QUASI-PERIODIC SOLUTIONS OF A DAMPED REVERSIBLE OSCILLATOR AT RESONANCE

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Abstract. We prove the existence of quasi-periodic solutions and of Aubry-Mather sets for a resonant reversible equation of the form $x'' + ax^+ - bx^- + \varphi(x) + f(x, x', t) = p(t)$; the functions p and f are 2π-periodic in t and the perturbation φ is bounded.

1. Introduction and result

In this paper we study the properties of the solutions of the differential equation

$$x'' + ax^+ - bx^- + \varphi(x) + f(x, x', t) = p(t), \tag{1.1}$$

where $a \neq b$ are two positive constants satisfying, for some $n \in \mathbf{N}$,

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{2}{n}, \tag{1.2}$$

and $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. The functions φ, p belong to the class $\mathcal{C}^2(\mathbf{R})$ and $f \in \mathcal{C}^2(\mathbf{R}^3)$. Moreover, p and f are 2π-periodic in the time variable.

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More precisely, we deal with the existence of Aubry-Mather sets and, as a consequence, of quasi-periodic solutions.

In the literature of the last twenty years, many results can be found on this topic (as well as for the related problems of the boundedness of all solutions and of the existence of unbounded or periodic solutions). These contributions share a common feature: they are based on careful asymptotic estimates of the Poincaré map associated to the planar system equivalent to (1.1). For a very nice survey on this subject, we refer to the recent contribution by Jean Mawhin [21]. In particular, important contributions are due to Alonso-Ortega [1] in the case $\varphi = f = 0$, to Ortega [22] in the case $a = b = n^2$, $f \equiv 0$ and φ a piecewise linear function, and to Liu [15] for $a = b = n^2$, $f \equiv 0$ and φ bounded.

Later, improvements in various directions have been obtained (among others) by Bonheure-Fabry-Smets [4], Fabry-Bonheure [3], Fabry-Mawhin [10] for the existence of 2π periodic and/or unbounded solutions, by the first author and Liu [5] as far as Aubry-Mather sets are concerned, and by Liu [16], Bonheure-Fabry [2] in the framework of the boundedness question. More recently, Fonda-Mawhin [12] have opened the way to generalizations to some first-order systems. It is worth noticing that in the proofs of the above quoted results the Hamiltonian structure of the problem plays a central role. In fact, the abstract results that are used concern area-preserving planar maps (cf. [22], [23]).

When $f \neq 0$, the planar system equivalent to (1.1) is not Hamiltonian in general. However, when some symmetry assumptions are satisfied, the system turns out to be reversible with respect to a suitable transformation (we refer to Section 2 for details). Thus, results can be obtained based on the variants of [22], [23] recently obtained by Liu-Song [18] and Chow-Pei [7], respectively. In particular, we refer to the contributions of Kunze-Kupper-Liu [13] and Liu-Wang [20] for the boundedness problem and to the works of Liu [17] and Liu-Wang [19] as far as the existence of quasi-periodic solutions is concerned. See also [14] and [24].

Our contribution is along the lines of the above cited papers. Indeed, we show that (not only linear but) also asymmetric resonant oscillators can be treated in the framework of reversible systems; at the same time, we deal with a not previously treated damping term (cf. Remark 3.1 for more details on this aspect). More precisely, under suitable symmetry conditions on f, p, we show with Theorem 1.1 that equation (1.1) fits into the framework of Aubry-Mather theory for reversible systems [7]. The boundedness problem for (1.1) has been treated in [20]; as far as the existence of unbounded or periodic solutions, we refer to [8], [9], [11].

In order to state our results, we list a set of hypotheses. The function φ satisfies the limit conditions

$$\lim_{x\to\pm\infty} \varphi(x) := \varphi(\pm\infty) \in \mathbf{R} \tag{1.3}$$

and

$$\lim_{|x|\to+\infty} x^2\varphi''(x) = 0. \tag{1.4}$$

Moreover, there exist $\sigma \in (0,1)$, $C_\pm > 0$ and $d_0 > 0$ such that

$$\lim_{x\to\pm\infty} |x|^{\sigma-1}\, x\,(\varphi(x) - \varphi(\pm\infty)) = C_\pm \tag{1.5}$$

