Differential and Integral Equations Volume 22, Numbers 9-10 (2009), 1019–1032

SOME EXISTENCE RESULTS FOR OPERATOR EQUATIONS INVOLVING DUALITY MAPPINGS ON SOBOLEV SPACES WITH VARIABLE EXPONENT

George Dinca
University of Bucharest, Faculty of Mathematics and Computer Science
St. Academiei, no. 14, RO - 010014 Bucharest, Romania
and
Romanian Academy, "Gheorghe Mihoc - Caius Iacob"
Institute of Mathematics Statistics and Applied Mathematics
Calea 13 Septembrie, no 13, Bucharest, Romania

Dedicated to Patrick Habets and Jean Mawhin
on the occasion of their 65th birthday with deep esteem and affection

1. Introduction

This paper is concerned with some existence results for equations of type $J_\varphi u = h$ and $J_\varphi u = N_f u$ where J_φ is a duality mapping on the Sobolev space with variable exponent $\big(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)}\big)$, $\|u\|_{1,p(\cdot)} = \| |\nabla u| \|_{p(\cdot)}$, corresponding to the gauge function φ, $h \in \big(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)}\big)^*$ and N_f is the Nemytskij operator generated by a Carathéodory function f which satisfies an appropriate growth condition ensuring that N_f may be viewed as acting from $(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)})$ into its dual.

It is well known (see e.g. [2], [3]) that the properties of duality mappings are heavily influenced by the geometry of the spaces where they are defined. In this respect, the geometric properties of the spaces $(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)})$ as established in [5] are used throughout this paper. Accordingly, if $p \in \mathcal{C}(\bar{\Omega})$ and $p(x) > 1$ for any $x \in \bar{\Omega}$, then $(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)})$ is a smooth Banach space.

It follows that, in this case, any duality mapping on $(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)})$ is single-valued. As the same space is also reflexive, we deduce that any duality mapping on $(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)})$ is surjective. In other words, in

AMS Subject Classifications: 35J2, 35J60.

this case, equation $J_\varphi u = h$ has a solution for any $h \in (W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)})^*$ (see Theorem 7 below).

Moreover, if $p \in \mathcal{C}(\bar{\Omega})$ and $p(x) \geq 2$ for all $x \in \bar{\Omega}$, then $(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)})$ is, in addition, uniformly convex (see Theorem 1 in [5]). Consequently, any duality mapping on $(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)})$ is a bijection of this space onto its dual. Consequently, equation $J_\varphi u = N_f u$ may be equivalently written as $u = J_\varphi^{-1} N_f u$. Sufficient conditions ensuring that this fixed point problem has a solution are given in Theorem 8 below, which is the main result of this paper.

Notice that, for the sake of simplicity, we assume throughout this paper that $\Omega \subset \mathbf{R}^N$, $N \geq 2$, is "smooth" i.e., Ω is a bounded open subset of $\mathbf{R}^N$, its boundary $\partial\Omega$ is an infinitely differentiable manifold of dimension $(n-1)$, and Ω lies in a same side of its boundary $\partial\Omega$.

2. The functional framework

2.1. **The space** $(W_0^{1,p(\cdot)}, \| \ \|_{1,p(\cdot)})$**.** Denote by

$$L_+^\infty(\Omega) = \Big\{ u \in L^\infty(\Omega) : 1 \leq u^- = \operatorname*{ess\,inf}_\Omega u \leq u^+ = \operatorname*{ess\,sup}_\Omega u < \infty \Big\}.$$

For $p \in L_+^\infty(\Omega)$, the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ (also known as $L^{p(x)}(\Omega)$) is defined by

$$L^{p(\cdot)}(\Omega) = \Big\{ u : \Omega \to \mathbf{R} : u \text{ is (Lebesgue) measurable and } \rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} \, \mathrm{d}x < \infty \Big\}$$

and it is endowed with the norm $\|u\|_{p(\cdot)} = \inf \big\{ \lambda > 0 : \rho_{p(\cdot)}\big(\frac{u}{\lambda}\big) \leq 1 \big\}$.

The next proposition illuminates the close relation between the norm $\| \ \|_{p(\cdot)}$ and the convex modular $\rho_{p(\cdot)}$:

Proposition 1 (Fan and Zhao [9], theorems 1.2 and 1.3)**.** *One has:*

(a) $\|u\|_{p(\cdot)} > 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}$;

(b) $\|u\|_{p(\cdot)} < 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^-} \geq \rho_{p(\cdot)}(u) \geq \|u\|_{p(\cdot)}^{p^+}$;

(c) $\|u\|_{p(\cdot)} = a > 0 \quad \Leftrightarrow \quad \rho_{p(\cdot)}\big(\frac{u}{a}\big) = 1$.

Proposition 1 has the following corollary:

Corollary 1. *One has:*

(a) $\|u\|_{p(\cdot)} < 1(=1;>1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1(=1;>1)$.
(b) $\|u\|_{p(\cdot)} < \rho_{p(\cdot)}(u) + 1$;

$$\rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^+} + 1, \quad \text{for any } u \in L^{p(\cdot)}(\Omega).$$

Next, we summarize some basic properties of the spaces $L^{p(\cdot)}(\Omega)$ (see, e.g. Edmunds and Rákosník [7], [8], Fan and Zhao [9], Fan and Zhang [10] Kováčik and Rákosník [11]):

Theorem 1.

