Differential and Integral Equations Volume 22, Numbers 9-10 (2009), 853–880

EXISTENCE AND MULTIPLICITY RESULTS FOR THE PRESCRIBED MEAN CURVATURE EQUATION VIA LOWER AND UPPER SOLUTIONS

Franco Obersnel and Pierpaolo Omari
Dipartimento di Matematica e Informatica, Università degli Studi di Trieste
Via A. Valerio 12/1, I-34127 Trieste, Italy

Dedicated to Patrick Habets and Jean Mawhin

Abstract. We discuss existence and multiplicity of bounded variation solutions of the mixed problem for the prescribed mean curvature equation

$$-\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = f(x,u) \ \text{ in } \Omega, \quad u=0 \ \text{ on } \Gamma_D, \quad \partial u/\partial\nu = 0 \ \text{ on } \Gamma_N,$$

where Γ_D is an open subset of $\partial\Omega$ and $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Our approach is based on variational techniques and a lower and upper solutions method specially developed for this problem.

1. Introduction

We are interested in the existence of solutions of the prescribed mean curvature problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = f(x,u) \ \text{ in } \Omega, \\ u=0 \ \text{ on } \Gamma_D, \quad \partial u/\partial\nu = 0 \ \text{ on } \Gamma_N. \end{cases} \tag{1.1}$$

The following hypotheses are assumed:

(h_0) Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) having a $C^{0,1}$ boundary $\partial\Omega$, with $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and Γ_D open in $\partial\Omega$,

and

(h_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions; i.e., for every $s \in \mathbb{R}$, $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable and, for almost every $x \in \Omega$, $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous.

Research supported by M.I.U.R, in the frame of the P.R.I.N. "Equazioni differenziali ordinarie e applicazioni".

AMS Subject Classifications: 35J25, 35J20, 35J60, 53A10, 49Q20, 58E15.

Due to geometric and physical motivations, such as the Plateau and the capillarity problems, the prescribed mean curvature equation with Dirichlet and Neumann boundary conditions, which respectively correspond to the choices $\Gamma_D = \partial\Omega$ and $\Gamma_N = \partial\Omega$ in (1.1), have been largely discussed in the literature in the frame of bounded variation solutions, starting with the classical works of De Giorgi, Miranda, Giusti, Finn, Emmer et alii. On the contrary, few works seem to have been devoted to the study of the mixed problem (see [15, 13]). In [15] solutions of (1.1) have been obtained as global minimizers in the space $BV(\Omega)$ of the functional

$$\mathcal{I}(v) = \int_\Omega \sqrt{1+|Dv|^2} + \int_{\Gamma_D} |v_{|\partial\Omega}| \, d\mathcal{H}_{N-1} - \int_\Omega F(x,v)\,dx,$$

where

$$\int_\Omega \sqrt{1+|Dv|^2} = \sup\Big\{ \int_\Omega \Big(v \sum_{i=1}^N \frac{\partial w_i}{\partial x_i} + w_{N+1}\Big)\,dx : w_i \in C^1_0(\Omega) \text{ for } i = 1,2,\dots,N+1 \text{ and } \Big\|\sum_{i=1}^{N+1} w_i^2\Big\|_{L^\infty(\Omega)} \le 1\Big\},$$

$v_{|\partial\Omega} \in L^1(\partial\Omega, \mathcal{H}_{N-1})$ is the trace of $v \in BV(\Omega)$ on $\partial\Omega$, $\mathcal{H}_{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and

$$F(x,s) = \int_0^s f(x,t)dt.$$

Recall that $BV(\Omega)$ is a Banach space with respect to the norm

$$\|v\|_{BV(\Omega)} = \int_\Omega |Dv| + \|v\|_{L^q(\Omega)},$$

where

$$\int_\Omega |Dv| = \sup\Big\{ \int_\Omega \Big(v \sum_{i=1}^N \frac{\partial w_i}{\partial x_i}\Big)\,dx : w_i \in C^1_0(\Omega) \text{ for } i = 1,2,\dots,N \text{ and } \Big\|\sum_{i=1}^{N} w_i^2\Big\|_{L^\infty(\Omega)} \le 1\Big\},$$

and $q \in [1, 1^*)$ is fixed, with $1^* = \frac{N}{N-1}$ if $N \geq 2$ and $1^* = \infty$ if $N = 1$. Set for convenience

$$\mathcal{J}(v) = \int_\Omega \sqrt{1 + |Dv|^2} + \int_{\Gamma_D} |v_{|\partial\Omega}| \, d\mathcal{H}_{N-1}.$$

The functional $\mathcal{J}$ is convex and, also by the continuity of the trace map [16, Theorem 2.11], it is (Lipschitz) continuous in $BV(\Omega)$. If we also assume for a while that there are constants $r \in (1, 1^*)$ and $a \geq 0$ and a function $b \in L^{\frac{r}{r-1}}(\Omega)$ such that

$$|f(x,s)| \leq a|s|^{r-1} + b(x) \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R},$$

then the functional $v \mapsto \int_\Omega F(x,v)\,dx$ is C^1 in $BV(\Omega)$ (cf. e.g. [7, Theorem 2.8]). Hence, any minimizer u of $\mathcal{I}$ in $BV(\Omega)$ satisfies

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \int_\Omega f(x,u)(v-u)\,dx \qquad \text{for every } v \in BV(\Omega); \tag{1.2}$$

i.e., u is a minimizer in $BV(\Omega)$ of the functional $v \mapsto \mathcal{J}(v) - \int_\Omega f(x,u)v\,dx$.

In this paper we look for solutions of (1.1), intended as functions $u \in BV(\Omega)$, with $f(\cdot, u) \in L^p(\Omega)$ for some $p > N$, satisfying the variational inequality (1.2), which are not necessarily global minimizers in $BV(\Omega)$ of $\mathcal{I}$. Indeed, our aim is to prove that the existence of a solution $u \in BV(\Omega)$ of (1.1) is implied by the existence of a lower solution α and an upper solution β of (1.1). These lower and upper solutions belong to $BV(\Omega)$, are defined in an appropriate BV-sense, inspired from [16, Section 12], and satisfy

$$\alpha(x) \leq \beta(x) \quad \text{a.e. in } \Omega.$$

The solution u is such that

$$\alpha(x) \leq u(x) \leq \beta(x) \quad \text{a.e. in } \Omega, \tag{1.3}$$

and

$$\mathcal{I}(u) = \min_{\substack{v \in BV(\Omega) \\ \alpha \leq v \leq \beta}} \mathcal{I}(v); \tag{1.4}$$

i.e., u minimizes $\mathcal{I}$ on the set $\{v \in BV(\Omega) : \alpha(x) \leq v(x) \leq \beta(x) \text{ a.e. in } \Omega\}$. This existence and localization result is the content of Theorem 2.4, which also provides further information on the structure of the solution set, specifically compactness and existence of extremal solutions. One question that is just partially faced in Theorem 2.4 concerns the regularity of the obtained solutions. Namely, we can prove the internal C^2-regularity of any solution u of (1.1) which satisfies (1.3), assuming that the functions $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$,

$\alpha, \beta : \bar{\Omega} \to \mathbb{R}$ are locally Lipschitz continuous and, for each $x \in \Omega$, $f(x, \cdot) : [\alpha(x), \beta(x)] \to \mathbb{R}$ is non-increasing.

We mention that the existence of classical solutions, in the presence of classical lower and upper solutions, for some special cases of problem (1.1) has been previously discussed in [26, 20, 18, 2].

We will show that Theorem 2.4 allows us in particular to prove the existence of at least one solution in $BV(\Omega) \cap L^\infty(\Omega)$ of the Dirichlet problem, assuming that the potential $F(x, s)$ of $f(x, s)$ has a desultorily sublinear behaviour at infinity with respect to s. This precisely means that there exist a constant $r_0 > 0$ and a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x, s) \operatorname{sgn}(s) \leq g(s) \operatorname{sgn}(s) \quad \text{for a.e. } x \in \Omega \text{ and every } |s| \geq r_0, \tag{1.5}$$

and

$$\liminf_{s \to \pm\infty} \frac{G(s)}{|s|} < \frac{2}{\operatorname{diam}(\Omega)}, \tag{1.6}$$

where $G(s) = \int_0^s g(t)dt$. It is worth mentioning that the existence of a solution is actually guaranteed by the following weaker local condition: there are constants $R > \frac{2}{\operatorname{diam}(\Omega)}$ and $S < -\frac{2}{\operatorname{diam}(\Omega)}$ such that

$$\frac{G(R) - G(s)}{R - s} \leq \frac{2}{\operatorname{diam}(\Omega)} \qquad \text{for every } s \in [R - \tfrac{2}{\operatorname{diam}(\Omega)}, R), \tag{1.7}$$

and

$$\frac{G(S) - G(s)}{S - s} \geq -\frac{2}{\operatorname{diam}(\Omega)} \qquad \text{for every } s \in (S, S + \tfrac{2}{\operatorname{diam}(\Omega)}]. \tag{1.8}$$

Indeed, this assumption yields the existence of a lower solution α and an upper solution β, with $S \leq \alpha(x) \leq S + \frac{2}{\operatorname{diam}(\Omega)}$ and $R - \frac{2}{\operatorname{diam}(\Omega)} \leq \beta(x) \leq R$ in Ω, which are constructed as solutions of suitable autonomous ordinary differential equations associated with (1.1). These solutions are generated by a shooting method and controlled by time-mapping and energy estimates. We point out that the conditions above are optimal for the existence of a bounded variation solution in the one-dimensional case: elementary examples in fact show that if $\frac{2}{\operatorname{diam}(\Omega)}$ is replaced by any larger constant, then the solvability is no longer ensured. Note also that $\frac{2}{\operatorname{diam}(\Omega)} \leq \frac{\operatorname{Per}(\Omega)}{\operatorname{meas}(\Omega)}$, where $\operatorname{Per}(\Omega)$ is the perimeter of Ω (cf. [16, page 5]), and in some cases $\frac{\operatorname{Per}(\Omega)}{\operatorname{meas}(\Omega)}$ is precisely the principal eigenvalue λ_1 of the minus 1-Laplacian with Dirichlet boundary conditions (cf. [6]).