and

$$|x| \geq d_0 \quad \Rightarrow \quad \varphi'(x) \leq 0. \tag{1.6}$$

Finally, there exists $M > 0$ such that

$$\begin{aligned} &\left| x^k\, y^l\, \frac{\partial^{k+l+m} f}{\partial x^k\, \partial y^l\, \partial t^m} \right| \leq M, \quad 0 \leq k+l+m \leq 2, \\ &\left| x^{k+1}\, y^l\, \frac{\partial^{k+l+m} f}{\partial x^k\, \partial y^l\, \partial t^m} \right| \leq M, \quad 0 \leq k+l+m \leq 1, \\ &\left| x^k\, y^{l+1}\, \frac{\partial^{k+l+m} f}{\partial x^k\, \partial y^l\, \partial t^m} \right| \leq M, \quad 0 \leq k+l+m \leq 1, \end{aligned} \tag{1.7}$$

for every $(x, y, t) \in \mathbf{R}^3$.

Observe that the cases $k + l + m = 0$ and $k + l + m = 1$ in the first condition in (1.7) follow from the second and third condition.

Now, let us denote by C the solution of

$$x'' + ax^+ - bx^- = 0 \tag{1.8}$$

satisfying the initial condition $x(0) = 1$, $x'(0) = 0$ and by $-S$ its derivative; it is well known that C and S are even and odd, respectively, and $2\pi/n$-periodic.

We are in position to state our main results. Without loss of generality, it is possible to set $n = 1$ in (1.2).

Theorem 1.1. *Assume that* (1.2), (1.3), (1.4), (1.5), (1.6), *and* (1.7) *hold true and that*

$$f(x, -y, -t) = f(x, y, t),\ p(-t) = p(t),\ \forall\ (x, y, t) \in \mathbf{R}^3. \tag{1.9}$$

For every $\theta \in \mathbf{R}$ let

$$\Xi(\theta) = \frac{1}{\pi}\sqrt{a}\Big(\frac{\varphi(+\infty)}{a} - \frac{\varphi(-\infty)}{b}\Big) - \frac{1}{2\pi}\int_0^{2\pi} p(\theta+t)C(t)\,dt \qquad (1.10)$$

and suppose that

$$\Xi(\theta) \geq 0, \quad \forall\ \theta \in \mathbf{R}. \qquad (1.11)$$

Then there exists $\epsilon_0 > 0$, such that for every $\omega \in (n, n+\epsilon_0)$, the equation (1.1) has a solution $(x_\omega(t), x'_\omega(t))$ of Mather type with rotation number ω. More precisely:

- *if $\omega = p/q$ is rational, the solutions $(x_\omega(t+2i\pi), x'_\omega(t+2i\pi))$, $1 \leq i \leq q-1$ are periodic solutions of period $2q\pi$; moreover, in this case*

$$\lim_{\omega\to n}\min_{t\in\mathbf{R}}\big(|x_\omega(t)| + |x'_\omega(t)|\big) = +\infty.$$

- *if ω is irrational, the solution $(x_\omega(t), x'_\omega(t))$ is either a usual quasi-periodic solution or a generalized one.*

We recall that a generalized quasi-periodic solution means a solution of (1.1) such that the closed set $\overline{\{(x_\omega(2i\pi), x'_\omega(2i\pi)) : i \in \mathbf{Z}\}}$ is a Denjoy's minimal set.

Remark 1.2. We observe that the conclusion of Theorem 1.1 holds true also when $C_\pm < 0$ and in assumptions (1.6) and (1.11) the inequalities are reversed.

Remark 1.3. Under the same assumptions of Theorem 1.1, except for (1.9), the existence of unbounded or periodic solutions to (1.1) follows from Corollary 1.1 in [8] and Corollary 3 in [9], respectively. See also [11].

We recall now the following definition, frequently used in the literature.

Definition 1.1. *Let $n \geq 0$ be an integer. We say a function $g(r,t,\theta)$ is $O_n(r^{-j})$, if it is smooth in (r,t), continuous in θ, periodic of period 2π in both t and θ, and*

$$\Big|r^{k+j}\frac{\partial^{k+l}}{\partial r^k \partial t^l} g\Big| \leq M, \quad 0 \leq k+l \leq n,$$

for some constant $M > 0$ and all (r,t,θ).

Moreover, we call a function $g(r,t,\vartheta)$ $o_n(r^{-j})$, if it is smooth in (r,t), continuous in θ, periodic of period 2π in both t and θ, and

$$\lim_{r\to+\infty}\Big|r^{k+j}\frac{\partial^{k+l} g}{\partial r^k \partial t^l}(r,t,\theta)\Big| = 0, \quad 0 \leq k+l \leq n,$$

uniformly in (t,θ).

