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HETEROCLINICS FOR NON-AUTONOMOUS SECOND-ORDER DIFFERENTIAL EQUATIONS

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Dedicated to Patrick Habets and Jean Mawhin on their sixty-fifth birthdays

Abstract. We investigate new conditions for the existence of heteroclinics connecting ± 1 for a non-autonomous equation of the form

$$\ddot{u} = a(t)f(u) \tag{1}$$

where $a(t)$ is a bounded positive function and $f(\pm 1) = 0$. Here $f = F'$, where F is a C^1 non-negative function such that $F(-1) = F(1) = 0$. We are interested mainly in the case where $a(t)$ approaches its positive limit, as $|t| \to \infty$, from above, but we allow also the ("asymptotically asymmetric") case where $|\lim_{t\to-\infty} a(t) - \lim_{t\to+\infty} a(t)|$ is a sufficiently small positive number. Variational methods are used in the proofs.

1. Introduction

Let us consider the autonomous scalar equation

$$\ddot{u} = a\,F'(u), \tag{2}$$

where

- (F) $F \in C^1(\mathbb{R},\mathbb{R})$ is a non-negative function, $F(-1) = F(1) = 0$ and $F > 0$ in $(-1,1)$ and
- (A_0) $a > 0$ is a constant.

The equation (2) has two equilibria, $u = \pm 1$, at the (same) zero level of the potential. As for solutions of (2) energy is conserved; that is,

$$\tfrac{1}{2}\dot{u}^2 - a\,F(u) = K \tag{3}$$

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for some constant K, it makes sense to look for heteroclinic solutions connecting -1 and 1, i.e., solutions such that

$$u(\pm\infty) := \lim_{x\to\pm\infty} u(x) = \pm 1 \quad \text{and} \quad \dot{u}(\pm\infty) := \lim_{x\to\pm\infty} \dot{u}(x) = 0$$

or the same properties with the roles of $+\infty$ and $-\infty$ reversed. In fact, for such solutions we must have $K = 0$ in (3) and they can be easily found by separation of variables. It is easily seen that they do not reach the equilibria ± 1 in finite time whenever there exists $c > 0$ so that

$$F(u) \leq c(u \pm 1)^2 \tag{4}$$

in a neighborhood of -1 and $+1$ respectively.

The heteroclinic of (2) that goes from -1 to 1 is unique up to translation. For future reference, we shall denote by z_a the solution of (3) such that

$$z_a(-\infty) = -1 \quad z_a(+\infty) = 1 \quad \text{and} \quad z_a(0) = 0. \tag{5}$$

Instead of using elementary integration techniques, z_a can be characterized by a variational property. Formally, (2) is the Euler-Lagrange equation of the functional

$$\mathbf{I}_a(u) := \int_{-\infty}^{+\infty} \Big(\frac{\dot{u}^2}{2} + a\,F(u)\Big)dt. \tag{6}$$

We may look for the heteroclinics of (2) as minimizers of $\mathbf{I}_a$ in the functional space

$$\mathcal{E} := \left\{u \in H^1_{loc}(\mathbb{R},\mathbb{R}) : u(\pm\infty) = \pm 1\right\}.$$

In fact it is not difficult to see that we may confine ourselves to functions taking values in $[-1,1]$ by simply assuming that F *is extended by* 0 *on* $(-\infty,-1)\cup(+1,+\infty)$, *which we assume hereafter.*

It can be shown that (see [1]) the following holds.

Theorem 1.1. *Let* $F \in C^1([-1,1],\mathbb{R})$, *extended by* 0 *outside the interval* $(-1,1)$, *satisfy the assumption* (F). *Then the functional* $\mathbf{I}_a$ *defined by* (6) *attains a minimum in* $\mathcal{E}$. *A minimizer is a heteroclinic solution of* (2) *connecting* -1 *and* 1.

It is not difficult to see that (see [5])

$$\mathbf{I}_a(z_a) = \sqrt{2a}\int_{-1}^{1}\sqrt{F(z)}\,dz \tag{7}$$

and

$$\mathbf{I}_k(z_a) = \frac{1}{2}(1+\frac{k}{a})\mathbf{I}_a(z_a). \tag{8}$$

In the same way, it is even easier to show that

$$\int_{-\infty}^{t_0} (\frac{\dot{u}^2}{2} + a\,F(u))dt$$

attains a minimum in the class of functions $u \in H^1_{loc}((-\infty, t_0))$ such that $u(-\infty) = -1$, $u(t_0) = 0$, the minimum being attained at $z_a(t - t_0)$ with value $\sqrt{2a}\int_{-1}^{0} \sqrt{F(z)}\,dz$. A similar remark applies to

$$\int_{t_0}^{+\infty} (\frac{\dot{u}^2}{2} + a\,F(u))dt.$$

Theorem 1.1 extends to some second-order non-autonomous differential equations. (See [3, 6, 7] for a variational approach to some properties of non-autonomous equations that are inherited, or not, from their autonomous counterparts.) Consider

$$\ddot{u} = a(t)f(u), \tag{9}$$

where a primitive F of $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the assumption (F) and $a \in L^\infty(\mathbb{R}, \mathbb{R})$ is such that

(A) there exist $a_1, a_2 \in \mathbb{R}$ so that $0 < a_1 \le a(t) \le a_2$ for all $t \in \mathbb{R}$.

Maybe the simplest way to look for heteroclinics of (9) that connect ± 1 is to consider the functional

$$\mathbf{J}(u) := \int_{-\infty}^{+\infty} \Big(\frac{\dot{u}^2}{2} + a(t)F(u)\Big)dt \tag{10}$$

and seek conditions that allow us to minimize it in $\mathcal{E}$. We have the following.

Theorem 1.2. *Assume that $f \in C(\mathbb{R}, \mathbb{R})$, $F' = f$ in $\mathbb{R}$ are such that F and $a \in L^\infty(\mathbb{R}, \mathbb{R})$ satisfy $(F) - (A)$. If in addition*

$$\lim_{|t|\to\infty} a(t) = a_2$$

and $a(t) < a_2$ in some subset with non-zero measure, then (9) has a heteroclinic solution from -1 to 1. This solution takes values in $[-1, 1]$.

A proof can be found in [1]. See [2] for related results.

