Differential and Integral Equations Volume 22, Numbers 9-10 (2009), 901–911

A NOTE ON MULTIPLE SOLUTIONS FOR SUBLINEAR ELLIPTIC SYSTEMS

Miguel Ramos
University of Lisbon, CMAF - Faculty of Science
Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

Abstract. We prove the existence of a sequence of solutions approaching zero for an elliptic system of the form $-\Delta u = |v|^{q-2}v + g(x,v)$, $-\Delta v = |u|^{p-2}u + f(x,u)$ in a bounded domain, under Dirichlet homogeneous boundary conditions. We assume that $1 < p, q < 2$ and that both $f(x,u)$ and $g(x,v)$ are small enough near the origin.

1. Introduction

Given a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, consider the semilinear elliptic equation

$$-\Delta u = |u|^{p-2}u + f(x), \qquad u \in H_0^1(\Omega), \tag{1.1}$$

with, say, $f \in L^2(\Omega)$. The case where $2 < p < 2N/(N-2)$ (superlinear and subcritical case) has been widely studied in the past decades. In case $f \equiv 0$ the problem admits an unbounded sequence of solutions ([3]). The case where $f \not\equiv 0$ was first considered independently by A. Bahri, H. Berestycki and M. Struwe ([5, 20]) in their pioneering papers in 1980, with subsequent improvements by A. Bahri, P.L. Lions and K. Tanaka ([6, 21]). P. Rabinowitz ([17]) stated a variational principle for dealing with such "Perturbation from Symmetry" problems in 1982. In particular, it is known that (1.1) admits an unbounded sequence of solutions in case $2 < p < (2N-2)/(N-2)$, and it remains an open problem to know whether one can allow for p the full subcritical range.

Since then a number of authors have considered these type of problems, which we write in the more general form

$$-\Delta u = |u|^{p-2}u + f(x,u), \qquad u \in H_0^1(\Omega), \tag{1.2}$$

AMS Subject Classifications: 35J50, 35J55, 58E05.

Research supported by Fundação para a Ciência e a Tecnologia, Financiamento Base 2008 - ISFL/1/209.

where $p \in]2, 2N/(N-2)[$ is assumed to be the dominating term at infinity and $f(x,s)$ is not assumed to be odd symmetric in s. For recent results we refer the reader to [8, 10, 11, 12, 18, 19, 23].

Consider now the case where $1 < p < 2$ (sublinear case). If $f \equiv 0$ it can proved that (1.2) admits a sequence of solutions approaching zero (cf. e.g. [2, 24, 25]). Quite recently, R. Kajikiya ([16, Theorem 1.1]) has obtained the same conclusion if $f \not\equiv 0$, assuming f is Hölder continuous and $|f(x,u)u| \leqslant C|u|^r$ in $\Omega \times [-\delta, \delta]$ for some $\delta, C > 0$, provided $r > p$ is sufficiently large, namely if

$$r > p + \frac{N(2-p)(2N - p(N-2))}{4p}. \tag{1.3}$$

It is not known whether this condition is sharp.

On the other hand, in the past years special attention has been devoted to the study of elliptic systems leading to strongly indefinite functionals (see for example [14] for a survey on the subject). In the context of superlinear elliptic systems with perturbed symmetry, we mention the papers [13, 15], which deal with potential systems of the form $-\Delta u = \partial_u F(x,u,v)$, $-\Delta v = -\partial_v F(x,u,v)$, for some smooth function $F(x,u,v)$.

In the present paper, we are concerned with strongly coupled elliptic systems of the form

$$-\Delta u = |v|^{q-2}v + g(x,v), \quad -\Delta v = |u|^{p-2}u + f(x,u), \quad u, v \in H_0^1(\Omega). \tag{1.4}$$

In the case where $f \equiv g \equiv 0$ and $p, q > 2$, it is shown in [4], among other results, that (1.4) admits an unbounded sequence of solutions provided $\frac{N}{2}(1 - \frac{1}{p} - \frac{1}{q}) < 1$. In [9], a similar conclusion was proved to hold for the perturbed problem with $f(x,u) = f(x) \in L^2(\Omega)$, $g(x,v) = g(x) \in L^2(\Omega)$ and $2 < p \leqslant q$ (say), provided $\frac{N}{2}(1 - \frac{1}{p} - \frac{1}{q}) < \frac{p-1}{p}$, improving an earlier result in [22].

