Differential and Integral Equations Volume 22, Numbers 9-10 (2009), 843–852

ON A CLASS OF 1-MONOTONE SOLUTIONS FOR A FORCED PENDULUM MODEL

Paul H. Rabinowitz
Department of Mathematics, University of Wisconsin–Madison
Madison, WI 53706

Dedicated to Jean Mawhin who always liked the pendulum

1. Introduction

As has been noted by Jean Mawhin [4], the pendulum has served as a paradigm for nonlinear analysis and dynamical systems. In a recent paper [1], elementary minimization arguments were used to find a variety of solutions of equations of forced pendulum type. The existence of these solutions is not new, but the approach is very simple and requires almost no facts from the theory of dynamical systems. The purpose of this note is to carry the analysis of [1] a step further to obtain another natural class of heteroclinic solutions. See the survey papers [3] and [4] by Mawhin for references and much more on the pendulum.

As in [1], the forced pendulum model equation

$$-u'' + V_u(x,u) = 0 \tag{1.1}$$

will be considered. Here V satisfies

- (V1) $V \in C^2(\mathbb{T}^2, \mathbb{R})$, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$;

i.e., V is 1-periodic in its arguments. We further assume

- (V2) $V(-t,z) = V(t,z),\ t,z \in \mathbb{R}$;

i.e., V is time reversible. As in [1], (V2) is not necessary, but it considerably simplifies the construction of solutions of (1.1) that undergo transitions.

Before describing the new class of heteroclinics that will be obtained for (1.1), we recall some results from [1] for (1.1). Let $E = W^{1,2}(\mathbb{T}^1)$ be the Hilbert space of 1-periodic functions for which $||u|| < \infty$, where

$$||u||^2 = \int_0^1 (|u'|^2 + u^2)dt.$$

AMS Subject Classifications: 34C37, 37J45.

For $u \in E$, set

$$L(u) = \frac{1}{2}|u'^2| + V(t,u).$$

Define

$$I(u) = \int_0^1 L(u)dt,$$

and set

$$c = \inf_{u \in E} I(u). \tag{1.2}$$

The first result gives periodic solutions of (1.1) as minima of I.

Theorem 1.3. *If V satisfies* (V1)-(V2), (1.1) *possesses an ordered set of C^2 solutions given by*

$$\mathcal{M}_0 = \{u \in E : I(u) = c\}.$$

Moreover, the members of $\mathcal{M}_0$ have an additional minimization property: if $u \in \mathcal{M}_0$, for any $a < b$ with $a, b \in \mathbb{R}$, u minimizes

$$\int_a^b L(u)dt,$$

over the class of $\phi \in W^{1,2}[a,b]$ with $\phi(a) = u(a)$ and $\phi(b) = u(b)$.

Remark 1.4. By $\mathcal{M}_0$ ordered, we mean if $v, w \in \mathcal{M}_0$, then either $v \equiv w, v < w$, or $v > w$.

Suppose $v, w \in \mathcal{M}_0$ with $v < w$ and there is no $\phi \in \mathcal{M}_0$ with $v < \phi < w$. Then we say v and w are a gap pair in $\mathcal{M}_0$. Perturbing V slightly in C^2 if need be shows

$$\mathcal{M}_0 = \{v + k : k \in \mathbb{Z}\},$$

for any $v \in \mathcal{M}_0$. Thus, gap pairs exist and, given such a gap pair, one can seek solutions of (1.1) that are heteroclinic from v to w. Such solutions can also be obtained by a minimization argument. However, the natural extension of I to this setting is

$$\int_{\mathbb{R}} L(u)dt, \tag{1.5}$$

and the integral in (1.5) will in general be infinite on the class of functions asymptotic to v and w as $t \to \pm\infty$. Therefore, I has to be replaced by an appropriately renormalized functional. As was shown in [1] using (V2), if $u \in W^{1,2}_{\text{loc}}(\mathbb{R})$, and $p \in \mathbb{Z}$,

$$\int_p^{p+1} L(u)dt \geq c,$$

with c given by (1.2). Therefore,

$$a_p(u) \equiv \int_p^{p+1} L(u)dt - c \geq 0,$$

for all $p \in \mathbb{Z}$ and this leads to the choice of the renormalized functional, $J(u)$, to treat (1.1):

$$J(u) = \sum_{p \in \mathbb{Z}} a_p(u),$$

and $J(u) \geq 0$ for any $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$.