We can partially improve this result. Let us consider the assumptions:

(A_1) There exists t_0 such that $a(t)$ is increasing (respectively decreasing) in $[t_0, +\infty)$ (respectively in $(-\infty, -t_0]$), $l := \lim_{|t|\to\infty} a(t)$ exists and

$$\lim_{|t|\to\infty} |t|(l - a(t)) = +\infty. \tag{11}$$

(F_1) There exists $\delta > 0$ and A, $B > 0$ such that

$$A(x \pm 1)^2 \leq F(x) \leq B(x \pm 1)^2 \quad \text{if} \quad 0 \leq 1 \pm x < \delta.$$

Theorem 1.3. *Assume $f \in C(\mathbb{R}, \mathbb{R})$, $F' = f$ in $\mathbb{R}$ are such that F and $a \in L^\infty(\mathbb{R}, \mathbb{R})$ with $\operatorname{essinf} a > 0$ satisfy (F), (F_1), (A_1). Then (9) has a heteroclinic solution from -1 to 1. This solution takes values in $(-1, 1)$.*

Proof. According to Theorem 1.1 in [6] (see also [7]) the boundary-value problems

$$\ddot{u} = a(t)f(u), \quad u(t_0) = -1, \; u(+\infty) = 1 \tag{12}$$

and

$$\ddot{u} = a(t)f(u), \quad u(-t_0) = 1, \; u(-\infty) = -1 \tag{13}$$

have solutions, say $\varphi(t)$ and $\psi(t)$, respectively, taking values in $[-1, 1]$. Let us define the functions

$$U(t) = \begin{cases} -1 & \text{if } t \leq t_0 \\ \varphi(t) & \text{if } t \geq t_0 \end{cases} \tag{14}$$

$$V(t) = \begin{cases} \psi(t) & \text{if } t \leq -t_0 \\ 1 & \text{if } t \geq -t_0. \end{cases} \tag{15}$$

In every interval $[-T, T]$ with $T > t_0$ the boundary-value problem

$$\ddot{u} = a(t)f(u), \quad u(-T) = -1, \; u(T) = 1 \tag{16}$$

has the lower solution $U(t)$ and the upper solution $V(t)$. Accordingly (see [4], Section II-4) (16) has a solution $u_T(t)$ such that $U(t) \leq u_T(t) \leq V(t)$, $-T \leq t \leq T$. The Ascoli-Arzelà theorem and a diagonal argument allows us to pass to the limit along some subsequence of values $T \to \infty$ and we obtain the desired solution of (9). □

Remark 1. If $F \in C^2(\mathbb{R})$ and $F''(\pm 1) > 0$ the same result is true replacing (11) with the weaker condition

$$\lim_{|t| \to +\infty} (l - a(t)) e^{2\mu_\pm |t|} = +\infty, \tag{17}$$

where

$$l := \lim_{t \to +\infty} a(t) \quad \text{and} \quad \mu_\pm = \sqrt{\eta F''(\pm 1)}.$$

See [6].

Now, contrasting with the behavior of $a(t)$ with respect to its limit in the two preceding theorems, the main purpose of this paper is to consider (9) in the situation where $a(t)$ approaches its limits at infinity from above. The simplest case corresponds to assumption (A) together with

$$\lim_{|t|\to\infty} a(t) = a_1, \tag{18}$$

but we shall also consider in the last section the case where

$$\lim_{t\to-\infty} a(t) = a_3, \quad \lim_{t\to+\infty} a(t) = a_1 \tag{19}$$

and $a_3 - a_1 > 0$ is sufficiently small. This will be referred to as the "asymptotically asymmetric" case. The results are stronger and the proofs simpler in case (18). Moreover, the proofs need only small changes to cover (19). So we focus our attention first in the simplest situation.

We have organized the remainder of the paper as follows: in the next section we present the main results pertaining to the simplest case; Sections 3 and 4 are devoted to proofs; in the final section we deal with the asymptotically asymmetric case. As we shall point out, our results in the asymptotically asymmetric case are in fact only of perturbative nature.

2. Main results

As a first step towards the heteroclinics of (9) we shall introduce a boundary value problem for (9) in bounded intervals $[-T, T]$, the boundary condition being

$$u(-T) = -1, \quad u(T) = 1, \tag{20}$$

and then we let $T \to +\infty$.

Also, as a complement of (F), for simplicity, *we assume that F has only one critical point,* 0 *in* $(-1, 1)$. Setting $F' = f$:

$$(f) \qquad uf(u) < 0 \quad \text{if} \quad 0 \neq u \in (-1, 1).$$

One can handle a more general setting, see [5]. Note however that no symmetry is implied by assumption (f).

With respect to $(9) - (20)$ we shall prove the following.

Theorem 2.1. *Assume that $f \in C(\mathbb{R}, \mathbb{R})$, $F' = f$ in $\mathbb{R}$, so that F and $a \in L^\infty(\mathbb{R}, \mathbb{R})$ satisfy $(F) - (f) - (A)$. If in addition*

$$\lim_{|t|\to\infty} a(t) = a_1$$

and $a(t) > a_1$ almost everywhere in some neighborhood of zero, then for all T sufficiently large (9) − (20) has at least three solutions. These solutions take values in $[-1, 1]$ and two of these are monotone.

Next we introduce some notation. Set

$$m = \mathbf{I}_{a_1}(z_{a_1}),\ m_- := \sqrt{2a_1}\int_{-1}^{0}\sqrt{F(z)}\,dz,\ m_+ := \sqrt{2a_1}\int_{0}^{1}\sqrt{F(z)}\,dz. \quad (21)$$

Clearly, $m = m_- + m_+$.

We also consider the following slightly stronger form of (F_1):

(F_2) There exists $\delta > 0$ and A, $B > 0$ such that

$$A(x \pm 1)^2 \leq f(x)(x \pm 1) \leq B(x \pm 1)^2 \quad \text{if} \quad 0 \leq 1 \pm x < \delta.$$

Remark 2. This condition implies that $u(t) \equiv \pm 1$ is the unique solution $-1 \leq u \leq 1$ to $\ddot{u} = a(t)f(u)$ satisfying an initial condition of the kind $u(t_0) = \pm 1$, $\dot{u}(t_0) = 0$.

Theorem 2.2. *Assume that $f \in C(\mathbb{R}, \mathbb{R})$, $F' = f$ in $\mathbb{R}$, so that F and $a \in L^\infty(\mathbb{R}, \mathbb{R})$ satisfy $(F) - (f) - (A) - (F_2)$ and*

$$\lim_{|t| \to \infty} a(t) = a_1$$

with $a(t) > a_1$ almost everywhere in some neighborhood of zero.