(a) *For* $p \in L_+^\infty(\Omega)$, $\big(L^{p(\cdot)}(\Omega),\ \|\ \|_{p(\cdot)}\big)$ *is a separable Banach space.*

(b) *In addition, for* $p \in L_+^\infty(\Omega)$ *with* $1 < p^-$, *one has :*

(b$_1$) $\big(L^{p(\cdot)}(\Omega),\ \|\ \|_{p(\cdot)}\big)$ *is uniformly convex and thus reflexive;*

(b$_2$) *the conjugate space of* $\big(L^{p(\cdot)}(\Omega),\ \|\ \|_{p(\cdot)}\big)$ *is*

$$\big(L^{p(\cdot)}(\Omega),\ \|\ \|_{p(\cdot)}\big)^* = \big(L^{p'(\cdot)}(\Omega),\ \|\ \|_{p'(\cdot)}\big),$$

where $p'(\cdot)$ *is given by* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, $p'(\cdot) \in L_+^\infty(\Omega)$ *and* $1 < p'^-$;

(b$_3$) *for any* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *one has*

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

(c) *If* $p_1, p_2 \in L_+^\infty(\Omega)$ *are such that* $p_1(x) \le p_2(x)$ *for a.e.* $x \in \Omega$, *then* $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ *and the embedding is continuous.*

(d) *Let* $p_1, p_2 \in L_+^\infty(\Omega)$ *and let* $f : \Omega \times \mathbf{R} \to \mathbf{R}$ *be a Carathéodory function. If* f *satisfies the growth condition*

$$|f(x,s)| \le c_1 |s|^{p_1(x)/p_2(x)} + a(x), \quad \text{a.e.} \quad x \in \Omega, \forall s \in \mathbf{R},$$

with $c_1 = \text{const.} > 0$ *and* $a \in L^{p_2(\cdot)}(\Omega), a(x) \ge 0$ *a.e.* $x \in \Omega$, *then the Nemytskij operator generated by* f, $(N_f u)(x) = f\big(x, u(x)\big)$ *a.e.* $x \in \Omega$, *is well defined from* $L^{p_1(\cdot)}(\Omega)$ *into* $L^{p_2(\cdot)}(\Omega)$, *continuous and bounded.*

For $p \in L_+^\infty(\Omega)$, the generalized Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \Big\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\Big\}, \quad |\nabla u|^2 = \sum_{i=0}^N \Big(\frac{\partial u}{\partial x_i}\Big)^2$$

and it is endowed with the norm $\|u\| = \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$, $\forall u \in W^{1,p(\cdot)}(\Omega)$.

Theorem 2.

(a) *If* $p \in L_+^\infty(\Omega)$, *then* $\left(W^{1,p(\cdot)}(\Omega), \| \ \|\right)$ *is a separable Banach space.*

(b) *If* $p_1, p_2 \in L_+^\infty(\Omega)$ *and* $p_1(x) \le p_2(x)$, *for almost every* $x \in \Omega$, *then* $W_2^{1,p_2(\cdot)}(\Omega)$ *is embedded into* $W^{1,p_1(\cdot)}(\Omega)$ *continuously.*

(c) *If* $p \in L_+^\infty$ *and* $1 < p^-$, *then* $W^{1,p(\cdot)}(\Omega, \| \ \|)$ *is uniformly convex and thus reflexive.*

(d) *Let* $p, q \in \mathcal{C}_+(\bar{\Omega}) = \left\{h \in \mathcal{C}(\bar{\Omega}) : h^- > 1\right\}$. *If*

$$q(x) < p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & if\ p(x) < N \\ +\infty & if\ p(x) \ge N, \end{cases}$$

then $W^{1,p(\cdot)}(\Omega)$ *is compactly embedded in* $L^{q(\cdot)}(\Omega)$.

For $p \in L_+^\infty(\Omega)$, we define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $(W^{1,p(\cdot)}(\Omega), \| \ \|)$ and $\overset{o}{W}{}^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$.

Theorem 3.

(a) *If* $p \in L_+^\infty(\Omega)$, *then* $\left(W_0^{1,p(\cdot)}(\Omega), \| \ \|\right)$ *is a separable Banach space. One has* $W_0^{1,p(\cdot)}(\Omega) \subset \overset{o}{W}{}^{1,p(\cdot)}(\Omega)$.

(b) *If* $p \in L_+^\infty(\Omega)$ *and* $1 < p^-$, *then* $\left(W_0^{1,p(\cdot)}(\Omega), \| \ \|\right)$ *is uniformly convex and thus reflexive.*

(c) *If* $p \in \mathcal{C}_+(\bar{\Omega})$, *then* $\left(W_0^{1,p(\cdot)}(\Omega), \| \ \|\right)$ *is compactly embedded in* $L^{q(\cdot)}(\Omega)$ *for any* $q \in \mathcal{C}_+(\bar{\Omega})$ *satisfying* $q(x) < p^*(x)$, $x \in \bar{\Omega}$.

(d) *(Poincaré's inequality) If* $p \in \mathcal{C}_+(\bar{\Omega})$, *there is a constant* $c > 0$ *such that* $\|u\|_{p(\cdot)} \le c \, \||\nabla u|\|_{p(\cdot)}$, *for all* $u \in \overset{o}{W}{}^{1,p(\cdot)}(\Omega)$.

