, g, respectively. Then the non-increasing rearrangement has the following properties.*

(1) *f^* is non-increasing and non-negative.*
(2) *If $|f(x)| \leq |g(x)|$ for almost all $x \in \mathbb{R}^N$, then $f^*(r) \leq g^*(r)$ for any $r > 0$.*
(3) *If $f \in L^\infty(\mathbb{R}^N)$, then*
$$\|f\|_{L^\infty} = \inf\{\lambda : \mu_f(\lambda) = 0\} = f^*(0).$$
(4) *If $\Phi : [0, \infty) \to [0, \infty)$ is a non-decreasing function, then*
$$\int_\Omega \Phi(|f(x)|)dx = \int_0^{|\Omega|} \Phi(f^*(r))dr.$$
(5) *If $\Phi : [0, \infty) \to [0, \infty)$ is a non-increasing function, then*
$$\big(\Phi(|f|)\big)^*(r) = \Phi\big(f^*(r)\big).$$
(6) *If $1 \leq p \leq \infty$, then*
$$\int_{\mathbb{R}^N} |f(x)g(x)|dx \leq \int_0^\infty f^*(r)g^*(r)dr \leq \|f\|_{L^p}\|g\|_{L^{p'}}.$$
(7) *O'Neil's inequality:*
$$(f * g)^{**}(r) \leq r f^{**}(r)g^{**}(r) + \int_r^\infty f^*(s)g^*(s)ds,$$
where $f^{**}(r) := \frac{1}{r}\int_0^r f^*(s)ds$.

See, for the proof, Lieb-Loss [14], Ziemer [24] and O'Neil [20]. In this paper, we in particular use $\Phi(r) = \exp[r]$; that is, for the rearrangement f^* of f, we have from (2) in Proposition 2.1 that
$$\int_\Omega \exp(|f(x)|)dx = \int_0^{|\Omega|} \exp(f^*(r))dr.$$

2.2. **Weak solutions for problem** (1.1)**.** Here we recall the definition of a variational solution and an entropy solution ([4, 6]).

Definition 2.2. *We say that u is a variational solution of* (1.1) *if $u \in W_0^{1,N}(\Omega)$ satisfies*
$$\int_\Omega |\nabla u|^{N-2}\nabla u \cdot \nabla\phi dx = \int_\Omega f\phi dx,$$
for every $\phi \in C_0^\infty(\Omega)$.

Definition 2.3 (Entropy solutions [4], [6]). *We say that* $u \in \mathcal{T}_0^{1,N}(\Omega)$ *if* $T_\lambda(u) \in W_0^{1,N}(\Omega)$ *for any* $\lambda > 0$*, where* T_λ *is the truncature operator defined by*

$$T_\lambda(s) := \begin{cases} s, & |s| \le \lambda, \\ \lambda \dfrac{s}{|s|}, & |s| > \lambda. \end{cases}$$

If $u \in \mathcal{T}_0^{1,N}(\Omega)$ *satisfies the entropy condition*

$$\int_\Omega |\nabla u|^{N-2}\nabla u \cdot \nabla\big(T_\lambda(u-\phi)\big)dx \le \int_\Omega T_\lambda(u-\phi) f dx,$$

for every $\lambda > 0$*,* $\phi \in C_0^\infty(\Omega)$*, then* u *is called an entropy solution to problem* (1.1)*.*

Remark 2.3.1. From the definition, any variational solution of (1.1) for $f \in L^1(\Omega)$ is an entropy solution. Boccardo et al. [6] and Bénilan et al. [4] showed the existence and uniqueness of the entropy solution of (1.1) for $f \in L^1(\Omega)$.

Proposition 2.4 (Talenti [21]). *Let* $f \in L^1(\Omega)$*. If* u *is a variational solution of* (1.1) *then we have*

$$u^*(r) \le \frac{1}{N\omega_N^{\frac{1}{N-1}}} \int_r^{|\Omega|} \frac{1}{s}\Big(\int_0^s f^*(\eta)d\eta\Big)^{\frac{1}{N-1}} ds, \tag{2.1}$$

where ω_N *is the measure of the unit sphere in* $\mathbb{R}^N$*.*

The outline of the proof of Proposition 2.4 is given in the following. For a detailed proof, see [21] and also [12], [8].

Proof of Proposition 2.4. We employ three inequalities to prove Proposition 2.4.

$$-\frac{d}{d\lambda}\int_{\{|u|>\lambda\}} |\nabla u| dx \le \Big(-\frac{d}{d\lambda}\int_{\{|u|>\lambda\}} |\nabla u|^N dx\Big)^{\frac{1}{N}} \big(-\mu'(\lambda)\big)^{\frac{N-1}{N}}, \tag{2.2}$$

$$-\frac{d}{d\lambda}\int_{\{|u|>\lambda\}} |\nabla u|^N dx \le \int_0^{\mu(\lambda)} f^*(s)ds, \tag{2.3}$$

$$\omega_N^{\frac{1}{N}}\big(N\mu(\lambda)\big)^{\frac{N-1}{N}} \le -\frac{d}{d\lambda}\int_{\{|u|>\lambda\}} |\nabla u| dx, \tag{2.4}$$

where we put $\mu(\lambda) = \mu_u(\lambda)$. To see that the inequality (2.2) holds, we start with Hölder's inequality, giving

$$\frac{1}{h}\int_{\{\lambda+h>|u|>\lambda\}}|\nabla u|dx \le \Big(\frac{1}{h}\int_{\{\lambda+h>|u|>\lambda\}}|\nabla u|^N dx\Big)^{\frac{1}{N}}\Big(\frac{\mu(\lambda)-\mu(\lambda+h)}{h}\Big)^{\frac{N-1}{N}}.$$

By lettinging $h \to 0$, (2.2) follows. Next, we prove the inequality (2.3). We choose the test function φ such that

$$\varphi(x) = \begin{cases} |u(x)| - \lambda, & x \in \{x;\, |u(x)| > \lambda\}, \\ 0, & x \in \{x;\, |u(x)| \le \lambda\}, \end{cases}$$

then we have from the definition of the variational solution,

$$\int_{\{|u|>\lambda\}} |\nabla u(x)|^N dx = \int_{\{|u|>\lambda\}} f(x)\big(|u(x)| - \lambda\big)dx.$$

Hence, it follows that

$$\begin{aligned}\int_{\{|u|>\lambda\}}&|\nabla u(x)|^N dx - \int_{\{|u|>\lambda+h\}} |\nabla u(x)|^N dx \\ &= \int_{\{|u|>\lambda\}} f(x)\big(|u(x)|-\lambda\big)dx - \int_{\{|u|>\lambda+h\}} f(x)\big(|u(x)|-\lambda-h\big)dx \\ &= h\int_{\{|u|>\lambda\}} f(x)dx - h\int_{\{\lambda+h\ge|u|>\lambda\}} f(x)\Big(1-\frac{|u(x)|-\lambda}{h}\Big)dx\end{aligned}$$

for all $h > 0$. Here we have

$$\frac{1}{h}\int_{\{\lambda+h\ge|u|>\lambda\}} |\nabla u(x)|^N dx \le \int_{\{|u|>\lambda\}} |f(x)|dx + \int_{\{\lambda+h>|u|>\lambda\}} |f(x)|dx.$$

Again by letting $h \to 0$, we obtain (2.3).

