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ON POSITIVE SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

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To Jean and Patrick

Abstract. In 1981, Peter Hess established a multiplicity result for solutions of boundary-value problems for nonlinear perturbations of the Laplace operator. The sufficient conditions given were later shown to be also necessary by Dancer and the second author. In this paper, we show that similar (and slightly more general) results hold when the Laplace operator is replaced by the $p-$Laplacian. Some applications to singular problems are given, as well.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p \in (1,\infty)$, and E be the usual Sobolev space $W_0^{1,p}(\Omega)$ with norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{1}{p}}.$$

Let f be a continuous function on $\mathbb{R}$ and assume throughout that f satisfies

($\mathcal{F}$) $f(0) \geq 0$, and there exist $0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_{m-1} < a_m$ such that for all $k = 1, \cdots, m-1$

$$\begin{cases} f(\cdot) \leq 0 & \text{on } (a_k, b_k) \\ f(\cdot) \geq 0 & \text{on } (b_k, a_{k+1}). \end{cases}$$

(See e.g. figure 1)

Motivated by the results in [1] (for the case $p = 2$), we establish sufficient conditions in order that the problem

$$\begin{cases} -\Delta_p u &= \lambda f(u) & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

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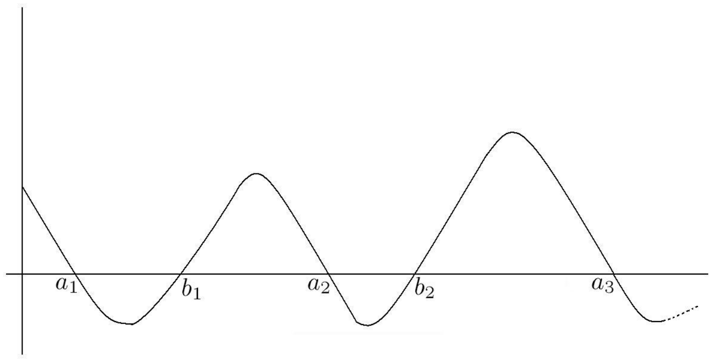

FIGURE 1

has, for $\lambda >> 1$, at least $m-1$ nonnegative weak solutions

$$\{u_1, \cdots, u_{m-1}\} \subset E \cap L^\infty(\Omega),$$

such that

$$a_k < \|u_k\|_\infty \leq a_{k+1},\ k = 1, \cdots, m-1.$$

The following are the results to be established in the paper.

Theorem 1.1. *Assume that the function f satisfies*

$$\int_{a_k}^{a_{k+1}} f(s)ds > 0 \text{ for all } k \in \{1, \cdots, m-1\}; \tag{1.2}$$

then, for all λ sufficiently large, (1.1) *has at least $m-1$ nonnegative solutions $\{u_1, \cdots, u_{m-1}\} \subset E \cap L^\infty(\Omega)$ such that $a_k < \|u_k\|_\infty \leq a_{k+1}$ for each $k = 1, \cdots, m-1$.*

We further have necessary conditions given by the following result.

Theorem 1.2. *Assume that*

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

has a nonnegative weak solution u such that $\|u\|_\infty \in (a_k, a_{k+1}]$, for some $k \in \{1, \cdots, m-1\}$, then for such k

$$\int_{a_k}^{a_{k+1}} f(s)ds > 0.$$

The proofs of these theorems will be given in the next two sections and follow ideas used in of the proofs in the papers [4] and [1], which have been suitably modified and expanded for the case being considered. In the final section, we provide some remarks and discuss applications to problems whose nonlinear terms may become singular at 0.

2. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. The proof follows Hess' [4] arguments very closely and we give it here only for the sake of being complete.

We first need a lemma which is a consequence of the weak maximum principle for the $p-$Laplace operator. This lemma is well known, but we give a short proof, nevertheless.

Lemma 2.1. *Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function such that there exists $s_0\geq 0$ such that $g(s)\geq 0$ if $s\in(-\infty,0)$ and $g(s)\leq 0$ if $s\geq s_0$. If u is a weak solution of*

$$-\triangle_p u = g(u) \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega, \tag{2.1}$$

then u is nonnegative almost everywhere and belongs to $L^\infty(\Omega)$. Moreover, $\|u\|_\infty\leq s_0$.

