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GROUND STATE SOLUTIONS FOR A SEMILINEAR PROBLEM WITH CRITICAL EXPONENT

Andrzej Szulkin [1]
Department of Mathematics, Stockholm University
S-106 91 Stockholm, Sweden

Tobias Weth
Institut fr Mathematik, Goethe-Universität Frankfurt
Robert-Mayer-Str. 10, D-60054 Frankfurt, Germany

Michel Willem [2]
Université Catholique de Louvain, Département de Mathématique
Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

To Patrick Habets and to Jean Mawhin with admiration and friendship.

Abstract. This work is devoted to the existence and qualitative properties of ground state solutions of the Dirchlet problem for the semilinear equation $-\Delta u - \lambda u = |u|^{2^*-2}u$ in a bounded domain. Here, 2^* is the critical Sobolev exponent, and the term ground state refers to minimizers of the corresponding energy within the set of nontrivial solutions. We focus on the indefinite case where λ is larger than the first Dirichlet eigenvalue of the Laplacian, and we present a particularly simple approach to the study of ground states.

1. Introduction

We consider the existence of solutions of the problem

$$(\mathcal{P}_\lambda) \qquad \begin{cases} -\Delta u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, and $2^* = 2N/(N-2)$ denotes the critical Sobolev exponent. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$ be the eigenvalues of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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When $0 < \lambda < \lambda_1$, the existence of (positive) ground state solutions was proved by Brezis and Nirenberg in the classical paper [9]. When $\lambda \geq \lambda_1$, the existence of nontrivial solutions was considered in [10] (see also [17, 12, 25]). This work is devoted to the existence of ground state solutions when $\lambda \geq \lambda_1$.

In Section 2, we give a minimax characterization of the ground states and we define the corresponding Nehari manifold. We use the manifold introduced by Pankov in [19] and we adapt to the critical case the method of [22]. Other reduction methods for variational problems are used in [11, 14, 20, 21].

In Section 3, we prove, under some assumptions, the existence of ground states when $\lambda \geq \lambda_1$ using the minimax characterization and the direct method of the calculus of variations. As observed in [22], this *analytical* minimax principle is much simpler than the usual *topological* minimax theorems (see e.g. [23]). Moreover, it is not necessary to use the deformation lemma or the Ekeland variational principle. Section 4 is devoted to the Morse index and to nodal properties of ground states. In Sections 5 and 6, we consider symmetry properties of ground states and radial ground states.

As has been pointed out above, our results on the existence of nontrivial solutions are not new. The novelty (besides the method) is that we can prove there exist *ground states* and characterize them by a minimax principle. Moreover, if Ω is a ball, then we show that for λ between λ_1 and λ_2 there are two solutions which respectively are a (nonradial) ground state and a radial ground state. These solutions have exactly two nodal domains.

2. Characterization of ground states

We consider the Hilbert space $X = H_0^1(\Omega)$ with scalar product

$$\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx,$$

and the induced norm $||\cdot||$. We fix a positive integer m and assume that $\lambda_m \leq \lambda < \lambda_{m+1}$. Moreover, we let $e_1, \ldots, e_m$ be normalized eigenfunctions corresponding to $\lambda_1, \ldots, \lambda_m$, and we consider the subspaces $Z := \text{span}(e_1, \ldots, e_m)$ and $Y := Z^\perp$ of X.

The solutions of $(\mathcal{P}_\lambda)$ are the critical points of the C^1-functional

$$\varphi(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda u^2) \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx,$$

defined on X. Let P (respectively Q) be the orthogonal projector onto Y (respectively Z) in X. Define the map

$$F : X \setminus \{0\} \to \mathbb{R} \times Z \simeq \mathbb{R}^{m+1}, \qquad F(u) = (\langle \nabla\varphi(u), u\rangle, Q\nabla\varphi(u)).$$

It is clear that, if u is a nontrivial critical point of φ, then $F(u) = 0$, and if $u \in Z \setminus \{0\}$, then $\langle \nabla\varphi(u), u\rangle < 0$.

