Differential and Integral Equations Volume 22, Numbers 5–6 (2009), 401–410

GRADIENT ESTIMATES FOR THE HEAT EQUATION IN THE EXTERIOR DOMAINS UNDER THE NEUMANN BOUNDARY CONDITION

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(Submitted by: Takashi Suzuki)

Abstract. We consider the Cauchy-Neumann problem for the heat equation in the exterior domain Ω of a compact set in $\mathbf{R}^N$ ($N \geq 2$). In this paper we give an estimate of the L^∞-norm of the gradient of the solutions.

1. Introduction

Let $N \geq 2$ and Ω be a smooth domain in $\mathbf{R}^N$ such that $K \equiv \mathbf{R}^N \setminus \Omega$ is a compact set in $\mathbf{R}^N$. In this paper we consider the Cauchy-Neumann problem for the heat equation,

$$\begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0,\infty), \\ \partial_\nu u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = \varphi(x) & \text{in} \quad \Omega, \quad \varphi \in L^p(\Omega), \end{cases} \tag{1.1}$$

and study the decay rate of the L^∞-norm of the gradient of the solution of (1.1) as $t \to \infty$. Here $1 \leq p \leq \infty$, $\partial_\nu = \partial/\partial\nu$, and ν is the outer unit normal vector to $\partial\Omega$.

For the case when $\Omega = \mathbf{R}^N$ or $\mathbf{R}^N_+$, the explicit representation of the fundamental solution of the heat equation implies that the solution u of (1.1) satisfies the inequality

$$\|(\nabla_x u)(t)\|_{L^\infty(\Omega)} \leq C_1 t^{-\frac{N}{2p}-\frac{1}{2}} \|\varphi\|_{L^p(\Omega)}, \quad t > 0, \tag{1.2}$$

for some constant C_1. Furthermore, it it well known that, if Ω is a smooth domain in $\mathbf{R}^N$, for any $T > 0$, there exists a constant C_2 such that

$$\|(\nabla_x u)(t)\|_{L^\infty(\Omega)} \leq C_2 t^{-\frac{N}{2p}-\frac{1}{2}} \|\varphi\|_{L^p(\Omega)}, \quad t \in (0,T). \tag{1.3}$$

Accepted for publication: October 2008.
AMS Subject Classifications: 35K05, 35K20.

This inequality is proved by the scale argument and the standard regularity theorem for parabolic equations (see e.g. Theorem 10.1 of Chapter IV in [8]). It is also proved by Theorem 3.1 in [1] and the L^p-L^q estimate for the heat semigroup (see e.g. (2.4)). Here we remark that the constants C_1 and C_2 are independent of p. (For recent results on the gradient estimates such as (1.3), see [1]–[4] and references therein.)

On the other hand, Li and Yau [9] studied the heat equation on Riemannian manifolds, and, in particular, proved that, if Ω is a convex domain in $\mathbf{R}^N$, any nonnegative solution u of (1.1) satisfies the inequality

$$\frac{|\nabla_x u|^2}{u^2} - \frac{\partial_t u}{u} \le C_3 \frac{1}{t}, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.4}$$

for some constant C_3. This inequality (1.4) with some fundamental inequalities for the heat equation gives the inequality (1.2) for the case when Ω is convex. (We remark that the inequality (1.4) holds for all sufficiently small $t > 0$ without the convexity of the domain Ω (see [12]).) On the other hand, Shibata and Shimizu [11] studied the decay properties of the Stokes semigroup under the Neumann boundary condition. Their arguments are applicable to the heat equation, and we see that the inequality (1.2) holds for the case when $N \ge 3$ and Ω is the exterior domain of a compact set. Recently, the author of this paper and Kabeya [7] considered the initial-boundary-value problem for the heat equation with a radially symmetric potential V in the exterior domain of a ball ($N \ge 2$),

$$\partial_t u = \Delta u - V(|x|)u, \tag{1.5}$$

where $V(r) = \omega r^{-2}(1 + o(1))$ as $r \to \infty$ for some constant $\omega \ge 0$. They proved that the inequality (1.2) does not necessarily hold for (1.5), and obtained the optimal decay rate of derivatives of the solutions by using the shape of the harmonic functions for $\Delta - V$. In particular, under the Neumann boundary condition, they proved that the inequality (1.2) holds if and only if

$$\text{(i)} \quad \omega \ge N-1 \qquad \text{or} \qquad \text{(ii)} \quad V(r) \equiv 0 \quad \text{on} \quad [0,\infty). \tag{1.6}$$

In this paper we study the decay rate of the gradient of the solution of (1.1) without the convexity or the radial symmetry of the domain Ω, and prove the inequality (1.2) for the case when $N \ge 2$ and Ω is the exterior smooth domain of a compact set. Our proof of the inequality (1.2) is completely different from the ones in [7] and [11], and is applicable to the case $N = 2$ (compare with [11]). Furthermore, the result is an extension of the result of [7] for the case (1.6)-(ii).