In view of the previous discussion, it becomes a natural question to seek for a similar multiplicity result for the system (1.4) in the sublinear case, namely if $1 < p, q < 2$. We give a partial result in this direction, much in the spirit of Kajikiya's one for the single equation case.

Theorem 1.1. *Let $1 < p, q < 2$ and $f, g : \overline{\Omega} \times [-\delta, \delta] \to \mathbb{R}$ be Hölder continuous functions ($\delta > 0$), C^1 with respect to the second variable, such that $f(x,0) \equiv 0$, $g(x,0) \equiv 0$ and, for some $C > 0$, $r, s > 0$,*

$$\left|\frac{\partial f}{\partial u}(x,u)u^2\right| \leqslant C|u|^r \quad \text{and} \quad \left|\frac{\partial g}{\partial v}(x,v)v^2\right| \leqslant C|v|^s \quad \forall x \in \Omega, |u| < \delta, |v| < \delta. \tag{1.5}$$

Then, if both r and s are sufficiently large, namely if

$$\min\{rq, sp\} > pq + \frac{N}{4}\,(\frac{1}{p} + \frac{1}{q} - 1)\{(p+q)N - pq(N-2)\}, \tag{1.6}$$

then problem (1.4) *admits a sequence of non-zero solutions whose* $C^2(\overline{\Omega})$*-norms converge to zero.*

We observe that if $f \equiv g$ and $p = q$, then (1.4) reduces to (1.2) and the condition in (1.6) is precisely the one in (1.3). (See also Remark 2.1 in the next section.)

The proof of Theorem 1.1 combines the arguments in [9, 16] and will be given in the next section. Solutions of (1.4) correspond to critical points of the strongly indefinite functional

$$I(u,v) := \int_\Omega (\langle \nabla u, \nabla v\rangle - \frac{1}{p}|u|^p - \frac{1}{q}|v|^q - F(x,u) - G(x,v)), \tag{1.7}$$

$(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, where

$$F(x,u) := \int_0^u f(x,\xi)\,d\xi \quad \text{and} \quad G(x,v) := \int_0^v g(x,\xi)\,d\xi.$$

A crucial step of the proof will consist in applying a reduction method in such a way that the resulting functional is bounded from below. This procedure allows us to handle the system (1.4) pretty much as the equation (1.2), except for one point. Indeed, in the sublinear case, (1.2) can be studied, leading to the same conclusions, by either applying a (slight variant of) Rabinowitz's perturbation method for superlinear problems, or by developing Bolle's continuation argument ([7, 8]); Kajikiya's proof relies on the latter idea. Concerning system (1.4), in the sublinear case the reduction method introduces a lack of differentiability which makes it difficult to apply Rabinowitz's perturbation arguments; this is in contrast with the superlinear case, where no such problem arises (cf. [9]). However, it turns out that this lack of differentiability is no longer apparent in Kajikiya's version of Bolle's argument, and therefore we will closely follow the proof in [16].

2. Proof of Theorem 1.1

Let $1 < p, q < 2$ and f, g satisfy (1.5). We may further assume that $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are locally Hölder continuous and, for some $C > 0$,

(a) $|f(x,u)u| \leqslant C$ and $|g(x,v)v| \leqslant C\ \forall u, v, x$;
(b) $|f(x,u)u| \leqslant C|u|^r$ and $|g(x,v)v| \leqslant C|v|^s\ \forall u, v, x$;

(c) $|f(x,u)u - 2F(x,u)| \leqslant \delta'|u|^p$ and $|g(x,v)v - 2G(x,v)| \leqslant \delta'|v|^q$ $\forall u, v, x$, where $\delta' > 0$ can be chosen arbitrarily small;

(d) $|u|^p/p + F(x,u)$ (respectively $|v|^q/q + G(x,v)$) is convex in u (respectively in v).