Remark 1. By Theorem 3(d), it follows that $\|u\|$ and $\|u\|_{1,p(\cdot)} = \||\nabla u|\|_{p(\cdot)}$ are equivalent norms on $\overset{o}{W}{}^{1,p(\cdot)}(\Omega)$ and from Theorem 3(c) we deduce that if $p \in \mathcal{C}_+(\bar{\Omega})$, then $\left(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)}\right)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$ for any $q \in \mathcal{C}_+(\bar{\Omega})$ satisfying $q(x) < p^*(x)$, $x \in \bar{\Omega}$.

In what follows, $W_0^{1,p(\cdot)}(\Omega)$ will be considered as endowed with the norm $\| \ \|_{1,p(\cdot)}$ and we will often write $W_0^{1,p(\cdot)}(\Omega)$ and $L^{p(\cdot)}(\Omega)$ instead of $\left(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)}\right)$ and $\left(L^{p(\cdot)}(\Omega), \| \ \|_{p(\cdot)}\right)$ respectively.

Theorem 4 (Dinca and Matei [5])**.**

(a) *If* $p \in L_+^\infty(\Omega)$ *and* $1 < p^-$, *then* $\left(L^{p(\cdot)}, \| \ \|_{p(\cdot)}\right)$ *is smooth. At any* $u \in L^{p(\cdot)}, u \ne 0$, *the gradient of the norm* $\| \ \|'_{p(\cdot)}(u)$ *is given by*

$$\left\langle \| \|_{p(\cdot)}^{'}(u), h \right\rangle = \frac{\displaystyle\int_\Omega p(x) \frac{|u(x)|^{p(x)-1}\, sgn\ u(x)}{\|u\|_{p(\cdot)}^{p(x)}} h(x) \mathrm{d}x}{\displaystyle\int_\Omega p(x) \frac{|u(x)|^{p(x)}}{\|u\|_{p(\cdot)}^{p(x)+1}} \mathrm{d}x}, \quad \forall h \in L^{p(\cdot)}(\Omega). \tag{2.1}$$

(b) *If $p \in \mathcal{C}(\bar{\Omega})$ and $p(x) > 1$, for all $x \in \bar{\Omega}$, then $\left(W_0^{1,p(\cdot)}(\Omega), \| \|_{1,p(\cdot)}\right)$ is smooth. At any $u \in W_0^{1,p(\cdot)}(\Omega)$, $u \neq 0$, the gradient of the norm $\| \|_{1,p(\cdot)}^{'}(u)$ is given by*

$$\left\langle \| \|_{1,p(\cdot)}^{'}(u), h \right\rangle = \frac{\displaystyle\int_\Omega p(x) \frac{|\nabla u(x)|^{p(x)-2}\, \nabla u \nabla h}{\|u\|_{1,p(\cdot)}^{p(x)}} \mathrm{d}x}{\displaystyle\int_\Omega p(x) \frac{|\nabla u(x)|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)+1}} \mathrm{d}x}, \quad \forall h \in W_0^{1,p(\cdot)}(\Omega). \tag{2.2}$$

(c) *If $p \in \mathcal{C}(\bar{\Omega})$ and $p(x) \geq 2$, for all $x \in \bar{\Omega}$, then $\left(W_0^{1,p(\cdot)}(\Omega), \| \|_{1,p(\cdot)}\right)$ is uniformly convex.*

2.2. **Duality mappings: basic definitions and surjectivity results.** In what follows, by gauge function we will understand a mapping $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ which is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(r) \to \infty$ as $r \to \infty$.

If X is a real Banach space, by duality mapping on X corresponding to the gauge function φ one understands the multivalued mapping $J_\varphi : X \to 2^{X^*}$ defined as follows:

$$\begin{aligned} J_\varphi 0_X &= \{0_{X^*}\} \\ J_\varphi x &= \varphi(\|x\|) \{x^* \in X^* : \langle x^*, x \rangle = \|x\|, \|x^*\| = 1\}, \quad x \neq 0_X. \end{aligned}$$

Clearly, J_φ may be equivalently defined as follows:

$$J_\varphi x = \{u^* \in X^* : \langle u^*, x \rangle = \varphi(\|x\|) \|x\|, \|u^*\| = \varphi(\|x\|)\}, \quad \text{for all} \quad x \in X. \tag{2.3}$$

According to the Hahn-Banach theorem, $D(J_\varphi) = \{x \in X : J_\varphi x \neq \emptyset\} = X$ and, according to an Asplund's result [1], at any $u \in X$,

$$J_\varphi u = \partial F(u) \qquad \text{where} \qquad F(u) = \int_0^{\|u\|} \varphi(t) \mathrm{d}t \tag{2.4}$$

and ∂F stands for the subdifferential of F in the sense of convex analysis. Since F is convex, it follows that $J_\varphi = \partial F$ is monotone.

From the definition of J_φ it appears that a duality mapping is single valued if and only if X is smooth; i.e., at any $x \in X, x \neq 0$, there is a unique support functional.