The main result of this paper is the following.

Theorem 1.1. *Let $N \geq 2$ and Ω be a smooth domain in $\mathbf{R}^N$ such that $K \equiv \mathbf{R}^N \setminus \Omega$ is a compact set in $\mathbf{R}^N$. Let u be a solution of the Cauchy-Neumann problem* (1.1) *such that $\varphi \in L^p(\Omega)$ with $1 \leq p \leq \infty$. Then there exists a constant C, independent of φ and p, such that*

$$\|(\nabla_x u)(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N}{2p}-\frac{1}{2}}\|\varphi\|_{L^p(\Omega)}, \quad t > 0. \tag{1.7}$$

Remark 1.1. Assume the same conditions as in Theorem 1.1. By the standard regularity theorems for parabolic equations and (1.7), we see that for any $j = 2, 3, \dots$, there exists a constant C, independent of p, such that

$$\|(\nabla_x^j u)(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N}{2p}-\frac{1}{2}}\|\varphi\|_{L^p(\Omega)}, \quad t \geq 1. \tag{1.8}$$

If Ω is the exterior domain of a ball and $1 \leq p < \infty$, the inequality (1.8) gives the optimal decay rate of $\|(\nabla_x^j u)(t)\|_{L^\infty(\Omega)}$ as $t \to \infty$ for some initial data $\varphi \in L^p(\Omega)$. See [7].

We prove Theorem 1.1 by using the rescaling argument used in [5] (see also (3.4)), which is useful for the study of the large-time behavior of the solutions of the heat equation and semilinear heat equations under the exponentially decaying condition on the initial data. For our problem (1.1), the initial data φ does not necessarily decay exponentially, however we can apply the rescaling argument to the function $v = G(t)\varphi - \Gamma(t)\varphi$, where

$$G(t)\varphi = u(t), \quad \Gamma(t)\varphi = (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} \exp\Big(-\frac{|x-y|^2}{4t}\Big)\varphi(y)dy. \tag{1.9}$$

Furthermore, in order to overcome the difficulty particular to the two-dimensional case, we introduce a quantity A_p and recall an estimate of the first eigenvalue of the operator $L\phi = \rho^{-1}\mathrm{div}(\rho\nabla_y\phi)$ with $\rho(y) = \exp(|y|^2/4)$ (see (2.8)). Then we can obtain an estimate of A_p, and prove that $\|v(t)\|_{L^\infty(\Omega)} = O(t^{-\frac{N}{2p}-\frac{1}{2}})$ as $t \to \infty$. This together with the standard regularity theorems for parabolic equations gives Theorem 1.1.

2. PRELIMINARIES

In this section we first recall some inequalities for the heat semigroups $G(t)$ and $\Gamma(t)$. Let $1 \leq p \leq q \leq \infty$. By (1.9), there exists a constant C_1 such that

$$\|\Gamma(t)\varphi\|_{L^q(\mathbf{R}^N)} \leq C_1 t^{-\frac{N}{2}(p^{-1}-q^{-1})}\|\varphi\|_{L^p(\mathbf{R}^N)}, \tag{2.1}$$

$$\|\nabla_x\Gamma(t)\varphi\|_{L^\infty(\mathbf{R}^N)} \le C_1 t^{-\frac{N}{2p}-\frac{1}{2}}\|\varphi\|_{L^p(\mathbf{R}^N)}, \tag{2.2}$$

for all $t > 0$. Next, let $G = G(x, y, t)$ be the Neumann heat kernel on the exterior domain Ω of a compact set. Then, by [12], there exists a positive constant C_2 such that

$$0 < G(x, y, t) \le C_2 t^{-\frac{N}{2}} \exp\Big(-\frac{|x-y|^2}{C_2 t}\Big), \quad x, y \in \Omega,\ t > 0. \tag{2.3}$$

This together with the Young inequality implies

$$\|G(t)\varphi\|_{L^q(\Omega)} \le C_3 t^{-\frac{N}{2}(p^{-1}-q^{-1})}\|\varphi\|_{L^p(\Omega)}, \quad t > 0, \tag{2.4}$$

for some constant C_3. Here we remark that the constants C_1 and C_3 are independent of p and q.