Otherwise, as in [16, 24], one replaces $f(x,u)$ and $g(x,v)$ by $f(x,u)\rho(u)$ and $g(x,v)\rho(v)$ respectively, where $\rho \in \mathcal{D}(\mathbb{R})$ is a smooth cut-off function such that $\rho(s) = 1$ if $|s| \leqslant \varepsilon$ and $\rho(s) = 0$ if $|s| \geqslant 2\varepsilon$, for a small $\varepsilon > 0$. If ε is sufficiently small then (a)–(d) hold, while the full statement in Theorem 1.1 shows that in this way we do not affect its conclusions.

Remark 2.1. In order to get (a)–(d), slight variants of (1.5) could have been considered. For example, in Theorem 1.1 we could as well have assumed that, for $x \in \Omega$, $|u| < \delta$ and $|v| < \delta$, $|f(x,u)u| \leqslant C|u|^r$, $\frac{\partial f}{\partial u}(x,u) \geqslant 0$, $|g(x,v)v| \leqslant C|u|^s$ and $\frac{\partial g}{\partial v}(x,v) \geqslant 0$.

Let $E := H_0^1(\Omega)$ be equipped with the usual norm $||u||^2 := \int_\Omega |\nabla u|^2$; we also denote $||u||_p := (\int_\Omega |u|^p)^{1/p}$ the usual L^p-norm.

Under our assumptions, the functional I defined in (1.7) is C^1 over $E \times E$. Moreover, if (u,v) is a solution of (1.4), by computing $2I(u,v) - I'(u,v)(u,v)$ and using (c) above we see that

$$\int_\Omega (|u|^p + |v|^q) \leqslant C|I(u,v)|, \qquad \forall (u,v) : I'(u,v) = 0, \tag{2.1}$$

for some $C > 0$. Similarly to [14, 16], one can also deduce an *a priori* bound for the L^∞-norms of the solutions.

Lemma 2.2. *There exists $C > 0$ such that*

$$||u||_\infty \leqslant C|I(u,v)|^{\frac{2q}{(p+q)N - pq(N-2)}} \quad \text{and} \quad ||v||_\infty \leqslant C|I(u,v)|^{\frac{2p}{(p+q)N - pq(N-2)}},$$

for every (u,v) such that $I'(u,v) = 0$.

Proof. Let $M(u,v)$ be given by

$$M(u,v) := \max_{x \in \overline{\Omega}} \max\{|u(x)|^{1/q}, |v(x)|^{1/p}\}. \tag{2.2}$$

Thanks to (2.1), it is sufficient to prove that

$$M(u,v) \leqslant C\Big(\int_\Omega |u|^p + \int_\Omega |v|^q\Big)^{\frac{2}{(p+q)N - pq(N-2)}} \quad \forall (u,v) : I'(u,v) = 0. \tag{2.3}$$

Now, let us assume that the reversed inequality holds for a sequence (u_j, v_j) of solutions of (1.4), with $C = C_j \to \infty$. Since, by elliptic regularity,

$\sup_j M(u_j, v_j) < \infty$, we must have that

$$\int_\Omega (|u_j|^p + |v_j|^q) \to 0,$$

and then, again by elliptic regularity, $M_j := M(u_j, v_j) \to 0$. Let $\lambda_j > 0$ be given by $\lambda_j^2 M_j^{pq-p-q} = 1$, so that also $\lambda_j \to 0$, and let the maximum in (2.2) for M_j be attained at some point $x_j \in \Omega$. By denoting

$$\widetilde{u}_j(x) := u_j(\lambda_j x + x_j)/M_j^q, \qquad \widetilde{v}_j(x) := v_j(\lambda_j x + x_j)/M_j^p,$$

we see that $(\widetilde{u}_j(0), \widetilde{v}_j(0)) \neq (0,0)$ and

$$-\Delta \widetilde{u}_j = |\widetilde{v}_j|^{q-2}\widetilde{v}_j + \mathrm{O}(|\widetilde{v}_j|^{q-1}), \quad -\Delta \widetilde{v}_j = |\widetilde{u}_j|^{p-2}\widetilde{u}_j + \mathrm{O}(|\widetilde{u}_j|^{p-1}),$$