Since X is smooth if and only if the norm of X is Gâteaux differentiable (cf. Diestel [4], chapters two, Section 1, Theorem 1), it follows from (2.4) that on a smooth real Banach space, the duality mapping corresponding to a gauge function φ is a single-valued operator $J_\varphi : X \to X^*$ defined as follows:

$$\begin{aligned} J_\varphi 0_X &= 0_{X^*} \\ J_\varphi x &= \varphi(\|x\|) \, \| \, \|' (x), \quad x \neq 0_X, \end{aligned} \tag{2.5}$$

where $\| \, \|' (x)$ stands for the gradient of the norm (in the Gâteaux sense) at $x \neq 0_X$.

Remark 2. If we insert in (2.5) the $\| \, \|' (x)$ given by (2.1) and (2.2) we obtain the form of the duality mappings on $\left(L^{p(\cdot)}(\Omega), \| \, \|_{p(\cdot)}\right)$ and $\left(W_0^{1,p(\cdot)}(\Omega), \| \, \|_{1,p(\cdot)}\right)$ respectively, under the hypotheses of Theorem 4.

Duality mappings are example of monotone operators having good surjectivity properties.

For example, it is a classical result (see e.g. [2], [3]) that X is reflexive if and only if any duality mapping J_φ on X is surjective: $X^* = \bigcup_{x \in X} J_\varphi x$ (respectively, $X^* = J_\varphi(X)$ in case X is smooth).

A Banach space is said to possess the Kadeč-Klee property if it is strictly convex and

$$\left[x_n \overset{w}{\rightharpoonup} x \quad \text{and} \quad \|x_n\| \to \|x\| \right] \quad \Rightarrow \quad x_n \overset{s}{\to} x.$$

Examples of Banach spaces possessing the Kadeč-Klee property are given by the following theorem (see e.g. Diestel [4], chapter 2, Section 2, Theorems 3 and 4(iii)).

Theorem 5. *Every locally uniformly convex space (in particular, every uniformly convex space) has the Kadeč-Klee property.*

Finally, we give (see e.g. [3]) the following.

Theorem 6. *If X is a real reflexive and smooth Banach space having the Kadeč-Klee property, then any duality mapping $J_\varphi : X \to X^*$ is bijective and has a continuous bounded and monotone inverse. Moreover, $J_\varphi^{-1} = \chi^{-1} J^*_{\varphi^{-1}}$*

where $J^*_{\varphi^{-1}} : X^* \to X^{**}$ *is the duality mapping on* X^* *corresponding to the gauge function* φ^{-1} *and* $\chi : X \to X^{**}$ *is the canonical isomorphism defined by* $\langle \chi(x), x^* \rangle = \langle x^*, x \rangle$, $\forall x \in X$, $\forall x^* \in X^*$.

3. Duality mappings on $\left(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)}\right)$: surjectivity results

From Theorems 4, 5, 6 and Remark 2 we deduce the following theorem.

Theorem 7. *Let* $p \in \mathcal{C}(\bar{\Omega})$. *Then one has*

(a) *If* $p(x) > 1$, *for all* $x \in \bar{\Omega}$, *any duality mapping* J_φ *on* $\left(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)}\right)$ *is single valued and surjective and defined by*

$$J_\varphi 0 = 0, \tag{3.1}$$

$$\langle J_\varphi u, h \rangle = \frac{\varphi(\|u\|_{1,p(\cdot)}) \displaystyle\int_\Omega p(x) \frac{|\nabla u(x)|^{p(x)-2} \nabla u \nabla h}{\|u\|_{1,p(\cdot)}^{p(x)}} \mathrm{d}x}{\displaystyle\int_\Omega p(x) \frac{|\nabla u(x)|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)+1}} \mathrm{d}x}, \quad \forall h \in W_0^{1,p(\cdot)}(\Omega),$$

if $u \neq 0$.

(b) *If* $p(x) \geq 2$, *for all* $x \in \bar{\Omega}$, *any duality mapping on* $\left(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)}\right)$ *is bijective and has a continuous inverse.*

Theorem 8. *Let* $p, q \in \mathcal{C}_+(\bar{\Omega})$ *satisfy*

$$q(x) < p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & p(x) < N \\ +\infty & p(x) \geq N \end{cases} \tag{3.2}$$

and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ *be a Carathéodory function which satisfies the growth condition*

$$|f(x,s)| \leq c_1 |s|^{\frac{q(x)}{q'(x)}} + a(x), \quad \text{for a.e.} \quad x \in \Omega, \text{ for every } s \in \mathbf{R} \tag{3.3}$$

with $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1, a \in L^{q'(\cdot)}(\Omega)$, $a(x) \geq 0$ *a.e.* $x \in \Omega$, c_1=*const.*> 0. *Denote by*

$$N_f : L^{q(\cdot)}(\Omega) \to L^{q'(\cdot)}(\Omega), \quad \left(N_f u\right)(x) = f(x, u(x)) \quad \text{for a.e.} \quad x \in \Omega,$$

the Nemytskij operator generated by f *and by* $J_\varphi : W_0^{1,p(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$ *the duality mapping on* $W_0^{1,p(\cdot)}(\Omega)$ *corresponding to the gauge function* φ.

Then for any gauge function φ with the property $\frac{\varphi(t)}{t^{q^+-1}} \to \infty$, as $t \to \infty$, the solution set of the equation

$$J_\varphi u = N_f u \tag{3.4}$$

is nonempty and compact in $W_0^{1,p(\cdot)}(\Omega)$.