(i) If

$$\frac{a_2}{a_1} < \frac{\min\{4m_- + m,\, 4m_+ + m\}}{m}, \quad (22)$$

then (9) has a heteroclinic that connects ± 1. The heteroclinic takes values in $(-1, 1)$.

(ii) If

$$\frac{a_2}{a_1} < 5, \quad (23)$$

then (9) has either a non-trivial homoclinic at ± 1 or a heteroclinic that connects ± 1. This solution takes values in $(-1, 1)$.

Remark. If $m_- = m_+$, in particular if F is even, the right-hand side of (22) is equal to 3.

Remark 3. Statement (i) is essentially a version of the main result of [5] under slightly weaker regularity assumptions.

Theorem 2.3. *Assume that* $f \in C(\mathbb{R},\mathbb{R})$, $F' = f$ *in* $\mathbb{R}$, *so that* F *and* $a \in L^\infty(\mathbb{R},\mathbb{R})$ *satisfy* $(F)-(f)-(A)-(F_2)$ *and*

$$\lim_{|t|\to\infty} a(t) = a_1$$

with $a(t) > a_1$ *almost everywhere in some neighborhood of zero.*

(*i*) *If*

$$(\sup F)\int_{-\infty}^{+\infty} (a(t)-a_1))\,dt < 2\min\{m_-, m_+\}, \tag{24}$$

then (9) *has a heteroclinic that connects* ± 1. *The heteroclinic takes values in* $(-1,1)$.

(*ii*) *If*

$$(\sup F)\int_{-\infty}^{+\infty} (a(t)-a_1)\,dt < 2m, \tag{25}$$

then (9) *has either a non-trivial homoclinic at* ± 1 *or a heteroclinic that connects* ± 1. *This solution takes values in* $(-1,1)$.

We end this section by stressing that in the presence of symmetries the existence of a (symmetric) heteroclinic of (9) is a much simpler matter than in the general case.

Indeed, let us assume that F *and* a *are even functions*. Then we may focus our attention on a boundary-value problem in the half line, by considering (9) together with the boundary conditions

$$u(0) = 0, \quad u(+\infty) = 1, \tag{26}$$

since, given a solution $u(t)$ of $(9)-(26)$, then the odd extension of u is a heteroclinic of (9).

Example 1. Suppose that $a \in L^\infty_{\text{loc}}(\mathbb{R})$ is bounded below: $a(t) \ge a > 0$ for all $t \in \mathbb{R}$. Then $(9)-(26)$ has a solution x such that $z_a \le x \le 1$.

In fact, for each $T > 0$ the two-point boundary-value problem

$$\ddot{x} = a(t)f(x), \quad x(0)=0,\ x(T)=1 \tag{27}$$

has a lower solution (z_a) and an upper solution (the constant 1). Therefore (27) has a solution x_T such that $z_a \le x_T \le 1$. Using the Ascoli-Arzelà theorem and a standard diagonal method we find that along some subsequence x_T converges as $T \to \infty$ to a solution x of $(9)-(26)$ as desired.

Example 2. In some instances $(9)-(26)$ has a solution despite the fact that $\liminf_{t\to+\infty} a(t) = 0$. Suppose that there exists $0 < \varepsilon \leq 1$ with the property

$$\sup_{0<x<\varepsilon} \frac{2x}{|f(x)|} \leq \inf_{0<t<\infty} a(t)\cosh^2 t.$$

Then $(9) - (26)$ has a solution x such that $\varepsilon \tanh t \leq x(t) \leq 1$.

In fact, it is easily seen that $\varepsilon \tanh t$ is a lower solution of (27) and we argue as in the preceding example. The fact that one obtains a solution with $x(+\infty) = 1$ is straightforward from the differential equation (9).

3. The boundary-value problem in a finite interval

In this section we study the boundary-value problem $(9) - (20)$.

We shall use the functional space

$$\mathcal{E}_T := \left\{u \in H^1(-T,T) : u(\pm T) = \pm 1\right\}.$$

In addition to $\mathbf{J}$ consider the functional

$$\mathbf{J}_T(u) := \int_{-T}^{T} \Big(\frac{\dot{u}^2}{2} + a(t)F(u)\Big)dt \tag{28}$$

which is well defined and C^1 in the linear manifold $\mathcal{E}_T$; its critical points are solutions to $(9) - (20)$. Clearly, $\mathbf{J}_T$ is coercive and weakly lower semicontinuous, so that

$$m_T := \inf_{\mathcal{E}_T} \mathbf{J}_T$$

is attained (at a solution of $(9) - (20)$). In fact we can say more.

Consider the open sets

$$\Omega_{T,+} = \{u \in \mathcal{E}_T : u(t) > 0\, \forall t \in [0,T]\}$$

and $\Omega_{T,-} = \{u \in \mathcal{E}_T : u(t) < 0\, \forall t \in [-T,0]\}$.

Recall that $m = \mathbf{I}_{a_1}(z_{a_1})$. The following lemma is fairly obvious.

Lemma 3.1. $\lim_{T\to\infty} m_T = m$.

Lemma 3.2. $\lim_{T\to\infty} \inf_{\Omega_{T,+}} \mathbf{J}_T = m = \lim_{T\to\infty} \inf_{\Omega_{T,-}} \mathbf{J}_T$.

Proof. Given $\varepsilon > 0$, take $d > 0$ so that $|z_{a_1}(\pm d) \mp 1| < \varepsilon$. Define $\tilde{z} \in \mathcal{E}$ by $\tilde{z}(t) = z_{a_1}$ if $|t| \leq d$, $\tilde{z} \equiv -1$ in $[-\infty, -d-\varepsilon)$, $\tilde{z} \equiv 1$ in $[d+\varepsilon, \infty)$ and $\tilde{z}$ linear on $[-d-\varepsilon, -d]$ and $[d, d+\varepsilon]$. Set $\tau_c\tilde{z}(t) := \tilde{z}(t-c)$ for all $t \in \mathbb{R}$. Then by Lebesgue's theorem

$$\lim_{c\to\infty} \mathbf{J}(\tau_c\tilde{z}) = \mathbf{I}_{a_1}(\tilde{z}) \leq m + (1 + 2a_2 \sup F)\varepsilon.$$