Next we give some inequalities for the function $v(t) = G(t)\varphi - \Gamma(t)\varphi$. By the definition of v, the function v satisfies

$$\begin{cases} \partial_t v = \Delta v & \text{in} \quad \Omega \times (0,\infty), \\ \partial_\nu v(x,t) = -\partial_\nu \Gamma(t)\varphi & \text{on} \quad \partial\Omega \times (0,\infty), \\ v(x,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.5}$$

Furthermore, by (2.2), for any $k, l = 0, 1, 2, \ldots$, there exists a constant C_4, independent of p, such that

$$\begin{aligned} \|\partial_t^l \partial_\nu v(t)\|_{C^k(\partial\Omega)} &= \|\partial_t^l \partial_\nu \Gamma(t)\varphi\|_{C^k(\partial\Omega)} \\ &\le C_4 t^{-\frac{N}{2p}-\frac{1}{2}-l}(1 + t^{-\frac{k}{2}})\|\varphi\|_{L^p(\Omega)} \quad \text{on} \quad \partial\Omega \times (0,\infty). \end{aligned} \tag{2.6}$$

On the other hand, by Green's formula and (2.5), we have

$$v(x,t) = \int_\Omega G\Big(x, y, \frac{t}{2}\Big) v\Big(y, \frac{t}{2}\Big) dy + \int_{\frac{t}{2}}^t \int_{\partial\Omega} G(x, y, t-s)\partial_\nu v(y,s)\, d\sigma ds$$

for all $t > 0$. Then, by (2.3) and (2.6), there exists a constant C_5, independent of p, such that

$$\|v(t)\|_{L^\infty(\Omega)} \le C_5 t^{-\frac{N}{2}}\|v(t/2)\|_{L^1(\Omega)} + C_5 t^{-\frac{N}{2p}-\frac{1}{2}}\|\varphi\|_{L^p(\Omega)}, \quad t > 0. \tag{2.7}$$

Next we recall a result on the eigenvalue problem for the operator

$$L\phi = \frac{1}{\rho}\operatorname{div}(\rho\nabla_y\phi), \quad \rho(y) = e^{|y|^2/4}.$$

Let $R > 0$ and $D(s) = \{|y| > Re^{-s/2}\}$, and consider the eigenvalue problem

$$(P) \quad -L\phi = \lambda\phi \quad \text{in} \quad D(s), \quad \partial_\nu\phi = 0 \quad \text{on} \quad \partial D(s), \quad \phi \in H^1(D(s), \rho dy).$$

Then there exists the first eigenvalue $\lambda(s)$ of the problem (P), and $\lambda(s)$ is characterized by

$$\lambda(s) = \inf \left\{ \int_{D(s)} |\nabla\phi|^2 \rho dy \ : \ \int_{D(s)} \phi^2 \rho dy = 1 \right\} \geq \frac{N}{2}. \tag{2.8}$$

This inequality is proved by use of the radial symmetry of the domain $D(s)$ (see Lemma 3.2 in [6]).

3. Proof of Theorem 1.1

We first consider the case $4N \leq p \leq \infty$ and assume $\varphi \in C_0(\Omega)$ to prove Theorem 1.1. For any $q \geq 1$, the inequalities (2.1) and (2.4) imply that

$$\|v(t)\|_{L^\infty(\Omega)} \leq c_1 t^{-\frac{N}{2q}} \|\varphi\|_{L^q(\Omega)}, \quad t > 0, \tag{3.1}$$

for some constant c_1. By (3.1) with $q = 1$ and $p \geq 4N > 2N$, we can put

$$A_p \equiv \sup_{t>0} t^{\frac{N}{2p}+\frac{1}{2}} \|v(t)\|_{L^\infty(\Omega)} / \|\varphi\|_{L^p(\Omega)} < \infty. \tag{3.2}$$

Then, by (2.5), (2.6), and (3.2), we can apply the standard regularity theorems for parabolic equations (for example, see Chapters III and IV in [8], or Chapter IV in [10]), and see that there exists a constant c_2 such that

$$\|(\nabla_x v)(t)\|_{L^\infty(\Omega)} \leq c_2(1 + A_p) t^{-\frac{N}{2p}-\frac{1}{2}} \|\varphi\|_{L^p(\Omega)}, \quad t \geq 1,$$

where c_2 is a constant independent of A_p. This inequality, together with (1.3) and (2.2), implies that

$$\|(\nabla_x v)(t)\|_{L^\infty(\Omega)} \leq c_3(1 + A_p) t^{-\frac{N}{2p}-\frac{1}{2}} \|\varphi\|_{L^p(\Omega)}, \quad t > 0, \tag{3.3}$$

for some constant c_3.

