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PERIODIC SOLUTIONS AND ASYMPTOTIC BEHAVIOR IN LIÉNARD SYSTEMS WITH p-LAPLACIAN OPERATORS

M. García-Huidobro
Departamento de Matemáticas, P. Universidad Católica de Chile
Casilla 306, Correo 22, Santiago, Chile

R. Manásevich
Centro de Modelamiento Matemático and Departamento de Ingeniería
Matemática, Universidad de Chile, Casilla 170, Correo 3, Santiago, Chile

J. R. Ward
Department of Mathematics, University of Alabama at Birmingham
Birmingham, AL 35294

Dedicated to Patrick Habets and Jean Mawhin on their sixty-fifth birthdays

Abstract. We first prove the existence of periodic solutions to systems of the form

$$(\phi_p(u'))' + \frac{d}{dt}(\nabla F(u)) + \nabla G(u) = e(t).$$

We then study the asymptotic behavior of all solutions to such systems, and give sufficient conditions for uniform ultimate boundedness of solutions.

1. Introduction

Let $p > 1$, and $\Phi_p(x) = \frac{1}{p}|x|^p$ for $x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$ and $\phi_p(x) = \nabla\Phi_p(x) = |x|^{p-2}x$. Here $N \geq 1$ is an integer, $\mathbb{R}$ denotes the real numbers and $|x| = (x_1^2+x_2^2+...+x_N^2)^{1/2}$ denotes the Euclidean norm on $\mathbb{R}^N$. The function ϕ_p is invertible and it is easily verified that $\phi_p^{-1} = \phi_q$, where $1/p + 1/q = 1$.

In this paper we will consider existence of $T-$periodic solutions and stability questions for systems of ordinary differential equations of the form

$$(\phi_p(u'))' + \frac{d}{dt}(\nabla F(u)) + \nabla G(u) = e(t) \tag{1.1}$$

where $F \in C^2(\mathbb{R}^N, \mathbb{R})$, $G \in C^1(\mathbb{R}^N, \mathbb{R})$, and $e : \mathbb{R} \to \mathbb{R}^N$ with appropriate properties. Equations of the form (1.1) are said to be of Liénard type.

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A solution of (1.1) is a function u defined on an interval J into $\mathbb{R}^N$ such that u is C^1 on J, $\phi_p(u'(t))$ is differentiable almost everywhere on J, and u satisfies (1.1) almost everywhere on J. A $T-$periodic solution is a solution u defined on $J = \mathbb{R}$ which is also $T-$periodic; i.e., $u(t+T) = u(t)$ for all $t \in \mathbb{R}$.

The case that especially interests us is (1.1) with the Hessian matrix $(\frac{\partial^2 F}{\partial x_i \partial x_j}(x)) = F''(x)$ uniformly positive (or negative) definite. The case of greatest interest is when $F''(x)$ is positive definite, that is, when damping is present.

A purpose of this paper is to find conditions on F,G, and e such that (1.1) has periodic solutions, and there is a bounded region, containing the periodic solutions, which all solutions eventually enter. Periodic solutions of p-Laplacian and similar equations and systems have been considered lately in several papers ([7], [11], [6], [16], and elsewhere). Periodic solutions of p-Laplacian Liénard equations are considered in [11] and [6], and the relations to this paper will be described later. Both [11] and [6] apply a continuation theorem based on degree theory, proved in [7]. We will also apply that continuation theorem here to prove the existence of periodic solutions. Using Liapunov's second method together with the Conley index we then proceed to investigate long-time behavior of all solutions to (1.1) and show that, under some natural conditions on G and e, damping implies the ultimate boundedness of solutions (Theorems 5 and 8). This seems to be the first time that such questions have been considered for differential equations involving the p-Laplacian. For the study of periodic solutions to (1.1) with $p = 2$, see, e.g., [18], [19], [13], [3], [20]. For the study of stability in systems of ordinary differential equations, see [5], [18], [19], [13], and [12].

Section 2 contains some preliminaries, including a continuation result from [7] that we will use to study existence of periodic solutions. In Section 3 we consider, for simplicity, the existence of $T-$periodic solutions to the scalar equation

$$(\phi_p(u'))' + cu' + g(u) = e(t). \tag{1.2}$$

In Section 4 we consider the existence of T-periodic solutions to (1.1). In Section 5 we return to scalar equations and study stability aspects for an equation more general than (1.2). We introduce a Liapunov function and use it to prove that the solutions are uniformly ultimately bounded. In Section 6 we study long-time behavior in the N-dimensional case with no forcing, $e = 0$. We provide sufficient conditions for asymptotic stability of the trivial solution. In Section 7 we consider long-time behavior of solutions to (1.1) with small forcing $e(t) = \mu w(t)$ where $w(t)$ is a bounded function and μ is a small positive parameter; indeed, by using the Conley index we are able to connect the asymptotic stability of the unforced equation with the solution behavior in the forced case for small μ. It would be interesting to remove this smallness condition.

2. Preliminaries

For any normed space X with norm $|\cdot|_X$ with $x_0 \in X$ and $r > 0$ let $B(x_0, r) = \{x \in X : |x - x_0|_X < r\}$. Let C_T be the Banach space of continuous functions $u : [0, T] \to \mathbb{R}^N$ satisfying $u(0) = u(T)$ with norm $\|u\|_0 := \max_{0 \le t \le T} |u(t)|$. Let C_T^1 be the Banach space of continuously differentiable $u \in C_T$ satisfying $u'(0) = u'(T)$ with norm $\|u\|_1 := \|u\|_0 + \|u'\|_0$ for $u \in C_T^1$. We let $L^2 = L^2(0, T)$ denote the Banach space of Lebesgue measurable functions mapping $[0, T]$ into $\mathbb{R}^N$, which are square integrable, with norm $\|u\|_{L^2} := (\int_0^T |u(t)|^2 \, dt)^{1/2}$. Let $s \ge 1$; a function $f : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ will be said to be L^s-Carathéodory provided:

(i) For all $x, y \in \mathbb{R}^N$ the function $t \mapsto f(t, x, y)$ is Lebesgue measurable.

(ii) For almost all $t \in [0, T]$, the function $(x, y) \mapsto f(t, x, y)$ is continuous.

(iii) For every $\rho \ge 0$ there is a Lebesgue measurable function $\alpha_\rho \in L^s(0, T)$ with $|f(t, x, y)| \le \alpha_\rho(t)$ almost everywhere for $|x|, |y| \le \rho$.

Let $f : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be L^s-Carathéodory, $s \ge 1$, and consider the boundary-value problem

$$(\phi_p(u'))' = f(t, u, u'), \tag{2.1}$$

$$u(0) = u(T), \; u'(0) = u'(T). \tag{2.2}$$

The following theorem is a special case of a continuation theorem proved in [7].

Theorem 1. *Assume Ω is a non-empty open bounded set in C_T^1 such that the following conditions hold:*

(1) *For each $\lambda \in (0, 1)$ the problem*

$$(\phi_p(u'))' = \lambda f(t, u, u'), \; u(0) = u(T), \; u'(0) = u'(T)$$

has no solution on $\partial\Omega$.

(2) *The equation*

$$\overline{f}(a) := \frac{1}{T} \int_0^T f(t, a, 0) dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}^N$.