Proof. We let $v=u^-=\max\{-u,0\}\in E$, then

$$\nabla v=\begin{cases} -\nabla u & u<0\\ 0 & u\geq 0.\end{cases}$$

Hence, since u is a weak solution of (2.1), we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx=\int_\Omega g(u)v dx.$$

This implies $\|v\|\leq 0$ and therefore $v=0$, and $u\geq 0$ almost everywhere on Ω. Next, choosing the test function $v=(u-s_0)^+=\max\{u-s_0,0\}\in E$ in the equation

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx=\int_\Omega g(u)v dx,$$

we have $\|v\|\leq 0$ and therefore $u\leq s_0$ almost everywhere; i.e., $\|u\|_\infty\leq s_0$. □

For $k = 2, \cdots, m$, define f_k as follows:

$$f_k(s) := \begin{cases} f(0) & s \le 0 \\ f(s) & 0 \le s \le a_k \\ 0 & s > a_k, \end{cases}$$

and let

$$F_k(s) = \int_0^s f_k(\sigma) d\sigma.$$

For any $\lambda \ge 0$, let the functional $\Phi_k(\lambda, \cdot) : E \to \mathbb{R}$ be defined by

$$\Phi_k(\lambda, u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \lambda \int_\Omega F_k(u) dx,$$

and denote by $K_k(\lambda)$ the set of critical points of $\Phi_k(\lambda, \cdot)$. Then, if u is in $K_k(\lambda)$, u is a weak solution of

$$\begin{cases} -\triangle_p u = \lambda f_k(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and by Lemma 2.1, u is nonnegative and $\|u\|_\infty \le a_k$. We have thus shown the following.

Lemma 2.2. *u is in $K_k(\lambda)$ if and only if u is a nonnegative weak solution of (1.1) and belongs to $L^\infty(\Omega)$ with $\|u\|_\infty \le a_k$.*

We next claim that $K_k(\lambda)$ is not empty. Since f_k is bounded and vanishes on (a_k, ∞), $\Phi_k(\lambda, \cdot)$ is coercive. Further, since the first summand defining $\Phi_k(\lambda, \cdot)$ is $\frac{1}{p}\| \cdot \|^p$, it is continuous and since it is a convex functional it is weakly lower semicontinuous. The second summand is weakly continuous, as follows from the compact embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$. Thus, there exists $u_k(\lambda)$ such that

$$\Phi_k(\lambda, u_k(\lambda)) = \inf\{\Phi_k(\lambda, v) : v \in E\}.$$

The following lemma shows that for $k = 2, \cdots, m$, $a_{k-1} < \|u_k\|_\infty \le a_k$ and therefore, (1.1) has at least $m - 1$ solutions when $\lambda > 0$ is sufficiently large.

Lemma 2.3. *For each $k = 2, \cdots, m$, there exists $\lambda_k > 0$, such that for all $\lambda > \lambda_k$, $u_k(\lambda) \notin K_{k-1}(\lambda)$.*

Proof. We show that there exist $\lambda_k > 0$ and $w \in E$, $w \ge 0$, and $\|w\|_\infty \le a_k$, such that

$$\Phi_k(\lambda, w) < \Phi_{k-1}(\lambda, u_{k-1}(\lambda)), \qquad \forall \lambda > \lambda_k.$$

This will imply the assertion.

Since f satisfies (1.2),

$$0 < \alpha := F(a_k) - \max\{F(s) : 0 \le s < a_{k-1}\},$$

where $F(s) = \int_0^s f(\sigma)d\sigma$. Then for all u in E satisfying $0 \le u \le a_{k-1}$ almost everywhere,

$$\int_\Omega F(a_k)dx \ge \int_\Omega F(u)dx + \alpha|\Omega|,$$

where $|\Omega|$ is the Lebesgue measure of Ω.

For $\delta > 0$, let $\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$. By Lebesgue's theorem, the measure $|\Omega_\delta| \to 0$ as $\delta \to 0$. On the other hand, for each $\delta > 0$, there exists $w_\delta \in C_0^\infty(\Omega)$ with $0 \le w_\delta \le a_k$, $w_\delta(x) = a_k$ for all $x \in \Omega \setminus \Omega_\delta$. Thus,

$$\begin{aligned}\int_\Omega F(w_\delta)dx &= \int_{\Omega\setminus\Omega_\delta} F(a_k)dx - \int_{\Omega_\delta} F(w_\delta)dx \\ &= \int_\Omega F(a_k)dx - \int_{\Omega_\delta} (F(a_k) + F(w_\delta))dx \ge \int_\Omega F(a_k)dx - 2C|\Omega_\delta|,\end{aligned}$$

where $C = \max\{|F(s)| : 0 \le s \le a_k\}$. Hence, for all u in E, $0 \le u \le a_{k-1}$,

$$\int_\Omega F(w_\delta)dx \ge \int_\Omega F(u)dx + \alpha|\Omega| - 2C|\Omega_\delta|.$$

Fix $\delta > 0$ such that $\eta := \alpha|\Omega| - 2C|\Omega_\delta| > 0$ and set $w := w_\delta$. Then, for all u, $0 \le u \le a_{k-1}$,

$$\begin{aligned}\Phi_k(\lambda, w) - \Phi_{k-1}(\lambda, u) &= \frac{1}{p}(\|w\|^p - \|u\|^p) - \lambda \int_\Omega (F(w) - F(u))dx \\ &\le \frac{1}{p}\|w\|^p - \lambda\eta < 0,\end{aligned}$$

provided $\lambda > 0$ is chosen sufficiently large. □

Thus, for all λ large enough, there are $m-1$ solutions $u_1(\lambda), \cdots, u_{m-1}(\lambda)$ as asserted.