When $n = 0$ we will simply write O and o instead of O_0 and o_0.

Example 1.1. Consider the function $f(t,x,y) = e(t)\alpha(x)\beta(y)$, with $e \in C^2([0,2\pi])$ and $\alpha, \beta \in C^2(\mathbf{R})$ such that

$$\alpha(x) = O_2(x^{-c}), \quad \beta(y) = O_2(y^{-c}), \quad c > 1.$$

It is straightforward to check that f satisfies assumption (1.7). Finally, it is easy to check that any function which grows at infinity like $M + \arctan \frac{1}{x^\alpha}$, with $0 < \alpha < 1$ and $M \in \mathbf{R}$, satisfies assumptions (1.3) to (1.6).

2. Preliminary changes of variables

In this section, we will introduce some changes of variables for the study of the first-order system

$$\begin{cases} x' = -y, \\ y' = ax^+ - bx^- + \varphi(x) + f(x,-y,t) - p(t), \end{cases} \tag{2.1}$$

which is clearly equivalent to (1.1). Let us observe that assumption (1.9) implies that system (2.1) is reversible with respect to the transformation $G : (x,y) \mapsto (x,-y)$; i.e., $P^{-1} = G \circ P \circ G$, where P is the Poincaré map associated to (2.1) (see [13])); we remark that this condition will be preserved along all the changes of coordinates we are going to introduce.

We point out that the transformations we will define are quite standard when studying asymptotic properties of the Poincaré map of (2.1) (see e.g. [13, 19]). Let us set

$$x = rC\left(\theta\right), \quad y = rS\left(\theta\right);$$

with the transformation $(r,\theta) \mapsto (x,y)$, with $r > 0$ and $\theta(\bmod 2\pi)$, system (2.1) becomes

$$\begin{cases} r' = a^{-1}\left[\varphi(rC(\theta)) + f(rC(\theta), -rS(\theta), t) - p(t)\right] S(\theta), \\ \theta' = 1 + a^{-1}r^{-1}\left[\varphi(rC(\theta)) + f(rC(\theta), -rS(\theta), t) - p(t)\right] C(\theta). \end{cases} \tag{2.2}$$

From assumptions (1.3) and (1.7) we deduce that

$$\lim_{r\to+\infty} \left\{1 + a^{-1}r^{-1}\left[\varphi(rC(\theta)) + f(rC(\theta), -rS(\theta), t) - p(t)\right] C(\theta)\right\} = 1 > 0, \tag{2.3}$$

uniformly in $\theta \in [0,2\pi]$. This implies that for every solution (r,θ) of (2.2) there exists $r^* > 0$ such that

$$\frac{d\theta}{dt} > 0, \quad \forall\, r \geq r^*.$$

Hence, the function θ is invertible and let us denote by t its inverse; as a consequence the function $(r(t(\cdot)), t(\cdot))$ is a solution of the system

$$\frac{dr}{d\theta} = g_1(r,t,\theta), \quad \frac{dt}{d\theta} = g_2(r,t,\theta), \tag{2.4}$$

where

$$g_1(r,t,\theta) = \frac{\Psi(r,t,\theta)}{1+\Phi(r,t,\theta)}, \qquad g_2(r,t,\theta) = \frac{1}{1+\Phi(r,t,\theta)} \tag{2.5}$$

and

$$\begin{aligned}\Phi(r,t,\theta) &= \frac{1}{ar}\left(\varphi\left(rC(\theta)\right) + f\left(rC(\theta), -rS(\theta), t\right) - p(t)\right)C(\theta),\\ \Psi(r,t,\theta) &= \frac{1}{a}\left(\varphi\left(rC(\theta)\right) + f\left(rC(\theta), -rS(\theta), t\right) - p(t)\right)S(\theta),\end{aligned} \tag{2.6}$$

for every $(r,t,\theta) \in (0,+\infty) \times \mathbf{R} \times [0,2\pi]$.

Lemma 2.1. *The functions g_1 and g_2 given in* (2.5)-(2.6) *satisfy*

$$\begin{aligned}g_1(r,t,\theta) &= \Psi(r,t,\theta) + O_1(1/r),\\ g_2(r,t,\theta) &= 1 - \Phi(r,t,\theta) + O_1(1/r^2),\end{aligned} \tag{2.7}$$

for $r \to +\infty$.