Next we put

$$w(y,s) = (1+t)^{\frac{N}{2p}} v(x,t), \quad y = (1+t)^{-\frac{1}{2}} x, \quad s = \log(1+t), \tag{3.4}$$

$$\Omega(s) = e^{-s/2}\Omega, \quad W = \bigcup_{s>0} (\Omega(s) \times \{s\}), \quad \partial W = \bigcup_{s>0} (\partial\Omega(s) \times \{s\}).$$

Then w satisfies

$$\partial_s w = \frac{1}{\rho} \operatorname{div}(\rho \nabla_y w) + \frac{N}{2p} w \quad \text{in} \quad W, \qquad w(y,0) = 0 \quad \text{in} \quad \Omega. \tag{3.5}$$

Furthermore, by (2.6), (3.1) with $q = p$, and (3.2), we have

$$|\partial_\nu w(y,s)| \leq c_4 (1 + s^{-1})^{\frac{N}{2p}+\frac{1}{2}} \|\varphi\|_{L^p(\Omega)} \quad \text{on} \quad \partial W, \tag{3.6}$$

$$|w(y,s)| \le c_4(1+s^{-1})^{\frac{N}{2p}}\|\varphi\|_{L^p(\Omega)} \quad \text{in} \quad W, \tag{3.7}$$

$$|w(y,s)| \le c_4(1+s^{-1})^{\frac{N}{2p}+\frac{1}{2}}A_p e^{-s/2}\|\varphi\|_{L^p(\Omega)} \quad \text{in} \quad W, \tag{3.8}$$

where c_4 is a constant. Then we prove the following two lemmas.

Lemma 3.1. *Let $\varphi \in C_0(\Omega)$, $4N \le p \le \infty$, and w be the function defined by* (3.4)*. Then, for any $s > 0$,*

$$w(s) \in H^1(\Omega(s), \rho dy). \tag{3.9}$$

Furthermore, there exists a constant C, independent of φ and p, such that

$$\sup_{0\le s\le 1}\int_{\Omega(s)} w(s)^2 \rho dy \le C\|\varphi\|^2_{L^p(\Omega)}. \tag{3.10}$$

Proof. Since $\varphi \in C_0(\Omega)$, by (1.9) and (2.3), we have (3.9). Next, by (3.5) and (3.7), there exists a constant C_1 such that

$$\begin{aligned}
\frac{d}{ds}\int_{\Omega(s)} w(s)^2\rho dy &= -\frac{1}{2}\int_{\partial\Omega(s)} w^2\rho\,(y\cdot\nu)d\sigma + 2\int_{\Omega(s)} w w_s \rho dy \qquad (3.11)\\
&\le C_1 e^{-Ns/2}(1+s^{-1})^{\frac{N}{p}}\|\varphi\|^2_{L^p(\Omega)}\\
&\qquad +2\int_{\Omega(s)} w\,\mathrm{div}(\rho\nabla w)dy + \frac{N}{p}\int_{\Omega(s)} w^2\rho dy\\
&\le C_1 e^{-Ns/2}(1+s^{-1})^{\frac{N}{p}}\|\varphi\|^2_{L^p(\Omega)} + 2\int_{\partial\Omega(s)} w\partial_\nu w\rho d\sigma\\
&\qquad -2\int_{\Omega(s)} |\nabla w|^2\rho dy + \frac{N}{p}\int_{\Omega(s)} w^2\rho dy
\end{aligned}$$

for all $s > 0$. Furthermore, by (3.6), (3.7), and (3.11), there exists a constant C_2 such that

$$\begin{aligned}
\frac{d}{ds}\int_{\Omega(s)} w(s)^2\rho dy &\le C_1 e^{-Ns/2}(1+s^{-1})^{\frac{N}{p}}\|\varphi\|^2_{L^p(\Omega)} \qquad (3.12)\\
&\quad + C_2 e^{-(N-1)s/2}(1+s^{-1})^{\frac{N}{p}+\frac{1}{2}}\|\varphi\|^2_{L^p(\Omega)} + \frac{N}{p}\int_{\Omega(s)} w^2\rho dy
\end{aligned}$$