To introduce a class of functions appropriate for finding heteroclinic solutions of (1.1), for $i \in \mathbb{Z}$ define

$$\begin{aligned} \Gamma_1(v,w) \quad = \quad & \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}) : ||u-v||_{L^2[i,i+1]} \to 0 \text{ as } i \to \infty \\ & \text{and } \ ||u-w||_{L^2[i,i+1]} \to 0 \text{ as } i \to \ \infty\}. \end{aligned}$$

Finally, set

$$c_1 \equiv c_1(v,w) = \inf_{u \in \Gamma_1(v,w)} J(u). \tag{1.6}$$

The following was shown in [1].

Theorem 1.7. *If V satisfies* $(V1)$-$(V2)$ *and v, w are a gap pair in $\mathcal{M}_0$, then*

1° $\mathcal{M}_1 \equiv \mathcal{M}_1(v,w) = \{u \in \Gamma_1(v,w) : J(u) = c_1\} \neq \emptyset$,
2° *any $U \in \mathcal{M}_1$ is a C^2 solution of* (1.1),
3° $v(t) < U(t) < U(t+1) < w(t)$, *i.e., U is* 1*-monotone in t,*
4° *$\mathcal{M}_1$ is an ordered set,*
5° *any $U \in \mathcal{M}_1$ is minimal* (*as in Theorem* 1.3).

The solutions of (1.1) given by Theorem 1.7 remain near v and w for all but a compact set of values of t and by 3° undergo one transition from v to w. One could also seek solutions of (1.1) which lie in the gap between v and w and make multiple transitions between v and w. The simplest such solutions would be homoclinics to v (or w) which spend at least a prescribed amount of time near w (or v). The existence of such solutions was shown in [1], the solutions being obtained as local minima of J via a constrained variational problem.

Another natural class of multi-transition solutions of (1.1) to pursue is the following: Suppose $v < w \leq \overline{v} < \overline{w}$ with v, w and $\overline{v}, \overline{w}$ gap pairs, e.g. $\overline{v} = v + 1$ and $\overline{w} = w + 1$. Then we can seek a 1-monotone solution of (1.1), i.e., $u(t+1) > u(t)$ which is heteroclinic from v to $\overline{w}$. In terms of the forced pendulum, in the simplest case this corresponds to a solution which is vertically upright at $t = -\infty$, moves e.g. counterclockwise as t increases, slows down as it approaches the vertically upright position again, slowly passes through that position, and finally makes one more rotation to asymptote in the vertically upright position at $t = \infty$. Our main goal in this note is to obtain the existence of such solutions using the arguments of [1] and [2]. This will be done in the next section.

2. The existence of 1-monotone 2-transition solutions

Let v, w and $\overline{v}, \overline{w}$ be gap pairs in $\mathcal{M}_0$ with $v < w \le \overline{v} < \overline{w}$. Under the further assumption that

$$\mathcal{M}_1(v,w) \text{ and } \mathcal{M}_1(\overline{v},\overline{w}) \text{ have gaps,} \tag{2.1}$$

we will show there are infinitely many solutions of (1.1) that are heteroclinic from v to $\overline{w}$. Note that (2.1) does not exclude the possibility of there being continua of periodic or heteroclinic solutions of (1.1) lying between v and $\overline{w}$. The new heteroclinic solutions will be obtained via a constrained minimization problem. A more complicated construction for a more difficult partial differential equation version of (1.1) was carried out in [2].