Proof. 1. We first prove the following estimates on Φ and Ψ:

$$\Phi(r,t,\theta) = O_1(1/r), \qquad \Psi(r,t,\theta) = O_1(1); \tag{2.8}$$

i.e.,

$$\begin{aligned}&\left|r^{k+1}\frac{\partial^{k+l}\Phi}{\partial r^k\,\partial t^l}(r,t,\theta)\right| \le M, \quad 0 \le k+l \le 1,\\ &\left|r^{k}\frac{\partial^{k+l}\Psi}{\partial r^k\,\partial t^l}(r,t,\theta)\right| \le M, \quad 0 \le k+l \le 1,\end{aligned} \tag{2.9}$$

for some constant $M > 0$. Let us check the validity of the estimates on Φ (the estimates on Ψ are completely analogous); when $k = 0$ and $l = 0$ or $l = 1$ the relations

$$|r\Phi(r,t,\theta)| \le M, \qquad \left|r\frac{\partial\Phi}{\partial t}(r,t,\theta)\right| \le M$$

plainly follow from the fact that C, p, p', φ, f and $\partial_t f$ are bounded (according to the periodicity and the continuity of p, p' and to assumptions (1.4) and (1.7)).

For $k = 1$ and $l = 0$, we have

$$\begin{aligned}\frac{\partial\Phi}{\partial r}(r,t,\theta) &= -\frac{1}{ar^2}(\varphi(rC(\theta)) + f(rC(\theta), -rS(\theta), t) - p(t))C(\theta)\\ &+ \frac{1}{ar}\Big(\varphi'(rC(\theta))C(t) + \frac{\partial f}{\partial x}(rC(\theta), -rS(\theta), t)C(t)\\ &\quad - \frac{\partial f}{\partial y}(rC(\theta), -rS(\theta), t)S(t)\Big)C(\theta) \qquad (2.10)\\ &= -\frac{1}{ar^2}(\varphi(rC(\theta)) + f(rC(\theta), -rS(\theta), t) - p(t))C(\theta)\\ &+ \frac{1}{ar^2}(\varphi'(rC(\theta))rC(t) + \frac{\partial f}{\partial x}(rC(\theta), -rS(\theta), t)rC(t)\\ &- \frac{\partial f}{\partial y}(rC(\theta), -rS(\theta), t)rS(t)\Big)C(\theta).\end{aligned}$$

Arguing as before, using again (1.4) and (1.7), from (2.10) it is easy to show that

$$\left|r^2\frac{\partial\Phi}{\partial r}(r,t,\theta)\right| \leq M$$

holds true.

2. Now, using (2.8) we prove the estimate on g_1 in (2.7). Indeed, we have to check that

$$\left|r^{k+1}\frac{\partial^{k+l}(g_1 - \Psi)}{\partial r^k\,\partial t^l}(r,t,\theta)\right| \leq M, \quad 0 \leq k + l \leq 1, \qquad (2.11)$$

where

$$(g_1 - \Psi)(r,t,\theta) = \Psi(r,t,\theta)\left((1 + \Phi(r,t,\theta))^{-1} - 1\right), \quad \forall\ r > 0,\ (t,\theta) \in \mathbf{R}^2.$$

To this aim, let us observe that (2.8) implies that

$$\left((1 + \Phi(r,t,\theta))^{-1} - 1\right) = O(1/r), \quad (1 + \Phi(r,t,\theta))^{-2} = O(1), \quad r \to +\infty. \qquad (2.12)$$

From these relations and (2.8) it is immediate to deduce that

$$\begin{aligned}|r(g_1 - \Psi)(r,t,\theta)| &= r|\Psi(r,t,\theta)|\left|(1 + \Phi(r,t,\theta))^{-1} - 1\right| = O(1),\\ \left|r\frac{\partial(g_1 - \Psi)}{\partial t}(r,t,\theta)\right| &\leq r\left|\frac{\partial\Psi}{\partial t}(r,t,\theta)\right|\ \left|(1 + \Phi(r,t,\theta))^{-1} - 1\right|\\ &+ |r\Psi(r,t,\theta)|\left|(1 + \Phi(r,t,\theta))^{-2}\right|\left|\frac{\partial\Phi}{\partial t}(r,t,\theta)\right| = O(1),\end{aligned}$$