for all $s > 0$. Since

$$\frac{N}{p}+\frac{1}{2} \le \frac{1}{4}+\frac{1}{2} < 1, \qquad w(y,0) = 0 \quad \text{in} \quad \mathbf{R}^N,$$

by (3.12), we have (3.10), and the proof of Lemma 3.1 is complete. □

Lemma 3.2. *Assume the same conditions as in Lemma* 3.1. *There exists a constant C, independent of φ and p, such that*

$$\int_{\Omega(s)} w^2 \rho dy \le C(1+A_p)e^{-s}\|\varphi\|^2_{L^p(\Omega)}, \quad s \ge 0. \tag{3.13}$$

Proof. Let $R>0$ such that $K \subset B(0,R)$. For any $s \ge 0$, we put $D(s) = \mathbf{R}^N \setminus B(0, Re^{-s/2})$. Since $D(s) \subset \Omega(s)$, by (2.8), we have

$$\int_{\Omega(s)} |\nabla w|^2 \rho dy \ge \int_{D(s)} |\nabla w|^2 \rho dy \ge \frac{N}{2}\int_{D(s)} w^2 \rho dy, \quad s>0. \tag{3.14}$$

Furthermore, by (3.7), there exists a constant C_1 such that

$$\begin{aligned}\int_{D(s)} w^2\rho dy &= \int_{\Omega(s)} w^2 \rho dy - \int_{\Omega(s)\setminus D(s)} w^2 \rho dy \\ &\ge \int_{\Omega(s)} w^2 \rho dy - C_1 e^{-Ns/2}\|\varphi\|^2_{L^p(\Omega)}\end{aligned}$$

for all $s \ge 1$. This together with (3.14) implies

$$\int_{\Omega(s)} |\nabla w|^2 \rho dy \ge \frac{N}{2}\int_{\Omega(s)} w^2\rho dy - \frac{C_1 N}{2} e^{-Ns/2}\|\varphi\|^2_{L^p(\Omega)}, \quad s \ge 1. \tag{3.15}$$

On the other hand, by (3.6), (3.8), and (3.11), there exists a constant C_2 such that

$$\begin{aligned}&\frac{d}{ds}\int_{\Omega(s)} w(s)^2 \rho dy \le C_2 e^{-Ns/2}\|\varphi\|^2_{L^p(\Omega)} \\ &+C_2 A_p e^{-Ns/2}\|\varphi\|^2_{L^p(\Omega)} - 2\int_{\Omega(s)} |\nabla w|^2\rho dy + \frac{N}{p}\int_{\Omega(s)} w^2 \rho dy\end{aligned}$$

for all $s \ge 1$. Then, by (3.15), there exists a constant C_3 such that

$$\begin{aligned}&\frac{d}{ds}\int_{\Omega(s)} w(s)^2\rho dy \\ &\le -N(1-p^{-1})\int_{\Omega(s)} w(s)^2 \rho dy + C_3 A_p e^{-Ns/2}\|\varphi\|^2_{L^p(\Omega)}\end{aligned} \tag{3.16}$$

for all $s \ge 1$. Since $N \ge 2$ and $N(1-p^{-1}) \ge 1$, the inequality (3.16) together with (3.10) implies (3.13), and the proof of Lemma 3.2 is complete. □

We go back to the proof of Theorem 1.1 for the case $4N \le p \le \infty$. By (3.13) and the Hölder inequality, there exists a constant c_5 such that

$$\begin{aligned}\int_{\Omega(s)} |w|dy &\le \Big(\int_{\Omega(s)} |w|^2\rho dy\Big)^{1/2}\Big(\int_{\Omega(s)} e^{-|y|^2/4}dy\Big)^{1/2} \\ &\le c_5(1+A_p)^{1/2}e^{-s/2}\|\varphi\|_{L^p(\Omega)}\end{aligned}$$

for all $s \ge 0$. Then, by (3.4), we have

$$\|v(t)\|_{L^1(\Omega)} \le c_5(1+A_p)^{1/2}(1+t)^{-\frac{N}{2p}+\frac{N}{2}-\frac{1}{2}}\|\varphi\|_{L^p(\Omega)}, \quad t > 0.$$