To introduce the constraints, first we define maps $r^{\pm} : \mathcal{M}_1(v,w) \to (0, w(0) - v(0))$ and $\overline{r}^{\pm} : \mathcal{M}_1(\overline{v},\overline{w}) \to (0, \overline{w}(0) - \overline{v}(0))$ via

$$r^-(u) = u(0) - v(0),\ r^+(u) = w(0) - u(0),$$

and

$$\overline{r}^-(u) = u(0) - \overline{v}(0),\ \overline{r}^+(u) = \overline{w}(0) - u(0).$$

By Theorem 1.7, each of the sets $\mathcal{M}_1(v,w)$, $\mathcal{M}_1(\overline{v},\overline{w})$ is ordered. Therefore, the functions $r^{\pm}$, $\overline{r}^{\pm}$ are strictly monotone. Moreover, since $\mathcal{M}_1(v,w)$ and $\mathcal{M}_1(\overline{v},\overline{w})$ possess gaps, so do their images under the maps $r^{\pm}, \overline{r}^{\pm}$. Choose $\rho = (\rho_1, \rho_2, \rho_3, \rho_4)$ as follows:

$$\begin{aligned}
\rho_1 &\in (0, w(0) - v(0)) \backslash r^-(\mathcal{M}_1(v,w)),\\
\rho_2 &\in (0, w(0) - v(0)) \backslash r^+(\mathcal{M}_1(v,w)),\\
\rho_3 &\in (0, \overline{w}(0) - \overline{v}(0)) \backslash \overline{r}^+(\mathcal{M}_1(\overline{v},\overline{w})),\\
\rho_4 &\in (0, \overline{w}(0) - \overline{v}(0)) \backslash \overline{r}^-(\mathcal{M}_1(\overline{v},\overline{w})).
\end{aligned}$$

Since $u \in \mathcal{M}_1(v,w)$ implies $u(t+j) \in \mathcal{M}_1(v,w)$ for all $j \in \mathbb{Z}$, there are gaps in $r^-(\mathcal{M}_1(v,w))$ arbitrarily close to 0. Hence, ρ_1 and similarly ρ_i, $i = 2,3,4$, can be chosen so that $|\rho|$ is as small as desired. Roughly speaking, the numbers ρ_i measure the size of the constraints. Next, let $m \in \mathbb{Z}^4$ with $m_{i+1} > m_i, 1 \le i \le 3$. The m_i's give the location of the constraints.

Next, we introduce the class of admissible functions for our minimization problem. Define

$$\Gamma_{m,\rho} \equiv \Gamma_{m,\rho}(v,\overline{w}) = \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}) : v \le u \le \overline{w} \text{ and } u(m_1) - v(m_1) \le \rho_1,\\ w(m_2) - u(m_2) \le \rho_2,\ \overline{v}(m_3) - u(m_3) \le \rho_3,\ u(m_4) - \overline{w}(m_4) \le \rho_4\}.$$

Finally, we set

$$b_{m,\rho} \equiv b_{m,\rho}(v,\overline{w}) = \inf_{u \in \Gamma_{m,\rho}} J(u). \tag{2.2}$$

It is straightforward to show that $b_{m,\rho} < \infty$.

Now, we are ready for our main result which parallels Theorem 1.7.

Theorem 2.3. *Let V satisfy $(V1)$-$(V2)$ and let $v < w \le \overline{v} < \overline{w}$ with v, w and $\overline{v}, \overline{w}$ gap pairs in $\mathcal{M}_0$. Assume that $\mathcal{M}_1(v,w)$ and $\mathcal{M}_1(\overline{v},\overline{w})$ have gaps. Let $m \in \mathbb{Z}^4$ with $m_{i+1} > m_i$, $1 \le i \le 3$, and let $\rho \in \mathbb{R}^4$ with $\rho_i > 0$, $1 \le i \le 4$. Then, if $\mathcal{M}(\Gamma_{m,\rho}) = \{u \in \Gamma_{m,\rho} : J(u) = b_{m,\rho}\}$,*