which correspond to (2.11) with $k = 0$. Finally, for $k = 1$ and $l = 0$, using again (2.8) and (2.12) we infer

$$\left| r^2 \frac{\partial(g_1 - \Psi)}{\partial r}(r,t,\theta) \right| \le r^2 \left| \frac{\partial \Psi}{\partial r}(r,t,\theta) \right| \; \left| (1 + \Phi(r,t,\theta))^{-1} - 1 \right|$$
$$+ \left| r^2 \Psi(r,t,\theta) \right| \left| (1 + \Phi(r,t,\theta))^{-2} \right| \left| \frac{\partial \Phi}{\partial r}(r,t,\theta) \right| = O(1).$$

In an analogous way it is possible to prove the estimate on g_2. □

In view of Lemma 2.1 we can write (2.4) as

$$\begin{cases} \dfrac{dr}{d\theta} = a^{-1} \left[\varphi(rC(\theta) + f(rC(\theta), -rS(\theta), t) - p(t) \right] S(\theta) + O_1(1/r), \\ \dfrac{dt}{d\theta} = 1 - a^{-1} r^{-1} \left[\varphi(rC(\theta) + f(rC(\theta), -rS(\theta), t) - p(t) \right] C(\theta) + O_1(\dfrac{1}{r^2}). \end{cases} \tag{2.13}$$

We remark that system (2.13) is reversible in θ with respect to the involution $(r,t) \mapsto (r,-t)$.

Let us now introduce another action variable, which is equivalent to r but takes into account the bounded perturbation φ. Indeed, for every $(r,\theta) \in (0,+\infty) \times [0,2\pi]$ let us define

$$K(r,\theta) = -a^{-1} \int_0^\theta \varphi\left(rC\left(\psi\right)\right) S\left(\psi\right) d\psi \tag{2.14}$$

and

$$\lambda(r,\theta) = r + K(r,\theta). \tag{2.15}$$

Let us observe that K is 2π-periodic in the second variable; moreover, from assumption (1.4) we deduce that

$$K(r,\theta) = O_1(1). \tag{2.16}$$

As a consequence, we can prove the following.

Lemma 2.2. *System* (2.13) *is transformed into*

$$\begin{cases} \dfrac{d\lambda}{d\theta} = a^{-1} \left[f(\lambda C(\theta), -\lambda S(\theta), t) - p(t) \right] S(\theta) + O_1(1/\lambda), \\ \dfrac{dt}{d\theta} = 1 - a^{-1} \lambda^{-1} \left[\varphi(\lambda C(\theta) + f(\lambda C(\theta), -\lambda S(\theta), t) - p(t) \right] C(\theta) + O_1(\dfrac{1}{\lambda^2}), \end{cases} \tag{2.17}$$

for $\lambda \to +\infty$. *Moreover,* (2.17) *is periodic in* θ *and reversible with respect to* $(\lambda, t) \mapsto (\lambda, -t)$.

Proof. The periodicity and the reversibility of (2.17) immediately follow from the periodicity of K and the symmetry assumption (1.9).

Now, from (2.16) we infer that

$$\lambda(r,\theta) = O_1(1); \tag{2.18}$$

from this relation and assumptions (1.4), (1.7) it is possible (arguing as in the proof of Lemma 2.1) to see that (2.17) holds true. □

As a final step, we can deduce some estimates on the perturbing terms containing f in (2.17); indeed, let us set

$$\begin{aligned} H_1(\lambda,t,\theta) &= a^{-1} f(\lambda C(\theta), -\lambda S(\theta), t) S(\theta),\\ H_2(\lambda,t,\theta) &= -a^{-1}\lambda^{-1} f(\lambda C(\theta), -\lambda S(\theta), t) C(\theta), \end{aligned}$$

for every $(\lambda,t,\theta) \in (0,+\infty) \times \mathbf{R} \times [0,2\pi]$. Using assumption (1.7) it is not difficult to prove (with arguments similar to the above) the following.

Lemma 2.3. *The functions H_1 and H_2 satisfy*

$$\begin{aligned} H_1(\lambda,t,\theta) &= O_1(1/\lambda),\\ H_2(\lambda,t,\theta) &= O_1(1/\lambda^2), \end{aligned}$$

for $\lambda \to +\infty$.