in $\Omega_j := \frac{1}{\lambda_j}(\Omega - x_j)$. Taking limits (and, if necessary, a reflection with respect to a hyperplane) yields a solution $(\widetilde{u}, \widetilde{v})$ of the limit problem

$$-\Delta \widetilde{u} = |\widetilde{v}|^{q-2}\widetilde{v}, \qquad -\Delta \widetilde{v} = |\widetilde{u}|^{p-2}\widetilde{u} \quad \text{in } \mathbb{R}^N, \tag{2.4}$$

such that $(\widetilde{u}(0), \widetilde{v}(0)) \neq (0,0)$. However, by construction

$$\int_{\Omega_j} (|\widetilde{u}_j|^p + |\widetilde{v}_j|^q) \to 0,$$

and this is a contradiction, proving (2.3). □

Remark 2.3. Using the fact that the limit system (2.4) has no non-zero solutions $\widetilde{u} \in L^p \cap L^\infty(\mathbb{R}^N)$, $\widetilde{v} \in L^q \cap L^\infty(\mathbb{R}^N)$, one can derive a stronger conclusion in Lemma 2.2. However, this would not ultimately improve the inequality in (1.6).

Next, we use a procedure analogous to the one in [9] for the superlinear case. We introduce the functional

$$J(u) := I(u + \psi_u, u - \psi_u) := \max_{\psi \in E} I(u + \psi, u - \psi), \qquad u \in E.$$

We point out that the map $\psi \mapsto I(u+\psi, u-\psi)$ is strictly concave, as follows from the property (d) above. This shows that J is well defined and that the map $u \mapsto \psi_u$ is continous (although not differentiable). We also consider the even symmetric functionals

$$I_0(u,v) := \int_\Omega \left(\langle \nabla u, \nabla v\rangle - \frac{1}{p}|u|^p - \frac{1}{q}|v|^q\right), \qquad (u,v) \in E \times E, \tag{2.5}$$

and $J_0(u) := \max_{\psi \in E} I_0(u + \psi, u - \psi)$. The relevant properties of J and J_0 are listed in the next proposition. We denote by $(\lambda_j)_{j \in \mathbb{N}}$ the non-decreasing sequence of eigenvalues of $(-\Delta, H_0^1(\Omega))$, by $(\varphi_j)_{j\in\mathbb{N}}$ a sequence of corresponding (normalized) eigenfunctions, and $E_k := \mathrm{span}\{\varphi_1, \dots, \varphi_{k+1}\}$.

Proposition 2.4.

(i) *$J \in C^1(E;\mathbb{R})$ and*

$$J'(u)\varphi = I'(u+\psi_u, u-\psi_u)(\varphi,\varphi), \qquad \forall u, \varphi \in E. \tag{2.6}$$

In particular, u is a critical point of J if and only if $(u+\psi_u, u-\psi_u)$ is a critical point of I.

(ii) $\inf_E J > -\infty$.

(iii) *J satisfies the Palais-Smale condition over E.*

(iv) *If $J'(u) = 0$, then*

$$\Big| \int_\Omega (F(x, u+\psi_u) + G(x, u-\psi_u)) \Big| \leqslant M \min\{|J(u)|^{\nu+1}, 1\},$$

for some $M > 0$ and

$$\nu := \frac{2\,(\min\{rq, sp\} - pq)}{(p+q)N - pq(N-2)}. \tag{2.7}$$

(v) *For every $u \in E$, $u \neq 0$, there exists a unique $\tau(u) > 0$ such that $J_0(su) < 0$ if $0 < s < \tau(u)$ and $J_0(su) > 0$ if $s > \tau(u)$.*

(vi) *Let $0 < \mu < 2$. There exists $c > 0$ such that, for every $k \geqslant 1$,*

$$\inf_{E_{k-1}^\perp} J_0 \geqslant \frac{-c}{k^\beta},$$

where

$$\beta := \frac{2}{N} \min\{\frac{\mu p}{2-p}, \frac{(2-\mu)q}{2-q}\}. \tag{2.8}$$