Lemma 2.1. *Let $u \in X \setminus \{0\}$ be such that $F(u) = 0$. Then $F'(u)$ is onto.*

Proof. It suffices to prove that, for every $(t, z) \in \mathbb{R} \times Z$, $(t, z) \neq 0$,

$$[F'(u)(tu + z)] \cdot (t, z) \neq 0.$$

Here, and in the following, we consider the scalar product on $\mathbb{R} \times Z$ defined by

$$(t_1, z_1) \cdot (t_2, z_2) = t_1 t_2 + \langle z_1, z_2\rangle \qquad \text{for } t_1, t_2 \in \mathbb{R},\ z_1, z_2 \in Z.$$

We modify an argument in [19]. Let $(t, z) \neq 0$. Since $\langle \nabla\varphi(u), u\rangle = \langle \nabla\varphi(u), z\rangle = 0$, we obtain

$$\begin{aligned}
&[F'(u)(tu + z)] \cdot (t, z)\\
&= t\varphi''(u)(tu + z, u) + t\langle \nabla\varphi(u), tu + z\rangle + \varphi''(u)(tu + z, z)\\
&= \varphi''(u)(tu + z, tu + z) - t\langle \nabla\varphi(u), tu + 2z\rangle\\
&= \int |\nabla z|^2 - \lambda z^2 dx - \int [(2^* - 1)(tu + z)^2 - tu(tu + 2z)]|u|^{2^*-2} dx\\
&= \int |\nabla z|^2 - \lambda z^2 dx - \int [(2^* - 2)t^2u^2 + 2(2^* - 2)tzu + (2^* - 1)z^2]|u|^{2^*-2} dx.
\end{aligned}$$

The last integrand is positive definite in t and $z(x)$ whenever $u(x) \neq 0$. If

$$\int |\nabla z|^2 - \lambda z^2 dx < 0,$$

the proof is complete. Assume that

$$\int |\nabla z|^2 - \lambda z^2 dx = 0.$$

Then either $\lambda = \lambda_m$ and z is an eigenfunction, or $z = 0$. If $z \neq 0$, then $z \neq 0$ almost everywhere on Ω and if $z = 0$, then $t \neq 0$. In both cases it is easy to see that $[F'(u)(tu + z)] \cdot (t, z) < 0$. □

By the preceding lemma, $\mathcal{M} := \{u \in X \setminus \{0\} : F(u) = 0\}$ is a $\mathcal{C}^1$-submanifold of X with codimension $m + 1$.

Theorem 2.2. *Let $u \in X$. Then u is a nontrivial critical point of φ if and only if*

$$u \in \mathcal{M}, \quad \varphi'(u)\Big|_{T_u\mathcal{M}} = 0. \tag{2.1}$$

Proof. The necessary condition is clear. Assume that u satisfies (2.1). By definition, $\varphi'(u)$ vanishes on the subspace $\mathbb{R}u \oplus Z$ and on $T_u\mathcal{M}$. Since, by the proof of Lemma 2.1, $\mathbb{R}u \oplus Z$ is transverse to $T_u\mathcal{M}$, $\varphi'(u)$ vanishes on X. □

By Proposition 2.3 and Lemma 2.8 in [22], for every $v \in Y\backslash\{0\}$, there exists a unique $(f(v), g(v)) \in]0, +\infty[\times Z$, continuously depending on v and such that

$$F(f(v)v + g(v)) = 0.$$

Moreover,

$$\varphi(f(v)v + g(v)) = \max_{t>0,\ w\in Z} \varphi(tv + w).$$

We define the value

$$c = c(\lambda) = \inf_{\mathcal{M}} = \inf_{\substack{v\in Y\\ v\neq 0}} \varphi(f(v)v + g(v)) = \inf_{\substack{v\in Y\\ v\neq 0}} \max_{\substack{t>0\\ w\in Z}} \varphi(tv + w).$$

Since $\mathcal{M}$ is a $\mathcal{C}^1$-submanifold of X, any $u \in \mathcal{M}$ with $\varphi(u) = c$ satisfies $\varphi'(u)\Big|_{T_u\mathcal{M}} = 0$, therefore it is a critical point of φ by Theorem 2.2. We call such a function u a *ground state solution*, since Theorem 2.2 implies that u minimizes φ within the set of nontrivial solutions of $(\mathcal{P}_\lambda)$.

3. Existence of ground states

Let us recall that the *best Sobolev constant*

$$S = \inf_{\substack{u\in H_0^1(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{2/2^*}}$$

is independent of Ω.

We shall use the following elementary lemma.

Lemma 3.1. *Let $A > 0$, $B > 0$. Then*

$$\max_{t>0} \left(A\frac{t^2}{2} - B\frac{t^{2^*}}{2^*}\right) = \frac{1}{N}\left(\frac{A}{B^{2/2^*}}\right)^{N/2}.$$

Let us also recall that $\lambda_m \leq \lambda < \lambda_{m+1}$.