An alternative condition which yields the existence of (arbitrarily small positive) lower solutions is the following: there is a constant $R > 0$ such that

$$f(x, s) \geq \mu_1 s \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [0, R], \tag{1.9}$$

where μ_1 is the principal eigenvalue of the minus Laplacian with Dirichlet boundary conditions. Similarly, the existence of (arbitrarily small negative) upper solutions follows from the condition: there is a constant $S < 0$ such that

$$f(x, s) \leq \mu_1 s \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [S, 0]. \tag{1.10}$$

The occurrence in these conditions of the principal eigenvalues of the minus 1-Laplacian and of the minus Laplacian can be loosely motivated observing that the curvature operator, near infinity and near zero, respectively behaves like the 1-Laplacian and the Laplacian.

The existence of infinitely many solutions can be also established by supposing, in addition to (1.5) and (1.6), that

$$\limsup_{s \to +\infty} \frac{F(x, s)}{|s|} > \frac{\operatorname{Per}(\Omega)}{\operatorname{meas}(\Omega)} \quad \text{uniformly a.e. in } \Omega. \tag{1.11}$$

We recall that, if $N = 1$, then $\frac{\operatorname{Per}(\Omega)}{\operatorname{meas}(\Omega)} = \frac{2}{\operatorname{diam}(\Omega)}$. This multiplicity result is achieved by constructing sequences $(\alpha_n)_n$ and $(\beta_n)_n$ of lower and upper solutions. For each n there is a solution u_n satisfying $\alpha_n(x) \leq u_n(x) \leq \beta_n(x)$ almost everywhere in Ω and

$$\mathcal{I}(u_n) = \min_{\substack{v \in BV(\Omega) \\ \alpha_n \leq v \leq \beta_n}} \mathcal{I}(v).$$

Condition (1.11) is then used to prove that

$$\lim_{n \to +\infty} \mathcal{I}(u_n) = -\infty.$$

These ideas have already been exploited for proving the existence of multiple solutions of the Dirichlet problem for elliptic p-Laplace type equations, with $p > 1$, and of the Dirichlet-periodic problem for parabolic equations (see e.g. [29, 10, 28, 25, 17, 27]). Conditions analogous to (1.6) and (1.11) seem to have been first introduced in [9] for the study of one-dimensional semilinear problems. On the other hand, this approach appears completely new in the context of the prescribed mean curvature equation and permits us to get variations and improvements of the classical results obtained, for example, in [23, 15], where the function $f(x, \cdot)$ was supposed to be nonincreasing in $\mathbb{R}$ for almost every fixed $x \in \Omega$. This last assumption, which implies the convexity of the functional $\mathcal{I}$, obviously rules out the possibility of dealing with functions f satisfying (1.5), (1.6) and (1.11), that likely give rise to infinitely many local minimum points of $\mathcal{I}$.

Another, much simpler, application of Theorem 2.4 can be given for the mixed problem (1.1), and in particular for the Neumann problem, just by using constant lower and upper solutions. Indeed, we can easily see that the existence of at least one solution follows from the condition

$$\liminf_{s\to\pm\infty} f(x,s)\,\mathrm{sgn}(s) < 0 \quad \text{uniformly a.e. in } \Omega, \tag{1.12}$$

while the existence of infinitely many solutions is achieved assuming condition (1.11) as well. For the Neumann problem this latter assumption can be replaced by

$$\limsup_{s\to+\infty} f(x,s) > 0 \quad \text{uniformly a.e. in } \Omega.$$

We just point out that even these very simple statements cannot be deduced from the results in [8, 11, 15, 13], where the function $f(x,\cdot)$ was supposed to be non-increasing, and sometimes even strictly decreasing, in $\mathbb{R}$ for almost every fixed $x \in \Omega$.

We conclude this introduction pointing out that while preparing this paper we learned of a very recent work by V.K. Le [21], where a lower and upper solution method in $BV(\Omega)$ has been developed for the prescribed mean curvature equation with Dirichlet boundary conditions. Even though, as far as the Dirichlet problem is concerned, our Theorem 2.4 is substantially equivalent to the existence result proved in [21, Theorems 3.2, 3.4, 3.5], nevertheless there is at least one significant difference: unlike [21], where pseudo-monotone operator theory is used in the proof, our approach is genuinely variational and yields the variational characterization of the solution given by (1.4). As we already noticed, this information is crucial in order to get the above cited multiplicity results. Furthermore, in this paper we exhibit non-trivial examples of construction of lower and upper solutions, as for instance those based on conditions (1.8), (1.7), (1.9), (1.10), and we provide some information about the interior regularity of the obtained solutions. Finally, unlike [21], we discuss here a quite general homogeneous mixed problem, including in particular the Neumann problem, which was not considered in [21]. Dealing with the Neumann problem presents in the proof a small additional difficulty of technical character, due to the lack of coerciveness of the functional $\mathcal{J}$ in $BV(\Omega)$ when $\Gamma_D = \emptyset$.

We finally list some notation that is used throughout this paper. For functions $u, v : E(\subseteq \mathbb{R}^N) \to \mathbb{R}$, where E has positive N-dimensional Lebesgue measure, we write $u \leq v$ if $u(x) \leq v(x)$ almost everywhere in E. We also define the functions $u \vee v$ and $u \wedge v$ by setting $(u \vee v)(x) = \max\{u(x), v(x)\}$ and $(u \wedge v)(x) = \min\{u(x), v(x)\}$ almost everywhere in E. As usual we write $u^+ = u \vee 0$ and $u^- = -(u \wedge 0)$. For every Caccioppoli set $B(\subseteq \Omega)$ (cf. [16, page 5]) we write

$$\mathrm{Per}(B; \Omega \cup \Gamma_D) = \int_\Omega |D\chi_B| + \int_{\Gamma_D} \chi_{B|\partial\Omega} d\mathcal{H}_{N-1},$$

where χ_B is the characteristic function of B. Of course, if $\Gamma_D = \partial\Omega$, then $\mathrm{Per}(B; \bar{\Omega})$ is the perimeter of B in $\mathbb{R}^N$ and will be denoted by $\mathrm{Per}(B)$.

2. A lower and upper solutions method

The following notion of lower and upper solutions is adopted. We say that a function $\alpha \in BV(\Omega)$ is a lower solution of (1.1) if there exist $\alpha_1, \ldots, \alpha_m \in BV(\Omega)$ such that

- $\alpha = \alpha_1 \vee \cdots \vee \alpha_m$;
- there is $p > N$ such that $f(\cdot, \alpha_i) \in L^p(\Omega)$ for every $i = 1, \ldots, m$;
- for each $i = 1, \ldots, m$,

$$\mathcal{J}(\alpha_i + z) - \mathcal{J}(\alpha_i) \geq \int_\Omega f(x, \alpha_i) z \, dx,$$

for every $z \in BV(\Omega)$ with $z \leq 0$.

Similarly, we say that a function $\beta \in BV(\Omega)$ is an upper solution of (1.1) if there exist $\beta_1, \ldots, \beta_n \in BV(\Omega)$ such that

- $\beta = \beta_1 \wedge \cdots \wedge \beta_n$;
- there is $p > N$ such that $f(\cdot, \beta_j) \in L^p(\Omega)$ for every $j = 1, \ldots, n$;
- for each $j = 1, \ldots, n$,

$$\mathcal{J}(\beta_j + z) - \mathcal{J}(\beta_j) \geq \int_\Omega f(x, \beta_j) z \, dx, \tag{2.1}$$

for every $z \in BV(\Omega)$ with $z \geq 0$.

We recall that by a solution of (1.1) we mean a function $u \in BV(\Omega)$, with $f(\cdot, u) \in L^p(\Omega)$ for some $p > N$, satisfying the variational inequality (1.2)

Remark 2.1. If we assume $m = 1$, then $\alpha \in BV(\Omega)$, with $f(\cdot, \alpha) \in L^p(\Omega)$, for some $p > N$, is a lower solution of (1.1) if and only if α minimizes the functional $v \mapsto \mathcal{J}(v) - \int_\Omega f(x, \alpha) v \, dx$ on the cone $\{v \in BV(\Omega) : v \leq \alpha\}$. Similarly, if we assume $n = 1$, then $\beta \in BV(\Omega)$, with $f(\cdot, \beta) \in L^p(\Omega)$, for some $p > N$, is an upper solution of (1.1) if and only if β minimizes the functional $v \mapsto \mathcal{J}(v) - \int_\Omega f(x, \beta) v \, dx$ on the cone $\{v \in BV(\Omega) : v \geq \beta\}$. This notion of lower and upper solutions has already been used in [16, Section 12] for dealing with classical solutions of the minimal surface equation.

The next inequality will be systematically used in the sequel (special cases can be found in [22, 8]).