(3) *The Brouwer degree $d_B(\overline{f}, \Omega \cap \mathbb{R}^N, 0) \neq 0$.*

Then problem (2.1), (2.2) *has a solution in* $\overline{\Omega}$.

In Sections 3 and 4 we will apply Theorem 1 to prove the existence of periodic solutions to (1.2) and (1.1). Let us here note that if f satisfies the conditions of Theorem 1 and, in addition, f is defined on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ and $T-$periodic in the first variable $t \in \mathbb{R}$, then any solution to the boundary-value problem (2.1), (2.2) on $[0,T]$ may be extended as a $T-$periodic C^1 function on all of the real line $\mathbb{R}$, and this $T-$periodic extension is a solution to the differential equation (2.1) on $\mathbb{R}$.

3. Periodic solutions in the scalar case

Here we consider the scalar boundary-value problem

$$(\phi_p(u'))' + cu' + g(u) = e(t), \; u(0) = u(T), \; u'(0) = u'(T). \tag{3.1}$$

For $e \in L^1(0,T)$ let $\overline{e} = \frac{1}{T}\int_0^T e(t)dt$.

Theorem 2. *Let* $p > 1$, $g \in C(\mathbb{R},\mathbb{R})$, $e \in L^2(0,T)$, *and* $c \in \mathbb{R}$. *Suppose:*

(i) *There is a number* $r \geq 0$ *such that* $(g(x) - \overline{e})x \geq 0$ (≤ 0) *for* $|x| \geq r$.

(ii) $c \neq 0$.

Then (3.1) *has a solution.*

Proof. Initially we assume the inequality in (i) is strict, so that $(g(x)-\overline{e})x > 0$ (or < 0) for $|x| \geq r$. We apply Theorem 1 with $f(t,u,u') = -cu' - g(u) + e(t)$. That is, we consider the family of boundary-value problems

$$(\phi_p(u'))' + \lambda cu' + \lambda g(u) = \lambda e(t), \; u(0) = u(T), \; u'(0) = u'(T) \tag{3.2}$$

for $\lambda \in (0,1)$. Suppose (u,λ) is a solution to (3.2), and multiply the differential equation by u' and integrate over $[0,T]$, to get

$$\int_0^T (\phi_p(u'))'(t)u'(t)dt + \lambda c \int_0^T \left|u'(t)\right|^2 dt = \lambda \int_0^T e(t)u'(t)dt \tag{3.3}$$

since

$$\int_0^T g(u(t))u'(t)dt = 0.$$

Now ϕ_p is a homeomorphism of $\mathbb{R}$ with inverse $\phi_p^{-1} = \phi_q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Thus $\phi_p(x)x = \phi_p(x)\phi_q(\phi_p(x))$, and hence, with $\Phi_q(x) = \frac{1}{q}|x|^q$ so $\Phi_q' = \phi_q$, we have

$$\begin{aligned}\int_0^T (\phi_p(u'))'(t)u'(t)dt &= \int_0^T (\phi_p(u'))'(t)\phi_q(\phi_p(u'(t))dt \\ &= \Phi_q(\phi_p(u'(T))) - \Phi_q(\phi_p(u'(0))) = 0.\end{aligned}$$

Thus (3.3) becomes

$$\lambda c \int_0^T \left|u'(t)\right|^2 dt = \lambda \int_0^T e(t)u'(t)dt.$$

Therefore, since $0 < \lambda < 1$,

$$|c| \left\|u'\right\|_{L^2}^2 = \left| \int_0^T e(t)u'(t)dt \right| \leq \|e\|_{L^2} \left\|u'\right\|_{L^2}$$

and

$$\left\|u'\right\|_{L^2} \leq \frac{1}{|c|} \|e\|_{L^2} . \tag{3.4}$$

Integrating the differential equation in (3.2) over $[0, T]$ and using periodicity, we get

$$\frac{1}{T} \int_0^T g(u(t))dt = \overline{e}.$$

By the integral mean value theorem there is a $z \in [0, T]$ such that $g(u(z)) = \overline{e}$. Since we are assuming that $(g(x) - \overline{e})x > 0$ for $|x| \geq r$, we conclude that $|u(z)| < r$. Thus, for $t \in [0, T]$,

$$u(t) = u(z) + \int_z^t u'(s)ds$$

and hence

$$|u(t)| < r + \sqrt{T} \left\|u'\right\|_{L^2} \leq r + \frac{\sqrt{T}}{|c|} \|e\|_{L^2} = b_0 \tag{3.5}$$

by the Cauchy-Schwarz inequality. We now need a bound on $u'(t)$. Since u is $T-$periodic, there is a number $w \in [0, T]$ at which $u'(w) = 0$. Integrating (3.2) we get

$$\phi_p(u'(t)) - \phi_p(u'(w)) + \lambda cu(t) - \lambda cu(w) + \lambda \int_w^t g(u(s))ds = \lambda \int_w^t e(s)ds$$

and

$$\begin{aligned}\phi_p(\left|u'(t)\right|) = \left|\phi_p(u'(t))\right| &\leq 2\,|c|\,\|u\|_0 + TM + \sqrt{T}\,\|e\|_{L^2} \\ &< 2\,|c|\,b_0 + TM + \sqrt{T}\,\|e\|_{L^2} := b_1\end{aligned}$$

where $M = \max\limits_{|x| \leq b_0} |g(x)|$. Therefore,

$$\left|u'(t)\right| < \phi_q(b_1). \tag{3.6}$$

By (3.5) and (3.6) we have that, for all $\lambda \in (0, 1)$,

$$\|u\|_1 < b_0 + \phi_q(b_1) := R_0(b_0, b_1). \tag{3.7}$$

Thus if we let $\Omega = \{u \in C^1_T : \|u\|_1 < R_0(b_0, b_1)\}$, then (3.2) has no solutions on $\partial\Omega$ for any $\lambda \in (0,1)$. This verifies condition (1) of Theorem 1. Now let $f(t,u,u') = -cu' - g(u) + e(t)$. We have

$$\overline{f}(a) = \frac{1}{T}\int_0^T f(t,a,0)dt = \frac{1}{T}\int_0^T (-g(a) + e(t))dt = -g(a) + \overline{e}.$$

For $|a| \geq R_0 > r$, we have $\overline{f}(a)a = (-g(a) + \overline{e})a < 0$ which verifies condition (2) of Theorem 1, and also shows that $d_B(\overline{f}, \Omega \cap \mathbb{R}^1, 0) = -1$, verifying condition (3) of Theorem 1. This proves the theorem in case $(g(x) - \overline{e})x > 0$ for $|x| \geq r$. We now prove it in case the inequality is weak.