Finally, since by the Fleming-Rishel formula ([12]),

$$\int_{\{|u|>\lambda\}} |\nabla u|dx = \int_\lambda^\infty P(\{x \in \Omega\,;\, |u(x)| > \eta\})d\eta,$$

where $P(\Omega)$ stands for the $(N-1)$-dimensional measure of the boundary of Ω, and De Giorgi's isoperimetric theorem ([8])

$$P(\{x \in \Omega\,;\, |u(x)| > \lambda\}) \ge \omega_N^{1/N}\big(N\mu(\lambda)\big)^{1-1/N},$$

we have the inequality (2.4).

We prove the inequality (2.1). From (2.2), (2.3) and (2.4), we have

$$N\omega_N^{1/(N-1)} \le -\frac{\mu'(\lambda)}{\mu(\lambda)}\Big(\int_0^{\mu(\lambda)} f^*(s)ds\Big)^{-\frac{1}{N-1}}. \tag{2.5}$$

We have by integration by parts over $(0, u^*(r))$ that

$$N\omega_N^{1/(N-1)} u^*(r) \le \int_0^{u^*(r)} -\frac{\mu'(\lambda)}{\mu(\lambda)} \Big(\int_0^{\mu(\lambda)} f^*(s) ds \Big)^{-\frac{1}{N-1}} d\lambda,$$

which yields

$$u^*(r) \le \frac{1}{N\omega_N^{1/(N-1)}} \int_r^{|\Omega|} \frac{1}{s} \Big(\int_0^s f^*(\eta) d\eta \Big)^{\frac{1}{N-1}} ds. \qquad \square$$

3. Proof of Proposition 1.1 and Remark

Since the problem (1.2) is linear, we estimate the solution using the fundamental solution of the Poisson equation.

Proof of Proposition 1.1. Proposition 1.1 was proved in [22]. Here we prove Proposition 1.1 in another way. Let $\operatorname{diam}\Omega := \sup_{x,y\in\Omega} |x-y|$, $R = \operatorname{diam}\Omega/2$ so that $\Omega \subset B_R$ for some ball of radius R. We extend the function $f \in L(\log L)^p(\Omega)$ to be zero in the outside of Ω and set the fundamental solution of the Poisson equation in 2 dimensions as follows:

$$\Phi(x) := \log \frac{2R}{|x|}.$$

Define

$$\bar{u}(x) := \frac{1}{2\pi} \int_\Omega \Phi(x-y)|f(y)|dy$$

so that

$$-\Delta \bar{u} = |f|, \quad x \in \mathbb{R}^2$$

in the sense of distributions. It follows from the maximum principle that $|u| \le \bar{u}$ on Ω. By the properties of the rearrangement in Proposition 2.1, we have

$$u^*(r) \le \big(\bar{u}\big)^*(r) = \frac{1}{2\pi} (\Phi * |f|)^*(r), \tag{3.1}$$

and

$$\int_\Omega \exp\Big[\Big\{ \alpha \frac{|u(x)|}{\|f\|_{L(\log L)^p}} \Big\}^{\frac{1}{1-p}} \Big] dx = \int_0^{|\Omega|} \exp\Big[\Big\{ \alpha \frac{u^*(r)}{\|f\|_{L(\log L)^p}} \Big\}^{\frac{1}{1-p}} \Big] dr. \tag{3.2}$$

From the inequality by O'Neil in Proposition 2.1, we have

$$\begin{aligned} (\Phi * |f|)^{**}(r) &\le r\Phi^{**}(r) f^{**}(r) + \int_r^\infty \Phi^*(s) f^*(s) ds \qquad (3.3) \\ &= \log\Big(\sqrt{\frac{4\pi e}{r}} R \Big) \int_0^r f^*(s) ds + \int_r^{|\Omega|} \log\Big(\sqrt{\frac{4\pi}{s}} R \Big) f^*(s) ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\log\Big(\frac{4\pi e R^2}{r}\Big)\int_0^r f^*(s)ds + \frac{1}{2}\int_r^{|\Omega|}\log\Big(\frac{4\pi R^2}{s}\Big)f^*(s)ds \\
&= \frac{1}{2}\log\Big(\frac{e|\Omega|}{r}\Big)\int_0^r f^*(s)ds + \frac{1}{2}\int_r^{|\Omega|}\log\Big(\frac{e|\Omega|}{s}\Big)f^*(s)ds \\
&\qquad\qquad + \frac{1}{2}\log\Big(\frac{4\pi R^2}{|\Omega|}\Big)\int_0^{|\Omega|} f^*(s)ds.
\end{aligned}
$$

Since $\log\big(\frac{e|\Omega|}{r}\big)$ is a decreasing function, we have

$$
\begin{aligned}
\log\Big(\frac{e|\Omega|}{r}\Big) &\le \log^{1-p}\Big(\frac{e|\Omega|}{r}\Big)\log^p\Big(\frac{e|\Omega|}{s}\Big) \quad (0<s<r),\\
\log\Big(\frac{e|\Omega|}{s}\Big) &\le \log^{1-p}\Big(\frac{e|\Omega|}{r}\Big)\log^p\Big(\frac{e|\Omega|}{s}\Big) \quad (r<s<|\Omega|).
\end{aligned}
\tag{3.4}
$$