Proposition 2.2. *For every $u, v \in BV(\Omega)$,*

$$\mathcal{J}(u \vee v) + \mathcal{J}(u \wedge v) \leq \mathcal{J}(u) + \mathcal{J}(v).$$

Proof. We first recall that $BV(\Omega)$ is a lattice (cf. [3]). Then, also using [19, Chapter II], we see that for every $u, v \in W^{1,1}(\Omega)$

$$\begin{aligned} &\int_\Omega \sqrt{1 + |\nabla(u \vee v)|^2} dx + \int_\Omega \sqrt{1 + |\nabla(u \wedge v)|^2} dx \\ &= \int_\Omega \sqrt{1 + |\nabla u|^2} dx + \int_\Omega \sqrt{1 + |\nabla v|^2} dx. \end{aligned}$$

Finally, the density and semicontinuity results in [1, page 491] and [16, Theorem 14.2] easily yield

$$\begin{aligned} &\int_\Omega \sqrt{1 + |D(u \vee v)|^2} + \int_{\Gamma_D} |(u \vee v)_{|\partial\Omega}| \, d\mathcal{H}_{N-1} \\ &\quad + \int_\Omega \sqrt{1 + |D(u \wedge v)|^2} + \int_{\Gamma_D} |(u \wedge v)_{|\partial\Omega}| \, d\mathcal{H}_{N-1} \\ &\leq \int_\Omega \sqrt{1 + |Du|^2} + \int_{\Gamma_D} |u_{|\partial\Omega}| \, d\mathcal{H}_{N-1} + \int_\Omega \sqrt{1 + Dv|^2} + \int_{\Gamma_D} |v_{|\partial\Omega}| \, d\mathcal{H}_{N-1}, \end{aligned}$$

for every $u, v \in BV(\Omega)$. □

Remark 2.3. A function $u \in BV(\Omega)$ is a solution of (1.1) if and only if it is simultaneously a lower solution of (1.1) with $m = 1$ and an upper solution of (1.1) with $n = 1$. Indeed, if u is a lower and an upper solution, then for every $z \in BV(\Omega)$ we have

$$\mathcal{J}(u + z^+) - \mathcal{J}(u) \geq \int_\Omega f(x, u) z^+ \, dx,$$

and

$$\mathcal{J}(u - z^-) - \mathcal{J}(u) \geq \int_\Omega f(x, u)(-z^-) \, dx.$$

Hence, recalling that $u \vee (u + z) = u + z^+$ and $u \wedge (u + z) = u - z^-$ and using Proposition 2.2, we get

$$\int_\Omega f(x, u) z \, dx \quad \leq \quad \mathcal{J}(u + z^+) + \mathcal{J}(u - z^-) - 2\mathcal{J}(u)$$

$$= \mathcal{J}(u \vee (u+z)) + \mathcal{J}(u \wedge (u+z)) - 2\mathcal{J}(u)$$

$$\leq \mathcal{J}(u+z) + \mathcal{J}(u) - 2\mathcal{J}(u) = \mathcal{J}(u+z) - \mathcal{J}(u).$$

Theorem 2.4. *Let $p > N$ be fixed. Assume (h_0), (h_1),*

- (h_2) *there exist a lower solution α and an upper solution β of* (1.1) *such that $\alpha \leq \beta$,*

and

- (h_3) *there is $\gamma \in L^p(\Omega)$ such that, for almost every $x \in \Omega$ and every $s \in \mathbb{R}$, with $\min_{i=1,\dots,m} \alpha_i(x) \leq s \leq \max_{j=1,\dots,n} \beta_j(x)$, $|f(x,s)| \leq \gamma(x)$.*

Then problem (1.1) *has at least one solution $u \in BV(\Omega)$ such that*

$$\alpha \leq u \leq \beta \qquad \text{and} \qquad \mathcal{I}(u) = \min_{\substack{v \in BV(\Omega) \\ \alpha \leq v \leq \beta}} \mathcal{I}(v),$$

where

$$\mathcal{I}(v) = \mathcal{J}(v) - \int_\Omega (F(x,v) - F(x,\alpha))\,dx.$$

Moreover, there exist solutions v, w of (1.1)*, with $\alpha \leq v \leq w \leq \beta$, such that every solution u of* (1.1)*, with $\alpha \leq u \leq \beta$, satisfies $v \leq u \leq w$.*
If we further suppose that

- (h_4) *for each $i = 1,\dots,m$ and $j = 1,\dots,n$, $\alpha_i, \beta_j \in C^{0,1}(\bar{\Omega})$,*

and

- (h_5) *$f \in C^{0,1}(E)$, where $E = \{(x,s) \in \bar{\Omega} \times \mathbb{R} : \min_{i=1,\dots,m} \alpha_i(x) \leq s \leq \max_{j=1,\dots,n} \beta_j(x)\}$, and, for each $x \in \Omega$, the function $f(x,\cdot)$ is non-increasing in $[\min_{i=1,\dots,m} \alpha_i(x), \max_{j=1,\dots,n} \beta_j(x)]$,*

then any solution $u \in BV(\Omega)$ of (1.1)*, with $\alpha \leq u \leq \beta$, belongs to $C^{2,\sigma}(\Omega)$ for every $\sigma \in (0,1)$.*

Remark 2.5. If $F(\cdot,\alpha) \in L^1(\Omega)$, or $F(\cdot,\beta) \in L^1(\Omega)$, then we define

$$\mathcal{I}(v) = \mathcal{J}(v) - \int_\Omega F(x,v)\,dx.$$

Proof. Our argument follows some lines that are standard in the context of semilinear problems (see e.g. [5]). Define $q = \frac{p}{p-1} \in (1, 1^*)$.
Step 1. A modified problem. Let us define for almost every $x \in \Omega$ and every $s \in \mathbb{R}$

$$Q(s) = \begin{cases} \frac{q}{2} s^2 & \text{if } |s| \leq 1, \\ |s|^q + \frac{q-2}{2} & \text{if } |s| > 1, \end{cases} \tag{2.2}$$

$$g(x,s) = f(x,s) + Q'(s),$$

$$h_i(x,s) = \begin{cases} g(x,\alpha_i(x)) & \text{if } s < \alpha_i(x), \\ g(x,s) & \text{if } s \ge \alpha_i(x), \end{cases}$$

$$k_j(x,s) = \begin{cases} g(x,\beta_j(x)) & \text{if } s > \beta_j(x), \\ g(x,s) & \text{if } s \le \beta_j(x), \end{cases}$$

for $i = 1,\dots,m$, $j = 1,\dots,n$, and

$$\ell(x,s) = \begin{cases} \max\limits_{i=1,\dots,m} h_i(x,s) + \arctan(\alpha(x) - s) & \text{if } s < \alpha(x), \\ g(x,s) & \text{if } \alpha(x) \le s \le \beta(x), \\ \min\limits_{j=1,\dots,n} k_j(x,s) - \arctan(s - \beta(x)) & \text{if } s > \beta(x). \end{cases} \tag{2.3}$$

Clearly, Q is of class C^1 and ℓ satisfies the Carathéodory conditions. Notice that, for every $s \in \mathbb{R}$,

$$Q(s) \ge |s|^q - 1. \tag{2.4}$$

Moreover, there exists a function $\lambda \in L^p(\Omega)$ such that, for almost every $x \in \Omega$ and every $s \in \mathbb{R}$,

$$|\ell(x,s)| \le \lambda(x), \tag{2.5}$$

and hence, setting $L(x,s) = \int_0^s \ell(x,t)dt$,

$$|L(x,s)| \le \lambda(x)|s|. \tag{2.6}$$

Let us consider the modified problem

$$\begin{cases} -\operatorname{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) + Q'(u) = \ell(x,u) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \Gamma_D, \quad \partial u/\partial \nu = 0 \ \text{ on } \Gamma_N. \end{cases} \tag{2.7}$$

A solution of (2.7) is a function $u \in BV(\Omega)$ such that

$$\mathcal{J}(v) + \int_\Omega Q(v)\,dx - \mathcal{J}(u) - \int_\Omega Q(u)\,dx \ge \int_\Omega \ell(x,u)(v-u)\,dx \quad \text{for every } v \in BV(\Omega). \tag{2.8}$$

Note that (2.8) is equivalent to

$$\mathcal{J}(v) + \int_\Omega Q'(u)v\,dx - \mathcal{J}(u) - \int_\Omega Q'(u)u\,dx \ge \int_\Omega \ell(x,u)(v-u)\,dx \quad \text{for every } v \in BV(\Omega). \tag{2.9}$$

Indeed, let u satisfy (2.9). As, by (2.2), the functional $v \mapsto \int_\Omega Q(v)\,dx$ is C^1 and convex in $L^q(\Omega)$ (cf. e.g. [7, Theorem 2.8]), we have

$$\int_\Omega Q(v)\,dx - \int_\Omega Q(u)\,dx \geq \int_\Omega Q'(u)(v-u)\,dx \qquad \text{for every } v \in BV(\Omega).$$

Summing up, we get (2.8). Conversely, if u satisfies (2.8), then the functional $v \mapsto \mathcal{J}(v) + \int_\Omega Q(v)\,dx - \int_\Omega \ell(x,u)v\,dx$ has a global minimum in $BV(\Omega)$ at u. As $\mathcal{J}$ is continuous and convex in $BV(\Omega)$ and, by (2.5), the functional $v \mapsto \int_\Omega \ell(x,u)v\,dx$ is C^1 in $L^q(\Omega)$, we conclude that

$$\begin{aligned}\mathcal{J}(v) - \mathcal{J}(u) \geq & - \int_\Omega Q'(u)(v-u)\,dx \\ & + \int_\Omega \ell(x,u)(v-u)\,dx \quad \text{for every } v \in BV(\Omega),\end{aligned}$$

and hence (2.9) holds.

Step 2. Existence of solutions of the modified problem (2.7)**.** Define a functional $\mathcal{K} : BV(\Omega) \to \mathbb{R}$ by setting

$$\mathcal{K}(v) = \mathcal{J}(v) + \int_\Omega Q(v)\,dx - \int_\Omega L(x,v)\,dx.$$

Claim 1: $\inf_{v \in BV(\Omega)} \mathcal{K}(v) > -\infty$ **and** $\lim_{\|v\|_{BV(\Omega)} \to +\infty} \mathcal{K}(v) = +\infty$**.** Using (2.4), (2.6) and standard inequalities, we can find constants $d_1, d_2 > 0$ such that

$$\begin{aligned}\mathcal{K}(v) &\geq \int_\Omega |Dv| + \int_{\Gamma_D} |v_{|\partial\Omega}|\,d\mathcal{H}_{N-1} + \|v\|^q_{L^q(\Omega)} - \text{meas}(\Omega) - \|\lambda\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)} \\ &\geq d_1\|v\|_{BV(\Omega)} - d_2,\end{aligned}$$

for every $v \in BV(\Omega)$. This yields the conclusions of Claim 1.

