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ASYMPTOTICS AND SYMMETRIES OF GROUND-STATE AND LEAST ENERGY NODAL SOLUTIONS FOR BOUNDARY-VALUE PROBLEMS WITH SLOWLY GROWING SUPERLINEARITIES

Denis Bonheure
Département de Mathématique, Université libre de Bruxelles, CP 214
Boulevard du Triomphe, B-1050 Bruxelles, Belgium

Vincent Bouchez
Institut de Mathématique pure et appliquée
Université Catholique de Louvain
Chemin du cyclotron 2, B-1348 Louvain-la-Neuve, Belgium

Christopher Grumiau
Institut de Mathématique, Université de Mons, Le Pentagone
6, Avenue du Champ de Mars, B-7000 Mons, Belgium

Dedicated to Patrick Habets and Jean Mawhin

Abstract. We study the problems

$$\begin{aligned} -\Delta u = f_\theta(u) \text{ in } \Omega, &\qquad u = 0 \text{ on } \partial\Omega, \\ -\Delta u + u = f_\theta(u) \text{ in } \Omega, &\qquad \partial_\nu u = 0 \text{ on } \partial\Omega, \end{aligned}$$

where f_θ is a slowly superlinearly growing nonlinearity, and Ω is a bounded domain. Namely, we are interested in generalizing the results obtained in [4], where the model nonlinearity $f_\theta(u) = |u|^{\theta-2}u$ was considered in the case of Dirichlet boundary conditions. We derive the asymptotic behaviour of ground state and least energy nodal solutions when $\theta \to 2$, leading to symmetry results for θ small. Our assumptions permit us to study some typical nonlinearities such as a superlinear perturbation of a small pure power or the sum of small powers and slowly exponentialy growing nonlinearities in dimension 2.

1. Introduction

In this paper, we study the boundary-value problems

$$Lu = f_\theta(u) \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega, \qquad (\mathcal{P}_\theta)$$

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where Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$, $\theta > 2$,

$$L = -\Delta \quad \text{if } Bu = u, \quad L = -\Delta + \text{id} \quad \text{if Bu} = \partial_\nu \text{u},$$

and $f_\theta \in C^1(\mathbb{R})$.

In a recent work [4], in collaboration with Jean Van Schaftingen, we considered the so-called *Lane Emden* problem

$$-\Delta u = \lambda_2 |u|^{\theta-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \qquad (\mathcal{Q}_\theta)$$

and studied least energy nodal solutions. We proved that, if $(u_\theta)_{\theta>2}$ is a family of such solutions, then

(i) as $\theta \to 2$, the accumulation points of $(u_\theta)_{\theta>2}$ are functions in the second eigenspace E_2 of $-\Delta$ minimizing the limit functional

$$J_* : E_2 \to \mathbb{R} : u \to \frac{\lambda_2}{4} \int_\Omega \left(u^2 - u^2 \log u^2\right)$$

on the manifold

$$\mathcal{N}_* := \{u \neq 0 : \langle \mathrm{d}J_*(u), u\rangle = 0\};$$

(ii) as $\theta \to 2$, u_θ respects the symmetries of its projection in E_2. In particular, if λ_2 is simple and T is an axis of symmetry of Ω, then u_θ respects the symmetry or antisymmetry of the eigenfunctions in E_2 with respect to T. In the degenerate case, the least energy nodal solutions still have the symmetries shared by all the second eigenfunctions. For instance, if Ω is a square, then u_θ is odd with respect to the center of the square;

(iii) a symmetry breaking may occur: there exists a rectangle Ω and $\theta > 2$ such that any least energy nodal solutions is neither symmetric nor antisymmetric with respect to the medians of Ω.

We are interested in generalizing those results for Problem $(\mathcal{P}_\theta)$ with a not necessarily homogeneous nonlinearity as well as considering the counterpart for ground state solutions.

Let H be either the space $H_0^1(\Omega)$ when we work with the Dirichlet boundary conditions, or the space $H^1(\Omega)$ when we work with the Neumann boundary conditions. We will denote by λ_i the i^{th} (without considering the multiplicity) eigenvalue of L in H and E_i the associated eigenspace. Let $\|\cdot\|$ be the norm given by $\sqrt{(\cdot|\cdot)}$ where $(u|v)$ is the inner product $\int_\Omega \nabla u \nabla v$ in the case of $H_0^1(\Omega)$ and $\int_\Omega \nabla u \nabla v + \int_\Omega uv$ in the case of $H^1(\Omega)$. We write $\langle \cdot, \cdot \rangle$ for the duality product.

Consider the energy functional

$$J_\theta : H \to \mathbb{R} : u \to \frac{1}{2}\|u\|^2 - \int_\Omega F_\theta(u),$$

where $F_\theta(t) := \int_0^t f_\theta(s)\,\mathrm{d}s$, whose critical points are solutions of Problem $(\mathcal{P}_\theta)$. Let us recall that *ground state* solutions of this problem are solutions u such that

$$J_\theta(u) \leqslant J_\theta(v) \tag{1.1}$$

for any nonzero solution v. Under simple conditions, ground state solutions are signed, as one can easily deduce from minimization arguments. In their seminal paper, Gidas, Ni and Nirenberg [8] showed, using the moving planes technique, that, on a convex domain, the ground state solution of Dirichlet boundary-value problems inherit all the symmetries of the domain. An alternative to derive symmetry properties of ground state solutions is to use rearrangements, see e.g. [16], which can sometimes be used in nonconvex domains where the moving planes technique is not efficient. Observe that both the moving plane approach and rearrangement techniques fail in the case of Neumann boundary conditions. In this case it is known that ground state may fail to be symmetric as for instance in the case of the BVP

$$-\Delta u + u = u^{p-1} \text{ in } B, \quad \partial_\nu u = 0 \text{ on } \partial B,$$

where B is a ball in $\mathbb{R}^N$, $N \geqslant 3$ and p is close enough to $2^* = 2N/(N-2)$. One can however expect that, in this model case, when $p > 2$ is close to 2, the ground state solution is unique. Indeed, this follows from the simplicity of the first eigenfunction and a straightforward use of the implicit function theorem. We point out that this argument has been used in dealing with related problems as e.g. in [11, 17, 20] while a direct approach has been used in [12, 5] to deal with the Hénon equation with Neumann boundary condition. Of course, symmetry properties follow directly from this uniqueness result. Here we will deduce symmetry properties from a simpler argument which does not rely directly on the uniqueness nor on the implicit function theorem.

Least energy nodal (l.e.n.) solutions are sign-changing solutions u satisying (1.1) for any sign-changing solution v. It is known that least energy nodal solutions do not in general inherit the symmetries of the domain. Aftalion and Pacella [2] proved for instance that, in the case of a Dirichlet BVP on a ball, a least energy nodal solution cannot be radial. On the other hand, Bartsch, Weth and Willem [3] deduced partial symmetry results in radial domains. The first tentative step towards considering nonradial domains has been performed in [4] where it was observed, among other things, that a uniqueness property also holds for l.e.n. solutions in the special case of $(\mathcal{Q}_\theta)$ when θ is small. Namely, l.e.n. solutions are unique up to their projections on the second eigenspace. This property was then used to deduce partial symmetries. We will use the same idea to treat nonhomogeneous nonlinearities.

Our paper is organized as follows. In the next section, we introduce our assumptions and state our main result. Section 3 is concerned with an abstract symmetry result around any eigenfunction, while Section 4 deals with symmetry around the first and second eigenfunctions. In the later case, we deduce, from an asymptotic analysis of ground state and l.e.n. solutions, sharp symmetry results. Finally, we present in Section 6 some numerical computations of ground state and l.e.n. solutions on particular cases, illustrating our statements.

2. Assumptions and statements

Let $M(\mathbb{R})$ denote the set of Borel-measurable functions from $\mathbb{R}$ to $\mathbb{R}$. For a function $h \in M(\mathbb{R})$, we refer respectively to a polynomial or an exponential growth condition by

(P_q) $\exists C > 0, \forall t \in \mathbb{R} : |h(t)| \leqslant C(|t|^q + 1)$,
(E_γ) $\exists C > 0, \forall t \in \mathbb{R} : |h(t)| \leqslant Ce^{\gamma t^2}$.

We can then introduce the classical subcritical growth conditions, namely

(A_1) $\forall \theta > 2, \exists\, q \in (2, 2^*)$ such that f_θ satisfies (P_{q-1}),
(A_1') $\forall \theta > 2, \exists\, \gamma > 0$ such that f_θ satisfies (E_γ),

where $2^* = \frac{2N}{N-2}$ if $N > 2$ and $+\infty$ if $N = 2$. As usual, the condition (A_1) is motivated by the classical Sobolev imbedding theorem for $N \geqslant 2$ while the latter is related to the Trudinger-Moser inequality when $N = 2$: whereas for all $\gamma > 0$ and all $u \in H^1(\Omega)$, we have $\int_\Omega e^{\gamma u^2} < +\infty$, there exists $\gamma_0 = \gamma_0(\Omega) > 0$ such that

$$\sup\Big\{ \int_\Omega e^{\gamma u^2} : \|u\| \leqslant 1\Big\} < +\infty$$

if and only if $\gamma < \gamma_0$ [1, 7, 18]. These assumptions ensure that the functional J_θ is well defined and of class C^1 on H. The following standard assumption (A_2) gives a mountain pass structure to the functional J_θ with a Nehari fibering method:

(A_2) (a) $\forall \theta > 2 : \lim_{t\to 0} \frac{f_\theta(t)}{|t|} = 0$,
(b) $\forall \theta > 2 : \lim_{|t|\to +\infty} \frac{f_\theta(t)}{t} = +\infty$,
(c) $\forall \theta > 2 : \forall t \neq 0 : \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{f_\theta(t)}{|t|}\Big) > 0$,

The *Nehari manifold* $\mathcal{N}_\theta$ and the *nodal Nehari set* $\mathcal{M}_\theta$ are defined by

$$\mathcal{N}_\theta := \Big\{u \in H\backslash\{0\} : \|u\|^2 = \int_\Omega f_\theta(u)u\Big\}, \quad \mathcal{M}_\theta := \big\{u \in H : u^\pm \in \mathcal{N}_\theta\big\},$$

where we respectively denote $\max\{u,0\}$ and $\min\{u,0\}$ by $u^\pm$. The ground state (respectively l.e.n.) solutions of $(\mathcal{P}_\theta)$ are the critical points of J_θ with minimum energy on $\mathcal{N}_\theta$ (respectively $\mathcal{M}_\theta$). The following theorem is due to M. Willem and J. Van Schaftingen in [19] and improves a result of A. Castro, J. Cossio and J.M. Neuberger in [6].