Applying the inequality (3.4) to (3.3), we obtain

$$
\begin{aligned}
(\Phi * |f|)^{**}(r) &\le \frac{1}{2}\log^{1-p}\Big(\frac{e|\Omega|}{r}\Big)\Big\{\int_0^{|\Omega|}\log^p\Big(\frac{e|\Omega|}{s}\Big)f^*(s)ds\Big\} \\
&\qquad + \frac{1}{2}\log\Big(\frac{4\pi R^2}{|\Omega|}\Big)\int_0^{|\Omega|} f^*(s)ds \\
&= \frac{1}{2}\log^{1-p}\Big(\frac{e|\Omega|}{r}\Big)\|f\|_{L(\log L)^p} + \frac{1}{2}\log\Big(\frac{4\pi R^2}{|\Omega|}\Big)\|f\|_{L^1}.
\end{aligned}
$$

Thus, we obtain

$$
u^*(r) \le \frac{1}{4\pi}\log^{1-p}\Big(\frac{e|\Omega|}{r}\Big)\|f\|_{L(\log L)^p} + \frac{1}{4\pi}\log\Big(\frac{4\pi R^2}{|\Omega|}\Big)\|f\|_{L^1}. \tag{3.5}
$$

Here, for any $0<\varepsilon<4\pi-\alpha$, there exists $0<r_\varepsilon$ such that, for $0<r<r_\varepsilon$,

$$
\frac{1}{4\pi}\log\Big(\frac{4\pi R^2}{|\Omega|}\Big)\|f\|_{L^1} < \frac{\varepsilon}{4\pi}\log^{1-p}\Big(\frac{e|\Omega|}{r}\Big)\|f\|_{L(\log L)^p}. \tag{3.6}
$$

The inequalities (3.1), (3.5), and (3.6) with (3.2), yield that

$$
\begin{aligned}
&\int_\Omega \exp\Big[\Big\{\alpha\frac{|u(x)|}{\|f\|_{L(\log L)^p}}\Big\}^{\frac{1}{1-p}}\Big]dx = \int_0^{|\Omega|}\exp\Big[\Big\{\alpha\frac{u^*(r)}{\|f\|_{L(\log L)^p}}\Big\}^{\frac{1}{1-p}}\Big]dr \\
&\le \int_0^{r_\varepsilon}\exp\Big[\Big\{\frac{\alpha}{4\pi}\log^{1-p}\Big(\frac{e|\Omega|}{r}\Big) + \frac{\varepsilon}{4\pi}\log^{1-p}\Big(\frac{e|\Omega|}{r}\Big)\Big\}^{\frac{1}{1-p}}\Big]dr + C \\
&\le \int_0^{r_\varepsilon}\exp\Big[\Big\{\frac{\alpha+\varepsilon}{4\pi}\log^{1-p}\Big(\frac{e|\Omega|}{r}\Big)\Big\}^{\frac{1}{1-p}}\Big]dr + C
\end{aligned}
$$

$$= \int_0^{r_\varepsilon} \left(\frac{e|\Omega|}{r}\right)^{(\frac{\alpha+\varepsilon}{4\pi})^{\frac{1}{1-p}}} dr + C \leq C(\alpha, p, \Omega).$$

This proves Proposition 1.1. □

We remark that, if $f \in L(\log L)^p(\Omega)$ and $0 \leq p < 1$ then u is not bounded in general. In fact, when $\Omega = B_1(0)$,

$$f(x) = \frac{1}{|x|^2 \log^2 \frac{e}{|x|}}, \quad u(x) = \log\log\frac{e}{|x|}$$

satisfies the problem (1.2) and $f \in L(\log L)^p(\Omega)$, $u \notin L^\infty(\Omega)$.

We consider the necessity of f being in $L\log L(\Omega)$ to the boundedness of u. We show that f being in $L\log L(\Omega)$ is unnecessary for the boundedness of u.

Proposition 3.1. *For $f \in L^1(\Omega)$, let u be a weak solution of* (1.2) *in the sense of distributions.*

(1) *If $f \notin L\log L(\Omega)$, $f \geq 0$ and there exists $x' \in \Omega$ such that $f(\cdot - x')$ is radially symmetric and non-increasing then u is not bounded.*
(2) *There exists $f \notin L\log L(\Omega)$ such that u is bounded.*

Remark 3.1.1. The assumption of radial symmetry in Proposition 3.1 (1) is essential. The result (1) implies that, for all radially symmetric and non-increasing functions at $x' \in \Omega$, the condition $f \in L\log L(\Omega)$ is necessary to the boundedness of u, while the result (2) implies that the condition $f \in L\log L(\Omega)$ is generally unnecessary to the boundedness of u.

Proof of Proposition 3.1. Proof of (1). Since Ω is an open set, for any $x' \in \Omega$, there exists $r > 0$ such that $B_r(x') \subset \Omega$. By the assumptions of Proposition 3.1, the function $f \notin L\log L$ has a singularity at x' and we see that

$$\int_{B_r(x')} \log\frac{2R}{|x'-y|} f(y)dy = \infty. \tag{3.7}$$

Now we define

$$f_r(x) := \begin{cases} f(x), & x \in B_r(x'), \\ 0, & x \notin B_r(x'), \end{cases}$$

$$v(x) := \frac{1}{2\pi}\int_\Omega \log\frac{2R}{|x-y|} f_r(y)dy$$

so that

$$-\Delta v = f_r \quad \text{in} \quad \mathbb{R}^2.$$

Since $0 \le f_r(x) \le f(x)$ and by the maximum principle, we have

$$\inf_{\Omega}(u-v) = \inf_{\partial\Omega}(u-v) = \sup_{\partial\Omega} v \ge 0;$$

namely we obtain $u \ge v$. From (3.7), v is not bounded and non-negative and thus, we conclude that u is not bounded.