Claim 2: there exists $\min_{v \in BV(\Omega)} \mathcal{K}(v)$**.** Let $(u_n)_n$ be a minimizing sequence. Claim 1 implies that $(u_n)_n$ is bounded in $BV(\Omega)$. Hence, there is a subsequence of $(u_n)_n$, which we still denote by $(u_n)_n$, and a function $u \in BV(\Omega)$ such that $\lim_{n\to+\infty} u_n = u$ in $L^q(\Omega)$. Moreover, we have

$$\liminf_{n\to+\infty} \mathcal{J}(u_n) \geq \mathcal{J}(u),$$

as $\mathcal{J}$ is lower semicontinuous with respect to the L^1-convergence in $BV(\Omega)$ (cf. e.g. [23, 15]), and

$$\lim_{n\to+\infty} \int_\Omega (Q(u_n) - L(x,u_n))\,dx = \int_\Omega (Q(u) - L(x,u))\,dx,$$

as the functionals $v \mapsto \int_\Omega Q(v)\,dx$ and $v \mapsto \int_\Omega L(x,v)\,dx$ are continuous in $L^q(\Omega)$. Hence, we conclude that

$$\inf_{v\in BV(\Omega)} \mathcal{K}(v) = \lim_{n\to+\infty} \mathcal{K}(u_n) \geq \mathcal{K}(u);$$

that is, $\mathcal{K}(u) = \min_{v\in BV(\Omega)} \mathcal{K}(v)$. This yields the conclusion of Claim 2.

Since any minimizer of $\mathcal{K}$ satisfies (2.8), we derive that (2.7) has at least one solution.

Step 3. Any solution u **of** (2.7) **satisfies** $\alpha \leq u \leq \beta$**.** Let us show that $u \leq \beta$; by a similar argument we get $u \geq \alpha$. Fix $j \in \{1,\dots,n\}$ and prove that $u \leq \beta_j$. Take $v = u \wedge \beta_j = u - (u-\beta_j)^+$ as a test function in (2.8). We obtain

$$\begin{aligned}
\mathcal{J}(u\wedge\beta_j) - \mathcal{J}(u) + \int_\Omega Q(u\wedge\beta_j)\,dx - \int_\Omega Q(u)\,dx \\
&\geq -\int_\Omega \ell(x,u)(u-\beta_j)^+dx \\
&\geq -\int_\Omega [k_j(x,u) - \arctan(u-\beta)](u-\beta_j)^+dx \\
&= -\int_\Omega g(x,\beta_j)(u-\beta_j)^+dx + \int_\Omega \arctan(u-\beta)(u-\beta_j)^+dx \\
&= -\int_\Omega f(x,\beta_j)(u-\beta_j)^+dx - \int_\Omega Q'(\beta_j)(u-\beta_j)^+dx \\
&\quad + \int_\Omega \arctan(u-\beta)(u-\beta_j)^+dx.
\end{aligned} \tag{2.10}$$

Taking $z = (u-\beta_j)^+$ as a test function in (2.1), we have, as $u \vee \beta_j = \beta_j + (u-\beta_j)^+$,

$$\mathcal{J}(u\vee\beta_j) - \mathcal{J}(\beta_j) \geq \int_\Omega f(x,\beta_j)(u-\beta_j)^+dx.$$

By the convexity of the functional $v \mapsto \int_\Omega Q(v)\,dx$, we get

$$\begin{aligned}
\mathcal{J}(u\vee\beta_j) - \mathcal{J}(\beta_j) + \int_\Omega Q(u\vee\beta_j)\,dx - \int_\Omega Q(\beta_j)\,dx \\
\geq \int_\Omega f(x,\beta_j)(u-\beta_j)^+dx + \int_\Omega Q'(\beta_j)(u-\beta_j)^+dx.
\end{aligned} \tag{2.11}$$

Summing up (2.10) and (2.11) and using both Proposition 2.2 and the identity

$$\int_\Omega Q(u \vee \beta_j)\, dx + \int_\Omega Q(u \wedge \beta_j)\, dx = \int_\Omega Q(u)\, dx + \int_\Omega Q(\beta_j)\, dx,$$

we find

$$\begin{aligned} 0 \;\geq\; & \mathcal{J}(u \wedge \beta_j) + \mathcal{J}(u \vee \beta_j) - \mathcal{J}(\beta_j) - \mathcal{J}(u) \\ & + \int_\Omega Q(u \vee \beta_j)\, dx + \int_\Omega Q(u \wedge \beta_j)\, dx - \int_\Omega Q(u)\, dx - \int_\Omega Q(\beta_j)\, dx \\ \geq\; & \int_\Omega \arctan(u - \beta)(u - \beta_j)^+ dx \geq \int_\Omega \arctan(u - \beta_j)(u - \beta_j)^+ dx \geq 0. \end{aligned}$$

This yields $(u - \beta_j)^+(x) = 0$ almost everywhere in Ω and therefore $u \leq \beta_j$.

Step 4. There is a solution u of (1.1) such that $\alpha \leq u \leq \beta$ and $\mathcal{I}(u) = \min\limits_{\substack{v \in BV(\Omega) \\ \alpha \leq v \leq \beta}} \mathcal{I}(v)$. Let u be a solution of (2.7). As u is such that $\alpha \leq u \leq \beta$, we have $\ell(x, u(x)) = f(x, u(x)) + Q'(u(x))$ almost everywhere in Ω and hence u satisfies by (2.9)

$$\begin{aligned} \mathcal{J}(v) \;+\; & \int_\Omega Q'(u) v\, dx - \mathcal{J}(u) - \int_\Omega Q'(u) u\, dx \\ \geq\; & \int_\Omega f(x, u)(v - u)\, dx + \int_\Omega Q'(u)(v - u)\, dx \quad \text{for every } v \in BV(\Omega); \end{aligned}$$

that is, u is a solution of (1.1). Further, as for almost every $x \in \Omega$ and every s, with $\alpha(x) \leq s \leq \beta(x)$, we have $L(x, s) = L(x, \alpha(x)) + F(x, s) - F(x, \alpha(x)) + Q(s)$, we get for every $v \in BV(\Omega)$, with $\alpha \leq v \leq \beta$,

$$\begin{aligned} \mathcal{K}(v) = \mathcal{J}(v) + \int_\Omega Q(v)\, dx - \int_\Omega L(x, \alpha)\, dx - \int_\Omega [F(x, v) - F(x, \alpha)]\, dx \\ - \int_\Omega Q(v)\, dx = \mathcal{I}(v) - \int_\Omega L(x, \alpha)\, dx. \end{aligned}$$

Since u minimizes $\mathcal{K}$, we conclude that u minimizes $\mathcal{I}$ on the set of all $v \in BV(\Omega)$, with $\alpha \leq v \leq \beta$.

Step 5. Existence of extremal solutions. Let us set

$$\mathcal{S} = \{u \in BV(\Omega) : u \text{ is a solution of (1.1) such that } \alpha \leq u \leq \beta\}.$$

Claim 1: $\mathcal{S}$ is a compact subset of $L^q(\Omega)$. Let $(u_n)_n \subseteq \mathcal{S}$. For each n, we have

$$\|u_n\|_{L^q(\Omega)} \leq \||\alpha| \vee |\beta|\|_{L^q(\Omega)},$$

and

$$\mathcal{J}(v) - \int_\Omega f(x, u_n)v\,dx \ge \mathcal{J}(u_n) - \int_\Omega f(x, u_n)u_n\,dx \quad \text{for every } v \in BV(\Omega).$$

Hence, taking $v = 0$, we get by (h_3)

$$\begin{aligned}\mathcal{J}(u_n) &\le \int_\Omega f(x, u_n)u_n\,dx \le \|f(\cdot, u_n)\|_{L^p(\Omega)}\|u_n\|_{L^q(\Omega)}\\ &\le \|\gamma\|_{L^p(\Omega)}\||\alpha| \vee |\beta|\|_{L^q(\Omega)}.\end{aligned}$$

This implies that $(u_n)_n$ is bounded in $BV(\Omega)$. Therefore, there is a subsequence of $(u_n)_n$, which we still denote by $(u_n)_n$, and a function $u \in BV(\Omega)$ such that $\lim_{n\to+\infty} u_n = u$ in $L^q(\Omega)$ and pointwise almost everywhere in Ω. Hence, we have $\alpha \le u \le \beta$ and

$$\mathcal{J}(u) \le \liminf_{n\to+\infty} \mathcal{J}(u_n).$$

Moreover, using $(u_n)_n \subseteq \mathcal{S}$ and (h_3) again, we get by (h_1) and the Lebesgue dominated convergence theorem

$$\lim_{n\to+\infty} \int_\Omega f(x, u_n)u_n\,dx = \int_\Omega f(x, u)u\,dx,$$

and

$$\lim_{n\to+\infty} \int_\Omega f(x, u_n)v\,dx = \int_\Omega f(x, u)v\,dx,$$

for every $v \in BV(\Omega)$. Thus we conclude that

$$\mathcal{J}(v) - \int_\Omega f(x, u)v\,dx \ge \mathcal{J}(u) - \int_\Omega f(x, u)u\,dx \quad \text{for every } v \in BV(\Omega).$$

Therefore, we have $u \in \mathcal{S}$. This proves the compactness of $\mathcal{S}$ in $L^q(\Omega)$.

Claim 2: $\mathcal{S}$ has a minimum element v and a maximum element w. Let us prove that there exists $\min \mathcal{S}$; a similar argument shows the existence of $\max \mathcal{S}$. For each $u \in \mathcal{S}$ define the closed subset of $L^q(\Omega)$

$$\mathcal{C}_u = \{v \in \mathcal{S} : v \le u\}.$$

The family $(\mathcal{C}_u)_{u\in\mathcal{S}}$ has the finite intersection property. Indeed, if $u_1, u_2 \in \mathcal{S}$, then $u_1 \wedge u_2$ is an upper solution of (1.1), with $\alpha \le u_1 \wedge u_2$. Hence, there is a solution u of (1.1), with $\alpha \le u \le u_1 \wedge u_2 \le \beta$; i.e., $u \in \mathcal{C}_{u_1} \cap \mathcal{C}_{u_2}$. The compactness of $\mathcal{S}$ implies that there exists $v \in \mathcal{S}$ such that $v \in \bigcap_{u\in\mathcal{S}} \mathcal{C}_u$; that is $v \le u$ for every $u \in \mathcal{S}$.