Theorem 1. *Under assumptions $(A_1) - (A_2)$, there exists a ground state solution and a least energy nodal solution of $(\mathcal{P}_\theta)$.*

In dimension 2, one can easily adapt the arguments of [19] in the framework of assumptions $(A_1') - (A_2)$.

We will always suppose that either assumptions $(A_1)-(A_2)$ hold or $N = 2$ and $(A_1') - (A_2)$ hold. We next consider slowly growing superlinearities:

(A_3) (a) $\exists\lambda, \forall t_0 \in \mathbb{R} : \lim_{(\theta,t)\to(2,t_0)} f_\theta(t) =: f_2(t_0) = \lambda t_0$, where λ is an eigenvalue of L,
(b) if f_θ satisfy (A_1) (respectively (A_1')), there exists $q \in (2, 2^*)$ (respectively $\gamma > 0$) and $h \in M(\mathbb{R})$ satisfying (P_q) (respectively (E_γ)) such that $\forall\theta > 2, \forall t \in \mathbb{R}, |f_\theta(t)| \leqslant h(t)$,

(A_4) (a) $\forall u_0 \neq 0 : \lim_{(s,t,\theta)\to(u_0,u_0,2)} \frac{f_\theta(t)-f_\theta(s)}{t-s} = \lambda$,
(b) there exists $q \in [2, 2^*]$ (respectively $\gamma > 0$) and $h_a, h_b \in M(\mathbb{R})$ satisfying (P_{q-2}) (respectively (A_1')) such that $\forall\theta > 2, \forall s \neq t \in \mathbb{R} : \left|\frac{f_\theta(t)-f_\theta(s)}{t-s}\right| \leqslant h_a(s) + h_b(t)$.

Assume $(u_\theta)_{\theta>2}$ is a bounded family of solutions bounded away from zero; i.e., there exist $a, b > 0$ such that $a \leqslant \|u_\theta\| \leqslant b$ uniformly in θ. Then, under the additional assumptions (A_3) and (A_4), for θ close enough to 2, u_θ respects the symmetries of its projection on the eigenspace associated with λ.

Theorem 2. *Let $(G_\alpha)_{\alpha\in E}$ be groups acting on H in such a way that, for every $g \in G_\alpha$ and for every $u \in H$,*

$$(i)\ g(E) = E, \quad (ii)\ g(E^\perp) = E^\perp, \quad (iii)\ g\alpha = \alpha, \quad (iv)\ J_\theta(gu) = J_\theta(u),$$

where E is the eigenspace associated to the eigenvalue λ. Under assumptions $(A_1) - (A_4)$, for all $R > 0$, if θ is close enough to 2, any solution

$$u_\theta \in \{u \in B_R : P_E(u) \notin B_{\frac{1}{R}}\}$$

of Problem $(\mathcal{P}_\theta)$ belongs to the fixator of G_{α_θ} where $\alpha_\theta := P_E u_\theta$.

This abstract result becomes of course more interesting once we can prove a priori bounds on a family of solutions. This is indeed the case for ground state solutions and l.e.n. solutions at least when more assumptions are imposed on the family $(f_\theta)_{\theta>2}$ of nonlinearities. First of all, we work with the classical super-quadraticity assumption

(A_5) $\forall\theta > 2, \forall t \in \mathbb{R} : f_\theta(t)t \geqslant \theta F_\theta(t)$.

Assumption (A_5) implies that $|F_\theta|$ is comparable to $|t|^\theta$ and ensures that, if $u \in \mathcal{N}_\theta$, then

$$J_\theta(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F_\theta(u) \geqslant \frac{1}{2}\|u\|^2 - \frac{1}{\theta}\int_\Omega f_\theta(u)u = \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2. \quad (2.1)$$

Consider the hypotheses

(A_6) (a) $\forall t \in \mathbb{R}, f_2(t) = \lambda t$, $\lambda \in \{\lambda_1, \lambda_2\}$, and $\lim_{(\theta,t)\to(2,t_0)} \frac{f_\theta(t)-f_2(t)}{\theta-2} =: f_*(t_0) \in \mathbb{R}$,

(b) $\exists\, q \in [2, 2^*]$ (respectively $\gamma > 0$) and $h \in M(\mathbb{R})$ satisfying (P_{q-1}) (respectively (E_γ)) such that $\left|\frac{f_\theta(t)-f_2(t)}{\theta-2}\right| \leqslant h(t)$, uniformly in θ and t,

(c) (i) $\lim_{t\to 0} \frac{f_*(t)}{t} < 0$, (ii) $\lim_{|t|\to+\infty} \frac{f_*(t)}{t} > 0$, (iii) $t \mapsto \frac{f_*(t)}{|t|}$ strictly increasing.

Roughly, it mainly means

$$f_\theta(t) = \lambda t + f_*(t)(\theta - 2) + g_\theta(t), \quad (2.2)$$

where $g_\theta(t)$ behaves like $o(\theta - 2)$. We then define the functional

$$J_* : E \to \mathbb{R} : u \mapsto -\int_\Omega F_*(u), \quad F_*(t) := \int_0^t f_*(s)\,\mathrm{d}s,$$

and the associated Nehari manifold $\mathcal{N}_* := \{u \in E \setminus \{0\} : \langle \mathrm{d}J_*(u), u\rangle = 0\}$. One can easily check that, assuming (A_6) holds, J_* is well defined and does have a mountain pass geometry. This assumption also implies that any possible accumulation points u_* of the family $(u_\theta)_{\theta>2}$ is a critical point of the functional J_*. In Section 4.3, we prove the boundedness of $(u_\theta)_{\theta>2}$ under the assumption

(A_7) (a) $\forall t_0 \in \mathbb{R} : \lim_{(\theta,t)\to(2,t_0)} \frac{f'_\theta(t)t^2 - f_\theta(t)t}{\theta-2} = H_*(t_0)$,

(b) $\exists q \in [2, 2^*]$ (respectively $\gamma > 0$) and $h \in M(\mathbb{R})$ satisfying (P_{q-1}) (respectively (E_γ)) such that $\forall\theta > 2, \forall t \in \mathbb{R} : \left|\frac{f'_\theta(t)t - f_\theta(t)}{\theta-2}\right| \leqslant h(t)$,

(c) there exists a critical point u_* of J_* such that $\int_\Omega H_*(u_*) \neq 0$.

Roughly, it mainly means, with regard to (2.2), that g'_θ behaves like $o(\theta-2)$ and that u_* is a nondegenerate critical point of J_*. Let us also note that $(A_3)(a)-(A_4)(a)-(A_6)(a)-(A_7)(a)$ can be rephrased as a convergence in θ locally uniform with respect to t.

At last, in Section 4.4, we prove that, if u_* is a weak accumulation point of $(u_\theta)_{\theta>2}$, then it is different from 0 under one of the following two assumptions:

(A_8) $\exists\, q\in[2,2^*], \forall\theta>2, \exists\, c_1(\theta), c_2(\theta)>0, \forall t\in\mathbb{R}: |f_\theta(t)|\leqslant\lambda|t|^{1+c_1(\theta)}+c_2(\theta)|t|^{p-1}$,
with $\lim_{\theta\to2} c_i(\theta)=0$ and $\exists\, k\in\mathbb{R}, \forall\theta>2, \frac{\log(1-c_2(\theta))}{c_1(\theta)}\geqslant k$,

(A'_8) $\exists\,\eta>0, \forall\theta<2+\eta, \forall t\in(-\eta,\eta)\setminus\{0\}: \frac{f_\theta(t)}{t}<\lambda$, with $\Omega\in C^2$ and $N\leqslant 5$.

Assumption (A'_8) requires some smoothness of the domain. However, when more information is known about its geometry, this smoothness requirement can be weakened as we can deal, e.g., with Lipschitz domains such as a product of intervals, see Remark 4.9 below.

Summing up, we will deduce the following result.

Theorem 3. *Let Ω be a domain in $\mathbb{R}^N$. Assume $(A_1)-(A_8)$ hold (with (A_1) or (A_8) possibly replaced by (A'_1) or (A'_8)) and $(u_\theta)_{\theta>2}$ is a family of ground state or l.e.n. solutions of Problem $(\mathcal{P}_\theta)$. Then, for θ close enough to 2,*

(i) *$(u_\theta)_{\theta>2}$ is bounded from above, bounded away from zero and if $\theta_n\to2$ and $u_{\theta_n}\rightharpoonup u_*$,*

$$\lim_{n\to+\infty}\left(\frac{J_{\theta_n}(u_{\theta_n})}{\theta_n-2}\right)=J_*(u_*)\leqslant J_*(v_*),$$

for any critical point $v_*\in E$ of J_* such that $\int_\Omega H_*(v_*)\neq0$;

(ii) *if $\lambda=\lambda_1$ or λ_2 with λ_2 simple, u_θ has respectively the symmetries of the first or the second eigenfunctions of L in H;*

(iii) *if $\lambda=\lambda_2$ and Ω is radially symmetric, u_θ is invariant by rotation with respect to $N-1$ directions and, if the functions f_θ are odd, antisymmetric with respect to the orthogonal one;*

(iv) *if $\lambda=\lambda_2$, Ω is a square and the functions f_θ are odd, u_θ is antisymmetric with respect to the center of the square.*

It is worth pointing out that the preceding result is not uniform with respect to the domain. Namely, we are able to build a sequence of rectangles $(R_n)_{n\in\mathbb{N}}$ approaching a square and a sequence of least energy nodal solutions which are neither symmetric nor antisymmetric with respect to the medians of R_n. This was already observed in [4] in the case of a homogeneous nonlinearity and Dirichlet boundary conditions. We show here that this symmetry breaking also occurs when dealing with Neumann boundary conditions. Of course in the case of a Dirichlet condition, in view of [4], it is not surprising that symmetry breaking can be observed with nonhomogeneous nonlinearities too.