Proof of (2). We assume for instance $\Omega = B_1(0)$. For sufficiently small $r > 0$, we define

$$A_r := \{(y_1, y_2) : 0 \le y_2 \le r\,,\ 0 \le y_1 \le e^{-1/y_2^2}\}, \qquad f(y) := e^{1/|y|^2}\chi_{A_r}(y).$$

From the maximum principle, we have

$$|u(x)| \le \bar{u}(x) := \frac{1}{2\pi}\int_{\Omega} \log\frac{2R}{|x-y|}|f(y)|dy.$$

We then prove that $f \in L^1(\Omega)$, $f \notin L\log L(\Omega)$ and $\bar{u}$ is bounded. First, we show $f \notin L\log L(\Omega)$. Indeed,

$$\|f\|_{L\log L} = \int_{\Omega}|f|\log(e+|f|)dy \ge \int_0^r\int_0^{e^{-1/y_2^2}} e^{1/|y|^2}\frac{1}{|y|^2}dy_1dy_2.$$

Here, changing variables $y_1 = s\tan\theta$, $y_2 = s$, we have

$$\begin{aligned}
&\int_0^r\int_0^{\tan^{-1}(\frac{1}{s}e^{-1/s^2})} \exp\Big[\frac{\cos^2\theta}{s^2}\Big]\frac{\cos^2\theta}{s^2}\frac{s}{\cos^2\theta}d\theta ds \\
&\quad\ge \int_0^r \tan^{-1}\Big(\frac{1}{s}e^{-1/s^2}\Big)\frac{1}{s}\exp\Big[\frac{\cos^2(\tan^{-1}(\frac{1}{s}e^{-1/s^2}))}{s^2}\Big]ds \\
&\quad\ge C\int_0^r \frac{1}{s}e^{-1/s^2}\frac{1}{s}\exp\Big[\frac{1}{s^2} - \frac{\sin^2(\frac{1}{s}e^{-1/s^2})}{s^2}\Big]ds \\
&\quad= C\int_0^r \frac{1}{s^2}\exp\Big[-\frac{\sin^2(\frac{1}{s}e^{-1/s^2})}{s^2}\Big]ds.
\end{aligned} \tag{3.8}$$

Since

$$\frac{\sin^2(\frac{1}{s}e^{-1/s^2})}{s^2} = \Big(\frac{\sin\left(\frac{1}{s}e^{-1/s^2}\right)}{\frac{1}{s}e^{-1/s^2}}\frac{e^{-1/s^2}}{s^2}\Big)^2,$$

we have

$$\lim_{s\to 0}\exp\Big[-\frac{\sin^2(\frac{1}{s}e^{-1/s^2})}{s^2}\Big] = 1.$$

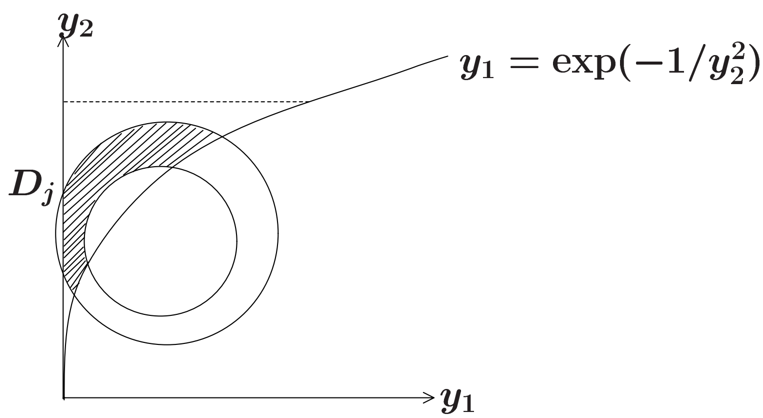

FIGURE 1. Figure of D_j

Thus, the right-hand side of (3.8) is unbounded; that is, $f \notin L\log L(\Omega)$. Next, we prove $f \in L^1(\Omega)$ in a similar argument:

$$\|f\|_{L^1} = \int_0^r \int_0^{e^{-1/y_2^2}} e^{1/|y|^2} dy_1 dy_2 = \int_0^r \int_0^{\tan^{-1}(\frac{1}{s}e^{-1/s^2})} e^{\frac{\cos^2\theta}{s^2}} \frac{s}{\cos^2\theta} d\theta ds$$
$$\leq \int_0^r \int_0^{\tan^{-1}(\frac{1}{s}e^{-1/s^2})} e^{\frac{1}{s^2}} \frac{s}{\cos^2\theta} d\theta ds = \int_0^r dr < \infty.$$

Thus, we conclude $f \in L^1(\Omega)$.

Finally, we prove $u \in L^\infty(\Omega)$. Since $|u| \leq \bar{u}$, it is enough to prove $\bar{u} \in L^\infty(\Omega)$. Moreover, since $f(x)$ has a singularity at the origin, we consider the case $0 < |x| < \frac{r}{2}$. To prove this, we divide A_r in the following way: for every $j \in \mathbb{N}$, we let

$$D_0(x) := \Big\{ y \in A_r : \frac{|y|}{2} < |x-y| \Big\},$$
$$D_j(x) := \Big\{ y \in A_r : \frac{|y|}{2^{j+1}} \leq |x-y| < \frac{|y|}{2^j} \Big\}.$$

Let

$$a_j := \frac{x_2}{1-1/4^j} - \frac{|x|}{(4^j-1)^{1/2}}, \quad b_j := \frac{x_2}{1-1/4^j} + \frac{|x|}{(4^j-1)^{1/2}}$$

and $E_j(x) := \{y \in A_r : a_j \le y_2 \le b_j\}$, so that $D_j(x) \subset E_j(x)$. Concerning the estimate of $\bar{u}$, we have

$$2\pi\bar{u}(x) = \int_\Omega |f(y)| \log \frac{2R}{|x-y|} dy \le \sum_{j=0}^{\infty} \int_{D_j} e^{1/|y|^2} \log \frac{2^{j+1}2R}{|y|} dy \tag{3.9}$$

$$\le \sum_{j=0}^{\infty} \Big(\int_{D_j} e^{1/|y|^2} \log \frac{2R}{|y|} dy + \int_{D_j} e^{1/|y|^2} \log 2^{j+1} dy \Big)$$

$$= \int_{A_r} e^{1/|y|^2} \log \frac{2R}{|y|} dy + \sum_{j=0}^{\infty} \int_{D_j} e^{1/|y|^2} \log 2^{j+1} dy.$$

Since

$$\int_{A_r} e^{1/|y|^2} \log \frac{2R}{|y|} dy \le \int_0^r \log \frac{2R}{s} ds = (1 + \log 2R) r - r \log r < \infty, \tag{3.10}$$

and

$$\int_{D_j} e^{1/|y|^2} dy \le \int_{E_j} e^{1/|y|^2} dy \le \int_{a_j}^{b_j} 1 ds = \frac{2|x|}{(4^j - 1)^{1/2}},$$

we have

$$\sum_{j=0}^{\infty} \int_{D_j} e^{1/|y|^2} \log 2^{j+1} dy \le C|x|. \tag{3.11}$$

From (3.9), (3.10) and (3.11), we conclude $u \in L^\infty(\Omega)$. □

4. Proof of Theorem 1.4 and Remark

The following lemma is a key estimate for the proof of Theorem 1.4.