Proof. The conclusion in (i) can be proved similarly to [1, page 133]. Indeed, using the definition of ψ_u we see that, given $\varepsilon > 0$,

$$\begin{aligned} &J(u+\varphi) - J(u) - I'(u+\psi_u, u-\psi_u)(\varphi,\varphi) \\ &\leqslant I(u+\varphi+\psi_{u+\varphi}, u+\varphi-\psi_{u+\varphi}) - I(u+\psi_{u+\varphi}, u-\psi_{u+\varphi}) \\ &\qquad - I'(u+\psi_u, u-\psi_u)(\varphi,\varphi) \\ &= \int_0^1 (I'(u+s\varphi+\psi_{u+\varphi}, u+s\varphi-\psi_{u+\varphi})(\varphi,\varphi) \\ &\qquad - I'(u+\psi_u, u-\psi_u)(\varphi,\varphi))\, ds \leqslant \varepsilon ||\varphi||, \end{aligned}$$

provided $||\varphi||$ is sufficiently small. In a similar way one shows that

$$J(u+\varphi) - J(u) - I'(u+\psi_u, u-\psi_u)(\varphi,\varphi) \geqslant -\varepsilon ||\varphi||,$$

and (2.6) follows. As for the final statement in (i), we merely observe that, by definition, $I'(u+\psi_u, u-\psi_u)(\varphi,-\varphi) = 0$ for all $u, \varphi \in E$.

As for (ii), we recall that

$$|F(x,u)| + |G(x,v)| \leqslant C \quad \forall x, u, v,$$

and we note that, by definition,

$$\inf_{u\in E} J(u) \geqslant \inf_{u\in E} I(u,u) > -\infty.$$

Concerning (iii), suppose $|J(u_n)| \leqslant C$ and $J'(u_n) \to 0$. Then $(\widetilde{u}_n, \widetilde{v}_n) := (u_n + \psi_{u_n}, u_n - \psi_{u_n})$ is a Palais-Smale sequence for the functional I, namely $|I(\widetilde{u}_n, \widetilde{v}_n)| \leqslant C$ and $I'(\widetilde{u}_n, \widetilde{v}_n) \to 0$. By computing $I'(\widetilde{u}_n, \widetilde{v}_n)(\widetilde{v}_n, \widetilde{u}_n)$ and using standard arguments we deduce that, up to a subsequence, $\widetilde{u}_n \to \widetilde{u}$ and $\widetilde{v}_n \to \widetilde{v}$ in E, for some $\widetilde{u}, \widetilde{v} \in E$, and the conclusion follows.

Property (iv) amounts to showing that

$$\Big|\int_\Omega (F(x,u) + G(x,v))\Big| \leqslant M \min\{|I(u,v)|^{\nu+1}, 1\}, \quad \forall (u,v) : I'(u,v) = 0. \tag{2.9}$$

Now, by using (2.1) and property (b) above,

$$\Big|\int_\Omega (F(x,u) + G(x,v))\Big| \leqslant C \int_\Omega (|u|^r + |v|^s) \leqslant C|I(u,v)|\,(||u||_\infty^{r-p} + ||v||_\infty^{s-q}),$$

and we derive (2.9) from Lemma 2.2.

In order to prove (v), let us fix $u \neq 0$ and denote

$$\theta(s) := J_0(su) = I_0(su+\psi_s, su-\psi_s), \qquad s > 0;$$

that is,

$$\theta(s) = s^2||u||^2 - ||\psi_s||^2 - \frac{1}{p}\int_\Omega |su+\psi_s|^p - \frac{1}{q}\int_\Omega |su-\psi_s|^q. \tag{2.10}$$

Since, by definition, $I_0'(su+\psi_s, su-\psi_s)(\psi_s, -\psi_s) = 0$, we see that

$$2||\psi_s||^2 = \int_\Omega |su-\psi_s|^{q-2}(su-\psi_s)\psi_s - \int_\Omega |su+\psi_s|^{p-2}(su+\psi_s)\psi_s. \tag{2.11}$$