In some sense, for ground state solutions, Theorem 3 provides an alternative to the moving plane method of Gidas, Ni and Nirenberg at least when the nonlinearity is weakly superlinear. Moreover, it works for Neumann conditions and on nonconvex domains. However, as mentioned earlier, the results are not so surprising as uniqueness of the ground state solutions for weakly superlinear problems is expected from the simplicity of the first eigenfunction.

For least energy nodal solutions, we obtain an alternative to the result of Bartsch, Weth and Willem in [3] for weakly superlinear problems. Moreover, our approach works for nonradial domains and leads to some symmetry breaking.

As the assumptions $(A_1) - (A_8)$ could seem quite technical and ad hoc, we give some explicit model examples in Section 6. For instance, we will analyse the following three cases where f_θ is given by

$$\lambda t|t|^{\theta-2} + (\theta-2)t|t|^{q-2}, \quad \lambda t(e^{t^2}-1)^{\theta-2}, \quad \lambda t \sum_{i=1}^{k} \alpha_i |t|^{\beta_i(\theta)},$$

i.e., the case of a subcritical superlinear $(2 < q < 2^*)$ perturbation of a slowly growing homogeneous superlinearity, of a slowly exponentially growing subcritical nonlinearity in dimension two, and of a sum of small powers. These cases will be numerically illustrated by using mountain pass algorithms.

In assumption (A_3) (a), we suppose that the limit of $f_\theta(t)$ is λt, with λ being precisely an eigenvalue. When looking at the Lane-Emden equation, one easily understands that this is not restrictive. Also, at least heuristically, one can see that, if $\lambda \neq \lambda_i$, any converging sequence of solutions goes to either 0 or infinity. That is why we have systematically studied the renormalized equation.

In the appendix, we provide another "geometrical" assumption implying the boundedness of the families of ground state solutions and least energy nodal solutions in H. We basically introduce a super-lower homogeneity assumption.

3. Symmetries around an eigenfunction

In this section, we prove that, under Assumptions (A_3)-(A_4), for θ close to 2, any bounded family $(u_\theta)_{\theta>2}$ of solutions uniformly bounded away from zero respects the symmetries of the family projected in E. This follows from a uniqueness result (up to projection) which is our next concern.

The main idea has been used in [4]. We include a sketch of the proof for completeness. We emphasize that the result holds at any eigenvalue λ_i of L.

Proposition 3.1. *Let $N \geqslant 3$. There exists $\eta > 0$ such that, if u solves*

$$Lu = a(x)u \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega, \qquad (\mathcal{R}_\theta)$$

and $\|a(x) - \lambda_i\|_{L^{N/2}} < \eta$, then, either $u = 0$ or $P_{E_i}u \neq 0$.

Proof. Let $N \geqslant 3$. Assume by contradiction that there exists a nontrivial solution u such that $P_{E_i}u = 0$. Let

$$w := P_{E_1\oplus\ldots\oplus E_{i-1}}u, \quad z := P_{(E_1\oplus\ldots\oplus E_i)^\perp}u.$$

Taking successively w and z as test functions and using the variational characterization of the i^{th} eigenvalue, we infer that $\exists\, C > 0$ such that

$$\begin{aligned}
\|w\|^2 &= \lambda_i \int_\Omega w^2 + \int_\Omega (a(x) - \lambda_i)uw \\
&\geqslant \frac{\lambda_i}{\lambda_{i-1}}\|w\|^2 - \|a(x) - \lambda_i\|_{L^{\frac{N}{2}}}\|w\|_{L^{2^*}}\|u\|_{L^{2^*}}, \\
&\geqslant \frac{\lambda_i}{\lambda_{i-1}}\|w\|^2 - C\|a(x) - \lambda_i\|_{L^{\frac{N}{2}}}\|w\|\|u\|, \\
\|z\|^2 &= \lambda_i \int_\Omega z^2 + \int_\Omega (a(x) - \lambda_i)uz \\
&\leqslant \frac{\lambda_i}{\lambda_{i+1}}\|z\|^2 + C\|a(x) - \lambda_i\|_{L^{\frac{N}{2}}}\|z\|\|u\|,
\end{aligned}$$

leading to the estimates

$$\begin{aligned}
\|w\| &\leqslant \frac{\lambda_{i-1}C}{\lambda_i - \lambda_{i-1}}\|a(x) - \lambda_i\|_{L^{\frac{N}{2}}}\|u\|, \\
\|z\| &\leqslant \frac{\lambda_{i+1}C}{\lambda_{i+1} - \lambda_i}\|a(x) - \lambda_i\|_{L^{\frac{N}{2}}}\|u\|.
\end{aligned}$$

The proof now easily follows. □

For $N = 2$, the same statement can be formulated with the $L^{\frac{N}{2}}$-norm replaced by the L^q-norm for any $1 < q < +\infty$.

As a consequence, we obtain the following.

Proposition 3.2. *Assume that* $(A_1)-(A_4)$ *hold and* $(u_{1,\theta})_{\theta>2}, (u_{2,\theta})_{\theta>2}$ *are families of solutions of* $(\mathcal{P}_\theta)$. *For every* $R > 0$, *there exists* $\theta_R > 2$ *such that, if*

$$\|u_{i,\theta}\| \leqslant R, \quad \|P_E(u_{i,\theta})\| \geqslant 1/R,$$

for $2 < \theta < \theta_R$, *then either* $u_{1,\theta} = u_{2,\theta}$ *or* $P_E u_{1,\theta} \neq P_E u_{2,\theta}$ *for every* $2 < \theta < \theta_R$, *where* E *corresponds to the eigenspace associated with the eigenvalue* λ *of assumption* (A_3).

Proof. Let $\theta_n \to 2$ and $(u_{i,\theta_n})_n$ be two sequences of solutions such that

$$\|u_{i,\theta_n}\| \leqslant R, \quad \|P_E(u_{i,\theta_n})\| \geqslant 1/R, \quad P_E u_{1,\theta_n} = P_E u_{2,\theta_n}.$$

Since those sequences are bounded in H, up to subsequences, there exist $\alpha_i \in H$ such that $u_{i,\theta_n} \rightharpoonup \alpha_i$. By assumption, we have $P_E\alpha_1 = P_E\alpha_2$, and by (A_3), a compactness argument and Lebesgue's dominated convergence theorem, we conclude easily that $\alpha_i \in E \setminus \{0\}$, so that $\alpha_1 = \alpha_2$.

Observe that $u_{1,\theta_n} - u_{2,\theta_n}$ solves Problem $(\mathcal{R}_\theta)$ where $a(x) = a_n(x)$ is defined by

$$a_n(x) = \begin{cases} \dfrac{f_{\theta_n}(u_{1,\theta_n}(x)) - f_{\theta_n}(u_{2,\theta_n}(x))}{u_{1,\theta_n}(x) - u_{2,\theta_n}(x)}, & \text{if } u_{1,\theta_n}(x) \neq u_{2,\theta_n}(x), \\ \lambda, & \text{otherwise.} \end{cases}$$

Assumption (A_4) ensures Lebesgue's dominated convergence theorem applies and, since $\alpha_i \neq 0$ almost everywhere, we deduce that $a_n \to \lambda$ in $L^{\frac{N}{2}}(\Omega)$. The proof now follows from Proposition 3.1. $\square$

Let us note that in the case of assumption (A_1') in dimension 2, the following lemma, proved in [5], plays the role of the standard compactness argument when assuming a subcritical polynomial growth.

Lemma 3.3. *Let* $u_n, v_n \in H^1(\Omega)$ *such that* $\|u_n\|, \|v_n\| \leqslant 1$. *If* $u_n \rightharpoonup u, v_n \rightharpoonup v$ *in* $H^1(\Omega)$ *and* $\gamma_n \to \gamma < \gamma_0$, *then, for every* $p \in [1, +\infty)$ *and every* $q \in [1, \frac{\gamma_0}{\gamma})$, $v_n^p e^{\gamma_n u_n^2} \to v^p e^{\gamma u^2}$ *in* $L^q(\Omega)$.

From what precedes, we can deduce an abstract symmetry result. Recall that a group action on H is a continuous application $G \times H \to H$ such that, for all $u \in H$ and for all $g, h \in G$,

$$(i)\ 1u = u, \quad (ii)\ (gh)u = g(hu), \quad (iii)\ u \mapsto gu \text{ is linear.}$$

We have assumed that G is a group with identity 1.

For instance, if $G = \{1, R\}$ where R is any reflection with respect to a hyperplane, $gu(x) := u(gx)$ and $gu(x) :=$ (detg)u(gx) define two actions whose fixed points are respectively the even and odd functions with respect to the hyperplane associated to R.

Theorem 3.4. *Let $(G_\alpha)_{\alpha\in E}$ be groups acting on H in such a way that, for every $g \in G_\alpha$ and for every $u \in H$,*

$$(i)\ g(E) = E, \quad (ii)\ g(E^\perp) = E^\perp, \quad (iii)\ g\alpha = \alpha, \quad (iv)\ J_\theta(gu) = J_\theta(u).$$

Then, under assumptions $(A_1) - (A_4)$, for all $R > 0$, if θ is close enough to 2, any solution u_θ of Problem $(\mathcal{P}_\theta)$ satisfying

$$\|u_\theta\| \leqslant R, \quad \|P_E(u_\theta)\| \geqslant 1/R$$

belongs to the fixator of G_{α_θ} where $\alpha_\theta := P_E u_\theta$.

Proof. It is sufficient to prove that if $T : H \to H$ is a continuous isomorphism satisfying

$$(i)\ T(E) = E,\ (ii)\ T(E^\perp) = E^\perp,\ (iii)\ T\alpha = \alpha,\ (iv)\ \forall u \in H,\ J_\theta(Tu) = J_\theta(u),$$

then, for θ close enough to 2, u_θ is in the fixator of T. Indeed, by Proposition 3.2, we only have to check that Tu_θ is a solution of Problem $(\mathcal{P}_\theta)$ and $P_E T u_\theta = \alpha$.