Lemma 4.1 (a priori estimate). *Let $N \ge 3$, $f \in L(\log L)^p(\Omega)\,(0 < p < N-1)$. If u is a variational solution of (1.1), then*

$$u^*(r) \le \begin{cases} \dfrac{1}{N\omega_N^{\frac{1}{N-1}}} \|f\|_{L(\log L)^p}^{\frac{1}{N-1}} \Big(\log \dfrac{|\Omega|}{r} \Big)^{1-\frac{p}{N-1}}, & 0 \le p \le 1, \\ \Big(\dfrac{1}{p(1-\frac{p-1}{N-2})^{N-2}} \Big)^{\frac{1}{N-1}} \dfrac{1}{N\omega_N^{\frac{1}{N-1}}} \|f\|_{L(\log L)^p}^{\frac{1}{N-1}} \Big(\log \dfrac{|\Omega|}{r} \Big)^{1-\frac{p}{N-1}}, & 1 < p < N-1. \end{cases}$$

Proof of Lemma 4.1. By the representation formula Proposition 2.4 and Hölder's inequality, for every index a satisfying $(p-1)/(N-1) \le a < (N-2)/(N-1)$, we have

$$u^*(r) \le \frac{1}{N\omega_N^{\frac{1}{N-1}}} \int_r^{|\Omega|} \frac{1}{s}\Big(\int_0^s f^*(\eta)d\eta\Big)^{\frac{1}{N-1}} ds \tag{4.1}$$
$$= \frac{1}{N\omega_N^{\frac{1}{N-1}}} \int_r^{|\Omega|} \frac{1}{s^{\frac{N-2}{N-1}} \log^a \frac{|\Omega|}{s}} \Big(\frac{\log^{a(N-1)} \frac{|\Omega|}{s}}{s} \int_0^s f^*(\eta)d\eta\Big)^{\frac{1}{N-1}} ds$$
$$\le \frac{1}{N\omega_N^{\frac{1}{N-1}}} \Big(\int_r^{|\Omega|} \frac{ds}{s \log^{\frac{N-1}{N-2}a} \frac{|\Omega|}{s}}\Big)^{\frac{N-2}{N-1}} \Big(\int_r^{|\Omega|} \frac{\log^{a(N-1)} \frac{|\Omega|}{s}}{s} \int_0^s f^*(\eta)d\eta ds\Big)^{\frac{1}{N-1}}.$$

Here we estimate the right-hand side of (4.1):

$$\int_r^{|\Omega|} \frac{ds}{s \log^{\frac{N-1}{N-2}a} \frac{|\Omega|}{s}} = \frac{\big(\log \frac{|\Omega|}{r}\big)^{1-\frac{N-1}{N-2}a}}{1-\frac{N-1}{N-2}a}. \tag{4.2}$$

Exchanging the order of integration, it follows that

$$\int_r^{|\Omega|} \int_0^s \frac{\log^{a(N-1)} \frac{|\Omega|}{s}}{s} f^*(\eta)d\eta ds \tag{4.3}$$
$$\le \int_0^r \int_r^{|\Omega|} \frac{\log^{a(N-1)} \frac{|\Omega|}{s}}{s} f^*(\eta) ds d\eta + \int_r^{|\Omega|} \int_\eta^{|\Omega|} \frac{\log^{a(N-1)} \frac{|\Omega|}{s}}{s} f^*(\eta) ds d\eta$$
$$\le \frac{1}{1+a(N-1)} \Big\{ \Big(\log \frac{|\Omega|}{r}\Big)^{1+a(N-1)} \int_0^r f^*(\eta)d\eta$$
$$+ \int_r^{|\Omega|} \Big(\log \frac{|\Omega|}{\eta}\Big)^{1+a(N-1)} f^*(\eta)d\eta \Big\}$$
$$\le \frac{1}{1+a(N-1)} \Big(\log \frac{|\Omega|}{r}\Big)^{1+a(N-1)-p} \int_0^{|\Omega|} \log^p \frac{|\Omega|}{\eta} f^*(\eta)d\eta$$
$$\le \frac{1}{1+a(N-1)} \Big(\log \frac{|\Omega|}{r}\Big)^{1+a(N-1)-p} \|f\|_{L(\log L)^p}.$$

From (4.1), (4.2) and (4.3), we conclude that

$$u^*(r) \le \frac{1}{N\omega_N^{\frac{1}{N-1}}} \Big(\frac{1}{1-\frac{N-1}{N-2}a}\Big)^{\frac{N-2}{N-1}} \Big(\frac{1}{1+a(N-1)}\Big)^{\frac{1}{N-1}}$$

$$\times \|f\|_{L(\log L)^p}^{\frac{1}{N-1}} \Big(\log \frac{|\Omega|}{r}\Big)^{1-\frac{p}{N-1}}. \tag{4.4}$$

To obtain a slightly sharper estimate of (4.4), we take an infimum for a over $(p-1)/(N-1) \le a < (N-2)/(N-1)$. By an elementary calculation, we have

$$u^*(r) \le \begin{cases} \dfrac{1}{N\omega_N^{\frac{1}{N-1}}} \|f\|_{L(\log L)^p}^{\frac{1}{N-1}} \Big(\log \dfrac{|\Omega|}{r}\Big)^{1-\frac{p}{N-1}}, \qquad 0 \le p \le 1, \\ \Big(\dfrac{1}{p\Big(1-\frac{p-1}{N-2}\Big)^{N-2}}\Big)^{\frac{1}{N-1}} \dfrac{1}{N\omega_N^{\frac{1}{N-1}}} \|f\|_{L(\log L)^p}^{\frac{1}{N-1}} \Big(\log \dfrac{|\Omega|}{r}\Big)^{1-\frac{p}{N-1}}, \\ \qquad 1 < p < N-1. \end{cases}$$

Thus we complete the proof of Lemma 4.1. □

We obtain Theorem 1.4 from Lemma 4.1.

Proof of Theorem 1.4. The existence of an entropy solution is considered in [6] and [4]. An outline of the proof in [6], [4] is as follows.

First, we define $f_n(x)$ as $T_n\big(f(x)\big)$, where $T_n(\cdot)$ is defined by (1.4). Then the Dirichlet problem

$$\begin{cases} -\operatorname{div}\big(|\nabla u_n|^{N-2}\nabla u_n\big) = f_n, \qquad x \in \Omega, \\ \qquad\qquad u_n = 0, \qquad x \in \partial\Omega \end{cases} \tag{4.5}$$

has a unique variational solution u_n, and $f_n \to f$ in $L^1(\Omega)$ and almost everywhere.