1° *$\mathcal{M}(\Gamma_{m,\rho}) \neq \emptyset$;*
2° *any $U \in \mathcal{M}(\Gamma_{m,\rho})$ is minimal in the non-constraint intervals : $(-\infty, m_1]$, $[m_i, m_{i+1}]$, $1 \le i \le 3$, and $[m_4, \infty)$;*
3° *any $U \in \mathcal{M}(\Gamma_{m,\rho})$ is a solution of (1.1) except possibly for $t = m_i$, $i \le i \le 4$;*
4° *if $\phi, \psi \in \mathcal{M}(\Gamma_{m,\rho})$ and $\phi(\xi) = \psi(\xi))$ for some ξ, then $\xi \in \{m_1, ..., m_4\}$ or $\phi \equiv \psi$ in one of the intervals in 2°;*
5° *there is a $\rho_0 > 0$ such that $|\rho| \le \rho_0$ implies $||U - v||_{C^2[i,i+1]} \to 0$ as $i \to -\infty$ and $||U - w||_{C^2[i,i+1]} \to 0$ as $i \to \infty$.*

Moreover, if $|\rho| \le \rho_0$, there is an $m_0 > 0$ such that, whenever $m_{i+1} - m_i \ge m_0, 1 \le i \le 3$,

6° *U is a solution of (1.1) on $\mathbb{R}$;*
7° *$\mathcal{M}(\Gamma_{m,\rho})$ is ordered;*
8° *any $U \in \mathcal{M}(\Gamma_{m,\rho})$ is strictly 1-monotone on $\mathbb{R}$.*

Proof. Let (u_k) be a minimizing sequence for (2.2). Then the form of J implies (u_k) is bounded in $W^{1,2}_{loc}(\mathbb{R})$. Therefore, there is a $U \in W^{1,2}_{loc}(\mathbb{R})$ such that, along a subsequence (which we can take to be the entire sequence), u_k converges to U weakly in $W^{1,2}_{loc}(\mathbb{R})$ and strongly in $L^{\infty}_{loc}(\mathbb{R})$. The latter convergence implies U satisfies the constraints of $\Gamma_{m,\rho}$. Therefore, $U \in \Gamma_{m,\rho}$ and

$$J(U) \ge b_{m,\rho}. \tag{2.4}$$

Next, observe that, for any $p, q \in \mathbb{Z}$, with $p \leq q$, $\sum_p^q a_i(u)$ is weakly lower semicontinuious in $W^{1,2}[p, q+1]$. Therefore,

$$0 \leq \sum_p^q a_i(U) \leq \varliminf_{k\to\infty} \sum_p^q a_i(u_k) \leq \varliminf_{k\to\infty} J(u_k) = b_{m,\rho}. \tag{2.5}$$

Letting $p \to -\infty$ *and* $q \to \infty$ in (2.5) shows

$$J(U) \leq b_{m,\rho}. \tag{2.6}$$

Combining (2.4) and (2.6) then gives 1°.

Next, to prove 2°, if it is false, there is an interval $[a, b]$ with

$$[a, b] \subset (-\infty, m_1] \cup [m_1, m_2] \cup [m_2, m_3] \cup [m_3, m_4] \cup [m_4, \infty),$$

and a function $\phi \in W^{1,2}[a, b]$ such that $\phi(a) = U(a), \phi(b) = U(b)$ and

$$\int_a^b L(\phi)dt < \int_a^b L(U)dt. \tag{2.7}$$

As in the proof of Theorem 3.1 of [1], using the minimality of v and $\overline{w}$, it can be assumed that $v \leq \phi \leq \overline{w}$. But then, replacing $U|_a^b$ by $\phi|_a^b$ yields a new function $U^* \in \Gamma_{m,\rho}$ and (2.7) shows $J(U^*) < J(U)$, contrary to 1°.