Some auxiliary results, which are of interest on their own, are needed for the proof.

Proposition 2. *Let $(X, \| \|_X)$ be a reflexive real Banach space, compactly embedded in the real Banach space $(Z, \| \|_Z)$. Denote by i the compact injection of X into Z. Then, there exists some $v_0 \in X$ with $\|v_0\|_X = 1$ such that $\|i\| = \|iv_0\|_Z$.*

Proof. Let $(v_n) \subset X$ be such that $\|v_n\|_X = 1$ for all $n \in \mathbf{N}$ and $\|iv_n\|_Z \to \|i\|$, as $n \to \infty$.

The space X being reflexive, there is a subsequence of (v_n), still denoted by (v_n), such that $v_n \rightharpoonup v_0$ (weakly in X). We have

$$\|v_0\|_X \le \liminf_{n\to\infty} \|v_n\|_X = 1. \tag{3.5}$$

On the other hand, because X is compactly embedded in Z, it holds that $iv_n \to iv_0$ (strongly in Z), which implies

$$\|iv_0\|_Z = \|i\|. \tag{3.6}$$

Moreover, from (3.5) and (3.6) we derive that $\|v_0\|_X = 1$. □

, The next theorem emphasizes another meaning of $\|i\|$, that is, being the first eigenvalue of a pair of duality mappings. We denote by $J_{X,\varphi} : X \to X^*$ and $J_{Z,\varphi} : Z \to Z^*$ the duality mappings (assumed to be single valued) on X, respectively on Z, corresponding to the same gauge function φ.

We say that $\lambda \in \mathbf{R}$ is an eigenvalue of the pair $(J_{X,\varphi}, J_{Z,\varphi})$ if there is some $u \in X \setminus \{0\}$ such that

$$J_{X,\varphi} u = \lambda J_{Z,\varphi} u. \tag{3.7}$$

Equality (3.7) is understood in the following sense:

$$\langle J_{X,\varphi} u, v\rangle_{X,X^*} = \lambda \langle J_{Z,\varphi}(iu), iv\rangle_{Z,Z^*}, \quad \text{for all} \quad v \in X. \tag{3.8}$$

The vector u in (3.7) is called the eigenvector of the pair $(J_{X,\varphi}, J_{Z,\varphi})$ corresponding to the eigenvalue λ.

Theorem 9. *The hypotheses on X and Z are as in Proposition 2, and we assume in addition that both of X and Z are with G-differentiable norm. Let $r > 1$ be a given number and $J_{X,(r-1)}, J_{Z,(r-1)}$ be the duality mappings* *on X and Z respectively, corresponding to the same gauge function $\varphi(t) = t^{r-1}, t \geq 0$. Then*

(a) $\|i\|^{-r}$ *is an eigenvalue of the pair* $\left(J_{X,(r-1)}, J_{Z,(r-1)}\right)$*;*

(b) *if λ is an eigenvalue of the pair* $\left(J_{X,(r-1)}, J_{Z,(r-1)}\right)$*, then $\lambda \geq \|i\|^{-r}$.*

Proof. (a) Consider $\Phi : X \to \mathbf{R}$ defined by

$$\Phi(v) = \frac{1}{r}\|v\|_X^r - \frac{\|i\|^{-r}}{r}\|iv\|_Z^r, \quad \text{for} \quad v \in X.$$

Let $v_0 \in X$ with $\|v_0\|_X = 1$ be such that $\|i\| = \|iv_0\|_Z$. Then $\Phi(v) \geq 0 = \Phi(v_0)$, for all $v \in X$. As a minimum point, v_0 satisfies the Euler equation $\Phi'(v_0) = 0$ and by a straightforward computation, for arbitrary $v \in X$, we have

$$\begin{aligned}
&\left\langle \Phi'(v_0), v\right\rangle_{X,X^*} \\
&= \left\langle \|v_0\|_X^{r-1} \| \ \|_X' (v_0), v\right\rangle_{X,X^*} - \|i\|^{-r}\left\langle \|iv_0\|_Z^{r-1} \| \ \|_Z' (iv_0), iv\right\rangle_{Z,Z^*} \\
&= \left\langle J_{X,(r-1)}v_0, v\right\rangle_{X,X^*} - \|i\|^{-r}\left\langle J_{Z,(r-1)}(iv_0), iv\right\rangle_{Z,Z^*} = 0,
\end{aligned}$$

showing that $\|i\|^{-r}$ is an eigenvalue of the pair $\left(J_{X,(r-1)}, J_{Z,(r-1)}\right)$, with v_0 a corresponding eigenvector.

(b) Let λ be an eigenvalue of the pair $\left(J_{X,(r-1)}, J_{Z,(r-1)}\right)$ and let $u \in X \setminus \{0\}$ be a corresponding eigenvector. Then

$$\|u\|_X^r = \left\langle J_{X,(r-1)}u, u\right\rangle_{X,X^*} = \lambda\left\langle J_{z,(r-1)}(iu), (iu)\right\rangle_{Z,Z^*} = \lambda\|iu\|_Z^r,$$

which implies

$$\lambda = \left(\frac{\|u\|_X}{\|iu\|_Z}\right)^r \geq \|i\|^{-r}. \qquad \square$$

Proof of Theorem 8. First we explain what we mean by a solution of equation (3.4).