Thus suppose that $(g(x) - \overline{e})x \geq 0$ for $|x| \geq r$. Let $n \in \mathbb{N}$ and define

$$g_n(x) := g(x) + \frac{x}{n(1 + |x|)}.$$

Then

$$\lim_{n\to\infty} g_n(x) = g(x)$$

uniformly on $\mathbb{R}$. We have that $(g_n(x) - x)x > 0$ for $|x| \geq r$, so there is a solution to (3.1) with g_n in place of g. That is, there is, for any $n \in \mathbb{N}$, a solution u_n to

$$(\phi_p(u'))' + cu' + g_n(u) = e(t),\ u(0) = u(T),\ u'(0) = u'(T). \tag{3.8}$$

We claim there is a constant $C_1 > 0$ such that $\|u_n\|_1 < C_1$ for all $n \in \mathbb{N}$. But the bounds obtained in the first part, under the assumption that $(g(x) - \overline{e})x > 0$ for $|x| \geq r$, must also hold here, so we have from (3.7) that $\|u_n\|_1 < R_0$ for all $n \in \mathbb{N}$. It follows that the sequence of functions $\{u_n : n \in \mathbb{N}\}$ is uniformly bounded and equicontinuous on $[0,T]$. By the Arzelà-Ascoli theorem there is a subsequence, which we renumber the same, which converges uniformly on $[0,T]$ to a function $u_0 \in C_T$. We claim u_0 is a solution to (3.1).

For each $n \in \mathbb{N}$ we have

$$(\phi_p(u_n'))' + cu_n' + g_n(u_n) = e(t),\ u_n(0) = u_n(T),\ u_n'(0) = u_n'(T).$$

We may extend $e(t)$ to $\widetilde{e}(t)$ and each $u_n(t)$ to $\widetilde{u}_n(t)$, $T-$periodic on $\mathbb{R}$. Then each $\widetilde{u}_n \in C^1(\mathbb{R})$ and $\widetilde{u}_n(t+T) = \widetilde{u}_n(t)$, $\widetilde{u}_n'(t+T) = \widetilde{u}_n'(t)$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$, and $\{\widetilde{u}_n\}$ converges uniformly on $\mathbb{R}$ to $\widetilde{u}_0$, the $T-$periodic extension of u_0. Now $\widetilde{u}_n$ satisfies

$$(\phi_p(\widetilde{u}_n'))' + c\widetilde{u}_n' + g_n(\widetilde{u}_n) = e(t) \text{ a.e. on } \mathbb{R}. \tag{3.9}$$

For each $n \in \mathbb{N}$ there is a number $\tau_n \in [0,T]$ such that $\widetilde{u}_n'(\tau_n) = 0$, so integrating (3.9) over (τ_n, τ) or (τ, τ_n) gives

$$\phi_p(\widetilde{u}_n'(\tau)) = c\widetilde{u}_n(\tau_n) - c\widetilde{u}_n(\tau) + \int_{\tau_n}^{\tau} (e(s) - g(\widetilde{u}_n(s)))ds$$

and

$$\widetilde{u}_n'(\tau) = \phi_q\Big[c\widetilde{u}_n(\tau_n) - c\widetilde{u}_n(\tau) + \int_{\tau_n}^{\tau} (e(s) - g(\widetilde{u}_n(s)))ds\Big]$$

so that

$$\widetilde{u}_n(t) = \widetilde{u}_n(0) + \int_0^t \phi_q\Big[c\widetilde{u}_n(\tau_n) - c\widetilde{u}_n(\tau) + \int_{\tau_n}^{\tau} (e(s) - g(\widetilde{u}_n(s)))ds\Big]d\tau. \quad (3.10)$$

Without loss of generality we may assume that $\tau_n \to \tau^* \in [0,T]$, so that $\widetilde{u}_n(\tau_n) \to \widetilde{u}_0(\tau^*)$. Letting $n \to \infty$ in (3.10) and using the uniform convergence of the sequence $\{\widetilde{u}_n\}$ we get, for $t \in \mathbb{R}$,

$$\widetilde{u}_0(t) = \widetilde{u}_0(0) + \int_0^t \phi_q\Big[c\widetilde{u}_0(\tau^*) - c\widetilde{u}_0(\tau) + \int_{\tau^*}^{\tau} (e(s) - g(\widetilde{u}_0(s)))ds\Big]d\tau$$

and from this it follows that $\widetilde{u}_0$ is $C^1$and, since $\widetilde{u}_0$ is also $T-$periodic, its continuous derivative $\widetilde{u}_0'$ must also be $T-$periodic. It follows that $\widetilde{u}_0$ restricted to $[0,T]$ is a solution to (3.1). Or, since $\widetilde{u}_0(t) = u_0(t)$ for $t \in [0,1]$, u_0 is a solution to (3.1). This proves the theorem. □

4. Periodic solutions in the system case

Here we consider the boundary-value problem

$$(\phi_p(u'))' + \frac{d}{dt}(\nabla F(u)) + \nabla G(u) = e(t),\ u(0) = u(T),\ u'(0) = u'(T), \quad (4.1)$$

where $G \in C^1(\mathbb{R}^N, \mathbb{R})$, $F \in C^2(\mathbb{R}^N, \mathbb{R})$, and $e \in L^2(0,T)$ with $\overline{e} = \frac{1}{T}\int_0^T e(t)dt$. In the following we let $F''(x)$ denote the Hessian matrix of $F(x)$.

Theorem 3. *Let $p > 1$, $G \in C^1(\mathbb{R}^N, \mathbb{R})$, $F \in C^2(\mathbb{R}^N, \mathbb{R})$, and $e \in L^2(0,T)$. Suppose:*

(i) *There is a number $r \geq 0$ and sets S and S' with $S \cup S' = \{1, 2, .., N\}$ and $S \cap S' = \varnothing$ such that $(\frac{\partial G}{\partial x_i}(x) - \overline{e_i})x_i \geq 0$ for $i \in S$ and $|x_i| \geq r$ and $(\frac{\partial G}{\partial x_i}(x) - \overline{e_i})x_i \leq 0$ for $i \in S'$ and $|x_i| \geq r$.*

(ii) *There is a constant $m > 0$ such that $< F''(x)y, y > \geq m\,|y|^2$ for all $x, y \in \mathbb{R}^N$.*

Then (4.1) *has a solution.*

Proof. Initially we assume the inequalities in (i) are strict, so that $(\frac{\partial G}{\partial x_i}(x)-\overline{e_i})x_i > 0$ for $i \in S$, $|x_i| \geq r$ and .$(\frac{\partial G}{\partial x_i}(x)-\overline{e_i})x_i < 0$ for $i \in S'$, $|x_i| \geq r$. We will apply Theorem 1 with $f(t,u,u') = -F''(u)u' - \nabla G(u) + e(t)$. That is, we consider the family of boundary-value problems

$$(\phi_p(u'))' + \lambda F''(u)u' + \lambda \nabla G(u) = \lambda e(t),\ u(0) = u(T),\ u'(0) = u'(T) \quad (4.2)$$

for $\lambda \in (0,1)$. Suppose (u,λ) is a solution to (4.2), and take the scalar product of each term in the differential equation with u' and integrate over $[0,T]$, to get

$$\int_0^T \langle (\phi_p(u'))'(t), u'(t)\rangle dt + \lambda \int_0^T \langle \frac{d}{dt}(\nabla F(u), u'(t)\rangle dt = \lambda \int_0^T \langle e(t), u'(t)\rangle dt \quad (4.3)$$

since $\int_0^T \langle \nabla G(u(t)), u'(t)\rangle dt = 0$. Now

$$\begin{aligned}\int_0^T \langle (\phi_p(u'))'(t), u'(t)\rangle dt &= \int_0^T \frac{d}{dt}\Phi_q(\phi_p(u'))dt \\ &= \Phi_q(\phi_p(u'(T))) - \Phi_q(\phi_p(u'(0))) = 0.\end{aligned}$$