On the other hand m_T decreases with T and $m_T \geq m$ for all $T > 0$. Hence, for large $c > 0$,

$$\mathbf{J}(\tau_c \tilde{z}) = \mathbf{J}_{c+d+\epsilon}(\tau_c \tilde{z}) \leq m + O(\varepsilon)$$

(here and in what follows we simply denote by $O(\varepsilon)$ the expression $(1 + 2a_2 \sup F)\varepsilon)$, showing that $\inf_{\Omega_{T,-}} \mathbf{J}_T = m$. The argument applies to $\Omega_{T,+}$ in an obvious manner. □

Lemma 3.3. *Let $a(t) > a_1$ in a neighborhood of zero. Then there exists $\Delta > 0$ such that for all $T > 0$ sufficiently large*

$$u \in \mathcal{E}_T \ \textit{ and } \ u(0) = 0 \ \Rightarrow \ \mathbf{J}_T(u) \geq m + \Delta.$$

Proof. Take a function $u \in \mathcal{E}_T$ with $u(0) = 0$. Set $C := m + 1$. There exists $L > 0$ such that $a(t) > a_1$ for $|t| \leq L$ and $8CL < 1$. Now, if

$$\int_{-T}^{T} \frac{\dot{u}^2}{2}\, dt > C,$$

it is enough to take $\Delta \leq 1$; so, let us suppose $T > L$ and

$$\int_{-T}^{T} \frac{\dot{u}^2}{2}\, dt \leq C.$$

We have

$$\begin{aligned}\mathbf{J}_T(u) &= \int_{-L}^{L} \Big(\frac{\dot{u}^2}{2} + a(t)\, F(u)\Big)\, dt + \int_{[-T,-L]\cup[L,T]} \Big(\frac{\dot{u}^2}{2} + a(t)\, F(u)\Big) dt \\ &\geq \int_{-T}^{T} \Big(\frac{\dot{u}^2}{2} + a_1\, F(u)\Big) dt + \int_{-L}^{L} (a(t) - a_1)\, F(u)\, dt.\end{aligned}$$

In the last integrand we have

$$|u(t)| \leq \max\{\int_{-L}^{0} |\dot{u}| dt,\ \int_{0}^{L} |\dot{u}| dt\} \leq \sqrt{2C}\sqrt{L} \leq \frac{1}{2},$$

so that, setting $\Delta_0 := \min_{|x|\leq 1/2} F(x)$,

$$\mathbf{J}_T(u) \geq m + \min\{1, \Delta_0 \int_{-L}^{L} (a(t) - a_1)\, dt\}.$$

□

Proof of Theorem 2.1 Consider $\inf_{\Omega_{T,+}} \mathbf{J}_T$. By the coerciveness and weak lower semicontinuity of $\mathbf{J}_T$ the infimum of J_T in the convex closed set $\{u \in \mathcal{E}_T : u(t) \geq 0\, \forall t \in [0,T]\}$ is attained, say at a function w. By Lemmas 3.3 and 3.2 it follows that, for large T, $w(0) > 0$. Suppose $w(t^*) = 0$ for some $t^* > 0$. Then if $\max_{[0,t*]} w = w(t_1)$ with $w(t) < w(t_1)$ for all $t \in (t_1, t^*]$

and $t_2 = \inf\{s \in [t^*, T] : w(t_1) = w(s)\}$, replacing $w|_{[t_1,t_2]}$ with the constant $w(t_1)$ we obtain a new function $\hat{w}$ such that $\mathbf{J}_T(\hat{w}) < \mathbf{J}_T(w)$. Hence $w \in \Omega_{T,+}$.

The same is true with respect to $\Omega_{T,-}$. Hence $\mathbf{J}_T$ possesses two local minima. Since $\mathbf{J}_T$ obviously satisfies the Palais-Smale condition, the mountain pass lemma [9] implies that $\mathbf{J}_T$ has a third critical point.

Finally, let v be the minimizer in $\Omega_{T,-}$. It is easy to see that v is negative, and therefore increasing, in $[-T, 0]$. If v is not monotone, then it attains a lowest (negative) local minimum, say at $t_1 > 0$ and a largest (positive) local maximum at $t_2 > 0$. Set $t_1' = \inf\{s < t_1 : \ v(s) = v(t_1)\}$ and $t_2' = \sup\{s > t_2 : \ v(s) = v(t_2)\}$. Replacing v with the constant $v(t_1)$ in $[t_1', t_1]$ or the constant $v(t_2)$ in $[t_2, t_2']$ (according to whether $F(v(t_1)) < F(v(t_2))$ or not) we obtain a monotone function $\hat{v} \in \Omega_{T,-}$ such that $\mathbf{J}_T(\hat{v}) < \mathbf{J}(v)$. The same argument applies to $\Omega_{T,+}$. □

4. Proof of the main theorems

Proof of Theorem 2.2. We shall use the mountain pass setting [9] in a form slightly different from the one we mentioned in the preceding section. Moreover, since the proofs are quite similar, we detail only the proof of Theorem 2.2. See Remark 4 below.

Note that assumption (22) may be written equivalently as

$$\frac{1}{2}(1 + \frac{a_2}{a_1})m < m + 2\ \min\{m_-,\ m_+\}.$$

So if (22) holds let us fix $\eta > 0$ so that

$$\frac{1}{2}(1 + \frac{a_2}{a_1})m + \eta < m + 2\ \min\{m_-,\ m_+\}. \tag{29}$$

On the other hand, if (23) holds, let us fix $\eta > 0$ so that

$$\frac{1}{2}(1 + \frac{a_2}{a_1})m + \frac{5\eta}{4} < 3m. \tag{30}$$

According to Lemma 3.3 and using the notation of the proof of Lemma 3.2 it is possible to choose $\Delta > 0$ and $T_0 = c + d + \varepsilon > 0$ such that, if $T > T_0$, $p = \tau_{-c}\tilde{z}$, $q = \tau_c\tilde{z}$,

$$\max\{\mathbf{J}_T(p) + \Delta,\ \ \mathbf{J}_T(q) + \Delta\} < \mathbf{J}_T(u)$$

whenever $u \in \mathcal{E}_T$, $u(0) = 0$.