From Lemma 2.3 we deduce that we can write (2.17) in the form

$$\begin{cases} \dfrac{d\lambda}{d\theta} = -a^{-1} p(t) S(\theta) + O_1(1/\lambda),\\ \dfrac{dt}{d\theta} = 1 + a^{-1}\lambda^{-1} \left[p(t) - \varphi(\lambda C(\theta)\right] C(\theta) + O_1(1/\lambda^2). \end{cases} \tag{2.19}$$

3. Proof of the main theorem

In this section, we will study the asymptotic behaviour of the Poincaré map P associated to (2.19); the first step is to obtain a C^1- development of P. From this, we will be able to apply some well-known abstract results which ensure the existence of quasi-periodic solutions.

In what follows, we shall argue according to a now well-established procedure (see e.g. [5, 16, 19]).

Let $\lambda^{-1} = \varepsilon\rho$, $\rho \in \left[\frac{1}{2}, 2\right]$, with a small parameter $\varepsilon > 0$; hence, $\varepsilon \to 0^+$ when $\lambda \to +\infty$.

Under the change of variables $(\lambda, t) \mapsto (\rho, t)$, system (2.19) is transformed into

$$\begin{cases} \dfrac{d\rho}{d\theta} = a^{-1}\varepsilon\rho^2 p(t)S(\theta) + \varepsilon^2 O_1(1), \\ \dfrac{dt}{d\theta} = 1 + a^{-1}\varepsilon\rho\left[p(t) - \varphi(\varepsilon^{-1}\rho^{-1}C(\theta))\right]C(\theta) + \varepsilon^2 O_1(1), \end{cases} \tag{3.1}$$

(for $\varepsilon \to 0^+$) which is reversible with respect to $(\rho, t) \mapsto (\rho, -t)$.

We denote by $(\rho(\cdot;\rho_0,t_0,\varepsilon), t(\cdot;\rho_0,t_0,\varepsilon))$ the solution of (3.1) satisfying the initial condition $(\rho(0), t(0)) = (\rho_0, t_0)$; in order to obtain an expression for the Poincaré map associated to (3.1) we make the ansatz

$$\rho(\theta;\rho_0,t_0,\varepsilon) = \rho_0 + \varepsilon F_1(\theta;\rho_0,t_0,\varepsilon), \quad t(\theta;\rho_0,t_0,\varepsilon) = t_0 + \theta + \varepsilon F_2(\theta;\rho_0,t_0,\varepsilon). \tag{3.2}$$

The relations (3.2) define functions

$$F_1 = F(\theta;\rho_0,t_0,\varepsilon), \quad F_2 = F_2(\theta;\rho_0,t_0,\varepsilon);$$

in particular, when $\theta = 0$ we have

$$F_1(0;\rho_0,t_0,\varepsilon) = F_2(0;\rho_0,t_0,\varepsilon) = 0. \tag{3.3}$$

Arguing as in [5], the study of (ρ, t) as functions of the initial data (ρ_0, t_0) and of the parameter ε leads to

$$F_1(\theta;\rho_0,t_0,\varepsilon) = O_1(1), \quad F_2(\theta;\rho_0,t_0,\varepsilon) = O_1(1). \tag{3.4}$$

Now, from (3.2), (3.3) and (3.1) we deduce that

$$\begin{aligned} F_1(2\pi;\rho_0,t_0,\varepsilon) &= \frac{1}{\varepsilon}\left(\rho(2\pi;\rho_0,t_0,\varepsilon) - \rho_0\right) \\ &= \frac{1}{\varepsilon}\Big(a^{-1}\varepsilon\int_0^{2\pi}\rho^2(2\pi;\rho_0,t_0,\varepsilon)p(t(2\pi;\rho_0,t_0,\varepsilon))S(\theta)\,d\theta + \varepsilon^2 O_1(1)\Big) \\ &= a^{-1}\int_0^{2\pi}(\rho_0 + \varepsilon F_1(2\pi;\rho_0,t_0,\varepsilon))^2 p(t_0 + 2\pi \\ &\qquad + \varepsilon F_2(2\pi;\rho_0,t_0,\varepsilon))S(\theta)\,d\theta + \varepsilon O_1(1). \end{aligned} \tag{3.5}$$

Using the periodicity and the continuity of p and p', together with (3.4) and some standard Taylor series arguments, from (3.5) we infer that

$$F_1(2\pi;\rho_0,t_0,\varepsilon) = a^{-1}{\rho_0}^2\int_0^{2\pi}p(t_0+\theta)S(\theta)d\theta + \varepsilon O_1(1). \tag{3.6}$$