Observe that (iv) implies $\langle \mathrm{d}J_\theta(Tu_\theta), v\rangle_H = \langle \mathrm{d}J_\theta(u_\theta), T^{-1}v\rangle_H$ for all $v \in H$, so that Tu_θ is a solution of $(\mathcal{P}_\theta)$. Since

$$Tu_\theta = TP_E u_\theta + TP_{E^\perp} u_\theta = P_E T u_\theta + P_{E^\perp} T u_\theta,$$

and conditions (i) and (ii) ensure that $TP_E u_\theta \in E$ and $TP_{E^\perp} u_\theta \in E^\perp$, we deduce that $P_E T u_\theta = TP_E u_\theta$. The conclusion now follows from condition (iii). □

4. Asymptotic behavior of ground state and l.e.n. solutions

In this section, we work out a priori estimates for ground state and l.e.n. solutions, and we analyze their asymptotic behaviour, when $\theta \to 2$. In the sequel, λ exclusively takes the values λ_1, when we discuss the case of ground state solutions, and λ_2 when dealing with l.e.n. solutions. Let us denote by $(u_\theta)_{\theta>2}$ a family of ground state (respectively l.e.n.) solutions.

4.1. **Limit functional.**

We first define the limit functional

$$J_* : E \to \mathbb{R} : u \to -\int_\Omega F_*(u),$$

where $F_*(t) := \int_0^t f_*(s)\,\mathrm{d}s$. It follows from (A_6) that this functional is well defined and of class C^1. The critical points u of this functional satisfy

$$\int_\Omega f_*(u)v = 0, \tag{4.1}$$

for every $v \in E$. The assumption $(A_6)(c)$ ensures a mountain pass geometry. Indeed, for each $u \neq 0$ there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_*$, and the function $u \mapsto t_u$ is continuous.

4.2. **Limit equation.** We next turn our attention to the candidate accumulation points of $(u_\theta)_{\theta>2}$, as $\theta \to 2$.

Proposition 4.1. *Let $(u_\theta)_{\theta>2}$ be a family of ground state (respectively l.e.n.) solutions of $(\mathcal{P}_\theta)$. If $\theta_n \to 2$ and $u_{\theta_n} \rightharpoonup u_*$ in H, then, under assumption $(A_1)-(A_2)$ and $(A_5)-(A_6)$, u_* solves*

$$\begin{cases} Lu_* = \lambda u_*, & \text{in } \Omega,\\ Bu_* = 0, & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega f_*(u_*)v = 0, & \forall v \in E. \end{cases}$$

Proof. Let $v \in H$. By Rellich's theorem (or Lemma 3.3) together with Lebesgue's dominated convergence theorem, we deduce, using (A_3), that

$$f_{\theta_n}(u_{\theta_n}) \to f_2(u_*) \text{ in } L^2(\Omega),$$

so that

$$(u_*|v) = \lim_{n\to\infty}(u_{\theta_n}|v) = \lim_{n\to\infty}\int_\Omega f_{\theta_n}(u_{\theta_n})v = \lambda\int_\Omega u_* v.$$

Henceforth, $u_* \in E$. To prove the second statement, taking $v \in E$ and multiplying the equation in $(\mathcal{P}_\theta)$ by v lead to

$$\int_\Omega (f_{\theta_n}(u_{\theta_n}) - \lambda u_{\theta_n})v = 0. \tag{4.2}$$

Finally, using again Lebesgue's dominated convergence theorem and (A_6), we conclude that

$$\lim_{n\to\infty}\int_\Omega \frac{(f_{\theta_n}(u_{\theta_n}) - \lambda u_{\theta_n})v}{\theta_n - 2} = \lambda\int_\Omega f_*(u_*)v.$$

Taking (4.2) into account, this completes the proof. □

Remark 4.2. Let us point out that our assumptions imply that if $u_\theta \in \mathcal{N}_\theta$ and $u_\theta \rightharpoonup u_*$ then $u_\theta \to u_*$ in H. Indeed,

$$\|u_\theta - u_*\|^2 = \|u_\theta\|^2 - 2(u_\theta|u_*) + \|u_*\|^2 = \int_\Omega f_\theta(u_\theta)u_\theta - 2(u_\theta|u_*) + \lambda \int_\Omega u_*^2$$

which converges to

$$\lambda \int_\Omega u_*^2 - 2(u_*, u_*) + \lambda \int_\Omega u_*^2 = 0.$$

4.3. **Upper bound.** The strategy we use to obtain an upper bound is similar to the one of [4]: we construct appropriate test functions v_θ leading to an estimate of the energy and the norm of g.s. (respectively l.e.n.) solutions of Problem $(\mathcal{P}_\theta)$. In the sequel we consider the function

$$h_\theta : \mathbb{R}_0^+ \to \mathbb{R} : t \mapsto \int_\Omega t^{-1} f_\theta(tv_\theta)v_\theta$$

associated to v_θ; we denote by $\hat{v}_\theta$ the projection of v_θ on $\mathcal{N}_\theta$ (respectively $\mathcal{M}_\theta$) along rays, i.e. $\hat{v}_\theta = t_\theta v_\theta$ (respectively $t_\theta^+ v_\theta^+ + t_\theta^- v_\theta^-$) where $h_\theta(t_\theta) = \|v_\theta\|^2$ (respectively $h_\theta(t_\theta^\pm) = \|v_\theta^\pm\|^2$).

Proposition 4.3. *Let $v_\theta := v_* + (\theta - 2)w$, where w is the unique function in $E^\perp$ satisfying*

$$Lw = f_*(v_*) + \lambda w \text{ in } \Omega, \quad B(w) = 0 \text{ on } \partial\Omega,$$

and v_* is a critical point of J_* such that $\int_\Omega H(v_*) \neq 0$. Then, under assumptions $(A_1) - (A_2)$ and $(A_5) - (A_7)$, the family $(\hat{v}_\theta)_{\theta>2}$ converges to v_*.

Proof. We focus on the case $\lambda = \lambda_1$, the argument being similar if $\lambda = \lambda_2$. Since v_* satisfies (4.1), by the Fredholm alternative, w is well defined. We then compute

$$\frac{\mathrm{d}}{\mathrm{d}t} h_\theta(t) = \int_\Omega t^{-1} f_\theta'(tv_\theta)v_\theta^2 - t^{-2} f_\theta(tv_\theta)v_\theta \tag{4.3}$$

and

$$\begin{aligned}
\|v_\theta\|^2 &= \int_\Omega (\lambda v_* + \lambda(\theta-2)w + (\theta-2)f_*(v_*))v_\theta \\
&= \int_\Omega (\lambda v_\theta + (\theta-2)f_*(v_*))v_\theta \\
&= \int_\Omega f_\theta(v_\theta)v_\theta + (\theta-2)\Big(\int_\Omega \frac{\lambda v_\theta - f_\theta(v_\theta)}{\theta-2} v_\theta + \int_\Omega f_*(v_*)v_\theta\Big) \\
&= \int_\Omega f_\theta(v_\theta)v_\theta + m_\theta(\theta-2),
\end{aligned}$$

where, by (A_6), $m_\theta \to 0$, as $\theta \to 2$. We therefore deduce, thanks to (A_2), that

$$t_\theta = h_\theta^{-1}\Big(\int_\Omega f_\theta(v_\theta)v_\theta + m_\theta(\theta-2)\Big).$$

By defining $k_\theta := \int_\Omega f_\theta(v_\theta)v_\theta$ such that $h_\theta^{-1}(k_\theta) = 1$, we need to prove that

$$\lim_{\theta\to 2} h_\theta^{-1}(k_\theta + m_\theta(\theta-2)) - h_\theta^{-1}(k_\theta) = 0,$$

which holds true as $(\theta-2)(h_\theta^{-1})'$ is uniformly bounded (with respect to θ) in a neighbourhood of k_θ by assumption (A_7) and the fact that $\int_\Omega H_*(v_*) \neq 0$. □

We now deduce from the previous proposition the desired upper bound.

Theorem 4.4. *Let $(u_\theta)_{\theta>2}$ be a family of ground state (respectively l.e.n.) solutions of Problem $(\mathcal{P}_\theta)$. Then, under assumptions $(A_1)-(A_2)$ and $(A_5)-(A_7)$, the family $(u_\theta)_{\theta>2}$ is bounded and if $\theta_n \to 2$ and $u_{\theta_n} \rightharpoonup u_*$, then we have*

$$\lim_{n\to+\infty}\Big(\frac{J_{\theta_n}(u_{\theta_n})}{\theta_n - 2}\Big) = J_*(u_*) \leqslant J_*(v_*),$$

for any critical point v_* of the functional J_* such that $\int_\Omega H_*(v_*) \neq 0$.

Proof. Let us consider the family $(\hat v_\theta)_{\theta>2}$ of the previous proposition. By definition of $\hat v_\theta$, we know that $\|\hat v_\theta\|^2 = \lambda\|\hat v_\theta\|_{L^2}^2 + o(\theta-2)$, so that

$$\frac{J_\theta(\hat v_\theta)}{\theta-2} = \frac{\int_\Omega(\frac{\lambda}{2}\hat v_\theta^2 - F_\theta(\hat v_\theta))}{\theta-2} + o(1) \xrightarrow[\theta\to 0]{} J_*(v_*),$$

where the last convergence is due to (A_6) and to the definitions of F_θ and F_*.