Thus for $\lambda \neq 0$ (4.3) becomes

$$\int_0^T \langle \frac{d}{dt}\nabla F(u), u'(t)\rangle dt = \int_0^T \langle e(t), u'(t)\rangle dt$$

or

$$\int_0^T < F''(u(t))u'(t), u'(t) > \leq \|e\|_{L^2} \left\|u'\right\|_{L^2} \quad (4.4)$$

implying by assumption (ii) that

$$\int_0^T m\left|u'(t)\right|^2 dt \leq \|e\|_{L^2} \left\|u'\right\|_{L^2}$$

and hence

$$\left\|u'\right\|_{L^2} \leq \|e\|_{L^2}/m := K_1. \quad (4.5)$$

Integrating the differential equation in (4.2) over $[0,T]$ and using periodicity, we get

$$\frac{1}{T}\int_0^T \nabla G(u(t))dt = \overline{e}$$

and thus

$$\frac{1}{T}\int_0^T \frac{\partial G}{\partial x_i}(u(t))dt = \overline{e_i}$$

for $i = 1, 2, ...N$. By the integral mean value theorem there is a $z_i \in [0,T]$ such that $\frac{\partial G}{\partial x_i}(u(z_i)) = \overline{e_i}$. Since we are assuming that $(\frac{\partial G}{\partial x_i}(x) - \overline{e_i})x_i \neq 0$ for $|x_i| \geq r$, we conclude that $|u_i(z_i)| < r$. Thus, for $t \in [0,T]$,

$$u_i(t) = u_i(z_i) + \int_{z_i}^{t} u_i'(s)ds$$

and hence

$$|u_i(t)| < r + \sqrt{T}\left\|u_i'\right\|_{L^2} \leq r + \sqrt{T}\left\|u'\right\|_{L^2} \leq r + \sqrt{T}K_1 \tag{4.6}$$

for $i = 1, 2, ..., N$ and $t \in [0,T]$. It now follows that, for $t \in [0,T]$,

$$|u(t)| < N(r + \sqrt{T}K_1) := b_0$$

and $\|u\|_0 < b_0$. We now need a bound on $u'(t)$. First of all, there is a point $w \in [0,T]$ such that $|u'(w)| \leq K_1/\sqrt{T}$. This follows from (4.5), since if $|u'(t)| > K_1/\sqrt{T}$ for all $t \in [0,T]$, then

$$\left\|u'\right\|_{L^2}^2 = \int_0^T \left|u'(t)\right|^2 dt > K_1^2$$

which contradicts (4.5).

Integrating (4.2) from w to t we get

$$\phi_p(u'(t)) - \phi_p(u'(w)) + \lambda\nabla F(u(t)) - \lambda\nabla F(u(w)) + \lambda\int_w^t \nabla G(u(s))ds$$
$$= \lambda\int_w^t e(s)ds$$

and

$$\begin{aligned}\left|\phi_p(u'(t))\right| &\leq B_0 + 2\left|F(u(t))\right| + TM + \sqrt{T}\left\|e\right\|_{L^2} \\ &< B_0 + 2C_0 + TM + \sqrt{T}\left\|e\right\|_{L^2} := b_1\end{aligned}$$

where

$$B_0 = \max_{|x|\leq K_1/\sqrt{T}}\left|\phi_p(x)\right|, \quad C_0 = \max_{|x|\leq b_0}\left|F(x)\right|, \quad \text{and } M = \max_{|x|\leq b_0}\left|\nabla G(x)\right|.$$

Therefore

$$\left|u'(t)\right| < b_1^{q-1}. \tag{4.7}$$

By (4.6) and (4.7) we have that for all $\lambda \in (0,1)$,

$$\|u\|_1 < b_0 + b_1^{q-1} := R_0. \tag{4.8}$$

Thus if we let $\Omega = \{u \in C^1_T : \|u\|_1 < R_0\}$, then (4.2) has no solutions on $\partial\Omega$ for any $\lambda \in (0,1)$. This verifies condition (1) of Theorem 1. Now since

$$f(t,u,u') = -\frac{d}{dt}\nabla F(u) - \nabla G(u) + e(t),$$

we have

$$\overline{f}(a) = \frac{1}{T}\int_0^T f(t,a,0)dt = \frac{1}{T}\int_0^T (-\nabla G(a) + e(t))dt = -\nabla G(a) + \overline{e}.$$

If $a = (a_1, ..., a_N)$ and $\overline{f}(a) = 0$, then

$$-\frac{\partial G_i}{\partial x_i}(a) + \overline{e}_i = 0$$

and therefore, by hypothesis,

$$|a_i| < r \tag{4.9}$$

for $i = 1, 2, ..., N$. Let $\Lambda = [-r, r] \times [-r, r] \times ... \times [-r, r]$ (N factors). If $x \in \Lambda$ then $|x| \leq Nr$ so that $\Lambda \subset \Omega \cap \mathbb{R}^N$. By (4.9) the equation $\overline{f}(a) = 0$ has no solutions in $\overline{\Omega} \cap \mathbb{R}^N \backslash \Lambda$ so by the excision property of Brouwer degree

$$d(\overline{f}, \Omega, 0) = d(\overline{f}, \Lambda, 0). \tag{4.10}$$

Let $\overline{f} = (\overline{f}_1, \overline{f}_2, ...\overline{f}_N)$ and $E_i = \{x \in \Lambda : x_i = r\}$ and $E'_i = \{x \in \Lambda : x_i = -r\}$. By hypothesis, if $x \in E_i$ and $x' \in E'_i$, then

$$\overline{f}_i(x)\overline{f}_i(x') = (-\frac{\partial G}{\partial x_i}(x) + \overline{e_i})(-\frac{\partial G}{\partial x_i}(x') + \overline{e_i}) < 0.$$

It follows from (4.10) and Miranda's theorem (cf. [8]) that

$$d_B(\overline{f}, \Lambda, 0) = d_B(\overline{f}, \Omega \cap \mathbb{R}^N, 0) = \pm 1$$

verifying condition (3) of Theorem 1. This proves the theorem in case the inequalities in (i) are strict. The proof in the case of weak inequalities now follows by a limit argument, as in the scalar case. □

The proofs of Theorems 2 and 3 contain ideas which go back at least to Faure [2], who observed that in some scalar cases with $p = 2$, $c \neq 0$ in (1.2) may be used to obtain an *a priori* L^2 bound on u'. This idea was used again for some systems in [13], section 5.6. Peng and Xu [11] use the same idea for some nonlinear Liénard systems related to (1.1) but with ϕ_p defined differently from here, letting $\phi_p(x) := (|x_1|^{p-2}\, x_1, ..., |x_N|^{p-2}\, x_N)$.They require $1 < p < 2$ in one of their results, and assume growth conditions on $G(x)$ or $\nabla G(x)$ in all their results. Liu [6] considers the Liénard p-Laplacian equation (1.1) in the scalar case $N = 1$, under the assumption that $g(x) = \nabla G(x)$ is decreasing; the same assumption is made in several recent papers. This is quite different from here. In this paper we are interested in finding conditions for periodic solutions and for some kind of stability or attracting region, which is not to be expected if $g(x)$ is decreasing. We turn to these questions in the next section.