Remark 2.8. Note that the above argument holds even for one of a, b being infinite.

Now, given the minimality of U in the non-constraint intervals, the remark following Proposition 2.3 of [1] shows U satisfies (1.1) in the interior of these intervals and 3° is satisfied.

To prove 4°, note first that, if $\phi, \psi \in \mathcal{M}(\Gamma_{m,\rho})$, then $\max(\phi, \psi)$ and $\min(\phi, \psi) \in \Gamma_{m,\rho}$. Therefore,

$$2b_{m,\rho} = J(\phi) + J(\psi) = J(\max(\phi, \psi)) + J(\min(\phi, \psi)) \geq 2b_{m,\rho}. \tag{2.9}$$

Hence, $J(\max(\phi, \psi)) = J(\min(\phi, \psi)) = b_{m,\rho}$ and $\max(\phi, \psi)$ and $\min(\phi, \psi)$ belong to $\mathcal{M}(\Gamma_{m,\rho})$. Thus, by 3°, $\max(\phi, \psi)$ and $\min(\phi, \psi)$ are solutions of (1.1) for $t \neq m_i$, $1 \leq i \leq 4$. In particular, if $\xi \in (m_i, m_{i+1})$ and $\phi(\xi) = \psi(\xi)$, then ξ is a global minimum of $\max(\phi, \psi) - \min(\phi, \psi)$ so $\max(\phi, \psi) = \min(\phi, \psi)$ and $(\max(\phi, \psi))' = (\min(\phi, \psi))'$ at $t = \xi$. Consequently, (1.1) implies $\phi \equiv \psi$ for $t \in [m_i, m_{i+1}]$. The same argument applies to the end intervals $(-\infty, m_1)$ and (m_4, ∞).

To establish the asymptotic behavior of assertion 5°, note first that, from [1], in particular the proof of Theorem 3.1, $J(U) < \infty$ and $v \leq U \leq \overline{w}$ implies there is a $\phi \in \mathcal{M}_0$ with $v \leq \phi \leq \overline{w}$ such that $||U - \phi||_{L^2[i,i+1]} \to 0$ as $i \to \infty$.

We claim $\phi = v$ which gives the first part of $5°$ and the second part follows in a similar way. To show that $\phi = v$, suppose not. Then $\phi = w$, for otherwise due to the constraint at m_1, $U(s) = w(s)$ for some smallest $s < m_1$ and $U(t) > w(t)$ for $t < s$ via $2°$. Set $\overline{U}(t) = \min(U(t), w(t)),\ t \le m_1$ *and* $\overline{U}(t) = U(t),\ t \ge m_1$. Then $\overline{U} \in \Gamma_{m,\rho}$ and by the choice of s, $\overline{U}(t) = w(t)$ for $t \le s$. Since

$$\sum_{-\infty}^{m_1-1} a_p(U) = \sum_{-\infty}^{m_1-1} a_p(\max(U,w)) + \sum_{-\infty}^{m_1-1} a_p(\overline{U}) \ge \sum_{-\infty}^{m_1-1} a_p(\overline{U}), \tag{2.10}$$

it follows that $J(U) \ge J(\overline{U})$. Thus, $\overline{U} \in \mathcal{M}(\Gamma_{m,\rho})$ and $\overline{U}(\xi) = U(\xi)$ for $t \le m_1$ and in particular $\overline{U}(t) = w(t)$ for $t \le m_1$. Thus, $\phi = w$.