We may assume in addition that

$$\eta < \Delta/2, \tag{31}$$

and with respect to (F_2),

$$\eta < 4\delta. \tag{32}$$

We suppose also that ε has been fixed so that

$$O(\varepsilon) < \eta/4. \tag{33}$$

Given $T > T_0$, let Γ_T denote the set of continuous paths $\gamma : [0,1] \to \mathcal{E}_T$ such that $\gamma(0) = p$, $\gamma(1) = q$. As for any such path there exists $s_0 \in [0,1]$ such that $\gamma(s_0)(0) = 0$; it follows that

$$k_T := \inf_{\gamma\in\Gamma_T} \max_{s\in[0,1]} \mathbf{J}_T(\gamma(s)) \geq \max\{\mathbf{J}_T(p) + \Delta,\, \mathbf{J}_T(q) + \Delta\}.$$

By the mountain pass theorem, k_T is a critical value of $\mathbf{J}_T$; let us denote by u_T the corresponding solution of (9) $-$ (20), so that

$$k_T = \mathbf{J}_T(u_T) \ \text{ and } \ \mathbf{J}'_T(u_T) = 0 \tag{34}$$

and in particular

$$\ddot{u}_T = a(t) f(u_T), \quad u_T(\pm T) = \pm 1. \tag{35}$$

Next we shall obtain estimates on the approximate solutions.

First, as we are assuming that F vanishes outside $[-1,1]$, it is clear from (35) that

$$-1 < u_T(t) < 1, \ \ \forall t \in (-T,T). \tag{36}$$

On the other hand, since k_T decreases with T, there is a number C such that

$$\int_{-T}^{T} \Big(\frac{\dot{u}_T^2}{2} + a(t)\, F(u_T)\Big)\, dt \leq C. \tag{37}$$

In fact we shall make use of a more specific upper bound. Since the family $\{\tau_s \tilde{z} : -c \leq s \leq c\}$ is a path connecting p and q, we have by the characterization of the critical level k_T

$$\begin{aligned} &\max\{\mathbf{J}_T(p) + \Delta,\ \ \mathbf{J}_T(q) + \Delta\} \leq k_T \qquad (38)\\ &\leq \max_{-c\leq s\leq c} \int_{-T}^{T} \Big(\frac{1}{2}(\frac{d}{dt}\tau_s \tilde{z})^2 + a(t)\, F(\tau_s \tilde{z})\Big)\, dt \leq \int_{-T}^{T} \Big(\frac{\dot{\tilde{z}}^2}{2} + a_2\, F(\tilde{z})\Big)\, dt \\ &= \int_{-d}^{d} \Big(\frac{\dot{z}_{a_1}^2}{2} + a_2\, F(z_{a_1})\Big)\, dt + O(\varepsilon) \leq \frac{1}{2}(1 + \frac{a_2}{a_1})m + O(\varepsilon). \end{aligned}$$

Remark 4. An alternative upper bound for k_T may be obtained in the following way:

$$k_T \leq \int_{-T}^{T} \left(\frac{\dot{\tilde{z}}^2}{2} + a_1\, F(\tilde{z})\right)\, dt + \int_{-T}^{T} (a(t) - a_1))\, F(\tilde{z})\, dt$$

$$\leq m + O(\varepsilon) + \sup F \int_{-\infty}^{+\infty} (a(t) - a_1))\, dt.$$

Along a sequence of values $T \to \infty$ we may assume by standard arguments that $u_T \to u$ in $C^1(I)$ for each compact interval and $\dot{u}_T \to \dot{u}$ weakly in $L^2(\mathbb{R})$.

Lemma 4.1.

$$\lim_{T\to\infty} \dot{u}_T(-T) = \lim_{T\to\infty} \dot{u}_T(T) = 0.$$

Proof. It is enough to prove that

$$\lim_{T\to\infty} \dot{u}_T(T) = 0.$$

Assume that along a sequence of T's tending to $+\infty$ we have $\dot{u}_T(T) \to k > 0$. Then a subsequence of the translates $\tilde{u}_T$, defined as $\tilde{u}_T(t) := u_T(t+T)$, converges uniformly on compact intervals to the solution w^* of (3) with $a = a_1$ such that $w^*(0) = 1$ and $\dot{w}^*(0) = k$. Since w^* attains the value -1 at a finite time $t_0 < 0$ with $\dot{w}^*(t_0) > 0$, this contradicts the fact that the u_T's take values in $[-1, 1]$. □

Since $|u_T| \leq 1$ and $\ddot{u}_T$ is uniformly bounded, there is a constant D such that for $T > T_0$

$$\|\dot{u}_T\|_\infty \leq D.$$

Lemma 4.2. *Let $u_T \to u$ as $T \to +\infty$ in C^1_{loc}. Then u is a solution of* (9),

$$\lim_{t\to-\infty} u(t),\quad \lim_{t\to\infty} u(t) \in \{-1,\, 1\}.$$

In particular, if u is constant then $u \equiv 1$ or $u \equiv -1$.

Proof. By Fatou's lemma,

$$\int_{-\infty}^{+\infty} F(u)\, dt < +\infty.$$

By an argument of Rabinowitz, see [8], Proposition 3.11, the assertion about the limits holds. □

In the sequel we shall use the following notation. Given an interval $K \subset \mathbb{R}$ and $w \in H^1(K)$, we set

$$\mathbf{J}_K(w) := \int_K \Big(\frac{\dot{w}^2}{2} + a(t)F(w)\Big)\, dt.$$

Lemma 4.3. *Assume that for all $T \geq T_0$ there exists L_T such that $u_T < 0$ in $[-T, L_T]$ and $u_T(L_T) \to -1$; then $\mathbf{J}_{[-T,L_T]}(u_T) \to 0$.*

Proof. First note that if a function $w \in H^1(a,b)$ is such that $|w \pm 1| < \delta$, then by (F_2)

$$\int_a^b \dot{w}(t)^2\,dt + \int_a^b a(t)f(w(t))(w(t) \pm 1)\,dt \geq \frac{2A}{B}\mathbf{J}_{[a,b]}(w). \qquad (39)$$

Now multiplying the equation in (35) by $u_T + 1$ and integrating in $[-T, L]$ (we write $L_T = L$) we obtain

$$\dot{u}_T(L)(u_T(L)+1) = \int_{-T}^{L} \dot{u}_T(t)^2\,dt + \int_{-T}^{L} a(t)f(u_T(t))(u_T(t)+1)\,dt.$$

We may suppose $-1 \leq u_T < -1+\delta$, since $u_T(L_T) \to -1$, $u_T \geq -1$, $\dot{u}_T(-T) \geq 0$ and u_T is a negative solution of (9), so that $\ddot{u}_T \geq 0$ on $[-T, L]$; hence u_T is increasing on$[-T, L]$. Now, by the above remark,

$$\mathbf{J}_{[-T,L]}(u_T) \leq \frac{B}{2A}\dot{u}_T(L)(u_T(L)+1). \qquad (40)$$

Clearly, $\dot{u}_T(L)$ is bounded, and this ends the proof. □

A similar result holds for $[L, T]$ in case $u_T(L) \to 1$.