Next, for some subsequence, $u_n \to u$, a unique entropy solution of (1.1). The convergence is understood as almost everywhere and locally in measure for the functions and their gradients.

We now apply Lemma 4.1 and the property of the rearrangement Proposition 2.1 to u_n, to see that

$$\begin{aligned} &\int_\Omega \exp\Big[\Big\{\frac{\alpha|u_n(x)|}{\|f\|_{L(\log L)^p}^{\frac{1}{N-1}}}\Big\}^{\frac{N-1}{N-1-p}}\Big]dx = \int_0^{|\Omega|} \exp\Big[\Big\{\frac{\alpha u_n^*(r)}{\|f\|_{L(\log L)^p}^{\frac{1}{N-1}}}\Big\}^{\frac{N-1}{N-1-p}}\Big]dr \\ &\le \int_0^{|\Omega|} \exp\Big[\Big(\frac{\alpha}{\beta}\Big)^{\frac{N-1}{N-1-p}} \log \frac{|\Omega|}{r}\Big]dr = \int_0^{|\Omega|} \Big(\frac{|\Omega|}{r}\Big)^{(\frac{\alpha}{\beta})^{\frac{N-1}{N-1-p}}} dr \\ &= \Big(1 - \Big(\frac{\alpha}{\beta}\Big)^{\frac{N-1}{N-1-p}}\Big)^{-1} |\Omega|. \end{aligned}$$

Thus the conclusion of Theorem 1.4 follows by Fatou's lemma. □

Remark 4.1.1. Recently, Misawa-Nagai-Ogawa ([17]) proved a similar result of a Brezis-Merle inequality [2] for N-Laplace parabolic equations.

We state the existence of solutions for the problem (1.1). Boccardo-Peral-Vazquez proved in [6] that if $f \in L\log L(\Omega)$ then the solution lies in the energy space $W_0^{1,N}(\Omega)$. In this case the entropy solution becomes the variational solution. Del Vecchio shows in [7] that if f is in the Lorentz space $L^{1,N'}(\Omega)$ (for the definition of $L^{1,N'}(\Omega)$, see [5] and [8]) then the solution becomes the variational solution, where $N' = N/(N-1)$.

We can also prove an existence theorem of a variational solution as a corollary of Lemma 4.1.

Proposition 4.2. *Let $f \in L(\log L)^{1/N'}(\Omega)$. Then there exists a unique variational solution of* (1.1).

Proof of Proposition 4.2. The uniqueness is obtained in [4]. We show the existence of the variational solution. Let f_n be the external force and u_n be a solution of (4.5) We will use Lemma 4.1 to prove the existence of a variational solution of problem (1.1) as a limit of u_n.

First, we prove that, for some subsequence of $\{u_n\}_{n=1}^{\infty}$, there exists $u \in W_0^{1,N}(\Omega)$ such that

$$u_n \to u \quad \text{weakly in } W_0^{1,N}(\Omega) \text{ and almost everywhere.}$$

Indeed, testing u_n in the equation (4.5), we have

$$\int_\Omega |\nabla u_n(x)|^N dx = \int_\Omega f_n(x)u_n(x)dx. \tag{4.6}$$

We apply Lemma 4.1 to the right-hand side of (4.6),

$$\begin{aligned}\int_\Omega f_n(x)u_n(x)dx &\le \int_0^{|\Omega|} f_n^*(r)u_n^*(r)dr \\ &= \int_0^{|\Omega|} f_n^*(r)\Big(\log\frac{e|\Omega|}{r}\Big)^{1/N'} u_n^*(r)\frac{1}{(\log\frac{e|\Omega|}{r})^{1/N'}}dr \\ &\le C\int_0^{|\Omega|} f^*(r)\Big(\log\frac{e|\Omega|}{r}\Big)^{1/N'}dr \le C',\end{aligned} \tag{4.7}$$

where C' is independent of n. From the estimate (4.7), we have the uniform bound $\|u_n\|_{W^{1,N}} \le C$. Thus, we conclude that for some subsequence $\{u_n\}_{n=1}^{\infty} \subset W_0^{1,N}(\Omega)$ there exists a weak limit $u \in W_0^{1,N}(\Omega)$ such that

$$u_n \to u \quad \text{weakly in } W_0^{1,N}(\Omega) \text{ and almost everywhere as } n \to \infty.$$

Moreover, u satisfies

$$u^*(r)\Big(\log\frac{e|\Omega|}{r}\Big)^{-1/N'}=\lim_{n\to\infty}u_n^*(r)\Big(\log\frac{e|\Omega|}{r}\Big)^{-1/N'}\le C. \tag{4.8}$$

By the following estimate:

$$\begin{aligned}&\Big|\int_\Omega\Big(|\nabla u(x)|^{N-2}\nabla u(x)\cdot\nabla v(x)-f(x)v(x)\Big)dx\Big| \qquad (4.9)\\ &=\Big|\int_\Omega\Big(\big(|\nabla u_n(x)|^{N-2}\nabla u_n(x)-|\nabla u(x)|^{N-2}\nabla u(x)\big)\cdot\nabla v(x)\\ &\qquad -(f(x)-f_n(x))v(x)\Big)dx\Big|\\ &\le\|u_n-u\|_{W_0^{1,N}(\Omega)}^{1/N'}\|v\|_{W_0^{1,N}(\Omega)}^{1/N}+C\|f_n-f\|_{L^1},\end{aligned}$$

the strong convergence of $u_n\to u$ implies that u is the variational solution of the problem (1.1). We then prove the strong convergence of $u_n\to u$ using Vitali's convergence theorem. We already know that u_n converges almost everywhere to u, so that we need to prove the following:

$$\lim_{n\to\infty}\int_\Omega f_n(x)\big(u_n(x)-u(x)\big)dx=0. \tag{4.10}$$

For any measurable set $E\subset\Omega$, we have

$$\begin{aligned}&\Big|\int_E f_n(x)(u_n(x)-u(x))dx\Big|\le\int_E|f(x)||u_n(x)-u(x)|dx\\ &\le\int_0^{|E|}f^*(r)\Big(\log\frac{e|\Omega|}{r}\Big)^{1/N'}\Big(\log\frac{e|\Omega|}{r}\Big)^{-1/N'}\big(u_n^*(r/2)+u^*(r/2)\big)dr\\ &\le C\int_0^{|E|}f^*(r)\Big(\log\frac{e|\Omega|}{r}\Big)^{1/N'}dr,\end{aligned}$$