Next, to prove that U asymptotes to w as $t \to \infty$ is not possible, set $\hat{U}(t) = U(t)$ for $t \le m_1,\ = v(t),\ t \ge m_1 + 1$, and choose $\hat{U}$ to minimize $a_{m_1}(u)$ over the class of $W^{1,2}[m_1, m_1{+}1]$ functions satisfying $u(m_1) = U(m_1)$ and $u(m_1 + 1) = v(m_1 + 1)$. Then there is a function $\gamma(s)$ with $\gamma(s) \to 0$ as $s \to 0$ such that

$$a_{m_1}(\hat{U}) \le \gamma(\rho_1). \tag{2.11}$$

Moreover, since $\hat{U} \in \Gamma_1(w, v)$ (where $\Gamma_1(w, v)$ is defined in the natural way), and (V2) holds,

$$J(\hat{U}) \ge c_1(w, v) = c_1(v, w). \tag{2.12}$$

Therefore,

$$\sum_{-\infty}^{m_1-1} a_p(U) = \sum_{-\infty}^{m_1} a_p(\hat{U}) - a_{m_1}(\hat{U}) \ge c_1(v, w) - \gamma(\rho_1). \tag{2.13}$$

Set $\Phi(t) = v(t),\ t \le m_1 - 1, \Phi(t) = U(t),\ t \ge m_1$. Further, define Φ in $[m_1 - 1, m_1]$ so that it minimizes $a_{m_1-1}(u)$ over the class of $W^{1,2}[m_1 - 1, m_1]$ functions such that $u(m_1 - 1) = v(m_1 - 1)$ and $u(m_1) = U(m_1)$. Hence, as for (2.11),

$$a_{m_1}(\Phi) \le \gamma(\rho_1). \tag{2.14}$$

Moreover, since $\Phi \in \Gamma_{m,\rho}$,

$$\sum_{-\infty}^{m_1-1} a_p(\Phi) \ge \sum_{-\infty}^{m_1-1} a_p(U). \tag{2.15}$$

Combining (2.13) - (2.15) yields

$$2\gamma(\rho_1) \ge c_1(v, w) > 0, \tag{2.16}$$

which is not possible for small ρ.

The next step in our proof is to establish that, for $m_{i+1} - m_i$ large, U is also a solution of (1.1) at the points m_i. To do so, a variant of arguments from [1]-[2] will be employed. Set

$$\Lambda_1(v,w) = \{u \in \Gamma_1(v,w) : u(0) = v(0) + \rho_1 \textit{ or } u(0) = w(0) - \rho_2\},$$

and set

$$d_1(v,w) = \inf_{u \in \Lambda_1} J(u).$$

Then by Proposition 9 of [1], $d_1(v,w)$ depends on ρ_1 and ρ_2 and satisfies

$$d_1(v,w) > c_1(v,w). \tag{2.17}$$

Set $\delta = d_1(v,w) - c_1(v,w)$ so $\delta > 0$. If U satisfies the constraints at m_1 and m_2 with a strict inequality, then classical regularity arguments as in [1] show U satisfies (1.1) at these points. Thus suppose that there is equality at one of m_1 or m_2. Set $f_1 = \min(U,w)$, $f_2 = \max(U,w)$, $g_1 = \min(f_2, \overline{v})$, and $g_2 = \max(f_2, \overline{v})$. Then $f_1 \in \Lambda_1(v,w)$, $g_2 \in \Gamma_1(\overline{v}, \overline{w})$ and, by (2.17),

$$\begin{aligned} J(U) = J(f_1) + J(f_2) &= J(f_1) + J(g_1) + J(g_2) \\ &\geq d_1(v,w) + J(g_1) + c_1(\overline{v}, \overline{w}). \end{aligned} \tag{2.18}$$

Set

$$X_1 = \{u \in \Gamma(w, \overline{v}) : u = w \text{ for } t \text{ near } -\infty \text{ and } u = \overline{v} \text{ for } t \text{ near } \infty\}.$$