Inequality (2.1) ensures the boundedness of $(u_\theta)_{\theta>2}$ and hence the weak convergence up to a subsequence. As observed in Remark 4.2, the convergence is strong. Observe also that

$$\|u_\theta\|^2 = \lambda\|u_\theta\|_{L^2}^2 + o(\theta-2).$$

Indeed, using Assumption (A_6) and Proposition 4.1, we infer

$$\frac{\|u_\theta\|^2 - \int_\Omega f_2(u_\theta)u_\theta}{\theta-2} = \int_\Omega \frac{f_\theta(u_\theta) - f_2(u_\theta)}{\theta-2}u_\theta \to \int_\Omega f_*(u_*)u_* = 0.$$

Hence, arguing as for the sequence $(\hat v_\theta)_{\theta>2}$, we infer that

$$\lim_{n\to+\infty}\frac{J_{\theta_n}(u_{\theta_n})}{\theta_n-2} = J_*(u_*)$$

and by definition of $(u_{\theta_n})_{n\in\mathbb{N}}$ we have $J_{\theta_n}(u_{\theta_n}) \leqslant J_{\theta_n}(\hat v_{\theta_n})$. This ends the proof. □

Remark 4.5. Assumption (A_6) essentially ensures that we can write

$$f_\theta(t) = \lambda t + f_*(t)(\theta - 2) + g_\theta(t)$$

where $g_\theta(\cdot) = o(\theta - 2)$. Then, if we rewrite the quotient $\frac{f'_\theta(t)t^2 - f_\theta(t)t}{\theta - 2}$ having this in mind, we obtain that a natural sufficient condition for (A_7) to hold is that $g'_\theta(\cdot) = o(\theta - 2)$ as well. In this case, the function H_* turns to be

$$f'_*(t)t^2 - f_*(t)t$$

and, since $u_* \in \mathcal{N}_*$, the condition $\int_\Omega H_*(u_*) \neq 0$ becomes $\int_\Omega f'_*(u_*)u_*^2 \neq 0$ which implies that $\langle J''_*(u_*), u_*, u_* \rangle \neq 0$. This last condition is the natural starting point of an approach based on the implicit function theorem or degree theory to ensure the existence of a continuum of solutions of $(\mathcal{P}_\theta)$ emanating from u_*. Let us emphasize that this remark is very rough as, indeed, we have supposed f_* to be derivable, while in some simple model example, it is not the case everywhere on $\mathbb{R}$.

4.4. **Lower bound.** We next consider two sets of conditions which imply the accumulation points u_* of a family $(u_\theta)_{\theta>2}$ of ground state (respectively l.e.n.) solutions are different from 0. While (A_8) allows us to consider any dimension, it is quite sensitive. On the other hand, (A'_8) requires some smoothness of the domain and the dimension to be small but allows for a broader class of nonlinearities.

Theorem 4.6. *Under assumption* $(A_1)-(A_2)$ *and* (A_8)*, any family* $(u_\theta)_{\theta>2}$ *of ground state (respectively l.e.n.) solutions is bounded away from* 0.

Proof. Let us give the proof in the case $\lambda = \lambda_2$, the other case being easier.

For each $\theta > 2$, we take $r := (u_\theta^+|e_1)/(|u_\theta||e_1)$ and define the function $\tilde{v}_\theta$ by $\tilde{v}_\theta = (1-r)u_\theta^+ + ru_\theta^-$. Let us then choose $t \geqslant 0$ such that $v_\theta := t\tilde{v}_\theta \in \mathcal{N}_\theta$. By construction, $v_\theta \in E_1^\perp \cap \mathcal{N}_\theta$. Since $\|v_\theta\|^2 = \int_\Omega f_\theta(v_\theta)v_\theta$ because $v_\theta \in \mathcal{N}_\theta$, we deduce by interpolation and assumption (A_8) that

$$\begin{aligned}\|v_\theta\|^2 \leqslant \lambda\Big(\int_\Omega |v_\theta|^2\Big)^{\frac{(1-\sigma)(c_1(\theta)+2)}{2}} \Big(\int_\Omega |v_\theta|^{2^*}\Big)^{\frac{\sigma(c_1(\theta)+2)}{2^*}} \\ + c_2(\theta)\Big(\int_\Omega |v_\theta|^2\Big)^{\frac{(1-s)p}{2}} \Big(\int_\Omega |v_\theta|^{2^*}\Big)^{\frac{sp}{2^*}}\end{aligned} \tag{4.4}$$

where $s, \sigma \in [0,1]$ are respectively equal to $\frac{p-2}{p} \cdot \frac{2^*}{2^*-2}$ and the same quantity with p replaced by $c_1(\theta) + 2$. Using Sobolev inequalities, the variational characterization of λ_2 and the fact that $v_\theta \in E_1^\perp$, we infer that

$$\|v_\theta\|^2 \leqslant \lambda^{1-\frac{(1-\sigma)(c_1(\theta)+2)}{2}} S^{\sigma(c_1(\theta)+2)} (\|v_\theta\|)^{c_1(\theta)+2} + c_2(\theta)\lambda^{\frac{(s-1)p}{2}} S^{sp} (\|v_\theta\|)^p$$

where S is the constant of the Sobolev inequalities. We deduce

$$1 \leqslant \alpha\beta \|v_\theta\|^{c_1(\theta)} + c_2(\theta) ab \|v_\theta\|^{p-2},$$

where $\alpha(\theta) := \lambda^{1-\frac{(1-\sigma)(c_1(\theta)+2)}{2}}$, $\beta(\theta) := S^{\sigma(c_1(\theta)+2)}$, $a := \lambda^{\frac{(s-1)p}{2}}$, $b := S^{sp}$, and so

$$\Big(\frac{1 - c_2(\theta) ab \|v_\theta\|^{p-2}}{\alpha\beta}\Big)^{\frac{1}{c_1(\theta)}} \leqslant \|v_\theta\|$$

which leads to the conclusion if the left-hand side of the inequality does not converge to 0. Since assumption (A_8) ensures that the numerator does not converge to 0, the claim follows. □

For the second theorem using assumption (A_8'), we need the following property, which has its own interest, about the optimal constant of the Poincaré inequality.

Proposition 4.7. *If $u \in H$, then one has either $\lambda_2 \|u^+\|_{L^2}^2 \leqslant \|u^+\|^2$ or $\lambda_2 \|u^-\|_{L^2}^2 \leqslant \|u^-\|^2$.*

Proof. Let us recall the Riesz-Fisher characterization of λ_2

$$\lambda_2 = \inf_{\dim F = 2} \sup_{u \in F \setminus \{0\}} \frac{\|u\|^2}{\|u\|_{L^2}^2}.$$

If we define $F := \{v \in E \setminus \{0\} : v = \alpha u^+ + \beta u^-, (\alpha, \beta) \neq (0,0)\}$, then

$$\lambda_2 \leqslant \max_{(\alpha,\beta)\neq(0,0)} \frac{\alpha^2 \|u^+\|^2 + \beta^2 \|u^-\|^2}{\alpha^2 \|u^+\|_{L^2}^2 + \beta^2 \|u^-\|_{L^2}^2} = \max \left\{ \frac{\|u^+\|^2}{\|u^+\|_{L^2}^2}, \frac{\|u^-\|^2}{\|u^-\|_{L^2}^2} \right\}.$$

Indeed, the last equality follows from

$$\frac{a+b}{c+d} \leqslant \frac{a}{c} \Leftrightarrow \frac{b}{d} \leqslant \frac{a}{c} \text{ for any } a, b, c, d > 0.$$

This completes the proof. □

In [14, Proposition 2.2], it is proved that a convergence in H_0^1 combined with the smoothness of Ω ensures uniform convergence in dimension $N \leqslant 5$. The arguments are presented for the Lane-Emden problem with a Dirichlet boundary condition and l.e.n. solutions but work in the same way in our framework.

Theorem 4.8. *Under assumption (A_1)-(A_2) and (A_8'), any bounded family $(u_\theta)_{\theta>2}$ of ground state (respectively l.e.n.) solutions is bounded away from 0.*

Proof. Let us denote by u_{θ_n}, where $\theta_n \to 2$, a weak convergent subsequence of the family $(u_\theta)_{\theta>2}$, and let u_* be its limit. By adapting [14, Proposition 2.2] combined with Remark 4.2, we know that u_{θ_n} converges to u_* uniformly in dimension $N \leqslant 5$. Let us show this convergence implies that $u_* \neq 0$ and more precisely that $\|u_{\theta_n}\|_{L^\infty} > \eta$ where η is defined in (A_8').

In the case $\lambda = \lambda_1$, as $u_{\theta_n} \in \mathcal{N}_{\theta_n}$, we have

$$\lambda_1 \int_\Omega u_{\theta_n}^2 \leqslant \|u_{\theta_n}\|^2 = \int_\Omega f_{\theta_n}(u_{\theta_n})u_{\theta_n} = \int_{\{u_{\theta_n}\neq 0\}} \frac{f_{\theta_n}(u_{\theta_n})}{u_{\theta_n}} u_{\theta_n}^2, \tag{4.5}$$

which is impossible if $\|u_{\theta_n}\|_{L^\infty} \leqslant \eta$.

In the case $\lambda = \lambda_2$, by Proposition 4.7 and since $u_{\theta_n}^\pm \in \mathcal{N}_{\theta_n}$, one of the following two inequalities hold:

$$\lambda_2 \int_\Omega (u_{\theta_n}^+)^2 \leqslant \|u_{\theta_n}^+\|^2 = \int_\Omega f_{\theta_n}(u_{\theta_n}^+)u_{\theta_n}^+ = \int_{\{u_{\theta_n}^+\neq 0\}} \frac{f_{\theta_n}(u_{\theta_n}^+)}{u_{\theta_n}^+} (u_{\theta_n}^+)^2, \tag{4.6}$$

$$\lambda_2 \int_\Omega (u_{\theta_n}^-)^2 \leqslant \|u_{\theta_n}^-\|^2 = \int_\Omega f_{\theta_n}(u_{\theta_n}^-)u_{\theta_n}^- = \int_{\{u_{\theta_n}^-\neq 0\}} \frac{f_{\theta_n}(u_{\theta_n}^-)}{u_{\theta_n}^-} (u_{\theta_n}^-)^2, \tag{4.7}$$

which is again impossible if $\|u_{\theta_n}\|_{L^\infty} \leqslant \eta$. □

Remark 4.9. It is worth pointing out that for polygonal domains in the plane, assumption (A_8') does not apply because of the lack of smoothness. However, at least for a square or a rectangle, one can give an alternative proof of the a priori estimates required by Theorem 4.8 which does not rely on the smoothness of the domain but on some extension arguments which work fine even for a product of intervals in $\mathbb{R}^N$.