By virtue of (3.2), Theorem 3 and Remark 1, $\left(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{1,p(\cdot)}\right)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$.

Denote by i the compact injection of $W_0^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ and let $i^* : L^{q'(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$, $i^*v = v \circ i$, for all $v \in L^{q'(\cdot)}(\Omega)$, be its adjoint (notice that i^* is also compact and $\|i\| = \|i^*\|$). By virtue of (3.3) and Theorem 1(d), the Nemytskij operator N_f is well defined from $L^{q(\cdot)}(\Omega)$

into $L^{q'(\cdot)}(\Omega)$, continuous and bounded. By a solution of (3.4) we mean an element $u \in W_0^{1,p(\cdot)}(\Omega)$ which satisfies

$$J_\varphi u = \left(i^* N_f i\right) u. \tag{3.9}$$

Notice that $\left(i^* N_f i\right) : W_0^{1,p(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega), \| \; \|_{1,p(\cdot)}\right)^*$ is compact.

Equality (3.9) equivalently rewrites as

$$\begin{aligned} \langle J_\varphi u, h\rangle &= \left\langle \left(i^* N_f i\right) u, h\right\rangle = \langle N_f(iu), ih\rangle \\ &= \int_\Omega f(x, u(x)) h(x) \mathrm{d}x, \quad \text{for all} \quad h \in W_0^{1,p(\cdot)}(\Omega). \end{aligned}$$

Finally, according to (3.1), we can say that by a solution of equation (3.9) we mean an element $u \in W_0^{1,p(\cdot)}(\Omega)$ which satisfies

$$\frac{\varphi(\|u\|_{1,p(\cdot)}) \displaystyle\int_\Omega p(x) \frac{|\nabla u|^{p(x)-2} \nabla u \nabla h}{\|u\|_{1,p(\cdot)}^{p(x)}} \mathrm{d}x}{\displaystyle\int_\Omega p(x) \frac{|\nabla u|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)+1}} \mathrm{d}x} = \int_\Omega f(x, u(x)) h(x) \mathrm{d}x,$$

for all $h \in W_0^{1,p(\cdot)}(\Omega)$. Let

$$\mathcal{S}(J_\varphi, N_f) = \left\{ u \in W_0^{1,p(\cdot)}(\Omega) : J_\varphi u = (i^* N_f i) u \right\} \tag{3.10}$$

be the solution set of equation (3.9).

Since, by Theorem 7(b), any duality mapping J_φ on $W_0^{1,p(\cdot)}(\Omega)$ is bijective and has a continuous inverse, one has:

$$\mathcal{S}(J_\varphi, N_f) = \mathrm{Fix}(K) = \left\{ u \in W_0^{1,p(\cdot)}(\Omega) : u = Ku \right\}, \; K = J_\varphi^{-1}(i^* N_f i). \tag{3.11}$$

Notice that $K : W_0^{1,p(\cdot)}(\Omega) \to W_0^{1,p(\cdot)}(\Omega)$ is compact.

Thus, proving that $\mathcal{S}(J_\varphi, N_f) \neq \emptyset$ reduces to proving that the compact operator K has a fixed point. We will show that by using the Leray-Schauder *a priori* estimate method, namely we will show that

$$\mathcal{B} = \left\{ u \in W_0^{1,p(\cdot)}(\Omega) : \exists t \in [0,1], \; u = tKu \right\}$$

is bounded. Since for $t = 0$ the only solution of equation $u = tKu$ is $u = 0$, it is enough to prove that the set $\left\{ u \in W_0^{1,p(\cdot)}(\Omega) : \exists t \in (0,1], \; u = tKu \right\}$ is bounded.

Indeed, let $u \in W_0^{1,p(\cdot)}(\Omega)$ satisfy $u = tJ_\varphi^{-1}(i^* N_f i)u$ for some $t \in (0,1]$ or, equivalently,

$$J_\varphi\big(\frac{u}{t}\big) = \big(i^* N_f i\big)u.$$

Since φ is increasing, we deduce that

$$\begin{aligned}\varphi(\|u\|_{1,p(\cdot)}) &\le \varphi\big(\frac{\|u\|_{1,p(\cdot)}}{t}\big) = \Big\|J_\varphi\big(\frac{u}{t}\big)\Big\|_{\left(W_0^{1,p(\cdot)}(\Omega)\right)^*} \\ &= \|(i^* N_f i)u\|_{\left(W_0^{1,p(\cdot)}(\Omega)\right)^*} \le \|i^*\| \ \|N_f(iu)\|_{L^{q'(\cdot)}(\Omega)}. \end{aligned} \tag{3.12}$$

In order to estimate $\|N_f(iu)\|_{L^{q'(\cdot)}(\Omega)}$, first we prove that, for any $v \in L^{q(\cdot)}(\Omega)$,

$$\|N_f(v)\|_{q'(\cdot)} < c_1 \|v\|_{q(\cdot)}^{q^+-1} + \big(2c_1 + \|a\|_{q'(\cdot)}\big). \tag{3.13}$$