Therefore, by (2.7), there exists a constant c_6 such that

$$\|v(t)\|_{L^\infty(\Omega)} \le c_6(1+A_p)^{1/2}t^{-\frac{N}{2p}-\frac{1}{2}}\|\varphi\|_{L^p(\Omega)}$$

for all $t \ge 1$. This together with (3.1) with $q = p$ implies that

$$\begin{aligned}A_p &\le \sup_{0<t<1} t^{\frac{N}{2p}+\frac{1}{2}}\frac{\|v(t)\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^p(\Omega)}} + \sup_{t\ge 1} t^{\frac{N}{2p}+\frac{1}{2}}\frac{\|v(t)\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^p(\Omega)}} \\ &\le c_7 + c_6(1+A_p)^{1/2}\end{aligned}$$

for some constant c_7. Therefore there exists a constant c_8, independent of $\|\varphi\|_{L^p(\Omega)}$, such that $A_p \le c_8$. Then, by (3.3), there exists a constant c_9 such that

$$\|(\nabla_x v)(t)\|_{L^\infty(\Omega)} \le c_9 t^{-\frac{N}{2p}-\frac{1}{2}}\|\varphi\|_{L^p(\Omega)}, \quad t > 0. \tag{3.17}$$

Furthermore, by (2.2) and (3.17), there exists a constant c_{10} such that

$$\|(\nabla_x u)(t)\|_{L^\infty(\Omega)} \le c_{10} t^{-\frac{N}{2p}-\frac{1}{2}}\|\varphi\|_{L^p(\Omega)}, \quad t > 0, \tag{3.18}$$

for all $\varphi \in C_0(\Omega)$. Here we remark that the constant c_{10} is independent of p. Therefore, since $C_0(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$, we obtain the inequality (1.7) for $4N \le p < \infty$.

Next we prove the inequality (1.7) with $p = \infty$. For $n = 1, 2, \ldots$, let χ_n be the characteristic function of the ball $B(0, n)$. Put $\varphi_n = \varphi\chi_n$ and $u_n = G(t)\varphi_n$. Then, since $\varphi_n \in L^p(\Omega)$ for all $p \ge 1$, by (1.7) with $4N \le p < \infty$, there exists a constant c_{11}, independent of p, such that

$$\|(\nabla_x u_n)(t)\|_{L^\infty(\Omega)} \le c_{11} t^{-\frac{N}{2p}-\frac{1}{2}}\|\varphi_n\|_{L^p(\Omega)}$$

for all $t > 0$, where $4N \le p < \infty$. This implies

$$\begin{aligned}\|(\nabla_x u_n)(t)\|_{L^\infty(\Omega)} &\le c_{11}\lim_{p\to\infty} t^{-\frac{N}{2p}-\frac{1}{2}}\|\varphi_n\|_{L^p(\Omega)} \\ &\le c_{11} t^{-\frac{1}{2}}\|\varphi_n\|_{L^\infty(\Omega)} \le c_{11} t^{-\frac{1}{2}}\|\varphi\|_{L^\infty(\Omega)}\end{aligned} \tag{3.19}$$

for all $t > 0$. On the other hand, by (2.3), we easily see that

$$\lim_{n\to\infty} \|u - u_n\|_{L^\infty((\Omega\cap B(0,R))\times(T,2T))} = 0$$

for all $R > 0$ and $T > 0$. Then, by the standard regularity estimates for parabolic equations (see e.g. Theorem 10.1 of Chapter IV in [8]), we have

$$\lim_{n\to\infty} \|\nabla u - \nabla u_n\|_{L^\infty((\Omega\cap B(0,R))\times(T,2T))} = 0$$

for all $R > 0$ and $T > 0$. This together with (3.19) implies

$$\|(\nabla_x u)(t)\|_{L^\infty(\Omega\cap B(0,R))} \le c_{11} t^{-\frac{1}{2}} \|\varphi\|_{L^\infty(\Omega)}$$

for all $R > 0$ and $t > 0$. Therefore, by the arbitrariness of R, we have the inequality (1.7) with $p = \infty$.

Finally we prove the inequality (1.7) with $1 \le p < 4N$. Let $q \ge 4N > p$. By (1.7) with $p \ge 4N$ and (2.4), there exist constants c_{12} and c_{13} such that

$$\|(\nabla_x u)(t)\|_{L^\infty(\Omega)} \le c_{12}(t/2)^{-\frac{N}{2q}-\frac{1}{2}} \|u(t/2)\|_{L^q(\Omega)} \le c_{13} t^{-\frac{N}{2p}-\frac{1}{2}} \|\varphi\|_{L^p(\Omega)}$$

for all $t > 0$, and obtain (1.7) with $1 \le p < 4N$. Therefore the proof of Theorem 1.1 is complete.

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