Recalling the definition of Ξ given in (1.10), we conclude that

$$F_1(2\pi;\rho_0,t_0,\varepsilon) = -\frac{2\pi}{a}{\rho_0}^2\,\Xi'(t_0) + \varepsilon O_1(1). \tag{3.7}$$

In an analogous way, using also (1.4), it is possible to prove that

$$F_2(2\pi;\rho_0,t_0,\varepsilon) = a^{-1}\rho_0\Big[\int_0^{2\pi} p(t_0+\theta)C(\theta)d\theta - \int_0^{2\pi}\varphi(\varepsilon^{-1}\rho_0^{-1}C(\theta))C(\theta)d\theta\Big] + \varepsilon O_1(1). \tag{3.8}$$

Hence, letting $\theta = 2\pi$ in (3.2) and using (3.7)-(3.8), we obtain the following asymptotic expansion for the Poincaré map

$$\begin{cases} \rho_1 = \rho_0 - \dfrac{2\pi}{a}\varepsilon{\rho_0}^2\ \Xi'(t_0) + \varepsilon^2 O_1(1) \\ t_1 = t_0 + 2\pi + a^{-1}\varepsilon\rho_0\Big[\displaystyle\int_0^{2\pi} p(t_0+\theta)C(\theta)d\theta \\ \qquad - \displaystyle\int_0^{2\pi}\varphi(\varepsilon^{-1}\rho_0^{-1}C(\theta))C(\theta)d\theta\Big] + \varepsilon^2 O_1(1), \end{cases} \tag{3.9}$$

where $(\rho_1, t_1) = P(\rho_0, t_0)$.

Now, in order to apply to P the abstract results for the existence of quasi-periodic or unbounded and periodic solutions, we need to estimate the integral containing φ in the second equation in (3.9). To this aim, let us define

$$\varphi_+(x) = \varphi(x) - \varphi(+\infty), \quad \forall\ x > 0$$
$$\varphi_-(x) = \varphi(x) - \varphi(-\infty), \quad \forall\ x < 0$$

and

$$E(t_0,\rho_0;\varepsilon) = \rho_0\int_0^{\frac{\pi}{2\sqrt{a}}}\varphi_+(\varepsilon^{-1}\rho_0^{-1}C(\theta))C(\theta)d\theta + \rho_0\int_{\frac{\pi}{2\sqrt{a}}}^{\pi}\varphi_-(\varepsilon^{-1}\rho_0^{-1}C(\theta))C(\theta)d\theta,$$

for every $(t_0,\rho_0;\varepsilon) \in \mathbf{R}\times[1/2,2]\times(0,+\infty)$.

We now state a lemma, based on the assumption (1.3), whose proof can be found in [5, Lemma 3].

Lemma 3.1. *We have*

$$\int_0^{2\pi}\varphi(\varepsilon^{-1}\rho_0^{-1}C(\theta))C(\theta)d\theta = 2\sqrt{a}\Big(\frac{\varphi(+\infty)}{a} - \frac{\varphi(-\infty)}{b}\Big) + 2\frac{E(t_0,\rho_0;\varepsilon)}{\rho_0}.$$

From Lemma 3.1 and (3.9) we plainly deduce the following expansion for the Poincaré map:

$$\begin{cases} \rho_1 = \rho_0 - \dfrac{2\pi}{a}\varepsilon{\rho_0}^2\ \Xi'(t_0) + \varepsilon^2 O_1(1) \\ t_1 = t_0 + 2\pi - \dfrac{2\pi}{a}\varepsilon\rho_0\ \Xi(t_0) - \dfrac{2}{a}\varepsilon E(t_0,\rho_0;\varepsilon) + \varepsilon^2 O_1(1). \end{cases} \tag{3.10}$$

In order to prove Theorem 1.1 we need a more careful estimate on the derivative of the function E with respect to ρ_0. This can be obtained (cf. [5, Lemma 4]) using (1.5) and (1.6).