5. Liapunov function for the scalar case

Let $p > 1$, $g, f \in C(\mathbb{R}, \mathbb{R})$, $e \in L^\infty(\mathbb{R}, \mathbb{R}) \cap C(\mathbb{R}, \mathbb{R})$, and $c \in \mathbb{R}$ with $c > 0$. Suppose:

(i^0) $g(x) \to \infty$ as $x \to \infty$ and $g(x) \to -\infty$ as $x \to -\infty$.
(ii^0) $f(x) \geq c > 0$ for all $x \in \mathbb{R}$.
(iii^0) There exists $K > 0$ such that $|g(x)| \geq K f(x)$ for all $|x|$ sufficiently large.

In this section we construct a Liapunov function to show that under the above conditions the solutions to

$$(\phi_p(u'))' + f(u)u' + g(u) = e(t) \tag{5.1}$$

are uniformly ultimately bounded (see the definition below). Set

$$u' = \phi_q(v), \quad v' = -f(u)\phi_q(v) - g(u) + e(t), \tag{5.2}$$

The system (5.2) is equivalent to (5.1).

Definition 4. *The solutions to* (5.2) *are said to be uniformly ultimately bounded provided there exists a* $B > 0$, *and corresponding to any* $\alpha > 0$, *there exists a* $T(\alpha) > 0$ *such that for any* $t_0 \in \mathbb{R}$, *if* $(u, v) = (u(t), v(t)$, $t \geq t_0$, *is a solution to* (5.2), *then* $|u(t_0)| + |v(t_0)| < \alpha$ *implies that* $|u(t)| + |v(t)| < B$ *for all* $t \geq t_0 + T(\alpha)$.

If the solutions to (5.2) are uniformly ultimately bounded then the equivalence of (5.2) and (5.1) implies the existence of a constant B', and, corresponding to any $\alpha > 0$, there exists $T'(\alpha)$ such that, for any $t_0 \in \mathbb{R}$, if $u = u(t), t \geq t_0$, is a solution to (5.1) then $|u(t_0)| + |u'(t_0)| < \alpha$ implies that $|u(t)| + |u'(t)| < B'$ for all $t \geq t_0 + T'(\alpha)$. In this case we say the solutions to (5.1) are uniformly ultimately bounded. We shall prove the following.

Theorem 5. *Let* $p > 1$, $g, f \in C(\mathbb{R}, \mathbb{R})$, $e \in L^\infty(\mathbb{R}, \mathbb{R}) \cap C(\mathbb{R}, \mathbb{R})$, *and* $c > 0$ *satisfy conditions* (i^0)-(iii^0) *listed above. Then the solutions to* (5.1) *are uniformly ultimately bounded.*

We will construct a Liapunov function $S(u, v)$ (see [18], [19]) which will be used to show that the solutions to (5.2) are uniformly ultimately bounded. We will apply a general result due to Yoshizawa ([18],[19]). We need the fact that the Liapunov function is continuous and locally Lipshitz. There must also be an $R_0 \geq 0$ such that $S(u,v)$ satisfies

(P_A) There exists an increasing, positive, continuous function $A(r)$, with $A(r) \to \infty$ as $r \to \infty$ such that

$$S(u,v) \leq A(|u|+|v|) \quad \text{for all } |u|+|v| \geq R_0.$$

(P_B) There exists an increasing, nonnegative, continuous function $B(r)$, with $B(r) \to \infty$ as $r \to \infty$ such that

$$S(u,v) \geq B(|u|+|v|) \quad \text{for all } |u|+|v| \geq R_0.$$

(P_C) There exists a positive continuous function $\mathcal{C}(r)$ such that

$$\frac{d}{dt}S(u,v) \leq -\mathcal{C}(|u|+|v|). \tag{5.3}$$

Proof. To construct S we first modify the functions Φ_q and G so the modifications are bounded below by linear functions. First, choose a and b so that

(1^0) $a > 0$ is such that

$$2|\phi_q(y)|||e||_\infty - c|\phi_q(y)|^2 < -K \quad \text{for all } |y| \geq a;$$

(2^0) $b > 0$ is such that

$$b \geq \frac{4Ka}{||e||_\infty}, \qquad ||e||_\infty\Big(\frac{\phi_q(a)}{K}+2\Big)+1 < g(x) \quad \text{for all } |x| \geq b.$$

Now let $r_1 > 0$ such that $\Phi_q(v) - 2K\,|v| \geq |v|$ whenever $|v| \geq r_1$ and let

$$m_1 = \min_{|v|\leq r_1}(\Phi_q(v) - 2K\,|v|).$$

We define $\widetilde{\Phi}_q(v) := \Phi_q(v) + |m_1| + r_1$. It is easy to check that

$$\widetilde{\Phi}_q(v) - 2K\,|v| \geq |v| \tag{5.4}$$

holds for all $v \in \mathbb{R}$.

Now $G(u)$ may be negative in a bounded region, but since

$$G(u) = \int_0^u g(s)ds$$

and $g(u) \to \pm\infty$ as $u \to \pm\infty$ it follows that $G(u)/\,|u| \to \infty$ as $|u| \to \infty$. Let $r_2 > 0$ such that $G(u) - \frac{4Ka}{b}\,|u| \geq |u|$ whenever $|u| \geq r_2$,

$$m_2 = \min_{|u|\leq r_2}(G(u) - \frac{4Ka}{b}\,|u|)$$

and define $\widetilde{G}(u) := G(u)+|m_2|+r_2+4Ka$. It is easy to check that $\widetilde{G}(u) \geq |u|$ for all $u \in \mathbb{R}$.

We define

$$S(u,v) := \begin{cases} \widetilde{\Phi}_q(v) + \widetilde{G}(u), & \text{if } v \geq a, \\ \widetilde{\Phi}_q(v) + \widetilde{G}(u) - 2Kv + 2Ka, & \text{if } |v| \leq a, \quad u \leq -b, \\ \widetilde{\Phi}_q(v) + \widetilde{G}(u) + 4Ka, & \text{if } v \leq -a, \quad u \leq -b, \\ \widetilde{\Phi}_q(v) + \widetilde{G}(u) - \frac{4Ka}{b}u, & \text{if } v \leq -a, \quad |u| \leq b, \\ \widetilde{\Phi}_q(v) + \widetilde{G}(u) - 4Ka, & \text{if } v \leq -a, \quad u \geq b, \\ \widetilde{\Phi}_q(v) + \widetilde{G}(u) + 2Kv - 2Ka, & \text{if } |v| \leq a, \quad u \geq b, \end{cases}$$

where as before $G(s) = \int_0^s g(t)dt$ and $\Phi_q(s) = \int_0^s \phi_q(t)dt$.

It can now be readily verified that conditions P_A and P_B hold. We will now verify that P_C holds.