Step 6. Regularity of the solutions. Assume (h_4) and (h_5). Without restriction we can suppose in (h_3) that $p > 2$ and, hence, $q = \frac{p}{p-1} < 2$. By

(2.2) and (2.3) the function $Q'-\ell$ belongs to $C^{0,1}(\bar\Omega\times\mathbb{R})$ and is such that, for each $x\in\bar\Omega$, $Q'(\cdot)-\ell(x,\cdot)$ is non-decreasing in $\mathbb{R}$. Let u be a solution of (1.1) such that $\alpha\le u\le\beta$; hence it satisfies (2.9). Note that $u\in BV(\Omega)\cap L^\infty(\Omega)$ and, by the convexity in $\mathbb{R}$ of $Q(\cdot)-L(x,\cdot)$ for each $x\in\bar\Omega$, we have

$$\mathcal{K}(u)=\min_{v\in BV(\Omega)}\mathcal{K}(v),$$

where

$$\mathcal{K}(v)=\mathcal{J}(v)+\int_\Omega (Q(v)-L(x,v))\,dx.$$

Hence, we can apply the results in [15, Section 3.B] (cf. also [12]) to conclude that $u\in C^{2,\sigma}(\Omega)$, for every $\sigma\in(0,1)$. □

3. Existence and multiplicity results

3.1. **The mixed and the Neumann problems.** We start by describing some applications of Theorem 2.4 to the general mixed problem (1.1) that basically use constant lower and upper solutions. Although quite elementary, our statements cannot be deduced from the classical results in [8, 11, 15, 13], where the function $f(x,\cdot)$ was supposed to be non-increasing in $\mathbb{R}$ for almost every fixed $x\in\Omega$.

Proposition 3.1. *Assume* (h_0),

(h_6) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ *satisfies the* L^p*-Carathéodory conditions for some* $p>N$*; i.e.,* f *is a Carathéodory function and, for each* $r>0$*, there exists* $\gamma_r\in L^p(\Omega)$ *such that* $|f(x,s)|\le\gamma_r(x)$ *for almost every* $x\in\Omega$ *and every* $s\in[-r,r]$,

and

(h_7) $\liminf_{s\to\pm\infty} f(x,s)\,\mathrm{sgn}(s)<0$ *uniformly almost everywhere in* Ω.

Then problem (1.1) *has at least one solution* $u\in BV(\Omega)\cap L^\infty(\Omega)$.

Proof. By assumption (h_7) there are constants α,β, with $\alpha\le 0\le\beta$, such that $f(x,\beta)\le 0\le f(x,\alpha)$ almost everywhere in Ω. For each $z\in BV(\Omega)$, with $z\ge 0$, we have, using (h_6) too,

$$\begin{aligned}\mathcal{J}(\beta+z)-\mathcal{J}(\beta)&=\int_\Omega\sqrt{1+|Dz|^2}+\int_{\Gamma_D}|\beta+z_{|\partial\Omega}|d\mathcal{H}_{N-1}-\int_{\Gamma_D}|\beta|d\mathcal{H}_{N-1}\\&\ge 0\ge\int_\Omega f(x,\beta)z\,dx.\end{aligned}$$

Hence, β is an upper solution of (2.7). Similarly we see that α is a lower solution of (1.1). Using (h_6) again, Theorem 2.4 yields the existence of a solution u of (1.1) such that $\alpha \le u \le \beta$. □

Remark 3.2. Assumption (h_7) actually implies the existence of two sequences of constant lower and upper solutions $(\alpha_n)_n$ and $(\beta_n)_n$ such that $\lim_{n\to+\infty} \alpha_n = -\infty$ and $\lim_{n\to+\infty} \beta_n = +\infty$. On the other hand, it is obvious that in Proposition 3.1 it is sufficient to assume, in place of (h_7), the following condition: there exist constants α, β, with $\alpha \le 0 \le \beta$, such that $f(x,\beta) \le 0 \le f(x,\alpha)$ almost everywhere in Ω. Moreover, when considering the Neumann problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = f(x,u) & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

it is not necessary to require $\alpha \le 0$ and $\beta \ge 0$. We also notice that, for the Neumann problem, a necessary condition for the existence of a solution $u \in BV(\Omega)$ is that $\int_\Omega f(x,u)\,dx = 0$, as easily follows from (1.2).

The next result provides the existence of infinitely many solutions when f exhibits an oscillatory behaviour at infinity.

Proposition 3.3. *Assume* (h_0), (h_6), (h_7) *and*

(h_8) *there exist Caccioppoli sets* $B_\pm \subseteq \Omega$ *such that*

$$\limsup_{s\to\pm\infty} \int_{B_\pm} \frac{F(x,s)}{|s|}\,dx > \mathrm{Per}(B_\pm;\Omega\cup\Gamma_D).$$

Then problem (1.1) *has a double sequence* $(u_m)_{m\in\mathbb{Z}}$ *of solutions such that, for each* $m \in \mathbb{Z}$, $u_m \in BV(\Omega)\cap L^\infty(\Omega)$ *and*

$$\lim_{m\to\pm\infty} \mathcal{I}(u_m) = -\infty, \quad \lim_{m\to-\infty} \operatorname*{ess\,inf}_\Omega\ u_m = -\infty, \quad \lim_{m\to+\infty} \operatorname*{ess\,sup}_\Omega\ u_m = +\infty.$$

Proof. From Remark 3.2 we know that condition (h_7) implies the existence of two sequences of constant lower and upper solutions $(\alpha_n)_n$ and $(\beta_n)_n$ such that $\lim_{n\to+\infty} \alpha_n = -\infty$ and $\lim_{n\to+\infty} \beta_n = +\infty$. Moreover, by (h_8), there is a sequence of constants $(c_n)_n$ such that $\lim_{n\to+\infty} c_n = +\infty$ and

$$\lim_{n\to+\infty} \int_{B_+} \frac{F(x,c_n)}{c_n}\,dx > \mathrm{Per}(B_+;\Omega\cup\Gamma_D).$$

Denoting by χ_{B_+} the characteristic function of B_+, we have

$$\begin{aligned}\mathcal{I}(c_n\chi_{B_+}) &= \mathcal{J}(c_n\chi_{B_+}) - \int_{B_+} F(x,c_n)\,dx\\ &\leq c_n\Big(\frac{\mathrm{meas}(\Omega)}{c_n} + \int_{B_+}|D\chi_{B_+}| + \int_{\Gamma_D}\chi_{B_+|\partial\Omega}\,d\mathcal{H}_{N-1} - \int_{B_+}\frac{F(x,c_n)}{c_n}\,dx\Big)\\ &= c_n\Big(\frac{\mathrm{meas}(\Omega)}{c_n} + \mathrm{Per}(B_+;\Omega\cup\Gamma_D) - \int_{B_+}\frac{F(x,c_n)}{c_n}\,dx\Big),\end{aligned}$$

and hence

$$\lim_{n\to+\infty}\mathcal{I}(c_n\chi_{B_+}) = -\infty.$$

Assume $\alpha_1 \leq 0 \leq \beta_1$. Theorem 2.4 yields the existence of a solution $u_1 \in BV(\Omega)\cap L^\infty(\Omega)$ of (1.1) such that $\alpha_1 \leq u_1 \leq \beta_1$ and

$$\mathcal{I}(u_1) = \min_{\substack{v\in BV(\Omega)\\ \alpha_1\leq v\leq\beta_1}}\mathcal{I}(v).$$

We can find a constant, say c_1, such that $c_1\chi_{B_+} \geq \alpha_1$ and $\mathcal{I}(c_1\chi_{B_+}) < \mathcal{I}(u_1)$. Pick an upper solution, say β_2, such that $\beta_2 \geq c_1\chi_{B_+}$. Theorem 2.4 yields the existence of a solution $u_2 \in BV(\Omega)\cap L^\infty(\Omega)$ of (1.1) such that $\alpha_1 \leq u_2 \leq \beta_2$ and

$$\mathcal{I}(u_2) = \min_{\substack{v\in BV(\Omega)\\ \alpha_1\leq v\leq\beta_2}}\mathcal{I}(v) \leq \mathcal{I}(c_1\chi_{B_+}) < \mathcal{I}(u_1).$$

Hence, we have in particular $u_1 \neq u_2$ and

$$\operatorname*{ess\,sup}_\Omega\ u_2 > \operatorname*{ess\,sup}_\Omega\ \beta_1 \geq \operatorname*{ess\,sup}_\Omega\ u_1.$$

Iterating this procedure we can construct a sequence $(u_n)_n$ of solutions of (1.1) such that, for each $n \in \mathbb{Z}^+$, $u_n \in BV(\Omega)\cap L^\infty(\Omega)$, $\mathcal{I}(u_{n+1}) < \mathcal{I}(u_n)$, $\lim_{n\to+\infty}\mathcal{I}(u_n) = -\infty$, and $\lim_{n\to+\infty}\ \operatorname*{ess\,sup}_\Omega\ u_n = +\infty$.

A similar argument yields the existence of a sequence $(u_m)_{m\in\mathbb{Z}^-}$ of solutions of (1.1) such that, for each $m \in \mathbb{Z}^-$, $u_m \in BV(\Omega)\cap L^\infty(\Omega)$, $\mathcal{I}(u_m) < \mathcal{I}(u_{m+1})$, $\lim_{m\to-\infty}\mathcal{I}(u_m) = -\infty$, and $\lim_{m\to-\infty}\ \operatorname*{ess\,inf}_\Omega\ u_m = -\infty$. □

Remark 3.4. It is clear from the proof of Proposition 3.3 that the existence of a sequence $(u_n)_{n\in\mathbb{Z}^+}$ of solutions of (1.1) such that, for each $n \in \mathbb{Z}^+$, $u_n \in BV(\Omega)\cap L^\infty(\Omega)$, $\lim_{n\to+\infty}\mathcal{I}(u_n) = -\infty$, and $\lim_{n\to+\infty}\ \operatorname*{ess\,sup}_\Omega\ u_n = +\infty$, can be obtained if conditions (h_7) and (h_8) in Proposition 3.3 hold only at $+\infty$ and there exists a lower solution $\alpha \in BV(\Omega)\cap L^\infty(\Omega)$ of (1.1) with $\alpha \leq 0$. The assumption $\alpha \leq 0$ can be omitted if $B_+ = \Omega$. Similarly the existence of a sequence $(u_m)_{m\in\mathbb{Z}^-}$ of solutions of (1.1) such that, for each $m \in \mathbb{Z}^-$, $u_m \in BV(\Omega)\cap L^\infty(\Omega)$, $\lim_{m\to-\infty} \mathcal{I}(u_m) = -\infty$, and $\lim_{m\to-\infty} \operatorname{ess\,inf}_\Omega u_m = -\infty$, can be obtained if conditions (h_7) and (h_8) in Proposition 3.3 hold only at $-\infty$ and there exists an upper solution $\beta \in BV(\Omega)\cap L^\infty(\Omega)$ of (1.1) with $\beta \geq 0$. The assumption $\beta \geq 0$ can be omitted if $B_- = \Omega$.