The following two lemmas may be proved using simple, standard arguments.

Lemma 4.4. *Let* $\mathcal{E}_{a,b,\alpha,\beta} := \left\{u \in H^1(a,b) : u(a) = \alpha,\ u(b) = \beta\right\}$. *Then*

$$\liminf_{(\alpha,\beta)\to(-1,1)} \min_{\mathcal{E}_{a,b,\alpha,\beta}} \mathbf{J}_{[a,b]} \geq m.$$

Lemma 4.5. *If the set of zeroes of* u_T *is bounded independently of* T *then* u *is a heteroclinic from* -1 *to* 1.

Lemma 4.6. *Let* $0 < k < 1$ *and* $w \in H^1(t_0, t_1)$ *with* $w(t_0) = 0$, $w(t_1) = -1+k$. *Then* $\mathbf{J}_{[t_0,t_1]}(w) \geq m_- - O(k)$ *where* $O(k) \to 0$ *as* $k \to 0$ *uniformly with respect to intervals* $[t_0,\, t_1]$.

Proof. Let $\tilde{w}$ be the extension of w to $[t_0, \infty)$ such that $\tilde{w} \equiv -1$ in $[t_1+k, \infty)$ and $\tilde{w}(t) = -1+k+t_1-t$ if $t \in [t_1, t_1+k]$. We have

$$\begin{aligned}&\mathbf{J}_{[t_0,t_1]}(w) + \frac{k}{2} + \int_{t_1}^{t_1+k} a(t)F(-1+k+t_1-t)\,dt \\ &\geq \int_{t_0}^{+\infty}\left(\frac{\dot{\tilde{w}}^2}{2} + a_1\,F(\tilde{w})\right)dt \geq m_-.\end{aligned}$$ □

To end the proof, in view of Lemma 4.5, we must discard that the zeroes are unbounded along the sequence. We shall study the case where c_T, the greatest zero of u_T, tends to $+\infty$. Similarly, one deals with the case where the smallest zero of u_T tends to $-\infty$.

Set $w_T(t) := u_T(t + c_T)$. Then w_T satisfies

$$w_T'' = a(t + c_T)f(w_T).$$

By the diagonal procedure we extract another subsequence such that $w_T \to w$, $w'' = a_1 f(w)$, $w(0) = 0$, $0 \le w \le 1$. Since the integrals $\int_0^{T-c_T} F(w_T)$ are uniformly bounded, and by Lemma 4.1 $T - c_T \to +\infty$, we have $w(+\infty) = 1$ and $w = z_{a_1}$ in $[0, \infty)$. Choose $l > 0$ so that

$$z_{a_1}(-l) < -1 + \frac{\eta A}{2BD}, \qquad z_{a_1}(l) > 1 - \frac{\eta A}{2BD}.$$

Then with $\lambda_T = -l + c_T$, $\mu_T = l + c_T$, we obtain, for sufficiently large T, $u_T(\lambda_T) < -1 + \frac{\eta A}{2BD}$, $u_T(\mu_T) > 1 - \frac{\eta A}{2BD}$ and

$$\mathbf{J}_{[\lambda_T,\mu_T]}(\tau_{c_T}(z_{a_1})) = \int_{-l}^{l} \left(\frac{\dot{z}_{a_1}^2}{2} + a(t + c_T)F(z_{a_1}) \right) dt < m + \eta/4.$$

Remember we may assume $\eta/4 < \delta$ (cf. (F_2)).

Now two cases are possible:

Case 1: *c_T is the only zero of u_T along the sequence.* Then if T is sufficiently large

$$|\mathbf{J}_{[\lambda_T,\mu_T]}(u_T) - \mathbf{J}_{[\lambda_T,\mu_T]}(\tau_{c_T}(z_{a_1}))| < \eta/4.$$

Also by the same arguments as in the proof of lemma 4.3 we deduce

$$\mathbf{J}_{[-T,\lambda_T]}(u_T) < \eta/4, \quad \mathbf{J}_{[\mu_T,T]}(u_T) < \eta/4.$$

It follows that $J_T(u_T) < m + \eta$ when T is large, contradicting the first inequality in (38) and (31).

Case 2: *there exists along the subsequence another zero $d_T < c_T$ and we may assume $u_T < 0$ in (d_T, c_T).*

By the preceding argument an appropriate translate of u_T approaches z_{a_1} uniformly in compact intervals, so that $c_T - d_T \to \infty$. Also, if λ_T' is the point in $[d_T, c_T]$ where $\min_{[d_T,c_T]} u_T$ is attained, then $\lim_{T\to\infty}(u_T(\lambda_T')) = -1$. In this case there exists an odd number of zeros. Let e_T be the smallest one. By the argument used in Case 1, we estimate

$$\mathbf{J}_{[\lambda_T',T]}(u_T) \ge m - \frac{\eta}{4}$$

for large T and then

$$\mathbf{J}_T(u_T) \ge m - \eta/4 + \mathbf{J}_{[d_T,\lambda_T']}(u_T) + \mathbf{J}_{[-T,e_T]}(u_T).$$

On the other hand

$$\mathbf{J}_{[-T,e_T]}(u_T) \geq \min_{u(-\infty)=-1,\ u(0)=0} \int_{-\infty}^{0} \Big(\frac{\dot{u}^2}{2} + a_1\, F(u)\Big) dt$$
$$= \int_{-\infty}^{0} \Big(\frac{\dot{z}_{a_1}^2}{2} + a_1\, F(z_{a_1})\Big) dt = m_-.$$

Using lemma 4.6 we obtain for a large T

$$\mathbf{J}_{[d_T,\lambda'_T]}(u_T) \geq m_- - \eta/4.$$

It follows that for large T $\mathbf{J}_T(u_T) \geq m - \eta/2 + 2m_-$, contradicting the last inequality in (38), (29) and (33).

The proof of Theorem 2.2, part (i), is now complete.