Lemma 3.2. *Suppose*

$$c < S^{N/2}/N. \tag{3.1}$$

Then there exists $v \in Y\backslash\{0\}$ *such that*

$$\max_{t>0,\ w\in Z} \varphi(tv+w) = \varphi(f(v)v+g(v)) = c.$$

Proof. Let $(v_n) \subset Y$ be such that $||v_n|| = 1$ and

$$\max_{t>0,\ w\in Z} \varphi(tv_n+w) \to c. \tag{3.2}$$

We can assume that

$$v_n \rightharpoonup v \text{ in } Y, \quad v_n \to v \text{ in } L^2(\Omega), \quad v_n \to v \text{ a.e. on } \Omega.$$

We define

$$A = \lim_{n\to\infty} \int_\Omega |\nabla(v_n - v)|^2 dx, \quad B = \lim_{n\to\infty} \int_\Omega |v_n - v|^{2^*} dx.$$

Using the Brezis-Lieb lemma (see [8] or [24]), we obtain, from (3.2), for every $t > 0$ and for every $w \in Z$,

$$\varphi(tv+w) + A\frac{t^2}{2} - B\frac{t^{2^*}}{2^*} \le c. \tag{3.3}$$

Assume that $v = 0$ and $B = 0$. Since $||v_n|| = 1$, it follows that $A = 1$ and, for every $t > 0$, $\frac{t^2}{2} \le c$, a contradiction.

Assume now that $B \neq 0$. We obtain from the Sobolev inequality that

$$\frac{S^{N/2}}{N} \le \frac{1}{N}\Big(\frac{A}{B^{2/2^*}}\Big)^{N/2} = \max_{t>0}\Big(A\frac{t^2}{2} - B\frac{t^{2^*}}{2^*}\Big). \tag{3.4}$$

If $v = 0$, it follows from (3.1), (3.3) and (3.4) that

$$\frac{S^{N/2}}{N} \le c < \frac{S^{N/2}}{N},$$

a contradiction. Thus, $v \neq 0$.

Let us define $h = (f(v))^{-1}g(v)$. Using the definition of c, we have

$$\begin{aligned} c \le \varphi(f(v)(v+h)) &= \max_{t>0} \varphi(t(v+h)) \qquad (3.5)\\ &= \frac{1}{N}\Big[\frac{\int_\Omega[|\nabla v|^2 + |\nabla h|^2 - \lambda(v^2+h^2)]dx}{(\int_\Omega |v+h|^{2^*}dx)^{2/2^*}}\Big]^{N/2}. \end{aligned}$$

Inequality (3.3) implies that

$$\max_{t>0}\Big[\varphi(t(v+h)) + A\frac{t^2}{2} - B\frac{t^{2^*}}{2^*}\Big] \tag{3.6}$$

$$= \quad \frac{1}{N}\left[\frac{A+\int_\Omega[|\nabla v|^2+|\nabla h|^2-\lambda(v^2+h^2)]dx}{(B+\int_\Omega|v+h|^{2^*}dx)^{2/2^*}}\right]^{N/2} \leq c.$$

It follows from (3.1) (3.4) (3.5) and (3.6) that

$$\begin{aligned}
&(Nc)^{2/N}\Big[B+\int_\Omega |v+h|^{2^*}dx\Big]^{2/2^*} \\
&< (Nc)^{2/N}\Big[B^{2/2^*}+\Big(\int_\Omega |v+h|^{2^*}dx\Big)^{2/2^*}\Big] \\
&< A+\int_\Omega\Big[|\nabla v|^2+|\nabla h|^2-\lambda(v^2+h^2)\Big]dx \\
&\leq (Nc)^{2/N}\Big[B+\int_\Omega |v+h|^{2^*}dx\Big]^{2/2^*},
\end{aligned}$$

a contradiction.

Thus, $B=0$ and we conclude from (3.3) that, by the definition of c,

$$c \leq \varphi(f(v)v+g(v)) \leq c. \qquad \square$$

In order to prove inequality (3.1), we need some preliminary results.

Lemma 3.3. *Let $w \in Z$ be such that $w=0$ on a nonempty open subset ω of Ω. Then $w=0$ on Ω.*

Proof. Suppose $w \neq 0$. We can assume that $w = w_1+\ldots+w_k$, where w_j is an eigenfunction corresponding to $\mu_j \in \{\lambda_1,\ldots,\lambda_m\}$. We can also assume that $\mu_1<\ldots<\mu_k$. Since, on ω,

$$0=\mu_k w+\Delta w=(\mu_k-\mu_1)w_1+\ldots+(\mu_k-\mu_{k-1})w_{k-1},$$

we can replace w_k by 0. After $k-1$ steps, we obtain an eigenfunction (corresponding to μ_1) vanishing on ω. This is impossible. $\square$

We may assume that $0\in\Omega$. Let $\psi\in\mathcal{D}(\Omega)$ be a radial test function such that $0\leq\psi\leq 1$ and $\psi\equiv 1$ in a neighbourhood of 0. Let us recall that the optimal function for the Sobolev inequality in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the instanton

$$U_\varepsilon(x)=[N(N-2)]^{(N-2)/4}\frac{\varepsilon^{(N-2)/2}}{(\varepsilon^2+|x|^2)^{(N-2)/2}}.$$