We find $\frac{d}{dt}S(u(t),v(t))$:

$$\frac{d}{dt}S(u(t),v(t)) =$$

$$\begin{array}{l} \phi_q(v)\big(e(t) - f(u)\phi_q(v)\big), \quad \text{if } v > a, \\ \phi_q(v)\big(e(t) - f(u)\phi_q(v)\big) \\ \qquad -2K\big(e(t) - g(u) - f(u)\phi_q(v)\big), \quad \text{if } |v| < a, \quad u < -b, \\ \phi_q(v)\big(e(t) - f(u)\phi_q(v)\big), \quad \text{if } v < -a, \quad u < -b, \\ \phi_q(v)\big(e(t) - f(u)\phi_q(v)\big) - \frac{4Ka}{b}\phi_q(v), \quad \text{if } v < -a, \quad |u| < b, \\ \phi_q(v)\big(e(t) - f(u)\phi_q(v)\big), \quad \text{if } v < -a, \quad u > b, \\ \phi_q(v)\big(e(t) - f(u)\phi_q(v)\big) \\ \qquad +2K\big(e(t) - g(u) - f(u)\phi_q(v)\big), \quad \text{if } |v| < a, \quad u > b. \end{array}$$

On the boundary points between the six regions, the time derivative of S must be calculated using the limit supremum of the difference quotient (see [18],[19]). However at such points this limit will simply be the larger of the limits of the derivative as found above, the limits taken from each side of the boundary line, and the inequalities we calculate below will still hold.

By (ii^0) and (1^0) it holds that whenever $v \geq a$ and $u \in \mathbb{R}$, then, using the fact that $f(u) \geq c > 0$,

$$\begin{aligned} \frac{d}{dt}S(u(t),v(t)) &= \phi_q(v)\big(e(t) - f(u)\phi_q(v)\big) \\ &\leq ||e||_\infty \phi_q(|v|) - c(\phi_q(|v|))^2 < -K. \end{aligned}$$

Also, for $|v| < a, \quad u < -b$, we have by (iii^0) and (2^0) that

$$\frac{d}{dt}S(u(t),v(t)) = \phi_q(v)\big(e(t) - f(u)\phi_q(v)\big) - 2K\big(e(t) - g(u) - f(u)\phi_q(v)\big)$$

$$= -f(u)\Big[\phi_q(v) - K\Big]^2 + K^2 f(u) + Ke(t)\Big(\frac{\phi_q(v)}{K} - 2\Big) + 2Kg(u)$$

$$\leq K\Big(Kf(u) + g(u) + ||e||_\infty\Big(\frac{\phi_q(a)}{K} + 2\Big) + g(u)\Big)$$

$$\leq K||e||_\infty\Big(\frac{\phi_q(a)}{K} + 2\Big) + g(u)\Big) < K \cdot (-1) = -K.$$

For $v < -a, \quad |u| < b$ we have that

$$\begin{aligned}\frac{d}{dt}S(u(t), v(t)) &= \phi_q(v)\big(e(t) - f(u)\phi_q(v)\big) - \frac{4Ka}{b}\phi_q(v)\\ &\leq \Big(\frac{4Ka}{b} + ||e||_\infty\Big)\phi_q(|v|) - f(u)(\phi_q(|v|))^2,\\ &\leq 2||e||_\infty \phi_q(|v|) - c(\phi_q(|v|))^2 < -K.\end{aligned}$$

Finally, for $|v| < a, u > b$ we have that

$$\frac{d}{dt}S(u(t), v(t)) = \phi_q(v)\big(e(t) - f(u)\phi_q(v)\big) + 2K\big(e(t) - g(u) - f(u)\phi_q(v)\big)$$

$$= -f(u)\Big[\phi_q(v) + K\Big]^2 + K^2 f(u) + Ke(t)\Big(\frac{\phi_q(v)}{K} + 2\Big) - 2Kg(u)$$

$$\leq K\Big(Kf(u) - g(u) + ||e||_\infty\Big(\frac{\phi_q(a)}{K} + 2\Big) - g(u)\Big) < -K.$$

The choice of a and b and the calculations above show that $\frac{d}{dt}S(u(t), v(t)) < -K$ and thus S has the property P_C with $\mathcal{C}(r) = -K$

It now follows from a theorem of Yoshizawa (Theorem 7 in [18], also stated as Theorem 8.10 in [19]) that the solutions of (5.2) are uniformly ultimately bounded. It follows that the solutions to (5.1) are also uniformly ultimately bounded. We have proved the theorem. □

6. Stability of the trivial solution in higher dimensions

We now consider long-term behavior for the system case,

$$(\phi_p(u'))' + \frac{d}{dt}(\nabla F(u)) + \nabla G(u) = 0, \tag{6.1}$$

where $p > 1$, $G \in C^1(\mathbb{R}^N, \mathbb{R})$, $F \in C^2(\mathbb{R}^N, \mathbb{R})$. We are interested in conditions that imply asymptotic stability of the trivial solution, partly in preparation for studying uniform ultimate boundedness in the forced equation. With this in mind, we make assumptions similar to those in Theorem 3. To be precise, we assume $G(x) > 0$ for all $x \in \mathbb{R}^N\backslash\{0\}$, $G(x) \to \infty$ as $|x| \to \infty$, $G(0) = 0$, $\nabla G(x) = 0$ if and only if $x = 0$, and $\nabla F(x) = 0$ if and only if $x = 0$. In addition, assume there is a constant $m > 0$ such that

$< F''(x)y, y > \geq m\,|y|^2$ for all $x, y \in \mathbb{R}^N$. We may rewrite (6.1) equivalently as

$$u' = \phi_q(v), \quad v' = -\frac{d}{dt}(\nabla F(u)) - \nabla G(u),$$

since $\phi_p^{-1}(x) = \phi_q(x)$, where $\frac{1}{p} + \frac{1}{q} = 1$. As before, for $x \in \mathbb{R}^N$ let

$$\Phi_q(x) = \frac{1}{q}\,|x|^q\,.$$

Define $S = S(u, v)$ by $S(u, v) = \Phi_q(v) + G(u)$. Note that if $S(u, v) > 0$ for all $(u, v) \neq (0, 0)$. We have for the derivative of S along solutions:

$$\begin{aligned}\frac{d}{dt}S &= \phi_q(v)\cdot v' + \nabla G(u)\cdot u' = \phi_q(v)\cdot(-F''(u)u' - \nabla G(u)) + \nabla G(u)\cdot\phi_q(v)\\ &= -\phi_q(v)\cdot F''(u)\phi_q(v) \leq -m\,|\phi_q(v)|^2 \leq 0.\end{aligned}$$

Let $R > 0$ and Ω_R be the set in $\mathbb{R}^N \times \mathbb{R}^N$ where $S(u, v) < R$. It is clear that Ω_R is bounded and $S(u, v) > 0$ for $(u, v) \in \Omega_R \backslash \{(0, 0)\}$. We have seen that $\frac{d}{dt}S \leq 0$. Let $\mathcal{A}$ be the set of all points in Ω_R where $\frac{d}{dt}S(u, v) = 0$, and let $\mathcal{E}$ be the largest invariant set in $\mathcal{A}$. LaSalle's invariance theorem states that every solution $(u(t), v(t)) \in \Omega_R$ tends to $\mathcal{E}$ as $t \to \infty$ (see, e.g., [5], Theorem VI, page 58). Now $\mathcal{A}$ consists of those points in Ω_R satisfying $\phi_q(v) = 0$ so that $v = 0$. The solutions in the largest invariant set in $\mathcal{A}$ must satisfy $u' = 0$, so $u = u_0$ is constant, and $0 = v' = -\nabla F(u_0)$. Thus $\nabla F(u_0) = 0$ and hence $u_0 = 0$. Thus $\mathcal{E}$ consists of only the origin $(0, 0)$. It follows that the origin is globally asymptotically stable. We have proved the following.