by inequality (4.8) and the fact that $|f_n|\le|f|$. Since C is independent of n and E and $f\in L(\log L)^{1/N'}(\Omega)$,

$$\lim_{|E|\to0}\int_E f_n(x)(u_n(x)-u(x))dx=0.$$

Using Vitali's convergence theorem, we conclude

$$\lim_{n\to0}\int_\Omega f_n(x)(u_n(x)-u(x))dx=0. \tag{4.11}$$

Finally, since $\Delta_N u \in W^{-1,N'}(\Omega)$ and by the estimate (4.11), we obtain

$$\int_\Omega |\nabla(u_n - u)(x)|^N dx = \langle -\Delta_N u_n + \Delta_N u,\, u_n - u\rangle \tag{4.12}$$
$$= \langle -\Delta_N u_n,\, u_n - u\rangle + \langle \Delta_N u,\, u_n - u\rangle$$
$$= \int_\Omega f_n(x)(u_n(x) - u(x))dx + \langle \Delta_N u,\, u_n - u\rangle \to 0 \quad (n \to \infty).$$

This completes the proof of Proposition 4.2. □

We remark about Theorem 1.4 that, when $f \in L(\log L)^{1/N'}(\Omega)$, the entropy solution u becomes the variational solution (see Proposition 4.2). Since $L\log L(\Omega) \subset L(\log L)^{1/N'}(\Omega)$, this fact implies the result of [6]. It is not clear if the inclusion relation between $L(\log L)^{1/N'}(\Omega)$ and $L^{1,N'}(\Omega)$ or the inclusion between the existence result by Del Vecchio and the result of ours holds.

5. Proof of Theorem 1.6

The following lemma is a key estimate for the proof of Theorem 1.6.

Lemma 5.1. *Let $p > 0$, $k, N \in \mathbb{N}$, $k \geq 2$, $N \geq 3$,*

$$f \in L(\log L)^{N-1}(\log^{[2]} L)^{N-2} \cdots (\log^{[k-1]} L)^{N-2}(\log^{[k]} L)^p(\Omega).$$

Then there exists a unique weak solution of (1.1) *and $C = C(p,k,N) > 0$ such that*

$$u^*(r) \leq \begin{cases} C\Big(\log^{[k]}\dfrac{|\Omega|}{r}\Big)^{\frac{N-2-p}{N-1}}, & 0 < p < N-2, \\ C\Big(\log^{[k+1]}\dfrac{|\Omega|}{r}\Big)^{\frac{N-2}{N-1}}, & p = N-2, \\ C, & p > N-2. \end{cases} \tag{5.1}$$

Proof of Lemma 5.1. The heart of the proof is similar to the proof of Lemma 4.1. The existence and uniqueness is obtained in Proposition 4.2. By Talenti's representation formula (Proposition 2.4) and Hölder's inequality, we have

$$u^*(r) \leq \frac{1}{N\omega_N^{\frac{1}{N-1}}} \int_r^{|\Omega|} \frac{1}{s}\Big(\int_0^s f^*(\eta)d\eta\Big)^{\frac{1}{N-1}} ds \tag{5.2}$$
$$= \frac{1}{N\omega_N^{\frac{1}{N-1}}} \int_r^{|\Omega|} \Big(\frac{1}{s\log\frac{|\Omega|}{s}\cdots\log^{[k-1]}\frac{|\Omega|}{s}(\log^{[k]}\frac{|\Omega|}{s})^{\frac{p}{N-2}}}\Big)^{\frac{N-2}{N-1}}$$

$$\times\Big(\frac{(\log\frac{|\Omega|}{s}\cdots\log^{[k-1]}\frac{|\Omega|}{s})^{N-2}(\log^{[k]}\frac{|\Omega|}{s})^p}{s}\int_0^s f^*(\eta)d\eta\Big)^{\frac{1}{N-1}}ds$$
$$\le\frac{1}{N\omega_N^{\frac{1}{N-1}}}\Big(\int_r^{|\Omega|}\frac{1}{s\log\frac{|\Omega|}{s}\cdots\log^{[k-1]}\frac{|\Omega|}{s}(\log^{[k]}\frac{|\Omega|}{s})^{\frac{p}{N-2}}}ds\Big)^{\frac{N-2}{N-1}}$$
$$\times\Big(\int_r^{|\Omega|}\frac{(\log\frac{|\Omega|}{s}\cdots\log^{[k-1]}\frac{|\Omega|}{s})^{N-2}(\log^{[k]}\frac{|\Omega|}{s})^p}{s}\int_0^s f^*(\eta)d\eta ds\Big)^{\frac{1}{N-1}}.$$

Integrating (5.2) by parts, we have

$$\int_r^{|\Omega|}\frac{1}{s\log\frac{|\Omega|}{s}\cdots\log^{[k-1]}\frac{|\Omega|}{s}(\log^{[k]}\frac{|\Omega|}{s})^{\frac{p}{N-2}}}ds \tag{5.3}$$
$$\le\begin{cases} C\Big(\log^{[k]}\dfrac{|\Omega|}{r}\Big)^{\frac{N-2-p}{N-2}}, & 0<p<N-2,\\ C\Big(\log^{[k+1]}\dfrac{|\Omega|}{r}\Big), & p=N-2,\\ C, & p>N-2\end{cases}$$

and

$$\int_r^{|\Omega|}\frac{(\log\frac{|\Omega|}{s}\cdots\log^{[k-1]}\frac{|\Omega|}{s})^{N-2}(\log^{[k]}\frac{|\Omega|}{s})^p}{s}\int_0^s f^*(\eta)d\eta ds$$
$$=\int_0^r\int_r^{|\Omega|}\frac{(\log\frac{|\Omega|}{s}\cdots\log^{[k-1]}\frac{|\Omega|}{s})^{N-2}(\log^{[k]}\frac{|\Omega|}{s})^p}{s}f^*(\eta)dsd\eta \tag{5.4}$$
$$+\int_r^{|\Omega|}\int_\eta^{|\Omega|}\frac{(\log\frac{|\Omega|}{s}\cdots\log^{[k-1]}\frac{|\Omega|}{s})^{N-2}(\log^{[k]}\frac{|\Omega|}{s})^p}{s}f^*(\eta)dsd\eta.$$