Lemma 3.4. *The truncated instanton $u_\varepsilon(x)=\psi(x)U_\varepsilon(x)$ satisfies the following estimates:*

a) $\displaystyle\int_\Omega |\nabla u_\varepsilon|^2dx = S^{N/2}+O(\varepsilon^{N-2}),$

b) $\displaystyle\int_\Omega u_\varepsilon^{2^*}dx = S^{N/2} + O(\varepsilon^N)$,

c) $\displaystyle\int_\Omega u_\varepsilon^{2^*-1}dx = O\big(\varepsilon^{\frac{N-2}{2}}\big), \int_\Omega u_\varepsilon dx = O\big(\varepsilon^{\frac{N-2}{2}}\big), \int_\Omega |\nabla u_\varepsilon| dx = O\big(\varepsilon^{\frac{N-2}{2}}\big)$,

d) $\displaystyle\int_\Omega u_\varepsilon^2 dx \geq \begin{cases} d\varepsilon^2|\ln\varepsilon| + O(\varepsilon^2), & \text{if } N=4,\\ d\varepsilon^2 + O(\varepsilon^{N-2}), & \text{if } N \geq 5,\end{cases}$

where d is a positive constant depending on N.

Proof. See [9] and [23, pages 35 and 52]. □

Lemma 3.5. *Let $N=4$ and $\lambda_m < \lambda < \lambda_{m+1}$ or let $N \geq 5$ and $\lambda_m \leq \lambda < \lambda_{m+1}$. Then $c(\lambda) < S^{N/2}/N$.*

Proof. It suffices to prove that, for $\varepsilon > 0$ small enough,

$$\max_{t>0,\ w\in Z} \varphi(tu_\varepsilon + w) < S^{N/2}/N. \tag{3.7}$$

Let $\omega = \Omega \backslash \operatorname{supp} \psi$. By Lemma 3.3, $||w||_{L^{2^*}(\omega)}$ defines a norm on Z. Since

$$\dim Z = m,$$

all the norms are equivalent on Z. We obtain, by convexity, for every $t>0$ and for every $w \in Z$,

$$\begin{aligned}\int_\Omega |tu_\varepsilon + w|^{2^*}dx &= \int_{\Omega\backslash\omega} |tu_\varepsilon + w|^{2^*}dx + \int_\omega |w|^{2^*}dx\\ &\geq t^{2^*}\int_\Omega u_\varepsilon^{2^*}dx + 2^* t^{2^*-1}\int_\Omega u_\varepsilon^{2^*-1}w dx + 2^* c_1 ||w||^{2^*}.\end{aligned}$$

It follows that

$$\begin{aligned}\varphi(tu_\varepsilon + w) \leq \varphi(tu_\varepsilon) + t\int_\Omega \nabla u_\varepsilon\cdot\nabla w - \lambda u_\varepsilon w dx\\ + \frac{1}{2}\int_\Omega |\nabla w|^2 - \lambda w^2 dx - t^{2^*-1}\int_\Omega u_\varepsilon^{2^*-1} w dx - c_1||w||^{2^*}.\end{aligned} \tag{3.8}$$

In particular, if $0 < \varepsilon < 1$,

$$\varphi(tu_\varepsilon + w) \leq A(t^2 + t||w|| + t^{2^*-1}||w||) - B(t^{2^*} + ||w||^{2^*}),$$

for some $A, B > 0$. Hence, there exists $R > 0$ such that, for $0 < \varepsilon < 1$, $t > R$ and $w \in Z$, $\varphi(tu_\varepsilon + w) \leq 0$. Inequality (3.8) implies that, for $t \leq R$,

$$\varphi(tu_\varepsilon + w) \leq \varphi(tu_\varepsilon) + O\big(\varepsilon^{\frac{N-2}{2}}\big)||w|| - c_1||w||^{2^*} \leq \varphi(tu_\varepsilon) + O\big(\varepsilon^{N\frac{N-2}{N+2}}\big).$$