4.5. **Conclusion.** Combining the previous estimates of this section with the abstract symmetry result of Section 3, we can complete the proof of Theorem 3. Of course, in the statement, the assumption (A_1) can be replaced by (A_1') in dimension 2 and (A_8) can be replaced by (A_8'). If λ_2 is not simple, then we can argue as in [4]. To this aim, we need to know the symmetries of second eigenfunctions of L with Dirichlet or Neumann boundary conditions. For radial domains (ball or annulus) and Dirichlet boundary conditions, it is proved in [15] that second eigenfunctions of $-\Delta$ are even with respect to $N-1$ directions and odd with respect to the orthogonal one. Moreover, it is unique up to rotations and a multiplicative factor. For Neumann boundary conditions it is proved in [13] that the second eigenfunctions are Schwarz foliated and therefore even with respect to $N-1$ orthogonal directions. The oddness with respect to the orthogonal direction and the uniqueness up to rotations and multiplicative factors also hold.

5. Symmetry breaking

In [4], for the Lane–Emden model $(\mathcal{Q}_\theta)$, the authors proved that the previous results cannot be extended to large θ when $\lambda = \lambda_2$. In fact the results are not uniform with respect to Ω and by adjusting the dimensions of a rectangle with respect to θ, the authors exhibited l.e.n. solutions that were neither symmetric nor antisymmetric with respect to the medians.

The main idea goes as follows. In [4], it is shown numerically that, on a square, the minimizers of J_* on $\mathcal{N}_*$, are symmetric with respect to the diagonal and not to the medians. It follows that l.e.n. solutions do accumulate on the second eigenfunctions which are symmetric with respect to the diagonals. On the contrary, on a rectangle, l.e.n. solutions do accumulate on the second eigenfunctions which are symmetric with respect to the medians because the second eigenvalue is simple. Taking a sequence of l.e.n. solutions on oblong rectangles approximating a square leads to a nonsymmetric regime.

Arguing like that requires us to extend the previous results to a more general second-order elliptic operator. Namely, one can first prove that the previous results concerning Problem $(\mathcal{P}_\theta)$ extend to the similar boundary-value problem with $-\Delta$ replaced by $-\mathrm{div}(A_\theta \nabla)$ where $A_\theta \in \mathcal{C}(\Omega, S^{N\times N})$, $S^{N\times N}$ being the set of symmetric $N \times N$ matrices, $A_0 = \mathrm{id}$ and $\theta \mapsto A_\theta$ being uniformly differentiable at $\theta = 0$. There are no changes in the previous proofs to obtain the same conclusions, except for the nonconvergence to 0. Indeed, the modifications in the proof of Theorem 4.6 are straightforward as soon as we assume the extra condition

$$\exists k \in \mathbb{R} : \forall \theta > 2 : \frac{\log(1 - c_2(\theta) - O(\theta - 2))}{c_1(\theta)} \geqslant k.$$

When dealing with (A_8') in Theorem 4.8 we impose the fact that $A_\theta - \mathrm{id}$ is semi positive definite. Then, we control the sign of the additional term in the right-hand side of (4.5) (or (4.6) and (4.7)):

$$\|u_{\theta_n}\|^2 = \int_{\{u_{\theta_n} \neq 0\}} \frac{f_{\theta_n}(u_{\theta_n})}{u_{\theta_n}} u_{\theta_n}^2 - \int_\Omega \nabla u_{\theta_n} (A_{\theta_n} - \mathrm{id}) \nabla \mathrm{u}_{\theta_n}.$$

We next quickly sketch the argument of [4] and check that the last extra assumption is fulfilled in this situation. Consider sequences $(\theta_n)_n$ and $(\nu_n)_n$ such that $0 < \nu_n = o(\theta_n - 2)$. Let R_{ν_n} be rectangles with sides of length 2 and $(1 - \nu_n)2$. We then consider the sequence of problems

$$Lu = f_{\theta_n}(u) \text{ in } R_{\nu_n}, \quad Bu = 0 \text{ on } \partial R_{\nu_n}. \qquad (\mathcal{P}_{\nu_n})$$

The change of variables $\tilde{u}(x,y) = u(x,(1-\nu)y)$ leads to the equivalent sequence of problems

$$L_n\tilde{u} = f_{\theta_n}(\tilde{u}) \text{ in } Q, \quad B\tilde{u} = 0 \text{ on } \partial Q,$$

where $L_n = -\partial_x^2 - (1-\nu_n)^{-2}\partial_y^2$ or $-\partial_x^2 - (1-\nu_n)^{-2}\partial_y^2 + Id$ and Q denotes the square $(-1,1)^2$. In both cases, $\lambda = \lambda_2(Q)$. Observe that the differential operator can be written in divergence form with

$$A_{\theta_n} = \begin{pmatrix} 1 & 0 \\ 0 & (1-\nu_n)^{-2} \end{pmatrix}$$

so that $A_{\theta_n} - \mathrm{id}$ is semi positive definite.

Theorem 5.1. *Assume $\lambda = \lambda_2$ and f_θ fulfills the assumptions $(A_1) - (A_8)$ (with (A_1) or (A_8) possibly replaced by (A_1') or (A_8')). Then, if Ω is a square and the minimizers u_* of J_* on $\mathcal{N}_*$ satisfy $\int_\Omega H_*(u_*) \neq 0$ and are neither symmetric nor antisymmetric with respect to the medians, there exists a rectangle R (close to Ω) and $\theta > 2$ such that any least energy nodal solutions of the problem*

$$Lu = f_\theta(u) \text{ in } R, \quad Bu = 0 \text{ on } \partial R$$

are neither symmetric nor antisymmetric with respect to the medians of R.

6. Examples

In this section, we give three examples of problems different from the Lane–Emden one, where the nonlinearity satisfies respectively $(A_1) - (A_8)$, $(A_1') - (A_8')$ and $(A_1) - (A_8')$.

We used the mountain pass algorithm (MPA) and modified mountain pass algorithm (MMPA) to compute one-signed and sign-changing solutions of Problem $(\mathcal{P}_\theta)$. The MPA was introduced and studied by Y. S. Choi and P. J. McKenna in [8] and J. Zhou and Y. Li in [21, 22]; the MMPA is due to D. G. Costa, Z. Ding and J. M. Neuberger [9].

All cases will be considered on the square $\Omega = (-1,1)^2$ in $\mathbb{R}^2$. Let us recall that $\lambda_1(D) = \frac{\pi^2}{2}$, $\lambda_2(D) = \frac{5\pi^2}{4}$, $\lambda_1(N) = 1$, $\lambda_2(N) = 1 + \frac{\pi^2}{4}$, where "D" means Dirichlet and "N" refers to Neumann.

The algorithm relies at each step on the finite element method (see e.g. [10]). The domain Ω is triangulated with a Delaunay condition. We used the Easymesh software to do it. The program stops when the gradient of the energy functional at the approximations has a norm strictly inferior to 1.0×10^{-2} or after 20.000 steps. We used the Java language to compute the algorithms and the Scilab software to graph numerical solutions.

We will also study the presence of a symmetry breaking as described in Theorem 5.1. To this aim, we compute, with the software Matlab, the function J_* on $\mathcal{N}_*$. At last, let us recall that $E_2 = \langle v_1, v_2\rangle$, where, in the case of Dirichlet and of Neummann conditions:

$$v_1(x,y) = \cos(\frac{\pi}{2}x)\sin(\pi y), \quad v_2(x,y) = \sin(\pi x)\cos(\frac{\pi}{2}y), \tag{D}$$

$$v_1(x,y) = \sin(\frac{\pi}{2}x), \quad v_2(x,y) = \sin(\frac{\pi}{2}y). \tag{N}$$

6.1. **Superlinear perturbation of Lane–Emden Problem.** The first example is a very natural nonlinearity in view of [4] where the Lane-Emden problem is handled. Let us consider the function

$$f_\theta(t) = \lambda t|t|^{\theta-2} + (\theta-2)t|t|^{q-2}$$

for some $q \in (2, 2^*)$. It is easily checked that

- $(A_1)-(A_2)$ are satisfied;
- (A_5) is satisfied;
- $(A_3)-(A_4), (A_6)-(A_7)$ are satisfied by $f_*(t) = \lambda t\log|t| + t|t|^{q-2}$, $H_*(t) = \lambda t^2 + (q-2)|t|^q$ and $h(t) = (2\lambda + q - 1)(|t|^s + 1)$, with $s = q-1$ or $q-2$ accordingly;
- (A_8) is satisfied for $c_1(\theta) = c_2(\theta) = \theta - 2$.

In order to illustrate this example numerically, let us consider the following problems:

$$Lu = \lambda u^3 + 2u^5 \text{ in } \Omega, \quad B(u) = 0 \text{ on } \partial\Omega, \tag{6.1}$$

where $\lambda = \lambda_1$ or λ_2. It is worth pointing out that instead of considering (6.1), we could have dealt with the model

$$Lu = u^3 + cu^5 \text{ in } \Omega, \quad B(u) = 0 \text{ on } \partial\Omega,$$

as there is an obvious bijection between the solutions of this last problem (with a precise choice of the constant c) and the solutions of (6.1). As already mentioned, from a theoretical point of view, dealing with $f_\theta(t)$ instead of $t|t|^{\theta-2} + (\theta-2)t|t|^{q-2}$ allows us to avoid recurrent rescalings that make the correct limit equation appear. From the numerical point of view, looking at the rescalled equation where the eigenvalue appears is the best choice to avoid very small or very large solutions that could lead to numerical inaccuracy and would not facilitate the visualization.

Figure 1 and Table 1 show the results of the numerical experiments: one-signed (respectively nodal) numerical solutions have the expected symmetries. While it is not certain that those solutions have least energy, all the other solutions that we have found numerically have a larger energy. Let us remark that for the ground state solutions in the Neumann case, the MPA gives the constant solution for θ close to 2 so that we do not consider it in our figures and tables. For both Dirichlet and Neumann boundary conditions, the nodal line seems to be a diagonal.