Remark 3.5. It is obvious that we can replace (h_7) in Proposition 3.3 with the following condition: there are two sequences $(\alpha_n)_n$ and $(\beta_n)_n$ of lower and upper solutions of (1.1) such that $\alpha_n, \beta_n \in BV(\Omega)\cap L^\infty(\Omega)$ for each n, $\lim_{n\to+\infty} \operatorname{ess\,sup}_\Omega \alpha_n = -\infty$ and $\lim_{n\to+\infty} \operatorname{ess\,inf}_\Omega \beta_n = +\infty$.

The result of Proposition 3.3 takes a much simpler form for the Neumann problem.

Proposition 3.6. *Assume* (h_0), (h_6) *and*

(h_9) $\liminf_{s\to+\infty} f(x,s) < 0 < \limsup_{s\to+\infty} f(x,s)$ *uniformly a.e. in* Ω.

Then the Neumann problem (3.1) *has a sequence* $(u_n)_n$ *of solutions such that* $u_n \in BV(\Omega)\cap L^\infty(\Omega)$ *for each* n *and* $\lim_{n\to+\infty} \operatorname{ess\,inf}_\Omega u_n = +\infty$.

Proof. Assumption (h_9) implies, as in Remark 3.2, the existence of sequences $(\alpha_n)_n$ and $(\beta_n)_n$ of constant lower and upper solutions of (3.1), with $\alpha_n \leq \beta_n$ for every n and $\lim_{n\to+\infty} \alpha_n = +\infty$. By Theorem 2.4, for each n there exists a solution u_n of (2.7) such that $\alpha_n \leq u_n \leq \beta_n$. $\square$

In order to construct lower and upper solutions of (1.1) the following two results turn out to be useful.

Lemma 3.7. *Assume*

(h_{10}) Ω *is a bounded domain in* $\mathbb{R}^N$ $(N \geq 1)$ *having a* C^1 *boundary* $\partial\Omega = \Gamma_D \cup \Gamma_N$, *with* $\Gamma_D \cap \Gamma_N = \emptyset$ *and both* Γ_D *and* Γ_N *open in* $\partial\Omega$,

and (h_1). *Let* $\beta \in W^{1,1}(\Omega)$ *be such that* $f(\cdot,\beta) \in L^p(\Omega)$ *for some* $p > N$, $\beta_{|\partial\Omega}(x) \geq 0$ *for* $\mathcal{H}_{N-1}$*-almost every* $x \in \Gamma_D$ *and*

$$\int_{\Gamma_D} z_{|\partial\Omega}\, d\mathcal{H}_{N-1} + \int_\Omega \nabla\beta\,\nabla z/\sqrt{1+|\nabla\beta|^2}\, dx \geq \int_\Omega f(x,\beta)z\, dx, \qquad (3.2)$$

for every $z \in W^{1,1}(\Omega)$ *with* $z \geq 0$. *Then* β *is an upper solution of* (1.1).

Proof. Let $z \in W^{1,1}(\Omega)$ be such that $z \geq 0$. Using the convexity in $\mathbb{R}^N$ of the function $a \mapsto \sqrt{1+|a|^2}$ and the assumption $\beta_{|\partial\Omega}(x) \geq 0$ for $\mathcal{H}_{N-1}$-almost every $x \in \Gamma_D$, we get from (3.2)

$$\begin{aligned}\int_\Omega f(x,\beta)z\,dx &\leq \int_\Omega \nabla\beta\,\nabla z/\sqrt{1+|\nabla\beta|^2}\,dx + \int_{\Gamma_D} z_{|\partial\Omega}\,d\mathcal{H}_{N-1}\\ &\leq \int_\Omega \sqrt{1+|\nabla(\beta+z)|^2}\,dx - \int_\Omega \sqrt{1+|\nabla\beta|^2}\,dx\\ &\quad + \int_{\Gamma_D} |(\beta+z)_{|\partial\Omega}|\,d\mathcal{H}_{N-1} - \int_{\Gamma_D} |\beta_{|\partial\Omega}|\,d\mathcal{H}_{N-1}\\ &= \mathcal{J}(\beta+z) - \mathcal{J}(\beta).\end{aligned}$$

Now, let $z \in BV(\Omega)$ be such that $z \geq 0$. By the results in [4, Section 3], [1, page 490], [16, Chapter 3], and [14, Section 4.4], there exists a sequence $(w_n)_n \subset W^{1,1}(\Omega)$ such that $w_n \geq \beta + z$ for every n, $\lim_{n\to+\infty} w_n = \beta + z$ in $L^q(\Omega)$ with $q = \frac{p}{p-1}$, and $\lim_{n\to+\infty} \mathcal{J}(w_n) = \mathcal{J}(\beta+z)$. Set, for each n, $z_n = w_n - \beta$; we have $z_n \in W^{1,1}(\Omega)$, $z_n \geq z \geq 0$, and $\lim_{n\to+\infty} z_n = z$ in $L^q(\Omega)$. Hence, we get

$$\begin{aligned}\mathcal{J}(\beta+z) &= \lim_{n\to+\infty} \mathcal{J}(\beta+z_n)\\ &\geq \lim_{n\to+\infty} \int_\Omega f(x,\beta)z_n\,dx + \mathcal{J}(\beta) = \int_\Omega f(x,\beta)z\,dx + \mathcal{J}(\beta);\end{aligned}$$

i.e., β is an upper solution of (1.1). $\square$

A similar result can be proved for lower solutions.

Lemma 3.8. *Assume* (h_{10}) *and* (h_1). *Let* $\alpha \in W^{1,1}(\Omega)$ *be such that* $f(\cdot,\alpha) \in L^p(\Omega)$ *for some* $p > N$, $\alpha_{|\partial\Omega}(x) \leq 0$ *for* $\mathcal{H}_{N-1}$*-almost every* $x \in \Gamma_D$ *and*

$$\int_{\Gamma_D} |z_{|\partial\Omega}|\,d\mathcal{H}_{N-1} + \int_\Omega \nabla\alpha\,\nabla z/\sqrt{1+|\nabla\alpha|^2}\,dx \geq \int_\Omega f(x,\alpha)z\,dx,$$

for every $z \in W^{1,1}(\Omega)$ *with* $z \leq 0$. *Then* α *is a lower solution of* (1.1).

3.2. **The Dirichlet problem.** Let us consider the Dirichlet problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.3}$$

In order to discuss the solvability of (3.3), we need some preliminaries.

Spectral constants. Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 1$). The principal eigenvalue of the Dirichlet problem for the minus 1-Laplacian is defined by (see e.g. [6])

$$\lambda_1 = \min\Big\{ \int_\Omega |Du| + \int_{\partial\Omega} |u_{|\partial\Omega}| \, d\mathcal{H}_{N-1} : u \in BV(\Omega), \|u\|_{L^1(\Omega)} = 1 \Big\}.$$

Denote by S^{N-1} the unit sphere in $\mathbb{R}^N$. For each $e \in S^{N-1}$ we set

$$a_e(\Omega) = \inf_{x\in\Omega} x\, e, \quad b_e(\Omega) = \sup_{x\in\Omega} x\, e, \quad L_e(\Omega) = b_e(\Omega) - a_e(\Omega). \tag{3.4}$$

As $L_e(\Omega)$ continuously depends on $e \in S^{N-1}$ and

$$\inf_{e\in S^{N-1}} L_e(\Omega) = \min_{e\in S^{N-1}} L_e(\Omega) > 0,$$

we define $\lambda_1^* = 2/\min_{e\in S^{N-1}} L_e(\Omega)$. If $N = 1$ and $\Omega = (a,b)$, then $\lambda_1^* = \frac{2}{b-a} = \lambda_1$; i.e., λ_1^* is the principal eigenvalue of the Dirichlet problem for the minus 1-Laplacian (cf. [6, Proposition 12]).

We also denote by μ_1 the principal eigenvalue of the Dirichlet problem for the minus Laplacian, defined by

$$\mu_1 = \min\Big\{ \int_\Omega |\nabla u|^2 \, dx : u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)} = 1 \Big\}.$$

Local conditions. In this subsection we show how, under some local assumptions relating the behaviour of f, or F, with the spectral constants λ_1^* and μ_1, it is possible to construct lower and upper solutions of (3.3), obtained as solutions of suitable comparison problems naturally associated with (3.3). The existence of such lower and upper solutions will then provide some existence and localization theorems for solutions of (3.3).

Lemma 3.9. *Assume*

(h_{11}) *Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) having a C^1 boundary $\partial\Omega$,*

(h_6) *and*

(h_{12}) *there exist $R \geq \frac{1}{\lambda_1^*}$ and $h : \mathbb{R} \to (0, +\infty)$ continuous such that*

$$f(x,s) \leq h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [R - \tfrac{1}{\lambda_1^*}, R],$$

and

$$\frac{H(R) - H(s)}{R - s} \leq \lambda_1^* \quad \text{for every } s \in [R - \tfrac{1}{\lambda_1^*}, R), \tag{3.5}$$

where $H(s) = \int_0^s h(t)\, dt$.

Then there exists an upper solution $\beta \in BV(\Omega)$ of (3.3) such that $\beta \in C^2(\Omega) \cap C^0(\bar\Omega)$ and $R - \frac{1}{\lambda_1^*} < \beta(x) \le R$ for every $x \in \Omega$.