For the proof of statement (ii) we note the following: in any case the limit u is a solution of the differential equation (9) and of course $\mathbf{J}(u) < +\infty$. Hence

$$\lim_{t\to-\infty} u(t), \quad \lim_{t\to\infty} u(t) \in \{-1,\, 1\}.$$

If the solution is constant, it cannot be zero. If, say, $u \equiv -1$, by the argument of the above Case 1 there are at least three zeros $e_T < d_T < c_T$ such that e_T, $d_T \to -\infty$ and $c_T \to +\infty$. Then similar estimates lead for large T to $\mathbf{J}_T(u_T) \geq 3m - \eta$. But this contradicts the last inequality in (38), (33) and (30).

Proof of Theorem 2.3. The proof is similar to the preceding one, the only difference being that we use the upper bound of k_T given in Remark 4 instead of the upper bound in (38). See the proof of Theorem 5.2 in the next section.

5. The asymptotically asymmetric case

While it is clear that (9) has no heteroclinics from -1 to 1 if $a(t)$ is non-constant and monotone, simple examples show that monotone heteroclinics are easy to find provided $a(t)$ takes three values only. In fact, given positive numbers a, b, c, T let us set

$$a(t) = \begin{cases} a & \text{if } t < 0 \\ b & \text{if } 0 < t < T \\ c & \text{if } t > T, \end{cases} \tag{41}$$

where $b < a \leq c$ or $c \leq a < b$. A monotone increasing heteroclinic $u(t)$ of (9) exists if we are able to glue together three pieces of solutions of autonomous equations, and in particular one finds

$$T = \int_{\xi}^{\eta} \frac{du}{\sqrt{2bF(u) + 2(a-b)F(\xi)}}, \tag{42}$$

where $\xi = u(0)$ and $\eta := u(T)$ are related by

$$F(\eta) = \gamma F(\xi), \ \ 0 < \gamma := \frac{a-b}{c-b} \le 1.$$

Assuming, for simplicity, that F has only a critical point in $(-1,1)$ and $F(u) = \alpha(u \pm 1)^2$ in a neighborhood of -1 or 1 respectively, η is well defined as a function of ξ as the greatest number satisfying the above condition and it is easy to check that $T = T(\xi)$ has the property

$$\lim_{\xi \to -1} T = +\infty, \quad \lim_{\xi \to 1} T = \frac{1}{\sqrt{2\alpha b}} \ln \left| \frac{\sqrt{b\gamma + (a-b)} - \sqrt{b\gamma}}{\sqrt{a} - \sqrt{b}} \right|.$$

Therefore, the admissible values of T are bounded below by some positive constant in the cases $b < a < c$ or $c < a < b$ (that is, $\gamma < 1$). In consequence, the increasing heteroclinic for a three-valued function of the form (41) exists in the asymmetric case only if a supplementary condition on the length of $[0,T]$ is required. Obviously, this restriction is needed for non-monotonic heteroclinics as well. (On the other hand, in case $b < a = c$ or $c = a < b$ there are no restrictions, in agreement with Example 1.)

In this section we discuss a setting where the situation (19) may be considered for more general functions $a(t)$, although our results are useful only if $|\lim_{t\to-\infty} a(t) - \lim_{t\to+\infty} a(t)|$ is a sufficiently small positive number.

To start, we keep the notation introduced in (21) and we analogously set

$$n = \mathbf{I}_{a_3}(z_{a_3}), \ \ n_- := \sqrt{2a_3}\int_{-1}^{0} \sqrt{F(z)}\,dz \ \ n_+ := \sqrt{2a_3}\int_{0}^{1} \sqrt{F(z)}\,dz. \tag{43}$$

Recall that for definiteness we consider $a_3 > a_1$. Let us define

$$H(t) = \begin{cases} a_3 & \text{if } t \le 0 \\ a_1 & \text{if } t > 0 \end{cases} \tag{44}$$

and consider the new assumption

(A_2) $a(t) \ge H(t)$ $\forall t \in \mathbb{R}$.

Lemma 5.1. *Assume* (A_2) *and*

(H) *there exist* $\alpha > 0$, $L > 0$ *such that* $8(n+\alpha)L < 1$ *and*

$$\Delta_0 \int_{-L}^{L} (a(t) - H(t))\,dt > n_+ - m_+ \tag{45}$$

where $\Delta_0 = \min_{|x|\le 1/2} F(x)$.

Then there exists $\Delta > 0$ *such that for all* $T > 0$ *sufficiently large*

$$u \in \mathcal{E}_T \ \ \textit{and} \ \ u(0) = 0 \ \Rightarrow \ \mathbf{J}_T(u) \ge n + \Delta.$$

Proof. Take a function $u \in \mathcal{E}_T$ with $u(0) = 0$. Set $C := n + \alpha$. Let $T > L$ and suppose $\int_{-T}^{T} \frac{\dot u^2}{2}\, dt \le C$. We have

$$\begin{aligned}\mathbf{J}_T(u) &= \int_{-L}^{L} \Big(\frac{\dot u^2}{2} + a(t)\, F(u)\Big)\, dt + \int_{[-T,-L]\cup[L,T]} \Big(\frac{\dot u^2}{2} + a(t)\, F(u)\Big) dt\\ &\ge \int_{-T}^{T} \Big(\frac{\dot u^2}{2} + H(t)\, F(u)\Big) dt + \int_{-L}^{L} (a(t) - H(t))\, F(u)\, dt\\ &\ge n_- + m_+ + \int_{-L}^{L} (a(t) - H(t))\, F(u)\, dt.\end{aligned}$$

In the last integrand we have $|u(t)| \le \frac{1}{2}$, so that using (H) and setting

$$\Delta = \min\{\alpha,\, \Delta_0 \int_{-L}^{L} (a(t) - H(t))\ dt - n_+ + m_+\}$$

we may conclude. □

Theorem 5.2. *Assume that* $f \in C(\mathbb{R}, \mathbb{R})$, $F' = f$ *in* $\mathbb{R}$ *are such that* F *and* $a \in L^\infty(\mathbb{R}, \mathbb{R})$ *satisfy* $(F) - (f) - (F_2) - (A_2) - (H)$ *and*

$$\lim_{t\to-\infty} a(t) = a_3, \quad \lim_{t\to+\infty} a(t) = a_1$$

with $a_3 > a_1$.

(*i*) *If*

$$(\sup F) \int_{-\infty}^{+\infty} (a(t) - H(t))\, dt < \min\{2n_+,\, m - n + 2n_-\}, \tag{46}$$

then (9) *has a heteroclinic that connects* ± 1.