Using (2.10)–(2.11) we conclude that

$$\begin{aligned} s\theta'(s) &= sI_0'(su+\psi_s, su-\psi_s)(u,u) \\ &= (\tfrac{2}{p} - 1)\int_\Omega |su+\psi_s|^p + (\tfrac{2}{q} - 1)\int_\Omega |su-\psi_s|^q > 0,\ \forall s > 0 : \theta(s) = 0. \end{aligned}$$

Thus, since $\theta(s) \geqslant I_0(su, su) \to +\infty$ as $s \to +\infty$, in order to prove (v) it is sufficient to show that

$$\theta(s) < 0 \qquad \forall s > 0 \text{ close to zero.} \tag{2.12}$$

For this, without loss of generality we may assume that $p \leqslant q$. For a small $\delta'' > 0$, if $s > 0$ is sufficiently close to zero we have that

$$s^2||u||^2 \leqslant \delta''||2su||_p^q.$$

Hence, denoting by C and C' some constants which depend only on p, q and Ω,

$$\begin{aligned} s^2||u||^2 &\leqslant \delta''\big(||su+\psi_s||_p + ||su-\psi_s||_p\big)^q \\ &\leqslant C\delta''\Big(\Big(\int_\Omega |su+\psi_s|^p\Big)^{q/p} + \Big(\int_\Omega |su-\psi_s|^p\Big)^{q/p}\Big) \\ &\leqslant C\delta''\Big(\int_\Omega |su+\psi_s|^p + \Big(\int_\Omega |su-\psi_s|^p\Big)^{q/p}\Big) \\ &\leqslant CC\delta''\Big(\int_\Omega |su+\psi_s|^p + \int_\Omega |su-\psi_s|^q\Big); \end{aligned}$$

in the third inequality we have used the fact that

$$\int_\Omega |su+\psi_s|^p \leqslant 1,$$

if s is small, while the last inequality follows from the imbedding $L^q(\Omega) \subset L^p(\Omega)$. By recalling (2.10), we get (2.12) by choosing a sufficiently small δ''.

At last, for the proof of (vi) we adapt an argument which is used, in the superlinear case, in [22, page 24]. For any $u \in E_{k-1}^\perp$,

$$u = \sum_{i>k} \alpha_i \varphi_i,$$

with

$$\|u\|^2 = \sum_{i>k} \alpha_i^2 \lambda_i < \infty,$$

let $u_0, v_0 \in E$ be given by

$$u_0 := \sum_{i>k} \frac{2\alpha_i}{1+\lambda_i^{\mu-1}}\,\varphi_i, \qquad v_0 := \sum_{i>k} \frac{2\alpha_i\lambda_i^{\mu-1}}{1+\lambda_i^{\mu-1}}\,\varphi_i.$$

We observe that $u_0 + v_0 = 2u$ and that indeed $||u_0||^2 \leqslant 4||u||^2 < \infty$. Let us denote

$$|||u_0|||_\mu^2 := \sum_{i>k} \frac{4\alpha_i^2}{(1+\lambda_i^{\mu-1})^2}\,\lambda_i^\mu.$$

By comparing coefficients, it is trivial to check that

$$\int_\Omega \langle \nabla u_0, \nabla v_0\rangle = |||u_0|||_\mu^2, \quad ||u_0||_2^2 \leqslant \frac{1}{\lambda_{k+1}^\mu}|||u_0|||_\mu^2, \quad ||v_0||_2^2 \leqslant \frac{1}{\lambda_{k+1}^{2-\mu}}|||u_0|||_\mu^2.$$