Indeed, since

$$|(N_f v)(x)| = |f(x, v(x))| \le c_1 |v(x)|^{\frac{q(x)}{q'(x)}} + a(x),$$

we deduce that

$$\|N_f v\|_{q'(\cdot)} \le \left\|c_1 |v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}} + a(\cdot)\right\|_{q'(\cdot)} \le c_1 \left\||v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}\right\|_{q'(\cdot)} + \|a\|_{q'(\cdot)}. \tag{3.14}$$

Now, let us prove that

$$\left\||v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}\right\|_{q'(\cdot)} < \|v\|_{q(\cdot)}^{q^+-1} + 2. \tag{3.15}$$

Indeed, one has

(i)

$$\|v(\cdot)\|_{q(\cdot)} \ge 1 \qquad \Rightarrow \qquad \left\||v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}\right\|_{q'(\cdot)} \le \|v\|_{q(\cdot)}^{q^+-1}. \tag{3.16}$$

This is seen as follows. Proving (3.16) is equivalent to proving that (see Corollary 1(a)) $\|v\|_{q(\cdot)} \ge 1$ implies

$$\rho_{q'(\cdot)}\Big(\frac{|v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}}{\|v\|_{q(\cdot)}^{q^+-1}}\Big) = \int_\Omega \frac{|v(x)|^{q(x)}}{\|v\|_{q(\cdot)}^{(q^+-1)q'(x)}} \mathrm{d}x \le 1.$$

This last inequality is justified as follows. Since $\|v\|_{q(\cdot)} \ge 1$ and

$$\begin{aligned}(q^+-1)q'(x) - q(x) &= q^+ q'(x) - (q(x) + q'(x)) = q^+ q'(x) - q(x)q'(x) \\ &= q'(x)(q^+ - q(x)) \ge 0,\end{aligned}$$

we have

$$\frac{|v(x)|^{q(x)}}{\|v\|_{q(\cdot)}^{(q^+-1)q'(x)}}=\frac{|v(x)|^{q(x)}}{\|v\|_{q(\cdot)}^{q(x)}}\cdot\frac{1}{\|v\|_{q(\cdot)}^{(q^+-1)q'(x)-q(x)}}\le\frac{|v(x)|^{q(x)}}{\|v\|_{q(\cdot)}^{q(x)}}.$$

Consequently, by Proposition 1(c),

$$\rho_{q'(\cdot)}\Big(\frac{|v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}}{\|v\|_{q(\cdot)}^{q^+-1}}\Big)\le\rho_{q(\cdot)}\Big(\frac{v}{\|v\|_{q(\cdot)}}\Big)=1.$$

(ii)

$$\|v\|_{q(\cdot)}<1 \qquad\Rightarrow\qquad \Big\|\,|v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}\Big\|_{q'(\cdot)}<2. \tag{3.17}$$

Indeed, by Corollary 1, one has

$$\Big\|\,|v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}\Big\|_{q'(\cdot)}<\rho_{q'(\cdot)}\big(|v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}\big)+1=\rho_{q(\cdot)}(v)+1<1+1=2.$$

Clearly, (3.15) is a consequence of (3.16) and (3.17), and (3.13) is a consequence of (3.14) and (3.15).

By writing (3.13) for $v=iu$, $u\in W_0^{1,p(\cdot)}(\Omega)$, we get

$$\|N_f(iu)\|_{q'(\cdot)}<c_1\|i\|^{q^+-1}\|u\|_{1,p(\cdot)}^{q^+-1}+\big(2c_1+\|a\|_{q'(\cdot)}\big). \tag{3.18}$$

In particular, if $u\in W_0^{1,p(\cdot)}(\Omega)$ and satisfies $u=tJ_\varphi^{-1}(i^*N_fi)u$ for some $t\in(0,1]$, we derive from (3.12) and (3.18) that

$$\varphi(\|u\|_{1,p(\cdot)})<c_1\|i\|^{q^+}\|u\|_{1,p(\cdot)}^{q^+-1}+\big(2c_1+\|a\|_{q'(\cdot)}\big)\|i\|. \tag{3.19}$$

Since $\dfrac{\varphi(t)}{t^{q^+-1}}\to\infty$ as $t\to\infty$, it follows from (3.19) that there is a constant $R>0$ such that $\|u\|_{1,p(\cdot)}<R$.

Once this a priori estimate is obtained, the invariance under homotopy of compact transforms of the Leray-Schauder degree gives us

$$d_{LS}(I-(tK),B(0,R),0)=d_{LS}(I,B(0,R),0)=1$$

for all $t\in[0,1]$, I denoting the identity over $W_0^{1,p(\cdot)}(\Omega)$.

Consequently, $\mathrm{Fix}(tK)\neq\emptyset$ and $\mathrm{Fix}(tK)\subset B(0,R)$ for any $t\in[0,1]$. Since $\mathrm{Fix}(tK)$ is closed and bounded, by standard arguments, the compactness of (tK) implies the compactness of $\mathrm{Fix}(tK)$. In particular, for $t=1$, $\mathrm{Fix}(K)$ is nonempty and compact. □

Corollary 2. *The same hypotheses as for Theorem 8 are assumed, except condition (3.2) which is replaced by*

$$q \in \mathcal{C}(\bar{\Omega}), \quad 1 < q(x) < p(x), \quad \forall x \in \bar{\Omega}. \tag{3.20}$$

Let $J_{(p^+-1)}$ be the duality mapping on $W_0^{1,p(\cdot)}(\Omega)$ corresponding to the gauge function $\varphi(t) = t^{p^+-1}, t \geq 0$. Then, the solution set of the equation

$$J_{(p^+-1)}u = N_f u \tag{3.21}$$

is nonempty and compact.