Observe that $g_1 \in X_1$. If $w = \overline{v}$, the argument that follows simplifies in an obvious way. Define

$$b_1 = \inf_{u \in X_1} J(u).$$

Then (2.18) implies

$$J(U) \geq d_1(v,w) + b_1 + c_1(\overline{v}, \overline{w}). \tag{2.19}$$

Let $v_1 \in \mathcal{M}_1(v,w)$. For $m_2 - m_1$ sufficiently large, there is a $j \in \mathbb{Z}$ such that $v_1(t+j)$ satisfies the constraints at m_1 and m_2 and

$$\inf_A a_{m_2}(u) \leq \delta/6, \tag{2.20}$$

where

$$A = \{u \in W^{1,2}[m_2, m_2{+}1] : u(m_2) = v_1(m_2{+}1) \textit{ and } u(m_2{+}1) = w(m_2{+}1)\}.$$

Note that 5° of Theorem 1.7 implies

$$\sum_{m_2}^{\infty} a_i(v_1(t+j)) \leq \delta/6. \tag{2.21}$$

Similarly let $\overline{v}_1 \in \mathcal{M}_1(\overline{v}, \overline{w})$. Then, for $m_4 - m_3$ sufficiently large, there is a $k \in \mathbb{Z}$ such that $\overline{v}_1(t+k)$ satisfies the constraints at m_3 and m_4 and

$$\inf_{\hat{A}} a_{m_3-1}(u) \leq \delta/6, \tag{2.22}$$

where

$\hat{A} = \{u \in W^{1,2}[m_3-1, m_3] : u(m_3-1) = \overline{v}(m_3-1) \text{ and } u(m_3) = \overline{v}_1(m_3+k)\}.$

As in (2.21), 5° of Theorem 1.7 shows

$$\sum_{-\infty}^{m_3} a_i(\overline{v}_1(t+k)) \leq \delta/6. \tag{2.23}$$

Finally, choose $\psi \in X_1$ so that

$$J(\psi) \leq b_1 + \delta/6. \tag{2.24}$$

By the definition of X_1, there is an $l \in \mathbb{N}$ such that $\psi = w$ for $t \leq -l$ and $\psi = \overline{v}$ for $t \geq l$. Choose $m_3 - m_2 \geq 2l + 1$. Then, for an appropriate $p \in \mathbb{Z}$, we can construct $U^* \in \Gamma_{m,\rho}$ by gluing $v_1(t+j)$ to $\psi(t+p)$ to $\overline{v}_1(t+k)$ and by (2.20), (2.22), and (2.24),

$$J(U^*) \leq c_1(v, w) + b_1 + c_1(\overline{v}, \overline{w}) + \delta/2. \tag{2.25}$$

But (2.25) is contrary to (2.19) and the definition of δ. Thus, there cannot be equality at $t = m_1, m_2$ and similarly at $t = m_3, m_4$. Thus, 6° is proved.

Now that we know the members of $\mathcal{M}(\Gamma_{m,\rho})$ are solutions of (1.1), the argument of 4° shows $\mathcal{M}(\Gamma_{m,\rho})$ is ordered, giving 7°.

Lastly, we prove that U is strictly 1-monotone in t. First, we claim this is true for $t \leq m_1 - 1$. Otherwise, there is an $a \leq m_1 - 1$ such that $U(a) \geq U(a+1)$. Due to 5°, there is a $\sigma \leq a$ such that $U(\sigma) = U(\sigma + 1)$. Set $W(t) = U(t-1),\ t < \sigma + 1$, and $W(t) = U(t),\ t \geq \sigma + 1$. Then $W \in \Gamma_{m,\rho}$ and

$$J(W) - J(U) = -\int_{\sigma}^{\sigma+1} (L(U) - c)dt < 0, \tag{2.26}$$

contrary to 2°. Next, suppose there is an $a \in (m_1 - 1, m_1]$ such that $U(a) \geq U(a+1)$. Then as above there is a $\sigma \leq a$ with $U(\sigma) = U(\sigma+1)$ with $\sigma > m_1 - 1$. Define $W(t)$ as earlier. Note that, by what has already been shown,

$$W(m_1) = U(m_1 - 1) < U(m_1) < v(m_1) + \rho_1, \tag{2.27}$$

so $W \in \Gamma_{m,\rho}$ and, as for the previous case, $J(W) < J(U)$, again contradicting 2°. Therefore, U is strictly 1-monotone for $t \leq m_1$ and similarly for $t \geq m_4$.