Indeed, for any $p > 1$, by Young's inequality,

$$\max_{s>0}(rs - \tfrac{1}{p}s^p) = \tfrac{p-1}{p} r^{\frac{p}{p-1}}.$$

Since $N(N-2)/(N+2) > 2$ whenever $N \geq 5$, we obtain, from Lemmas 3.1 and 3.4, for $\varepsilon > 0$ small enough,

$$\max_{t>0,\ w\in Z} \varphi(tu_\varepsilon + w) \leq \frac{1}{N}\Big(\frac{\int_\Omega |\nabla u_\varepsilon|^2 - \lambda u_\varepsilon^2 dx}{\big(\int_\Omega u_\varepsilon^{2^*} dx\big)^{2/2^*}}\Big)^{N/2} + O\big(\varepsilon^{N\frac{N-2}{N+2}}\big)$$

$$\leq \frac{1}{N}\left(S - \lambda d\varepsilon^2 S^{(2-N)/2} + O(\varepsilon^{N-2})\right)^{N/2} + O\left(\varepsilon^{N\frac{N-2}{N+2}}\right) < \frac{S^{N/2}}{N}.$$

Assume now that $N = 4$ and $\lambda_m < \lambda < \lambda_{m+1}$. Inequality (3.8) implies that, for $t \leq R$,

$$\varphi(tu_\varepsilon + w) \leq \varphi(tu_\varepsilon) + O(\varepsilon)||w|| - c_2||w||^2 \leq \varphi(tu_\varepsilon) + O(\varepsilon^2).$$

We obtain, from Lemma 3.4, for $\varepsilon > 0$ small enough,

$$\max_{t>0,\ w\in Z} \varphi(tu_\varepsilon + w) \leq \frac{1}{4}\Big(\frac{\int_\Omega |\nabla u_\varepsilon|^2 - \lambda u_\varepsilon^2 dx}{\big(\int_\Omega u_\varepsilon^{2^*} dx\big)^{2/2^*}}\Big)^2 + O(\varepsilon^2)$$

$$= \frac{1}{4}\left(S - \lambda d\varepsilon^2 |\ln\varepsilon| S^{-1} + O(\varepsilon^2)\right)^2 + O(\varepsilon^2) < \frac{S^2}{4}. \qquad \square$$

Theorem 3.6. *Let $N = 4$ and $\lambda_m < \lambda < \lambda_{m+1}$ or $N \geq 5$ and $\lambda_m \leq \lambda < \lambda_{m+1}$. Then there exists a solution u of $(\mathcal{P}_\lambda)$ such that $\varphi(u) = c$.*

Proof. Lemma 3.2 and Lemma 3.5 imply the existence of $u \in \mathcal{M}$ such that $\varphi(u) = c$. In particular $\varphi'(u)\Big|_{T_u\mathcal{M}} = 0$. It follows from Theorem 2.2 that $\varphi'(u) = 0$. $\square$

Remarks. a) When $\lambda_m < \lambda < \lambda_{m+1}$, the inequality (3.7) is contained in [10]. When $\lambda_m = \lambda$, this inequality seems new; however, see [17], where a similar result may be found for a maximum defined in a somewhat different way (the subspace Z is modified in order to make it orthogonal to u_ε).

b) Under the assumptions of Theorem 3.6, the existence of a nontrivial solution of $(\mathcal{P}_\lambda)$ was proved in [10] (see also [25] and [17]).

c) A nontrivial solution also exists if $N = 4$ and $\lambda = \lambda_1$. This is a direct consequence of a more general multiplicity result in [12]. However, the existence of a ground state solution remains an open question in this case.

4. Nodal properties of the ground states

We need the following elementary lemma.

Lemma 4.1. *Let $\mathcal{M}$ be a $\mathcal{C}^1$-submanifold of the Hilbert space X, $\varphi \in \mathcal{C}^2(X, \mathbb{R})$ and $u \in \mathcal{M}$ be such that $\varphi'(u) = 0$ and $\varphi(u) = \inf_{\mathcal{M}} \varphi$. Then, for every $h \in T_u\mathcal{M}$,*

$$\varphi''(u)(h, h) \geq 0.$$

Proof. Let $\gamma :]-1, 1[\to \mathcal{M}$ be a $\mathcal{C}^1$-curve such that $\gamma(0) = u$ and $\dot{\gamma}(0) = h$. Since

$$\frac{d}{dt} \varphi \circ \gamma(t) = \langle \nabla \varphi(\gamma(t)), \dot{\gamma}(t) \rangle,$$

we obtain

$$\begin{aligned} \frac{d^2}{dt^2} \varphi \circ \gamma(0) &= \lim_{\substack{t \to 0 \\ t \neq 0}} \frac{1}{t} \langle \nabla \varphi(\gamma(t)), \dot{\gamma}(t) \rangle \\ &= \lim_{\substack{t \to 0 \\ t \neq 0}} \langle \frac{\nabla \varphi(\gamma(t)) - \nabla \varphi(\gamma(0))}{t}, \dot{\gamma}(t) \rangle = \varphi''(\gamma(0))(\dot{\gamma}(0), \dot{\gamma}(0)), \end{aligned}$$

and

$$0 \leq \frac{d^2}{dt^2} \varphi \circ \gamma(0) = \varphi''(u)(h, h).$$

□

The *Morse index* of a critical point u of φ will be denoted by $M(u)$.