Now, since $u_0 + v_0 = 2u$, we can write $u_0 = u + \psi$, $v_0 = u - \psi$ for some $\psi \in E$ and so, by definition of J_0, $J_0(u) \geqslant I_0(u_0, v_0)$. By recalling (2.5), the imbeddings $L^2(\Omega) \subset L^p(\Omega)$, $L^2(\Omega) \subset L^q(\Omega)$, and the well-known fact that $\lambda_{k+1} \geqslant ck^{2/N}$ for some $c > 0$, the above estimates imply that

$$J_0(u) \geqslant t_0^2 - \frac{c_1}{k^{\mu p/N}}t_0^p - \frac{c_2}{k^{(2-\mu)q/N}}t_0^q := \theta_0(t_0),$$

for some $c_1, c_2 > 0$ and $t_0 := |||u_0|||_\mu$. An elementary computation shows that

$$\inf_{t>0} \theta_0(t) \geqslant \frac{-c}{k^\beta},$$

with β given by (2.8), and this proves (vi). □

Now, we can complete the proof of Theorem 1.1. Following a procedure similar to the one in [7, 8, 16], let

$$I(t,u,v) := \int_\Omega \Big(\langle \nabla u, \nabla v\rangle - \frac{1}{p}|u|^p - \frac{1}{q}|v|^q - tF(x,u) - tG(x,v)\Big),$$

$0 \leqslant t \leqslant 1, u, v \in E$ and, as before,

$$J(t,u) := \max_{\psi\in E} I(t, u+\psi, u-\psi) = I(t, u+\psi_{u,t}, u-\psi_{u,t}).$$

In spite of the non-differentiabiliy of $\psi_{u,t}$ with respect to either u or t, similarly to Proposition 2.4 (i) we have that J is C^1 in $[0,1]\times E$ and, for any t, u, φ,

$$J_u'(t,u)\varphi = I_u'(t, u+\psi_{u,t}, u-\psi_{u,t})(\varphi,\varphi)$$

and

$$J_t'(t,u) = -\int_\Omega (F(x, u+\psi_{u,t}) + G(x, u-\psi_{u,t})).$$

We apply Theorem 2.3 in [16] for the functional J. Its basic assumptions (A1)–(A6) are satisfied; this is essentially a consequence of properties (ii)–(v) in Proposition 2.4. On the other hand, for $k \geqslant 1$ let

$$S^k := \{u \in E_k : ||u|| = 1\}, \quad \Gamma_k := \{\gamma \in C(S^k; E) : \gamma \text{ is odd }\},$$

and

$$c_k := \inf_{\gamma\in\Gamma_k} \max_{u\in S^k} J(0,\gamma(u)) < 0.$$

Using Proposition 2.4 (vi) we see that

$$|c_k| \leqslant \frac{c}{k^\beta},$$

where $c > 0$ and β is given in (2.8), for an arbitrary $0 < \mu < 2$, yet to be chosen. As a consequence (cf. [16, Lemma 3.10]), the conclusion of Theorem 1.1 will follow once we show that it is possible to find some $\mu \in]0,2[$ in such a way that $\beta > 1/\nu$, where ν was introduced in Proposition 2.4 (iv).

Using the expressions in (2.7)–(2.8), it remains thus to show that

$$A := \frac{\min\{rq, sp\} - pq}{(p+q)N - pq(N-2)}\ \frac{4}{N} > \max\{\frac{2-p}{\mu p}, \frac{2-q}{(2-\mu)q}\},$$

or, in an equivalent way, that

$$\exists 0 < \mu < 2: \qquad \frac{2-p}{Ap} < \mu < \frac{2Aq-2+q}{Aq}.$$

This, in turn, amounts to asking that

$$\frac{2-p}{Ap} < \frac{2Aq-2+q}{Aq} \qquad \text{and} \qquad \frac{2-p}{Ap} < 2. \tag{2.13}$$

Now, we can write the first inequality in (2.13) as

$$A > \frac{1}{p} + \frac{1}{q} - 1, \tag{2.14}$$

which is precisely our assumption (1.6); while the second inequality in (2.13) can be written as

$$A > \frac{1}{p} - \frac{1}{2},$$

and this follows from (2.14). As we mentioned before, Theorem 1.1 is now a straightforward consequence of [16, Theorem 2.3].

REFERENCES

[1] H. Amann, *Saddle points and multiple solutions of differential equations,* Math. Z., 169 (1979), 127–166.

[2] A. Ambrosetti and M. Badiale, *The dual variational principle and elliptic problems with discontinuous nonlinearities,* J. Math. Anal. Appl., 140 (1989), 363–373.