Proof. Consider the initial-value problem

$$-\left(v'/\sqrt{1+|v'|^2}\right)' = h(v), \quad v(0)=R,\, v'(0)=0. \tag{3.6}$$

Let $v \in C^2(-\omega,\omega)$ be an even non-extendible solution of (3.6). Then $v_{|_{[0,\omega)}}$ is decreasing, concave and satisfies the energy relation

$$1 - \frac{1}{\sqrt{1+|v'(t)|^2}} = H(R) - H(v(t)), \tag{3.7}$$

in $[0,\omega)$. Define $T = \sup\left\{t \in [0,\omega) : v(t) > R - \frac{1}{\lambda_1^*}\right\}$. Since $v_{|_{[0,\omega)}}$ is decreasing and concave, we have $T < +\infty$. Then we set

$$v(T) = \lim_{t\to T} v(t) \ge R - \tfrac{1}{\lambda_1^*}.$$

Note that $v \in C^2(-T,T) \cap C^0([-T,T])$ and it may be that $v'(T) = -\infty$. Using (3.7), (3.5) and the fact that the function $t \mapsto (1-t)/\sqrt{2t-t^2}$ is decreasing in $(0,1]$, we get

$$\begin{aligned}
T &= \int_0^T -v'(t)\frac{1-(H(R)-H(v(t)))}{\sqrt{2\left(H(R)-H(v(t))\right)-\left(H(R)-H(v(t))\right)^2}}\,dt \\
&= \int_{v(T)}^{v(0)} \frac{1-(H(R)-H(s))}{\sqrt{2\left(H(R)-H(s)\right)-\left(H(R)-H(s)\right)^2}}\,ds \\
&\ge \int_{R-\frac{1}{\lambda_1^*}}^{R} \frac{1-\lambda_1^*(R-s)}{\sqrt{2\lambda_1^*(R-s)-\left(\lambda_1^*(R-s)\right)^2}}\,ds = \frac{1}{\lambda_1^*}\int_0^1 \frac{1-t}{\sqrt{2t-t^2}}\,dt = \frac{1}{\lambda_1^*}.
\end{aligned}$$

Let $\hat e \in S^{N-1}$ be such that $L_{\hat e}(\Omega) = \min_{e\in S^{N-1}} L_e(\Omega)$ and set, for every $x \in \bar\Omega$,

$$\beta(x) = v(x\,\hat e - \tfrac12(a_{\hat e}(\Omega) + b_{\hat e}(\Omega))),$$

where $L_{\hat e}(\Omega)$, $a_{\hat e}(\Omega)$ and $b_{\hat e}(\Omega)$ are defined in (3.4). Observe that $\beta \in C^2(\Omega)\cap C^0(\bar\Omega)$ and $R - \frac{1}{\lambda_1^*} < \beta(x) \le R$ for every $x \in \Omega$. As

$$\int_{-T}^{T} |v'|\,dt = 2(v(0)-v(T)),$$

we see that $v \in W^{1,1}(-T,T)$ and hence $\beta \in W^{1,1}(\Omega)$. Note also that $\nabla\beta/\sqrt{1+|\nabla\beta|^2} \in C^1(\bar\Omega)$ and $f(\cdot,\beta) \in L^p(\Omega)$. Moreover, we have

$$-\operatorname{div}\Big(\nabla\beta/\sqrt{1+|\nabla\beta|^2}\Big) = -v''/(1+|v'|^2)^{\frac{3}{2}} = h(v) \geq f(x,\beta), \tag{3.8}$$

almost everywhere in Ω. Take $z \in W^{1,1}(\Omega)$ such that $z \geq 0$. Multiplying (3.8) by z and integrating by parts, we easily get

$$\int_{\partial\Omega} z_{|\partial\Omega}\, d\mathcal{H}_{N-1} + \int_\Omega \nabla z\, \nabla\beta/\sqrt{1+|\nabla\beta|^2}\, dx \geq \int_\Omega f(x,\beta) z\, dx.$$

Lemma 3.7 implies that β is an upper solution of (3.3). □

Remark 3.10. If in (h_{12}) the stronger condition

$$\frac{H(R)-H(s)}{R-s} < \lambda_1^* \quad \text{for every } s \in [R-\tfrac{1}{\lambda_1^*}, R)$$

holds, then $\beta \in C^2(\bar\Omega)$. Indeed, in this case $T > \frac{1}{\lambda_1^*} = \frac{1}{2}L_{\hat e}(\Omega)$ and

$$v \in C^2([-\tfrac{1}{2}L_{\hat e}(\Omega), \tfrac{1}{2}L_{\hat e}(\Omega)]).$$

Similarly, the following symmetric result can be proved.

Lemma 3.11. *Assume* (h_{11}), (h_6) *and*

(h_{13}) *there exist* $S \leq -\frac{1}{\lambda_1^*}$ *and* $k : \mathbb{R} \to (-\infty, 0)$ *continuous such that*

$$f(x,s) \geq k(s) \quad \textit{for a.e. } x \in \Omega \textit{ and every } s \in [S, S+\tfrac{1}{\lambda_1^*}],$$

and

$$\frac{K(S)-K(s)}{S-s} \geq -\lambda_1^* \quad \textit{for every } s \in (S, S+\tfrac{1}{\lambda_1^*}],$$

where $K(s) = \int_0^s k(t)\, dt$.

Then there exists a lower solution $\alpha \in BV(\Omega)$ *of* (3.3) *such that* $\alpha \in C^2(\Omega) \cap C^0(\bar\Omega)$ *and* $S \leq \alpha(x) < S + \frac{1}{\lambda_1^*}$ *for every* $x \in \Omega$.

Remark 3.12. If in (h_{13}) the stronger condition

$$\frac{K(S)-K(s)}{S-s} > -\lambda_1^* \quad \text{for every } s \in (S, S+\tfrac{1}{\lambda_1^*}]$$

holds, then $\alpha \in C^2(\bar\Omega)$.

Lemma 3.13. *Assume*

(h_{14}) *Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) having a $C^{2,\sigma}$ boundary $\partial\Omega$, for some $\sigma \in (0,1]$,*

(h_6) *and*

(h_{15}) *there exists $S > 0$ such that*

$$f(x,s) \geq \mu_1 s \quad \text{for almost every } x \in \Omega \text{ and every } s \in [0,S].$$

Then, for each $\varepsilon > 0$ there exists a lower solution $\alpha \in C^{2,\sigma}(\bar{\Omega})$ of (3.3) *such that $\alpha(x) > 0$ for every $x \in \Omega$, $\frac{\partial\alpha}{\partial\nu}(x) < 0$ for every $x \in \partial\Omega$, where ν is the unit outer normal to Ω at $x \in \partial\Omega$, and $\|\alpha\|_{C^{2,\sigma}(\bar{\Omega})} < \varepsilon$.*

Proof. From the results in [24] we infer the existence of $\mu_0 \in (0,\mu_1)$ such that for every $\mu \in (\mu_0,\mu_1)$ there exists $\alpha_\mu \in C^{2,\sigma}(\bar{\Omega})$ satisfying

$$-\text{div}\Big(\nabla\alpha_\mu/\sqrt{1+|\nabla\alpha_\mu|^2}\Big) = \mu\alpha_\mu \text{ in } \Omega, \quad \alpha_\mu = 0 \text{ on } \partial\Omega,$$

$\lim_{\mu\to\mu_1} \alpha_\mu = 0$ in $C^{2,\sigma}(\bar{\Omega})$, and $\alpha_\mu \geq \varphi_1$, for some positive eigenfunction φ_1 corresponding to μ_1. Hence, using (h_{15}) and Lemma 3.8, we easily see that any α_μ, with μ sufficiently close to μ_1, is a lower solution of (3.3) having the desired properties. □

Similarly, the following symmetric result can be proved.

Lemma 3.14. *Assume* (h_{14}), (h_6) *and*

(h_{16}) *there exists $R < 0$ such that*

$$f(x,s) \leq \mu_1 s \quad \text{for almost every } x \in \Omega \text{ and every } s \in [R,0].$$

Then, for each $\varepsilon > 0$ there exists an upper solution $\beta \in C^{2,\sigma}(\bar{\Omega})$ of (3.3) *such that $\beta(x) < 0$ for every $x \in \Omega$, $\frac{\partial\beta}{\partial\nu}(x) > 0$ for every $x \in \partial\Omega$, where ν is the unit outer normal to Ω at $x \in \partial\Omega$, and $\|\beta\|_{C^{2,\sigma}(\bar{\Omega})} < \varepsilon$.*

From the preceding lemmas the following existence results can be deduced.

Proposition 3.15. *Assume* (h_{11}), (h_6), (h_{12}) *and* (h_{13}). *Then problem* (3.3) *has at least one solution u such that $S \leq u \leq R$.*

Proof. By Lemma 3.11 and Lemma 3.9 there exist a lower solution α and an upper solution β of (3.3) such that $S \leq \alpha \leq 0 \leq \beta \leq R$. Theorem 2.4 yields the existence of a solution u of (3.3) such that $\alpha \leq u \leq \beta$. □

Remark 3.16. In the one-dimensional case, i.e., if $N = 1$ and $\Omega = (a,b)$, conditions (h_{12}) and (h_{13}) are sharp, in the sense that if $\lambda_1^* = \frac{2}{b-a}$ is replaced by any constant $\lambda > \lambda_1^*$, then the solvability of (3.3) is no longer guaranteed. This is shown by the elementary example

$$-(u'/\sqrt{1+u'^2})' = \lambda, \quad u(a) = u(b) = 0,$$

which has no solutions $u \in BV(\Omega)$ for any $\lambda > \frac{2}{b-a}$.

Proposition 3.17. *Assume* (h_{14}), (h_6), (h_{15}) *and* (h_{12}). *Then problem* (3.3) *has at least one solution* u *such that* $0 < u(x) \le R$ *for almost every* $x \in \Omega$.