	Initial function	$\min u$	$\max u$	$J(u^+)$	$J(u^-)$
$\lambda_1(D)$	$\sin(\frac{\pi(x+1)}{2})\sin(\frac{\pi(y+1)}{2})$	0.0	1.29	1.47	0.0
$\lambda_2(D)$	$\sin(\pi(x+1))\sin(2\pi(y+1))$	-1.4	1.4	1.7	1.7
$\lambda_2(N)$	$\cos(\pi(y+1))$	-1.31	1.33	0.78	0.77

TABLE 1. Characteristics of the approximate solutions.

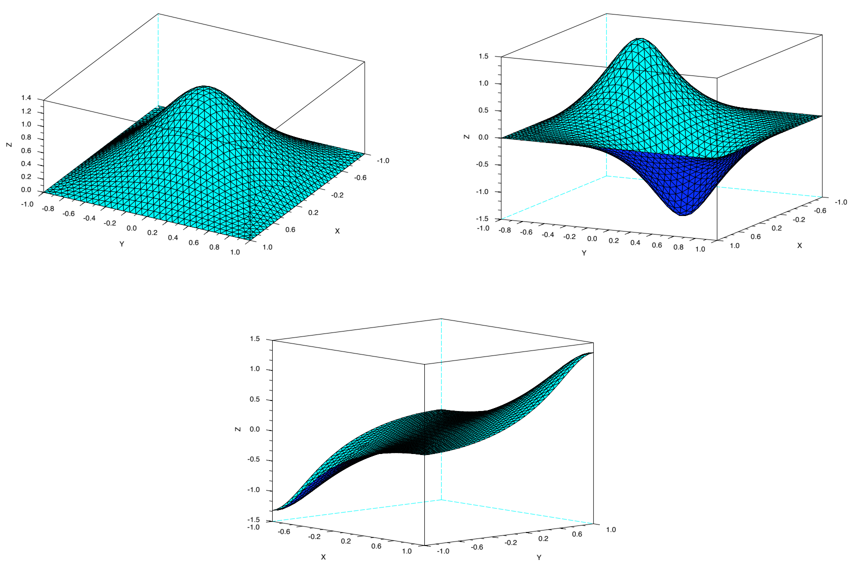

FIGURE 1. Numerical solutions in the case of a superlinear perturbation of the Lane Emden problem.

In the case of l.e.n. solutions, minimizers of J_* on $\mathcal{N}_*$ are symmetric with respect to a diagonal. Indeed, any second eigenfunctions can be written as a multiple of $v_\alpha = \cos(\alpha)v_1 + \sin(\alpha)v_2$, for $\alpha \in [0, 2\pi]$, where v_1 and v_2 have been defined previously. Figure 2 displays the graph of the function

$$S_* : [0, 2\pi] \to \mathbb{R} : \alpha \mapsto J_*(t_\alpha v_\alpha),$$

where $t_\alpha u_\alpha \in \mathcal{N}_*$. As the minima are attained for $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$, the corresponding minimizers are symmetric functions with respect to a diagonal.

As a consequence of the previous computation, the symmetry breaking described in Theorem 5.1 occurs in this example.

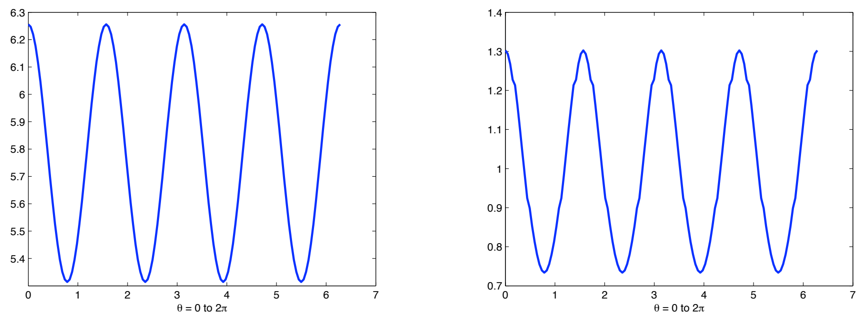

FIGURE 2. Computation of S_* for the superlinear perturbation of the Lane Emden problem. The graph on the left (respectively right) corresponds to Dirichlet (respectively Neumann) boundary conditions.

6.2. **Exponential growth.** The second example is a nonlinearity with an exponential growth in dimension 2: $f_\theta(t) = \lambda t(e^{t^2} - 1)^{\theta - 2}$. One can check that

- $(A_1') - (A_2)$ are satisfied for some $\gamma > 0$;
- (A_5) is satisfied as, indeed, we have
$$\begin{aligned}(f_\theta(t)t)' - (\theta F_\theta(t))' &= t(f_\theta(t))' - (\theta - 1) f_\theta(t) \\ &= \lambda t(e^{t^2} - 1)^{\theta-2}\Big(2(\theta - 2)\frac{t^2 e^{t^2}}{e^{t^2} - 1} + 2 - \theta\Big)\end{aligned}$$
  which proves the claim for t such that $g(t) := t^2 \frac{e^{t^2}}{e^{t^2}-1} \geqslant \frac{1}{2}$ (then observe that $g > \frac{1}{2}$ on $\mathbb{R}$);
- $(A_3)-(A_4), (A_6)-(A_7)$ are satisfied by $f_*(t) = \lambda t \log(e^{t^2} - 1)$, $H_*(t) = 2\lambda \frac{t^4 e^{t^2}}{e^{t^2}-1}$ (extended by 0 at 0) and $h(t) = C(\delta) e^{(\delta+1)t^2}$, for some $\delta > 0$ and $C(\delta) > 0$;
- (A_8') (except the smoothness of the domain, see Remark 4.9) is satisfied by taking for example $\eta = \sqrt{\log(a)}$ with $1 < a < 2$.

We numerically illustrate this example with the model problem

$$Lu = \lambda u(e^{u^2} - 1)^{0.5} \text{ in } \Omega, \quad B(u) = 0 \text{ on } \partial\Omega.$$

Figure 3 and Table 2 show the results of the numerical experiments. The same comments as in the previous example can be rephrased. In particular, in the case of l.e.n. solutions, minimizers of J_* on $\mathcal{N}_*$ are symmetric with respect to a diagonal as shown in Figure 4 and the symmetry breaking described in Theorem 5.1 still occurs in this example.

	Initial function	$\min u$	$\max u$	$J(u^+)$	$J(u^-)$
$\lambda_1(D)$	$\sin(\frac{\pi(x+1)}{2})\sin(\frac{\pi(y+1)}{2})$	0.0	1.11	3.05	0.0
$\lambda_2(D)$	$\sin(\pi(x+1))\sin(2\pi(y+1))$	-1.2	1.2	2.79	2.79
$\lambda_2(N)$	$\cos(\pi(y+1))$	-1.25	1.26	0.9	0.91

TABLE 2. Characteristics of the approximate solutions.

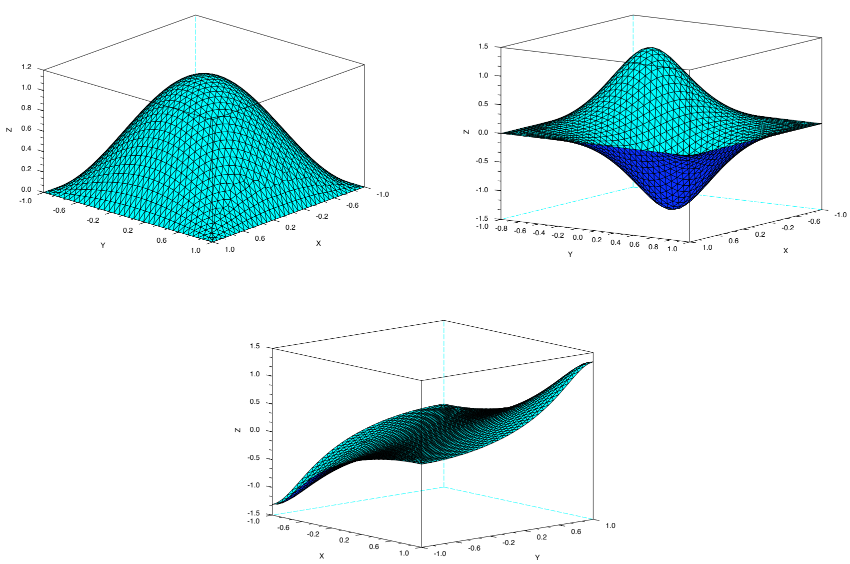

FIGURE 3. Numerical solutions in the case of an exponential growth.

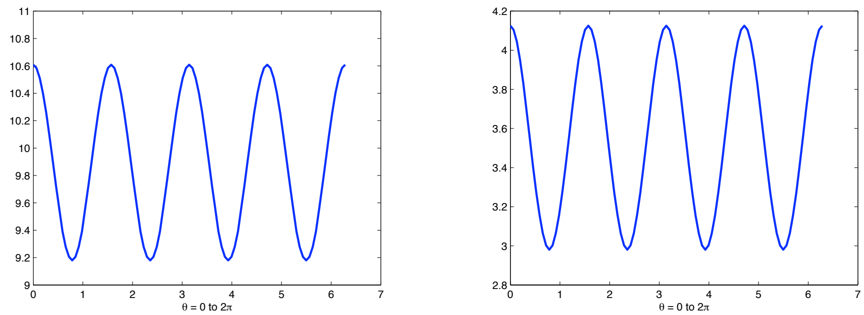

FIGURE 4. Computation of S_* in the case of an exponential growth. The graph on the left (respectively right) corresponds to Dirichlet (respectively Neumann) boundary conditions.