Theorem 6. *Let $p > 1$, $G \in C^1(\mathbb{R}^N, \mathbb{R})$, $F \in C^2(\mathbb{R}^N, \mathbb{R})$. Assume $G(x) > 0$, for all $x \in \mathbb{R}^N \backslash \{0\}$, $G(0) = 0$, $G(x) \to \infty$ as $|x| \to \infty$, and $\nabla F(x) = 0$ if and only if $x = 0$. Also assume there is a constant $m > 0$ such that $< F''(x)y, y > \geq m\,|y|^2$ for all $x, y \in \mathbb{R}^N$. Then the origin is a globally asymptotically stable solution to* (6.1).

7. CONLEY INDEX AND ULTIMATE BOUNDEDNESS

In this section we will consider the long-time behavior of the non-autonomous differential equation

$$(\phi_p(u'))' + \frac{d}{dt}(\nabla F(u)) + \nabla G(u) = \mu w(t), \tag{7.1}$$

where $p > 1$, $G \in C^1(\mathbb{R}^N, \mathbb{R})$, $F \in C^2(\mathbb{R}^N, \mathbb{R})$, $w \in C(\mathbb{R}, \mathbb{R}^N)$ and $\mu \in [0, 1]$. Assume that F and G satisfy the conditions in Theorem 6. We assume that w is bounded; i.e., there is a constant $M \geq 0$ such that for all $t \in \mathbb{R}$,

$|w(t)| \leq M$. We will also assume that w is uniformly continuous on $\mathbb{R}$. Topological methods will be used to study asymptotic behavior of solutions to (7.1). In particular, we will apply Conley index [1], [14],[9] to associated skew product flows [15] to study (7.1). We first apply Conley index ideas to the unforced differential system (6.1). We will continue to assume all of the conditions on F and G made in Theorem 6. We prove that the solutions are ultimately bounded.

Definition 7. *The solutions to* (7.1) *are ultimately bounded for bound B if there exists a $B > 0$ and a $T > 0$ such that, for every solution* $(u(t), v(t)) = (u(t; t_0, u_0, v_0), v(t; t_0, u_0, v_0))$ *of* (7.1), $|u(t)| + |v(t)| < B$ *for all $t \geq t_0 + T$, where B is independent of the particular solution while T may depend on each solution.*

We shall prove the following theorem.

Theorem 8. *Let* $p > 1$, $G \in C^1(\mathbb{R}^N, \mathbb{R})$, $F \in C^2(\mathbb{R}^N, \mathbb{R})$, *satisfy the conditions assumed in Theorem 6. Let* $h \in C(\mathbb{R}, \mathbb{R}^N) \cap L^\infty(\mathbb{R}, \mathbb{R}^N)$ *and* $\mu \in [0, 1]$. *Further assume that w is uniformly continuous on $\mathbb{R}$. Then there is a $B > 0$ and $\mu_0 > 0$ such that for all $0 \leq \mu < \mu_0$ solutions to* (7.1) *are ultimately bounded for bound B. If in addition the function w is periodic then the solutions are equiultimately bounded for bound B, for all $0 \leq \mu < \mu_0$.*

Proof. We first apply Conley index methods to the autonomous system (6.1). The system of differential equations (6.1) defines a dynamical system on $\mathbb{R}^N \times \mathbb{R}^N$ which we will identify with $\mathbb{R}^{2N}$, writing either $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$ or $(u, v) = x \in \mathbb{R}^{2N}$ as may add to clarity. Thus let $x = (u, v)$ and

$$W(x) := (\phi_q(v), -F''(u)\phi_q(v) - \nabla G(u)).$$

The initial-value problem

$$x'(t) = W(x(t)), \ x(0) = x_0 \tag{7.2}$$

defines a local dynamical system $\widetilde{\pi}$ on $\mathbb{R}^{2N}$. The equilibrium $x = 0$ is asymptotically stable, and has therefore Conley index Σ^0, the two-point pointed space; see Conley [1] or [14]. We should like to define a critical point at infinity, which should be a repeller. However, it is not clear that solutions extend in backward time all the way to $t = -\infty$. We thus modify (7.2) to

$$x'(t) = \frac{W(x(t))}{1 + |W(x(t))|}, \quad x(0) = x_0 \tag{7.3}$$

which has the same orbits as does (7.2), but solutions are defined for all time. Let π denote the dynamical system defined by (7.3). Let $\mathbb{R}^{2N}_*$ be the one-point compactification of $\mathbb{R}^{2N}$, which consists of $\mathbb{R}^{2N} \cup \{\infty\}$, with $U \subset \mathbb{R}^{2N}_*$ an open set if either (i) U is an open subset of $\mathbb{R}^{2N}$, or (ii), $\mathbb{R}^{2N}_* \backslash U$ is a compact subset of $\mathbb{R}^{2N}$. We define the flow π^* on $\mathbb{R}^{2N}_*$ by (i) $\pi^*(x,t) = \pi(x,t)$ if $x \in \mathbb{R}^{2N}$, and (ii) $\pi(\infty, t) = \infty$ (see [4], page 54).

There are two equilibrium points for π^*, the origin 0 and the point ∞. The origin 0 is an attractor and ∞ is a repeller, and the Conley indices are $h(\pi^*, 0) = \Sigma^0$ and $h(\pi^*, \infty) = \Sigma^{2N}$, the pointed $2N$-sphere. If $x \in \mathbb{R}^{2N}_* \backslash \{0, \infty\}$ then the orbit through x connects the points ∞ and 0. The pair $(0, \infty)$ is said to be an attractor-repeller decomposition of $\mathbb{R}^{2N}_*$ (see [1], [9], [10]).

We now consider the non-autonomous differential equation (7.1) and prove Theorem 8. First rewrite the system as before,

$$u' = \phi_q(v), \quad v' = -\frac{d}{dt}(\nabla F(u)) - \nabla G(u) + \mu w(t). \tag{7.4}$$

We will associate (7.4) with a skew product flow, as we now explain. Let $g \in C(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$ and assume that g satisfies a uniform local Lipshitz condition; that is, for each $r > 0$ there is a $K(r)$ such that

$$|g(t,x) - g(t,y)| \le K(r)\,|x - y|$$

holds for $x, y \in B(0,r)$ and $t \in \mathbb{R}$. Assume also that $g(t,x)$ is bounded and uniformly continuous on sets of the form $\mathbb{R} \times K$, K compact in $\mathbb{R}^m$. We let, for $\tau \in \mathbb{R}$, $g_\tau \in C\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$ be defined by $g_\tau(t,x) = g(\tau + t, x)$. Let $C_c(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$ denote $C(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$ with the topology of uniform convergence on compact sets and and let $H(g)$ denote the closure in $C_c(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$ of the set of time translates $\{g_\tau : \tau \in \mathbb{R}\}$. $H(g)$ is called the hull of g. The hull $H(g)$ is compact in $C_c(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$. If $g^* \in H(g)$ then the solutions to initial-value problems

$$x'(t) = g^*(t, x(t)),\ x(0) = x_0 \tag{7.5}$$

are unique, for all $\tau \in \mathbb{R}$ and $x_0 \in \mathbb{R}^m$. Let $x(t; x_0, g^*)$ denote the solution to (7.5). We may define a local dynamical system or flow π on $\mathbb{R}^m \times H(g)$ by

$$\pi((x_0, g^*), t) := (u(t; x_0, g^*), g^*_t)) \tag{7.6}$$

(see [15]). The flow $\beta : H(g) \times \mathbb{R} \to H(g)$ defined by $\beta(g^*, t) = g^*_t$ is called the Bebutov flow on $H(g)$. The skew product flow enables us to study the nonautonomous system as a dynamical system in a meaningful way. It is to be noted that $H(g)$ is an isolated compact invariant set for the flow β. Its only isolating block is itself, and it has an empty exit set. Thus its Conley index $h(\beta, H(g))$ is (homotopic to) the pointed space $[H(g), Z] := H(g)^0$ where Z is a point disjoint from $H(g)$.