Proof. By Lemma 3.9 and Lemma 3.13 we can find an upper solution β and a lower solution α with $0 < \alpha(x) \le \beta(x) \le R$ for every $x \in \Omega$. To this end we notice that, under (h_{14}), we have either $\beta(x) > 0$ or $\frac{\partial \beta}{\partial \nu}(x) < 0$ for each $x \in \partial\Omega$. Theorem 2.4 yields the existence of a solution u of (3.3) such that $\alpha \le u \le \beta$. □

A similar proof yields the following result.

Proposition 3.18. *Assume* (h_{14}), (h_6), (h_{13}) *and* (h_{16}). *Then problem* (3.3) *has at least one solution* u *such that* $S \le u(x) < 0$ *for almost every* $x \in \Omega$.

Asymptotic conditions. In this section we show how, under some asymptotic assumptions on F, it is possible to construct sequences of lower and upper solutions of (3.3). The existence of such lower and upper solutions will then provide existence and multiplicity theorems for solutions of (3.3).

Lemma 3.19. *Assume* (h_{11}), (h_6) *and*

(h_{17}) *there exist* $r_0 > 0$ *and* $h : \mathbb{R} \to \mathbb{R}$ *continuous such that*

$$f(x,s) \le h(s) \quad \text{for almost every } x \in \Omega \text{ and every } s \ge r_0$$

and

$$\liminf_{s \to +\infty} \frac{H(s)}{s} < \lambda_1^*,$$

where $H(s) = \int_0^s h(t)\,dt$.

Then there exists a sequence $(\beta_n)_n$ *of upper solutions of* (3.3) *such that, for each* n, $\beta_n \in C^2(\bar\Omega)$, $\beta_n(x) < \beta_{n+1}(x)$ *for every* $x \in \bar\Omega$ *and*

$$\lim_{n \to +\infty} (\min_{\bar\Omega} \beta_n) = +\infty.$$

Proof. Suppose first that $\sup\{s > 0 : h(s) \le 0\} = +\infty$. Then by Remark 3.2 there exists an increasing sequence $(\beta_n)_n$ of constant upper solutions with $\lim_{n\to+\infty} \beta_n = +\infty$. Therefore, we may assume $h(s) > 0$ in $[r_0, +\infty)$. Possibly replacing $h(s)$ with $h(r_0)$ in $(-\infty, r_0)$, we can further suppose that

$h(s) > 0$ in $\mathbb{R}$. Hence, we can find an increasing sequence $(R_n)_n$ such that $\lim_{n\to+\infty} R_n = +\infty$ and, for every n,

$$r_0 + \tfrac{1}{\lambda_1^*} < R_n < R_{n+1} - \tfrac{1}{\lambda_1^*},$$

and

$$H(R_n) - H(s) < \lambda_1^*(R_n - s) \quad \text{in } [0, R_n).$$

By Lemma 3.9 and Remark 3.10, for each n there exists an upper solution β_n of (3.3) such that $\beta_n \in C^2(\bar{\Omega})$ and $R_{n-1} < R_n - \frac{1}{\lambda_1^*} \le \beta_n \le R_n$. □

Similarly, the following symmetric result can be proved.

Lemma 3.20. *Assume* (h_{11}), (h_6) *and*

(h_{18}) *there exist* $s_0 < 0$ *and* $k : \mathbb{R} \to \mathbb{R}$ *continuous such that*

$$f(x,s) \ge k(s) \quad \text{for almost every } x \in \Omega \text{ and every } s \le s_0$$

and

$$\liminf_{s\to-\infty} \frac{K(s)}{|s|} < \lambda_1^*,$$

where $K(s) = \int_0^s k(t)\,dt$.

Then there exists a sequence $(\alpha_n)_n$ *of lower solutions of* (3.3) *such that, for each* n, $\alpha_n \in C^2(\bar{\Omega})$, $\alpha_n(x) > \alpha_{n+1}(x)$ *for every* $x \in \bar{\Omega}$ *and*

$$\lim_{n\to+\infty} (\max_{\bar{\Omega}} \alpha_n) = -\infty.$$

Proposition 3.21. *Assume* (h_{11}), (h_6) *and*

(h_{19}) *there exist* $r_0 > 0$ *and* $g : \mathbb{R} \to \mathbb{R}$ *continuous such that*

$$f(x,s)\operatorname{sgn}(s) \le g(s)\operatorname{sgn}(s) \quad \text{for almost every } x \in \Omega \text{ and every } |s| \ge r_0$$

and

$$\liminf_{s\to\pm\infty} \frac{G(s)}{|s|} < \lambda_1^*,$$

where $G(s) = \int_0^s g(t)\,dt$.

Then problem (3.3) *has at least one solution* $u \in BV(\Omega) \cap L^\infty(\Omega)$.

Proof. By Lemma 3.20 and Lemma 3.19 there exist a lower solution α and an upper solution β of (3.3) such that $\alpha, \beta \in C^2(\bar{\Omega})$ and $\alpha \le \beta$. Theorem 2.4 yields the existence of a solution u of (3.3) such that $\alpha \le u \le \beta$. □

Proposition 3.22. *Assume* (h_{11}), (h_6), (h_{19}) *and* (h_8). *Then problem* (3.3) *has a double sequence* $(u_m)_{m\in\mathbb{Z}}$ *of solutions such that, for each* $m \in \mathbb{Z}$, $u_m \in BV(\Omega) \cap L^\infty(\Omega)$ *and*

$$\lim_{m\to\pm\infty} \mathcal{I}(u_m) = -\infty, \lim_{m\to-\infty} \operatorname*{ess\,inf}_{\Omega}\, u_m = -\infty, \lim_{m\to+\infty} \operatorname*{ess\,sup}_{\Omega}\, u_m = +\infty.$$

Proof. By Lemma 3.20 and Lemma 3.19 there are sequences $(\alpha_n)_n$ and $(\beta_n)_n$ of lower and upper solutions of (3.3) such that, for each n, $\alpha_n, \beta_n \in C^2(\bar{\Omega})$, $\alpha_n \le \beta_n$, $\lim_{n\to+\infty} \max_{\bar{\Omega}} \alpha_n = -\infty$ and $\lim_{n\to+\infty} \min_{\bar{\Omega}} \beta_n = +\infty$. As noticed in Remark 3.5, we can apply Proposition 3.3 to get the conclusion. □

Remark 3.23. As already noticed in Remark 3.4, the existence of a sequence $(u_n)_{n\in\mathbb{Z}^+}$ of solutions of (3.3), such that, for each $n \in \mathbb{Z}^+$, $u_n \in BV(\Omega) \cap L^\infty(\Omega)$, $\lim_{n\to+\infty} \mathcal{I}(u_n) = -\infty$, and $\lim_{n\to+\infty} \operatorname*{ess\,sup}_{\Omega}\, u_n = +\infty$ can be obtained if conditions (h_{19}) and (h_8), with $B_+ = \Omega$, in Proposition 3.22 hold only at $+\infty$ and there exists a lower solution $\alpha \in BV(\Omega) \cap L^\infty(\Omega)$ of (1.1). The existence of such a lower solution is for instance guaranteed, via Lemma 3.13, by the condition

$$\liminf_{s\to 0^+} \frac{f(x,s)}{s} > \mu_1 \quad \text{uniformly a.e. in } \Omega,$$

provided that (h_{14}) holds. In this case, the obtained solutions are positive.

Similarly, the existence of a sequence $(u_m)_{m\in\mathbb{Z}^-}$ of solutions of (1.1) such that, for each $m \in \mathbb{Z}^-$, $u_m \in BV(\Omega) \cap L^\infty(\Omega)$, $\lim_{m\to-\infty} \mathcal{I}(u_m) = -\infty$, and $\lim_{m\to-\infty} \operatorname*{ess\,inf}_{\Omega}\, u_m = -\infty$, can be obtained if conditions (h_{19}) and (h_8), with $B_- = \Omega$, in Proposition 3.22 hold only at $-\infty$ and there exists an upper solution $\beta \in BV(\Omega) \cap L^\infty(\Omega)$ of (1.1). The existence of such an upper solution is for instance guaranteed, via Lemma 3.14, by the condition

$$\liminf_{s\to 0^-} \frac{f(x,s)}{s} > \mu_1 \quad \text{uniformly a.e. in } \Omega,$$

provided that (h_{14}) holds. In this case the obtained solutions are negative.

Remark 3.24. Assumption (h_8) is implied by the following

(h_{20}) there exist $r_0 > 0$ and $\ell : \mathbb{R} \to \mathbb{R}$ continuous such that

$$f(x,s)\operatorname{sgn}(s) \ge \ell(s)\operatorname{sgn}(s) \quad \text{for a.e. } x \in \Omega \text{ and every } |s| \ge r_0,$$

and

$$\limsup_{s\to\pm\infty} \frac{L(s)}{|s|} > \frac{\operatorname{Per}(\Omega)}{\operatorname{meas}(\Omega)},$$

where $L(s) = \int_0^s \ell(t)\,dt$.

Remark 3.25. In some situations (see [6, Proposition 11]) $\frac{\mathrm{Per}(\Omega)}{\mathrm{meas}(\Omega)}$ is the principal eigenvalue of the minus 1-Laplacian. In particular this is true if $N = 1$ and $\Omega = (a, b)$. In this case we have $\frac{\mathrm{Per}(\Omega)}{\mathrm{meas}(\Omega)} = \frac{2}{b-a} = \lambda_1^*$, so that conditions (h_{19}) and (h_{20}) are optimal.

Example 3.1. Let $F : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that

$$F(s) = |s| \ln(\ln(|s|))\Big(1 + \sin\big(\ln(\ln(\ln|s|))\big)\Big) + \frac{|s|}{\ln|s|},$$

for every $|s| \geq e^e$. It is easily seen that

$$0 = \liminf_{s\to\pm\infty} \frac{F(s)}{|s|} < \limsup_{s\to\pm\infty} \frac{F(s)}{|s|} = +\infty.$$

Accordingly, the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, given by $f(x, s) = F'(s)$ for every $x \in \Omega$ and $s \in \mathbb{R}$, satisfies conditions (h_{19}) and (h_{20}). Note that, as $F'(s)\mathrm{sgn}(s) > 0$ for every $|s| \geq e^e$, there exist neither large positive constant upper solutions nor large negative constant lower solutions of (3.3).

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