Since

$$\int_r^{|\Omega|}\frac{(\log\frac{|\Omega|}{s}\cdots\log^{[k-1]}\frac{|\Omega|}{s})^{N-2}(\log^{[k]}\frac{|\Omega|}{s})^p}{s}ds$$
$$\sim\log^{N-1}\frac{|\Omega|}{r}\Big(\log^{[2]}\frac{|\Omega|}{r}\cdots\log^{[k-1]}\frac{|\Omega|}{r}\Big)^{N-2}(\log^{[k]}\frac{|\Omega|}{r})^p,$$

and by equality (5.4), we obtain

$$\int_r^{|\Omega|}\frac{(\log\frac{|\Omega|}{s}\cdots\log^{[k-1]}\frac{|\Omega|}{s})^{N-2}(\log^{[k]}\frac{|\Omega|}{s})^p}{s}\int_0^s f^*(\eta)d\eta ds \tag{5.5}$$
$$\le C\log^{N-1}\frac{|\Omega|}{r}\Big(\log^{[2]}\frac{|\Omega|}{r}\cdots\log^{[k-1]}\frac{|\Omega|}{r}\Big)^{N-2}(\log^{[k]}\frac{|\Omega|}{r})^p\int_0^r f^*(\eta)d\eta$$

$$+ C' \int_r^{|\Omega|} \log^{N-1} \frac{|\Omega|}{\eta} \Big(\log^{[2]} \frac{|\Omega|}{\eta} \cdots \log^{[k-1]} \frac{|\Omega|}{\eta}\Big)^{N-2} (\log^{[k]} \frac{|\Omega|}{\eta})^p f^*(\eta) d\eta$$
$$\leq C'' \|f\|_{L(\log L)^{N-1}(\log^{[2]} L)^{N-2} \cdots (\log^{[k-1]} L)^{N-2} (\log^{[k]} L)^p}.$$

Inequalities (5.3), (5.5) and (5.2) imply (5.1). Thus we complete the proof of Lemma 5.1. □

Proof of Theorem 1.6. We start with the following inequality:

$$\alpha \log^{[k]} \frac{|\Omega|}{r} \leq \log^{[k]} \Big(\frac{|\Omega|}{r}\Big)^\alpha \tag{5.6}$$

for all $0 < \alpha \leq 1$. Indeed, by definition of $\log^{[k]}$, we have

$$\alpha \log^{[k]} \frac{|\Omega|}{r} = \log \Big(1 + \log^{[k-1]} \frac{|\Omega|}{r}\Big)^\alpha.$$

Since $(1+x)^\alpha \leq 1 + \alpha x$ for all $0 < \alpha \leq 1$ and $0 < x$, we obtain

$$\alpha \log^{[k]} \frac{|\Omega|}{r} \leq \log\big(1 + \alpha \log^{[k-1]} \frac{|\Omega|}{r}\big).$$

If we repeat this argument, (5.6) is proved.

The assertion of (3) in Theorem 1.6 is an immediate consequence of Lemma 5.1. We show (1) and (2) in Theorem 1.6.

Proof of Theorem 1.6 (1). Let $0 < p < N-2$. By Lemma 5.1, we have

$$\int_\Omega e_k \Big[\Big(\frac{\alpha |u(x)|}{\|f\|_{L(\log L)^{N-1}(\log^{[2]} L)^{N-2} \cdots (\log^{[k-1]} L)^{N-2} (\log^{[k]} L)^p}}\Big)^{\frac{N-1}{N-2-p}}\Big] dx$$
$$= \int_0^{|\Omega|} e_k \Big[\Big(\frac{\alpha u^*(r)}{\|f\|_{L(\log L)^{N-1}(\log^{[2]} L)^{N-2} \cdots (\log^{[k-1]} L)^{N-2} (\log^{[k]} L)^p}}\Big)^{\frac{N-1}{N-2-p}}\Big] dr$$
$$\leq \int_0^{|\Omega|} e_k \Big[\Big(\alpha C \Big(\log^{[k]} \frac{|\Omega|}{r}\Big)^{\frac{N-2-p}{N-1}}\Big)^{\frac{N-1}{N-2-p}}\Big] dr.$$

Choosing $\alpha < 1/C$, and using (5.6), we have

$$\int_\Omega e_k \Big[\Big(\frac{\alpha |u(x)|}{\|f\|_{L(\log L)^{N-1}(\log^{[2]} L)^{N-2} \cdots (\log^{[k-1]} L)^{N-2} (\log^{[k]} L)^p}}\Big)^{\frac{N-1}{N-2-p}}\Big] dx$$
$$\leq \int_0^{|\Omega|} e_k \Big[\log^{[k]} \Big(\frac{|\Omega|}{r}\Big)^{(\alpha C)^{(N-1)/(N-2-p)}}\Big] dr$$
$$= \int_0^{|\Omega|} \Big(\frac{|\Omega|}{r}\Big)^{(\alpha C)^{(N-1)/(N-2-p)}} dr < \infty.$$

Proof of Theorem 1.6 (2). Let $p = N-2$. By Lemma 5.1 and (5.6),

$$\begin{aligned}
&\int_\Omega e_{k+1}\Big[\Big(\frac{\alpha|u(x)|}{\|f\|_{L(\log L)^{N-1}(\log^{[2]}L)^{N-2}\cdots(\log^{[k]}L)^{N-2}}}\Big)^{\frac{N-1}{N-2}}\Big]dx \\
&\quad = \int_0^{|\Omega|} e_{k+1}\Big[\Big(\frac{\alpha u^*(r)}{\|f\|_{L(\log L)^{N-1}(\log^{[2]}L)^{N-2}\cdots(\log^{[k]}L)^{N-2}}}\Big)^{\frac{N-1}{N-2}}\Big]dx \\
&\quad = \int_0^{|\Omega|} e_{k+1}\Big[\Big(\alpha C\Big(\log^{[k+1]}\frac{|\Omega|}{r}\Big)^{\frac{N-2}{N-1}}\Big)^{\frac{N-1}{N-2}}\Big]dx \\
&\quad \le \int_0^{|\Omega|} e_{k+1}\Big[\log^{[k+1]}\Big(\frac{|\Omega|}{r}\Big)^{(\alpha C)^{(N-1)/(N-2)}}\Big]dr \\
&\quad = \int_0^{|\Omega|}\Big(\frac{|\Omega|}{r}\Big)^{(\alpha C)^{(N-1)/(N-2)}} dr < \infty
\end{aligned}$$

for small α. This completes the proof of Theorem 1.6. □

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