We have assumed that $w = w(t)$ is bounded and uniformly continuous on $\mathbb{R}$. It follows that the hull of w, $H(w)$, is compact in the space of bounded continuous functions with the sup norm. Let $\mu \in [0, 1]$. We consider the family of initial-value problems

$$u' = \phi_q(v), \quad v' = -F''(u)\phi_q(v) - \nabla G(u) + \mu w^*(t) \tag{7.7}$$
$$u(0) = u_0, v(0) = v_0,$$

where $\mu \in [0, 1]$ and $w^* \in H(w)$. Let $x = (u, v)$ and

$$F^*(x, t, \mu) := (\phi_q(v), F''(u)\phi_q(v) - \nabla G(u) + \mu w^*(t)).$$

We define a flow π_μ on $\mathbb{R}^N \times \mathbb{R}^N \times H(w)$ as follows:

$$\pi_\mu((x_0, w^*), t) = ((x(t; x_0, \mu), w_t^*)$$

where $x_0 = (u_0, v_0)$ are the initial conditions. It is not hard to show that π_μ is continuous in μ (see [1], [14]). Notice that when $\mu = 0$ we have the product flow on $\mathbb{R}^{2N} \times H(p)$ given by

$$\pi_0((x_0, w^*), t) = ((x(t; x_0, 0), w_t^*)$$

where $x(t; x_0, 0)$ is a solution to the autonomous system (7.2), which obtains when $\mu = 0$ in (7.7). We see that $\pi_0 = \widetilde{\pi} \times \beta$, where $\widetilde{\pi}$ is the flow defined above on $\mathbb{R}^{2N}$ and β is the Bebutov flow on $H(w)$. We extend the flows π_μ to flows π_μ^* on $\mathbb{R}_*^{2N} \times H(w)$ analogously to what we did with $\widetilde{\pi}$ in the $\mu = 0$ case. We now analyze the flows π_μ^*. First of all, $\pi_0^* = \pi^* \times \beta$ which has the two isolated invariant sets $\{0\} \times H(w)$ and $\{\infty\} \times H(w)$. Now $h(\pi_0^*, \{0\} \times H(w))$is homotopic to the smash product $h(\pi^*, \{0\}) \wedge h(\beta, H(w)) = \Sigma^0 \wedge H(w)^0 = H(w)^0$. It follows that $\{0\} \times H(w)$ is an attractor. The set $\{\infty\} \times H(w)$ is the associated repeller with index $h(\pi_0^*, \{\infty\} \times H(w)) = \Sigma^{2N} \wedge H(w)^0$. The sets $\{0\} \times H(w)$ and $\{\infty\} \times H(w)$ are attractor-repeller pairs and form an attractor-repeller decomposition of the space $\mathbb{R}_*^{2N} \times H(w)$ for the flow π_0^*. Now attractor-repeller decompositions continue [10], so there are disjoint compact isolating neighborhoods N_0, N_∞ of $\{0\} \times H(w)$ and $\{\infty\} \times H(w)$, respectively, such that for $0 \le \mu < \mu_0$ there are maximal invariant sets $I_\mu^0 \subset N_0$ and $I_\mu^\infty \subset N_\infty$ which form attractor-repeller pair decompositions of $\mathbb{R}_*^{2N} \times H(p)$ for π_μ^*. Thus the orbit through every point $(x_0, w_0) \in \mathbb{R}_*^{2N} \times H(w) \backslash \{\{0\} \times H(w), \{\infty\} \times H(w)\}$ eventually enters N_0 and approaches I_μ^0. The neighborhood N_0 is necessarily of the form $N_0 = N_{00} \times H(w)$ where N_{00} is a bounded region in $\mathbb{R}^{2N}$ with $N_{00} \subset B(0, R_1)$

for some $R_1 > 0$. Thus we can conclude that for any $x_0 \in \mathbb{R}^{2N}$ and $0 \leq \mu < \mu_0$ the solution $x(t; x_0, \mu w) = (u(t), v(t))$ to (7.4) with $(u(0), v(0)) = (u_0, v_0) = x_0$ eventually enters $B(0, R_1)$ and stays there. This proves the first part of Theorem 8. Now assume that w is a periodic function. A theorem of Yoshizawa implies that if initial-value problems for (7.4) are unique, w is periodic, and solutions are ultimately bounded, then solutions are equiultimately bounded ([18], page 85 and [19], Theorem 8.5). This completes the proof of Theorem 8 □

Earlier in Section 4 we provided sufficient conditions on F and G that guarantee a periodic solution exists for (7.1) when the forcing is periodic. If $w = w(t)$ is almost-periodic the preceding result may be applied to prove the existence of a solution defined and bounded on $\mathbb{R}$. We have the following.

Theorem 9. *Suppose the conditions on F and G assumed in Theorem 8 hold. Let $w = w(t)$ be an almost-periodic function. Let $\mu_0 > 0$ be the number guaranteed by Theorem* 8. *Then for each $\mu \in [0, \mu_0)$ the system* (7.1) *has a full bounded solution $(u_0(t), v_0(t))$ with $(u_0(t), v_0(t)) \in \overline{B}(0, R_1)$ for all $t \in \mathbb{R}$.*

Proof. Let $X(t) := (u(t), v(t))$ be any solution of (7.1). There is a time $T \geq 0$ such that $X(t) \in B(0, R_1)$ for all $t \geq T$. Now because $w = w(t)$ is almost periodic, there is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $w_n(t) := w(t + t_n) \to w(t)$ as $n \to \infty$, uniformly on $\mathbb{R}$. Let $X_n(t) := (u_n(t), v_n(t)) = (u(t + t_n), v(t + t_n))$. Then we have

$$u_n' = \phi_q(v_n), \quad v_n' = -F''(u_n)\phi_q(v_n) - \nabla G(u_n) + \mu w_n(t)$$

and $|u_n(t)|^2 + |v_n(t)|^2 \leq R_1^2$ for $t \geq T - t_n$. From the differential equation we have that the $(u_n'(t), v_n'(t))$ are also uniformly bounded for $t \geq T - t_n$. It follows from standard limiting arguments that there is a subsequence of $\{(u_n, v_n)\}$ uniformly convergent on compact sets to a function $(u_0(t), v_0(t))$ with $|u_0(t)|^2 + |v_0(t)|^2 \leq R_1^2$ for $t \in \mathbb{R}$ and for all $t \in \mathbb{R}$

$$u_0' = \phi_q(v_0), \quad v_0' = -F''(u_0)\phi_q(v_0) - \nabla G(u_0) + \mu w(t).$$

This proves the theorem. □

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