(*ii*) *If*

$$(\sup F) \int_{-\infty}^{+\infty} (a(t) - H(t))\, dt < n + m, \tag{47}$$

then (9) *has either a non-trivial homoclinic at* ± 1 *or a heteroclinic that connects* ± 1. *This solution takes values in* $(-1, 1)$.

Example 3. Functions $a(t)$ with a certain structure do satisfy the assumptions of the above theorem. Fix $a_1 > 0$ and a real function φ such that $\varphi - a_1 \in \mathcal{L} := L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\varphi \ge a_1$, $\varphi > a_1$ in a neighborhood of zero and $\lim_{|t|\to+\infty} \varphi(t) = a_1$. Define $K(t) = 1$ if $t < 0$, $K(t) = 0$ if $t > 0$. Let

f be a function such that $f - K \in \mathcal{L}$, $f \geq K$ almost everywhere in $\mathbb{R}$ and $\lim_{t\to-\infty} f(t) = 1$, $\lim_{t\to+\infty} f(t) = 0$. Put

$$H(t) = a_1 + \lambda K(t), \quad a(t) = \varphi(t) + \lambda f(t).$$

Choose $\alpha > 0$ and $L > 0$ so that $8(m+\alpha)L < 1$. With $a_3 = a_1 + \lambda$, (45) becomes

$$\Delta_0 \int_{-L}^{L} (a(t) - H(t))\, dt > (\sqrt{2(a_1+\lambda)} - \sqrt{2a_1}\,) \int_0^1 \sqrt{F(z)}\, dz$$

while (46) reads

$$(\sup F) \int_{-\infty}^{+\infty} (a(t) - H(t))\, dt < \min\Big\{2\sqrt{2(a_1+\lambda)} \int_0^1 \sqrt{F(z)}\, dz,$$

$$2\sqrt{2(a_1+\lambda)} \int_{-1}^{0} \sqrt{F(z)}\, dz + (\sqrt{2(a_1)} - \sqrt{2a_1+\lambda}\,) \int_{-1}^{1} \sqrt{F(z)}\, dz\Big\}.$$

Then, provided that

$$(\sup F) \int_{-\infty}^{+\infty} (\varphi(t) - a_1)\, dt < \min\{2m_+,\, 2m_-\}$$

it is clearly possible to choose $\lambda > 0$ sufficiently small so that the function $a(t)$ satisfies assumptions (H) and (46). In a similar way one can construct functions that satisfy (H) and (47).

Proof. We briefly outline the proof of (i), which is similar to that of theorem 2.2. Let us fix $\eta > 0$ such that

$$(\sup F) \int_{-\infty}^{+\infty} (a(t) - H(t))\, dt + \eta < \min\{2n_+,\, m - n + 2n_-\} \tag{48}$$

$$\eta < \min\{\Delta/2, 4\delta\}. \tag{49}$$

Here $\Delta > 0$ is related to the mountain pass setting for $\mathbf{J}_T$, $T > L$ as in Section 3. Functions p and q may be now defined as $p = \tau_{-c}\tilde{z}_{a_3}$, $q = \tau_c \tilde{z}_{a_3}$. Noticing that

$$\mathbf{I}_{a_1}(z_{a_3}) = \frac{1}{2}(1 + \frac{a_1}{a_3})n < n$$

with an appropriate choice of d, ε and c we may assume $\mathbf{J}_T(q) \leq \mathbf{J}_T(p)$ and the following estimates hold:

$$\mathbf{J}_T(p) > n - \Delta/2, \quad \mathbf{J}_T(p) + \Delta < \mathbf{J}_T(u) \ \text{ if } \ u \in \mathcal{E}_T,\ u(0) = 0. \tag{50}$$

We suppose also that ε has been fixed so that

$$O(\varepsilon) < \eta/4, \tag{51}$$

where $O(\varepsilon)$ appears in the computation below.

Just as in Section 3, we obtain the critical level k_T and it may be estimated in the following way. There exists $s_0 \in [-c, c]$ such that

$$\begin{aligned} n + \Delta \le k_T &\le \int_{-T}^{T} \Big(\frac{\dot{\tilde{z}}(t-s_0)^2}{2} + a(t)F(\tilde{z}(t-s_0)) \Big) dt \\ &= \int_{-T}^{T} \Big(\frac{\dot{\tilde{z}}(t-s_0)^2}{2} + H(t)F(\tilde{z}(t-s_0)) \Big) dt + \int_{-T}^{T} (a(t)-H(t))F(\tilde{z}(t-s_0)) dt \\ &\le \int_{-T}^{T} \Big(\frac{\dot{\tilde{z}}(t-s_0)^2}{2} + a_3 F(\tilde{z}(t-s_0)) \Big) dt + \int_{-T}^{T} (a(t)-H(t))F(\tilde{z}(t-t_0)) dt \\ &= \int_{-T}^{T} \left(\frac{\dot{\tilde{z}}(t)^2}{2} + a_3 F(\tilde{z}(t)) \right) dt + \int_{-T}^{T} (a(t)-H(t))F(\tilde{z}(t-t_0)) dt \\ &\le n + O(\varepsilon) + \sup F \int_{-\infty}^{+\infty} (a(t)-H(t))\, dt. \end{aligned} \tag{52}$$

As in the preceding section, we consider the convergence of the family of approximate mountain pass solutions (u_T). Again we now distinguish among the cases:

Case 1: *The greatest zero c_T of u_T goes to ∞.* In this case, if c_T is the only zero, arguing as in the preceding case we obtain for large T

$$J_T(u_T) \le m + \eta,$$

contradicting the first inequality in (52) and (49). If there exists a zero $d_T < c_T$ we obtain for large T

$$J_T(u_T) \ge m + 2n_- - \eta/2$$

contradicting (48) and the last inequality in (52).

Case 2: *The smallest zero e_T of u_T goes to $-\infty$.* Then if this is the only zero we obtain a contradiction as in the previous case with n in the place of m. So assume that there exists a next zero d_T. Then following the arguments of Section 3 we obtain for sufficiently large T

$$J_T(u_T) \ge n + 2n_+ - \eta/2$$

again contradicting (48) and the last inequality in (52). □

A final remark is in order: the example at the beginning of this section suggests that it must be possible to reach the conclusion of Theorem 5.2 under weaker assumptions.

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