Lemma 3.2. *There exist $c_0 > 0$, $c_0' > 0$ (depending on $C_\pm$ and σ given in (1.5)) and $\varepsilon^* > 0$ such that for every $\varepsilon \in (0,\varepsilon^*)$ we have*

$$\int_0^{\frac{\pi}{2\sqrt{a}}} \varphi_+(\varepsilon^{-1}\rho_0^{-1}C(\theta))C(\theta)d\theta - \int_0^{\frac{\pi}{2\sqrt{a}}} \varepsilon^{-1}\rho_0^{-1}\varphi_+'(\varepsilon^{-1}\rho_0^{-1}C(\theta))C^2(\theta)d\theta \geq c_0\varepsilon^\sigma + \varepsilon O(1), \tag{3.11}$$

$$\int_{\frac{\pi}{2\sqrt{a}}}^{\pi} \varphi_-(\varepsilon^{-1}\rho_0^{-1}C(\theta))C(\theta)d\theta - \int_{\frac{\pi}{2\sqrt{a}}}^{\pi} \varepsilon^{-1}\rho_0^{-1}\varphi_-'(\varepsilon^{-1}\rho_0^{-1}C(\theta))C^2(\theta)d\theta \geq c_0\varepsilon^\sigma + \varepsilon O(1), \tag{3.12}$$

for $\varepsilon \to 0^+$.

Proof of Theorem 1.1. We will apply the abstract result on the existence of quasi-periodic solutions proved in [7] in the context of Aubry-Mather theory for reversible systems.

We only need to show that the Poincaré map (3.10) has the monotone twist property; i.e.,

$$\frac{\partial t_1}{\partial \rho_0}(\rho_0, t_0) < 0. \tag{3.13}$$

From (3.10) we infer that

$$\frac{\partial t_1}{\partial \rho_0} = -\frac{2\pi}{a}\varepsilon\Xi(t_0) - \frac{2}{a}\varepsilon\frac{\partial E}{\partial \rho_0} + \varepsilon^2 O(1). \tag{3.14}$$

Lemma 3.2 shows that

$$\frac{\partial E}{\partial \rho_0} \geq (c_0 + c_0')\varepsilon^\sigma + \varepsilon O(1), \tag{3.15}$$

uniformly for $(t_0,\rho_0) \in [0,2\pi] \times \left[\frac{1}{2}, 2\right]$, when $\varepsilon \in (0,\varepsilon^*)$. Hence, from (3.14)-(3.15) and (1.11) we deduce that there exists $\varepsilon^{**} > 0$ such that if $\varepsilon \in (0,\varepsilon^{**})$

we have

$$\begin{aligned}\frac{\partial t_1}{\partial \rho_0} &= \varepsilon\Big(- 2\pi a^{-1}\Xi(t_0) - 2a^{-1}\frac{\partial E}{\partial \rho_0} + \varepsilon O(1)\Big) \\ &\leq \varepsilon\left(-2\pi a^{-1}\Xi(t_0) - 2a^{-1}(c_0 + c_0')\varepsilon^{\sigma} + \varepsilon O(1)\right) < 0.\end{aligned}$$

This proves the validity of (3.13). □

Remark 3.1. Some final remarks are in order.

(i) Damping terms of the form $f(x, x', t)$ are not found in literature relative to the existence of quasi-periodic solutions; only some particular cases have been considered (among others) in [17]. We point out that the classical Liénard equation, where $f(x, x', t) = f(x)x'$, cannot be treated in the presence of an asymmetric oscillator; indeed (we refer to Section 5 in [13]), the reversibility of the Poincaré map can be obtained when the first-order planar system $z' = Z(t, z)$, $z = (x, y)$, associated to the given equation, satisfies either $Z_1(t, x, y) = Z_1(-t, -x, y)$, $Z_2(-t, -x, y) = -Z_2(t, x, y)$ or $Z_1(-t, x, -y) = -Z_1(t, x, y)$, $Z_2(t, x, y) = -Z_2(-t, x, -y)$. Hence, if the damping term is inserted in the linear, thus odd, oscillator (as in [17], [19], [24]), then it is sufficient to require oddness in the x-variable, and in this case it is possible to treat the classical Liénard equation. Since in this paper we deal with the asymmetric (not odd) oscillator, the reversibility must be reached through the evenness in the y-variable, which prevents (even if we had not assumed (1.7)) the treatment of the classical Liénard equation.

(ii) Once the study of the damped asymmetric reversible oscillator is accomplished, one may raise the question whether it is possible to extend the results of this paper to the case when the asymmetric term is replaced by an isochronous potential (as in [2],[3],[4]) and to the case of planar first-order systems at resonance (as in [11], [12]). This will be the subject of further research.

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