Theorem 4.2. *Let $\lambda_m \leq \lambda < \lambda_{m+1}$ and let $u \in \mathcal{M}$ be such that $\varphi(u) = c$. Then $\varphi'(u) = 0$ and $M(u) = m + 1$.*

Proof. Since $c = \inf_{\mathcal{M}} \varphi$, it is clear that $\varphi'(u)\Big|_{T_u\mathcal{M}} = 0$. By Theorem 2.2, u is a critical point of φ. The preceding lemma implies that, for every $h \in T_u\mathcal{M} = (F'(u))^{-1}(0)$,

$$\varphi''(u)(h, h) \geq 0.$$

We conclude that $M(u) \leq m + 1$. It follows from the proof of Lemma 2.1 that $M(u) \geq m + 1$. □

The *number of nodal regions* of a (classical) solution u of $(\mathcal{P}_\lambda)$ is denoted by $\text{nod}(u)$.

The following result is due to Benci and Fortunato [6]. We recall the proof for the sake of completeness.

Theorem 4.3. *Let u be a nontrivial critical point of φ. Then* $\text{nod}(u) \leq M(u)$.

Proof. By elliptic regularity theory, u is a classical solution of $(\mathcal{P}_\lambda)$. Let $\Omega_1, ..., \Omega_n$ be the nodal domains of u and define $u_j = \chi_{\Omega_j} u,\ j = 1, ..., n$. Theorem IX.17 and Remarque 20 in [7] imply that $u_j \in H_0^1(\Omega)$. We obtain

$$\begin{aligned} \varphi''(u)(u_j, u_j) &= \int_\Omega |\nabla u_j|^2 - \lambda u_j^2 dx - (2^* - 1) \int_\Omega |u|^{2^*-2} |u_j|^2 dx \\ &< \int_\Omega |\nabla u_j|^2 - \lambda u_j^2 dx - \int_\Omega |u_j|^{2^*} dx \\ &= \langle \varphi'(u_j), u_j \rangle = 0, \end{aligned}$$

and therefore $n \leq M(u)$. □

Corollary 4.4. *Let $\lambda_m \leq \lambda < \lambda_{m+1}$ and let $u \in \mathcal{M}$ be such that $\varphi(u) = c$. Then* $\mathrm{nod}(u) \leq m + 1$*. If moreover $m = 1$, then* $\mathrm{nod}(u) = 2$*.*

Proof. The above theorems imply that

$$\mathrm{nod}(u) \leq M(u) = m + 1.$$

If $m = 1$, then $\lambda_1 \leq \lambda < \lambda_2$, and it is easy to verify that $2 \leq \mathrm{nod}(u)$. □

Remark. It was proved in [6] that, under the assumptions of Theorem 3.6, there exists a nontrivial solution of $(\mathcal{P}_\lambda)$ such that

$$\mathrm{nod}(u) \leq M(u) \leq m + 1.$$

5. Symmetry properties

In this section, we assume that Ω is a *radial* bounded domain, so it is a ball or an annulus in $\mathbb{R}^N$ centered at the origin. We focus on the case $\lambda_1 \leq \lambda < \lambda_2$, and we discuss the shape of ground state solutions of $(\mathcal{P}_\lambda)$. We recall that a function u defined on a radial domain is said to be *foliated Schwarz symmetric* if there is a unit vector $p \in \mathbb{R}^N$, $|p| = 1$, such that $u(x)$ only depends on $r = |x|$ and $\theta = \arccos\left(\frac{x}{|x|} \cdot p\right)$ and u is nonincreasing in θ.

Theorem 5.1. *Let $\lambda_1 \leq \lambda < \lambda_2$ and let $u \in \mathcal{M}$ be such that $\varphi(u) = c = \inf_{\mathcal{M}} \varphi$. Then u is foliated Schwarz symmetric. Moreover, if the underlying domain Ω is a ball, then u is nonradial.*

Proof. Let, as before, e_1 denote the first eigenfunction of the Dirichlet Laplacian in Ω. In the present case we have

$$\begin{aligned} \mathcal{M} = \Big\{ u \in X \setminus \{0\} : &\int_\Omega (|\nabla u|^2 - \lambda u^2 - |u|^{2^*})\, dx = 0 \\ &= \int_\Omega (\nabla u \cdot \nabla e_1 - \lambda u e_1 - |u|^{2^*-2} u e_1)\, dx \Big\} \end{aligned}$$

$$= \Big\{ u \in X \setminus \{0\} \;:\; \int_\Omega (|\nabla u|^2 - \lambda u^2 - |u|^{2^*})\, dx = 0$$
$$= \int_\Omega [(\lambda_1 - \lambda) u e_1 - |u|^{2^*-2} u e_1]\, dx \Big\}.$$