[3] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications,* J. Funct. Anal., 14 (1973), 349–381.

[4] S. Angenent and R. van der Vorst, *A superquadratic indefinite elliptic system and its Morse-Conley-Floer homology,* Math. Z., 231 (1999), 203–248.

[5] A. Bahri and H. Berestycki, *A perturbation method in critical point theory and applications,* Trans. Amer. Math. Soc., 267 (1981), 1–32.
[6] A. Bahri and P.L. Lions, *Morse-index of some min-max critical points. I. Application to multiplicity results,* Comm. Pure Appl. Math., 41 (1988), 1027–1037.
[7] Ph. Bolle, *On the Bolza problem,* J. Differential Equations, 152 (1999), 274–288.
[8] Ph. Bolle, N. Ghoussoub, and H. Tehrani, *The multiplicity of solutions to non-homogeneous boundary value problems,* Manuscripta Math., 101 (2000), 325–350.
[9] D. Bonheure and M. Ramos, *Multiple critical points of perturbed symmetric strongly indefinite functionals,* Ann. I. H. Poincaré AN, 26 (2009), 675–688. See also *Erratum* Ann. I. H. Poincaré AN, 26 (2009), 1049–1054.
[10] A.M. Candela and A. Salvatore, *Some applications of a perturbative method to elliptic equations with non-homogeneous boundary conditions,* Nonlinear Anal. TMA, 53 (2003), 299–317.
[11] A. Castro and M. Clapp, *Upper estimates for the energy of solutions of nonhomogeneous boundary value problems,* Proc. Amer. Math. Soc., 134 (2006), 167–175.
[12] C. Chambers and N. Ghoussoub, *Deformation from symmetry and multiplicity of solutions in non-homogeneous problems,* Discrete Contin. Dyn. Syst., 8 (2002), 267–281.
[13] M. Clapp, Y. Ding, and S. Hernández-Linares, *Strongly indefinite functionals with perturbed symmetries and multiple solutions of nonsymmetric elliptic systems,* Electronic J. Differential Equations, 2004 (2004), 1–18.
[14] D.G. de Figueiredo, *Nonlinear elliptic systems,* An. Acad. Brasil. Ciênc., 72 (2000), 453–469.
[15] D.G. de Figueiredo and Y.H. Ding, *Strongly indefinite functionals and multiple solutions of elliptic systems,* Trans. Amer. Math. Soc., 355 (2003), 2973–2989.
[16] R. Kajikiya, *Multiple solutions of sublinear Lane-Emden elliptic equations,* Calc. Var. Partial Differential Equations, 26 (2006), 29–48.
[17] P. Rabinowitz, *Multiple critical points of perturbed symmetric functionals,* Trans. Amer. Math. Soc., 272 (1982), 753–770.
[18] M. Ramos, H. Tehrani, *Perturbation from symmetry for indefinite semilinear elliptic equations,* Manuscripta Math., 128 (2009), 297–314.
[19] M. Ramos, H. Tavares, and W. Zou, *A Bahri-Lions Theorem Revisited*, Adv. Math., to appear.
[20] M. Struwe, *Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems,* Manuscripta Math., 32 (1980), 335–364.
[21] K. Tanaka, *Morse indices at critical points related to the symmetric mountain pass theorem and applications,* Comm. Partial Differential Equations, 14 (1989), 99–128.
[22] C. Tarsi, *Perturbation from symmetry and multiplicity of solutions for strongly indefinite elliptic systems,* Adv. Nonlinear Stud., 7 (2007), 1–30.
[23] H.T. Tehrani, *Infinitely many solutions for an indefinite semilinear elliptic problem in* $\mathbb{R}^N$*,* Advances in Differential Equations, 5 (2000), 1571–1596.
[24] Z.Q. Wang, *Nonlinear boundary value problems with concave nonlinearities near the origin,* NoDEA Nonlinear Differential Equations Appl., 8 (2001), 15–33.
[25] M. Willem, "Minimax Theorems," Birkhäuser, Boston, 1996.