Proof. By arguing as in the proof of Theorem 8 we obtain that any $u \in W_0^{1,p(\cdot)}(\Omega)$ which satisfies $u = tJ^{-1}_{p^+-1}(i^* N i)u$ for some $t \in (0,1]$ also satisfies

$$\|u\|^{p^+-1}_{p(\cdot)} \leq c_1 \|i\|^{q^+} \|u\|^{q^+-1}_{p(\cdot)} + c_2, \quad c_2 = \|i\| \left(2c_1 + \|a\|_{q'(\cdot)}\right) \tag{3.22}$$

(which is nothing but inequality (3.19) written for $\varphi(t) = t^{p^+-1}$).

Since from (3.20) we infer that $q^+ < p^+$, inequality (3.22) implies the boundedness of the set

$$\left\{u \in W_0^{1,p(\cdot)}(\Omega) : \exists t \in [0,1],\ u = tJ^{-1}_{(p^+-1)}(i^* N_f i)u\right\}. \qquad \square$$

Remark 3. If condition (3.20) is replaced by the weaker condition

$$q \in \mathcal{C}(\bar{\Omega}), \quad 1 < q(x) \leq p(x), \quad \forall x \in \bar{\Omega}, \tag{3.23}$$

then

(a) if $q^+ < p^+$, the conclusion of the Corollary 2 remains valid;
(b) if $q^+ = p^+$, then (3.22) becomes

$$\left(1 - c_1 \|i\|^{p+}\right) \|u\|^{p^+-1}_{1,p(x)} \leq c_2$$

and a sufficient condition ensuring that the conclusion of Corollary 2 remains true is $c_1 < \|i\|^{-p^+}$.

In particular, for $q(x) = p(x)$, for all $x \in \bar{\Omega}$ we have the following result.

Theorem 10. *Let $p \in \mathcal{C}(\bar{\Omega})$ be such that $p(x) \geq 2$, for all $x \in \bar{\Omega}$. Let $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function which satisfies the growth condition*

$$|f(x,s)| \leq c_1 |s|^{\frac{p(x)}{p'(x)}} + a(x), \quad a.e. \quad x \in \Omega, \ \textit{for every } s \in \mathbf{R}$$

with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, $a \in L^{p'(\cdot)}$, $a(x) \geq 0$, and $c_1 \in \left(0, \|i\|^{-p^+}\right)$, where i stands for the compact injection of $W_0^{1,p(\cdot)}(\Omega)$ into $L^{p'(\cdot)}(\Omega)$.

Denote by $N_f : L^{p(\cdot)}(\Omega) \to L^{p'(\cdot)}(\Omega)$, $(N_f u) = f(x, u(x))$ *the Nemytskij operator generated by* f *and by* $J_{(p^+-1)}$ *the duality mapping on* $W_0^{1,p(\cdot)}(\Omega)$ *corresponding to the gauge function* $\varphi(t) = t^{p^+-1}$. *Then, the solution set of the equation*

$$J_{(p^+-1)}u = N_f u$$

is nonempty and compact in $W_0^{1,p(\cdot)}(\Omega)$.

Acknowledgements. The author thanks the anonymous reviewer for his valuable suggestions which contributed to improving the final form of this paper.

REFERENCES

[1] E. Asplund, *Positivity of duality mappings,* Bull. Amer. Math. Soc., 73 (1967), 200–203.

[2] F.E. Browder, "Problèmes Non-Linéaires," Les Presses de l'Université de Montréal, 1964.

[3] I. Cioranescu, "Duality Mapping in Nonlinear Functional Analysis," Publishing House of Romanian Academy, Bucharest, 1974 (in Romanian).

[4] J. Diestel, "Geometry of Banach spaces," LNM Springer-Verlag, 485, 1975.

[5] G. Dinca and P. Matei, *Geometry of Sobolev spaces with variable exponent: Smoothness and uniform convexity,* C. R. Acad. Sci. Paris, Ser. I 347 (2009), 885-889.

[6] G. Dinca, P. Jebelean, and J. Mahwin, *Variational and Topological methods for Dirichlet problems with p-Laplacian,* Portugaliae Mathematica, Volume 58 (2001), 339–377.

[7] D.E. Edmunds and J. Rákosník, *Sobolev embeddings with variable exponent,* Studia Math., 143 (2000), 267–293.

[8] D.E. Edmunds and J. Rákosník, *Sobolev embeddings with variable exponent, II,* Math. Nachr., 246-247(2002), 53–67.

[9] X. Fan and D. Zhad, *On the space* $L^{p(x)(\Omega)}$ *and* $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., 263 (2001), 424–446.

[10] X. Fan and Q.H. Zhang, *Existence of solutions for* $p(x)$*-Laplacian Dirichlet problem,* Nonlinear Analysis, 52 (2003), 1843–1852.

[11] O. Kováčik and J. Rákosník, *On spaces* $L^{p(x)(\Omega)}$ *and* $W^{m,p(x)}(\Omega)$, Czech. Math. J., 41 (1991), 592–618.