Now, let $u \in \mathcal{M}$ be such that $\varphi(u) = c$. We pick $x_0 \in \mathrm{int}(\Omega)$, $x_0 \neq 0$, with

$$u(x_0) = \max\{u(x) \;:\; x \in \Omega, |x| = |x_0|\},$$

and put $p := \frac{x_0}{|x_0|}$. Moreover, we let $\mathcal{H}_p$ denote the family of all closed halfspaces H in $\mathbb{R}^N$ such that 0 lies in the hyperplane ∂H and p is contained in the interior of H. For $H \in \mathcal{H}_p$ we consider the reflection $\sigma_H : \mathbb{R}^N \to \mathbb{R}^N$ with respect to the boundary of H. We claim that, for every $H \in \mathcal{H}_p$,

$$u(x) \geq u(\sigma_H(x)) \qquad \text{for all } x \in H \cap \Omega. \tag{5.1}$$

To prove this, we fix $H \in \mathcal{H}_p$ and recall a simple rearrangement, namely the *polarization* of u with respect to H defined by

$$u_H(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & x \in \Omega \cap H \\ \min\{u(x), u(\sigma_H(x))\}, & x \in \Omega \setminus H. \end{cases}$$

It is well known and fairly easy to prove that

$$\int_\Omega |\nabla u_H|^2\, dx = \int_\Omega |\nabla u|^2\, dx \quad \text{and} \quad \int_\Omega |u_H|^p\, dx = \int_\Omega |u|^p\, dx,$$

for every $p \in [1, \infty]$. Moreover, since e_1 is a radial function, we also have

$$\int_\Omega u_H e_1\, dx = \int_\Omega u e_1\, dx \;\text{ and }\; \int_\Omega |u_H|^{p-2} u_H e_1\, dx = \int_\Omega |u|^{p-2} u e_1\, dx,$$

for every $p \in [1, \infty]$ (for details on these invariance properties, see e.g. [5, Section 2]). Consequently, $u_H \in \mathcal{M}$ and $\varphi(u_H) = \varphi(u) = c$, which implies that both u and u_H are critical points of φ and, hence, classical solutions of $(\mathcal{P}_\lambda)$. Therefore, $w := u_H - u$ is a nonnegative function in $\Omega \cap H$ which solves the Dirichlet problem

$$-\Delta w = q(x) w \quad \text{in } \mathrm{int}(H \cap \Omega), \qquad w = 0 \quad \text{on } \partial\Omega \cap H,$$

where

$$q(x) = \lambda + (2^* - 1) \int_0^1 |(1-t)u(x) + t u_H(x)|^{2^*-2}\, dt \qquad \text{for } x \in \Omega \cap H.$$

Since $q \in L^\infty(\Omega \cap H)$, the strong maximum principle implies that either $w \equiv 0$ or $w > 0$ in $\mathrm{int}(H \cap \Omega)$. The latter case is ruled out since $x_0 \in \mathrm{int}(H \cap \Omega)$

and $w(x_0) = u_H(x_0) - u(x_0) = 0$ by the choice of x_0. We, therefore, obtain $w \equiv 0$, hence, $u = u_H$ and (5.1) holds.

Now, from (5.1) it follows by continuity that u is symmetric with respect to every hyperplane containing p, so it is axially symmetric with respect to p. Hence, $u(x)$ only depends on $r = |x|$ and $\theta = \arccos\left(\frac{x}{|x|} \cdot p\right)$. Moreover, it also follows from (5.1) that u is nonincreasing in θ. We conclude that u is foliated Schwarz symmetric.

Finally, if Ω is a ball, a result of Aftalion and Pacella (see [1]) implies that every radial solution of $(\mathcal{P}_\lambda)$ has Morse index greater than or equal to $N+1$, whereas u has Morse index 2 by Theorem 4.2. Hence, u is nonradial in this case. $\square$

Remarks. a) In the radial setting considered here, the Dirichlet eigenvalues of the Laplacian satisfy $\lambda_1 < \lambda_2 = \cdots = \lambda_{N+1} < \lambda_{N+2}$. The proof of Theorem 5.1 does not extend to the case $\lambda \geq \lambda_{N+1}$, since in this case the set $\mathcal{M}$ is not invariant with respect to polarization.

b) In dimensions $3 \leq N \leq 6$ where the critical nonlinearity in $(\mathcal{P}_\lambda)$ has a convex derivative, Theorem 5.1 can also be deduced from Theorem 4.2 above and Theorem 1.1 in [18], the latter being a rather general result relating Morse index estimates to foliated Schwarz symmetry. However, the proof given here is simpler and works for every $N \geq 3$.