6.3. **Sum of powers.** In this example, we index the family by an extra parameter ε and rename $f_{\theta_\varepsilon}, J_{\theta_\varepsilon}$ by $f_\varepsilon, J_\varepsilon$. We consider the functions

$$f_\varepsilon(t) = \lambda t\Big(\sum_{i=1}^{k} \alpha_i |t|^{\beta_i(\varepsilon)}\Big)$$

with $\alpha_i, \beta_i(\varepsilon) > 0$ such that $\min \beta_i(\varepsilon) = \beta_1(\varepsilon)$, $\max \beta_i(\varepsilon) = \beta_k(\varepsilon)$, $\sum_i \alpha_i = 1$, $\lim_{\varepsilon\to 0} \beta_i(\varepsilon) = 0$ and $\lim_{\varepsilon\to 0} \frac{\beta_k(\varepsilon)}{\beta_1(\varepsilon)} = 1$. It is easy to check that

- $(A_1) - (A_2)$ are satisfied with $p = \beta_k(\varepsilon)$;
- (A_5) is satisfied with $\theta_\varepsilon = \beta_1(\varepsilon) + 2$;
- $(A_3) - (A_4), (A_6) - (A_7)$ are satisfied by $f_*(t) = \lambda t \log|t|$, $H_*(t) = \lambda t^2$ and $p = 1 + \delta$ for some $\delta > 0$;
- (A_8') (except the smoothness of the domain, see Remark 4.9) is satisfied for some $\eta < 1$.

Let us point out that in this example, we have a better control on the Nehari manifold, namely

$$\Big(\frac{1}{2} - \frac{1}{\beta_1(\varepsilon)+2}\Big)\|u\|^2 \leqslant J_\varepsilon(u) \leqslant \Big(\frac{1}{2} - \frac{1}{\beta_k(\varepsilon)+2}\Big)\|u\|^2$$

for every $u \in \mathcal{N}_\varepsilon$. This enables a very simple proof of the boundedness of ground state (respectively l.e.n.) solutions.

We illustrate this example numerically by considering

$$Lu = \lambda u\Big(\tfrac{1}{4}|u|^{0.25} + \tfrac{1}{4}|u|^{0.5} + \tfrac{1}{2}|u|\Big) \text{ in } \Omega, \quad B(u) = 0 \text{ on } \partial\Omega.$$

Figure 5 and Table 3 show the results of the numerical experiments. In Table 3, the initial functions are respectively $u_{1,D}(x,y) = 20\cos\left(\frac{\pi}{2}(x^2+y^2)^{0.5}\right)$, $u_{2,D}(x,y) = -800\cos\left(\frac{\pi}{2}(x^2+y^2)^{0.5}\right)\cos\left(\pi(x^2+y^2)^{0.5}\right)$ and $u_{2,N}(x,y) = \cos(\pi y)$. Observe that we can do the same comments as in the previous cases.

	Initial Function	$\min u$	$\max u$	$J(u^+)$	$J(u^-)$
$\lambda_1(D)$	$u_{1,D}$	0.0	1.39	4.79	0.0
$\lambda_2(D)$	$u_{2,D}$	-1.46	1.46	4.65	4.65
$\lambda_2(N)$	$u_{2,N}$	-1.55	1.55	0.4	0.5

TABLE 3. Characteristics of the approximate solutions.

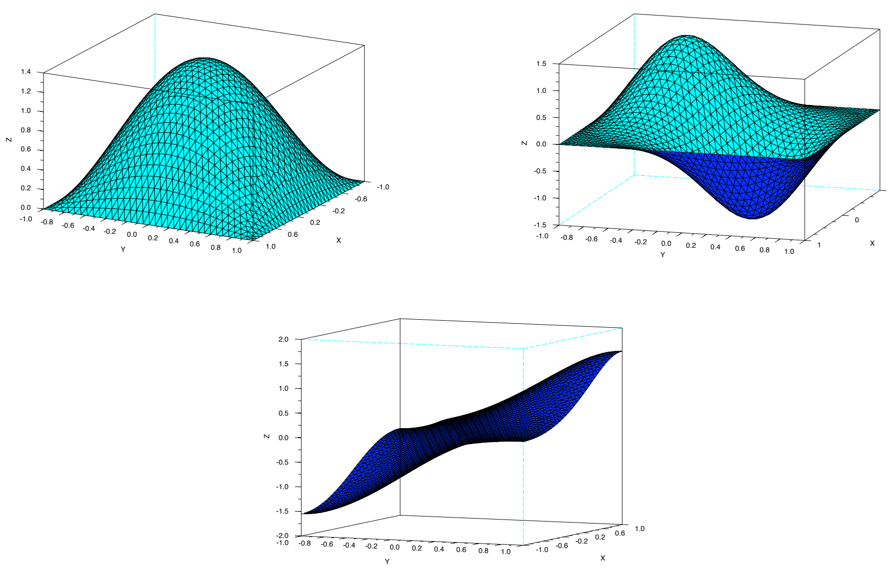

FIGURE 5. Numerical solutions in the case of a sum of powers.

Since the limit functional for this example is the same as the limit functional for the Lane Emden problem, the symmetry breaking phenomenum occurs.

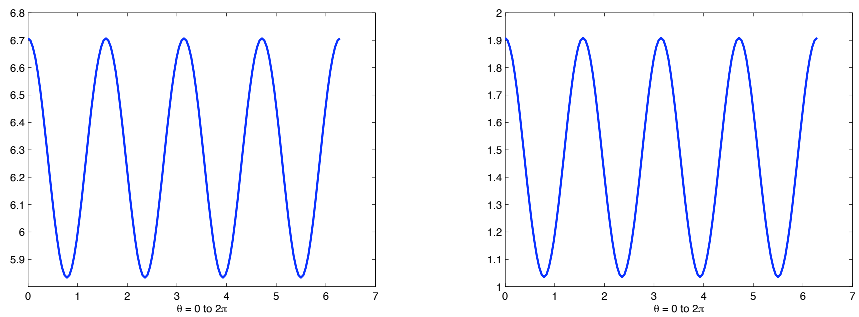

FIGURE 6. Computation of S_* in the case of a sum of powers. The graph on the left (respectively right) corresponds to Dirichlet (respectively Neumann) boundary conditions.

APPENDIX A. SUPER-LOWER HOMOGENEITY:

In this appendix, we provide an assumption of "*Super-lower homogeneity*" that leads to a naturel boundedness of any family of ground state (respectively l.e.n.) solutions of Problem $(\mathcal{P}_\theta)$. This result has to be compared with Theorem 4.4: its assumptions cover other nonlinearities than assumption (A_7).

Proposition A.1. *Let us assume assumptions* $(A_6)(a), (b)$ *and*

$$\exists\, \rho \geqslant 0, \forall s \in \mathbb{R}, \forall t \geqslant \rho : f_\theta(ts) \geqslant \sigma(s) t^{\theta-1} f_\theta(s),$$

where $\sigma(s)$ *is the sign of* s*. Then, there exists a sequence* $v_\theta \in \mathcal{N}_\theta$ *(respectively* $\mathcal{M}_\theta$*) bounded in* H*.*

Proof. We only consider the case of $\mathcal{M}_\theta$, the other one being similar. It is sufficient to prove that if $\theta_n \to 2$ then there exists a bounded sequence $v_{\theta_n} \in \mathcal{M}_{\theta_n}$.

Take e_2, a second eigenfunction of the operator L in H, and define v_θ as $t^+_{\theta_n} e^+_2 + t^-_{\theta_n} e^-_2$ where $t^\pm_{\theta_n}$ are such that $t^\pm_{\theta_n} e^\pm_2 \in \mathcal{N}_\theta$. We can suppose without loss of generality that $t^\pm_{\theta_n} \geqslant \rho$ for all n. The assumption ensures that

$$(t^\pm_{\theta_n})^2 \|e^\pm_2\|^2 = \int_\Omega f_{\theta_n}(t^\pm_{\theta_n} e^\pm_2) t^\pm_{\theta_n} e^\pm_2 \geqslant \int_\Omega (t^\pm_{\theta_n})^{\theta_n} f_{\theta_n}(e^\pm_2) e^\pm_2$$

which implies that

$$(t^\pm_{\theta_n})^{\theta_n - 2} \leqslant \frac{\|e^\pm_2\|^2}{\int_\Omega f_{\theta_n}(e^\pm_2) e^\pm_2}.$$

As $\|e_2^\pm\|^2 = \lambda_2 \|e_2^\pm\|_2^2$ and $f_{\theta_n}(e_2^\pm) \to \lambda e_2^\pm$, we just need to study the convergence of the last term in the expression

$$-\frac{\log(\|e_2^\pm\|^2) - \log(\int_\Omega f_{\theta_n}(e_2^\pm)e_2^\pm)}{\|e_2^\pm\|^2 - \int_\Omega f_{\theta_n}(e_2^\pm)e_2^\pm} \cdot \int_\Omega \frac{f_{\theta_n}(e_2^\pm)e_2^\pm - \lambda_2(e_2^\pm)^2}{\theta_n - 2}$$

since the first one converges to $-\|e_2^\pm\|^{-2}$. The assumptions $(A_6)(a),(b)$ ensuring that the last term converges to $\int_\Omega f_*(e_2^\pm)e_2^\pm$, we deduce the boundedness of $(t_{\theta_n}^\pm)_n$ and hence of v_{θ_n}. □

Observe now that Proposition A.1 leads in a straightforward way to an alternative assumption ensuring the boundedness of ground state (respectively l.e.n.) solutions of Problem $(\mathcal{P}_\theta)$. With this alternative, we can for instance treat the case of the sum of powers without an assumption on $\beta_i(\varepsilon)\beta_1(\varepsilon)^{-1}$: if $t \geqslant 1$ we have

$$\begin{aligned}\lambda \sum_{i=1}^n \Big(\alpha_i (ts)^{\beta_i(\varepsilon)} ts\Big) &\geqslant \lambda \min_{i\in\{1,\dots,n\}} (t^{\beta_i(\varepsilon)+1}) \Big(\sum_{i=1}^n \alpha_i s^{\beta_i(\varepsilon)+1}\Big) \\ &= \lambda t^{(\beta_1(\varepsilon)+1)} \Big(\sum_{i=1}^n \alpha_i s^{\beta_i(\varepsilon)} s\Big) \qquad \text{(A.1)}\end{aligned}$$

and we can conclude recalling that $\theta_\varepsilon = \beta_1(\varepsilon) + 2$.

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