In fact, since $U(m_1-1)<U(m_1)$, the last argument extends to show U is strictly 1-monotone for $t \le m_2-1$ and $t \ge m_3+1$.

It remains to treat $t \in (m_2-1, m_3+1)$. Observe that, by 5°, there is a $t^* \in \mathbb{R}$ such that $U(t^*) = w(t^*)$. Since $|\rho|$ is small, the constraint at m_4 implies $t < m_4$. Moreover, t^* is unique, for otherwise there is a pair $s^* < t^*$ such that $U(s^*) = w(s^*), U(t^*) = w(t^*)$. But then, by Theorem 1.2,

$$\int_{s^*}^{t^*} L(U)dt > \int_{s^*}^{t^*} L(w)dt, \tag{2.28}$$

and replacing $U|_{s^*}^{t^*}$ by $w|_{s^*}^{t^*}$ produces a function $U^* \in \Gamma_{m,\rho}$ such that $J(U^*) < J(U)$, contrary to $U \in \mathcal{M}(\Gamma_{m,\rho})$. It follows that $U(t) < w(t)$ for $t < t^*$ and $U(t) > w(t)$ for $t > t^*$. Now, suppose that $U(s) = U(s+1)$ for some $s \in [m_2, m_3]$. By the properties of t^*, $t^* < s$ or $t^* > s+1$. If, e.g., $t > s+1$, define a new function $U^* \in \Gamma_{m,\rho}$ by gluing $U|_{-\infty}^{s}$ to $U|_{s+1}^{t^*}$ (shifted 1 unit to the left) glued to $w|_{t^*-1}^{t^*}$ glued to $U|_{t^*}^{\infty}$. Then, as in (2.26), $J(U^*) < J(U)$, a contradiction. Likewise, if $t^* < s$, create U^* by gluing $U|_{-\infty}^{t^*}$ to $w|_{t^*}^{t^*+1}$ to $U|_{t^*}^{s}$ (shifted 1 unit to the right) to $U|_{s+1}^{\infty}$ to obtain a similar contradiction. Thus, U is strictly 1-monotone for $t \in [m_2, m_3]$ and as earlier this argument extends to $t \in [m_2-1, m_2) \cup [m_3, m_3+1)$.

The proof of Theorem 2.3 is complete.

Remark 2.29. The arguments used in the proof of Theorem 2.3 work equally well if we have a finite number of gap pairs, say $j+1$ in $\mathcal{M}_0$: $v_0.w_0, ..., v_j, w_j$ and seek a solution of (1.1) which shadows members of $\mathcal{M}_1(v_0,w_0), ..., \mathcal{M}_1(v_j,w_j)$, provided that the sets $\mathcal{M}_1(v_0,w_0), ..., \mathcal{M}_1(v_j,w_j)$ have gaps. However, to construct solutions which have an infinite number of transitions requires more work. See [2] for results of that type.

REFERENCES

[1] P.H. Rabinowitz, *The calculus of variations and the forced pendulum, Hamiltonian dynamical systems and applications,* (W. Craig, editor), Proceedings of the Advanced Study Institute on Hamiltonian Dynamical Systems and Applications, NATO Science for Peace and Security Series B: Springer - Verlag, (2008) XVI, pages 367–390.

[2] P.H. Rabinowitz and E. Stredulinsky, in progress.

[3] Jean Mawhin, *Global results for the forced pendulum equation,* Handbook of differential equations, Elsevier/North-Holland, Amsterdam, (2004) pages 533–589.

[4] Jean Mawhin, *The forced pendulum: a paradigm for nonlinear analysis and dynamical systems,* Exposition Math., 6 (1988), 271–287.