6. Radial ground states

As in the preceding section, we assume Ω is a ball or an annulus centered at the origin. Let $X_r = H^1_{0,r}(\Omega)$ be the subspace of $H^1_0(\Omega)$ consisting of radial functions and let $\varphi_r := \varphi|_{X_r}$. Since the group $O(N)$ of orthogonal transformations acts isometrically on X by means of the formula $gu(x) = u(g^{-1}x)$ (where $g \in O(N)$) and X_r is the space of fixed points of this action, it follows from the principle of symmetric criticality [23, Theorem 1.28] that critical points of φ_r are radial solutions of $(\mathcal{P}_\lambda)$. Denote the eigenvalues of $-\Delta$ to which there correspond eigenfunctions in X_r by λ_i^r. It is well known that all λ_i^r are simple (in X_r), $\lambda_1 = \lambda_1^r$ and $\lambda_2 = ... = \lambda_{N+1} < \lambda_2^r$. An easy inspection shows that the results of Sections 2-4 remain valid in X_r (with λ_m replaced by λ_m^r). Let $\mathcal{M}_r := \mathcal{M} \cap X_r$ and $c_r := \inf_{\mathcal{M}_r} \varphi_r$.

Theorem 6.1. *(i) Let $N = 4$ and $\lambda_m^r < \lambda < \lambda_{m+1}^r$ or $N \geq 5$ and $\lambda_m^r \leq \lambda < \lambda_{m+1}^r$. Then there exists u_r such that $\varphi_r(u_r) = c_r$. Moreover, each $u_r \in \mathcal{M}_r$ such that $\varphi_r(u_r) = c_r$ is a radial solution of $(\mathcal{P}_\lambda)$.*
(ii) Let $\lambda_m^r \leq \lambda < \lambda_{m+1}^r$ and let $u_r \in \mathcal{M}_r$ be such that $\varphi_r(u_r) = c_r$. Then $\mathrm{nod}(u_r) = m + 1$.

Proof. We only have to prove that $\mathrm{nod}(u_r) \geq m+1$. The function u_r is an eigenfunction of

$$\begin{cases} -\Delta v - V(x)v = \mu_n v & \text{in } \Omega, \\ v \text{ is a radial function}, & \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $V = |u_r|^{2^*-2}$ is a radially symmetric function and $\mu_n = \lambda \geq \lambda_m^r$ is the n-th eigenvalue of the operator $-\Delta - V$ in X_r. Since $V \geq 0$ and $V \not\equiv 0$, we have that $\mu_n < \lambda_n^r$. It follows that $m < n$. By Sturm-Liouville theory, $\mathrm{nod}(u_r) = n \geq m+1$. □

If Ω is a ball and $\lambda_1 \leq \lambda < \lambda_2$, then by Theorem 5.1, any $u \in \mathcal{M}$ such that $\varphi(u) = c$ is nonradial. Hence, we obtain the following:

Corollary 6.2. *Suppose Ω is a ball and $\lambda_1 < \lambda < \lambda_2$ if $N = 4$, $\lambda_1 \leq \lambda < \lambda_2$ if $N \geq 5$. Then the ground state solution u obtained in Theorem 3.6 and the radial ground state solution u_r obtained in Theorem 6.1 are distinct. Moreover, $c < c_r$ and both u and u_r have exactly two nodal domains.*

Remarks. a) If Ω is a ball, existence of nontrivial solutions of $(\mathcal{P}_\lambda)$ for each $\lambda > 0$ and $N \geq 4$ has been shown in [15]. However, the solutions obtained there, although symmetric, are not radial and need not be ground states.

b) Let Ω be the unit ball in $\mathbb{R}^N$, $N = 4$ or 5. According to [3] and [4], the bifurcation branch arising from λ_2^r tends asymptotically to λ_1 $(= \lambda_1^r)$ as the L^∞-norm of the solution tends to infinity. It is shown in [2], by a computer assisted proof, that this branch does not cross λ_1 when $N = 4$. This corresponds to our assumption $\lambda_1 < \lambda < \lambda_2^r$, when $N = 4$. On the other hand, when $N = 5$, this branch crosses λ_1 (see [16]).

c) Let Ω be a ball in $\mathbb{R}^3$. By Theorem 1 in [13], for every $\lambda \geq \lambda_1$ there exists a nodal solution of $(\mathcal{P}_\lambda)$. Again, these solutions, although symmetric, are not radial and need not be ground states.

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