3. Proof of Theorem 1.2

It follows easily that it will suffice to consider the case that $k = 1$. We also assume here that $f(0) > 0$ and remove this condition later. Throughout this section, all weak solutions u of (1.1) are in class $C^1(\Omega)$ because of regularity results in [6]. We first establish a strong maximum principle for weak solutions of equation (1.1). (Note that a similar strong maximum principle (which is not applicable here) was established in [8].)

Lemma 3.1. *Let* $u \in C^1(\Omega)$ *be a nonnegative weak solution of* (1.3). *If* $f(0) > 0$, *then* u *is positive in* Ω.

Proof. Assume there exists x_0 in Ω such that $u(x_0) = 0$. Let D be a ball contained in Ω such that $x_0 \in \partial D$. Denote by y_0 and r the center and the radius of D, respectively, and let $g \leq f$ be a strictly decreasing continuous function defined on $[0, \infty)$ such that $g(0) = f(0) > 0$ and $\gamma := g(\frac{a_1}{2}) = \inf\{g(s) : 0 \leq s \leq \frac{a_1}{2}\} > 0$. Now, define a function b on D as

$$b(x) = \epsilon\big(e^{-|\frac{x-y_0}{r}|^2} - e^{-1}\big),$$

where ϵ is sufficiently small such that

$$\sup_{x \in D} |\mathrm{div}(|\nabla b(x)|^{p-2} \nabla b(x))| \leq \gamma.$$

Then b is a subsolution of

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = g(u) & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases}$$

It follows that, for all $\varphi \geq 0$ in $W_0^{1,p}(D)$,

$$\int_D (|\nabla b|^{p-2}\nabla b - |\nabla u|^{p-2}\nabla u) \cdot \nabla \varphi dx \leq \int_D (g(b) - g(u))\varphi dx.$$

Choosing $\varphi = (b-u)^+$, and using the fact that the $p-$Laplace operator is monotone and g is strictly decreasing, we obtain

$$\begin{aligned} 0 &\leq \int_{D^+} (|\nabla b|^{p-2}\nabla b - |\nabla u|^{p-2}\nabla u) \cdot \nabla(b-u) dx \\ &\leq \int_{D^+} (g(b) - g(u))(b-u) dx \leq 0, \end{aligned}$$

where $D^+ = \{x \in D : b(x) > u(x)\}$. Therefore, D^+ is empty or equivalently $u \geq b$ in D. Since $u(x_0) = b(x_0) = 0$, and $b > 0$ in D we have that the normal derivative with respect to the boundary of D satisfies $\partial_\nu u(x_0) \leq \partial_\nu b(x_0) < 0$, implying that $|\nabla u(x_0)| \neq 0$, which contradicts the fact that $u(x_0) = 0$ is a minimum value of u in Ω. □

Let B be an open ball centered at 0 and containing Ω. Define the function α on Ω by

$$\alpha(x) = \begin{cases} u(x) & x \in \bar{\Omega} \\ 0 & x \in \bar{B} \setminus \Omega. \end{cases}$$

Since Ω is a domain with smooth boundary, $\alpha \in W_0^{1,p}(B)$. We also have the following.

Lemma 3.2. *α is a subsolution of*

$$\begin{cases} -\Delta_p u = f(u) & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases} \tag{3.1}$$

Proof. For each n in $\mathbb{N}$, define $v_n(x) = n\min\{u(x), \frac{1}{n}\},\ x \in \Omega$, where u is as in Lemma 3.1. Then $\nabla u \cdot \nabla v_n$ is nonnegative and, by Lemma 3.1, $\{v_n\}$ converges to 1 pointwise in Ω. Let $w \geq 0$ be in $C_0^\infty(B)$. Then, wv_n is in $W_0^{1,p}(\Omega)$ and since u is a weak solution of (1.1),

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(wv_n)dx = \int_\Omega f(u)wv_n dx,$$

or

$$\int_\Omega w|\nabla u|^{p-2}\nabla u \cdot \nabla v_n dx + \int_\Omega v_n|\nabla u|^{p-2}\nabla u \cdot \nabla w dx = \int_\Omega f(u)wv_n dx.$$

Now, applying Lebesgue's dominated convergence theorem and noting that $0 \leq v_n \leq 1$ for all n, we have

$$\begin{aligned} &\int_B |\nabla \alpha|^{p-2}\nabla\alpha \cdot \nabla w dx = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w dx \\ &= \lim_{n\to\infty}\int_\Omega v_n|\nabla u|^{p-2}\nabla u \cdot \nabla w dx \\ &= \lim_{n\to\infty}\int_\Omega \left(f(u)wv_n - w|\nabla u|^{p-2}\nabla u \cdot \nabla v_n\right) dx \\ &\leq \int_\Omega f(u)w dx \leq \int_B f(\alpha)w dx, \end{aligned}$$

proving the lemma. □

The following special case may be used to demonstrate Theorem 1.2, which is an extension of a result of [1].

Theorem 3.1. *Assume that $f(0)$ is positive. If (1.1) has a nonnegative weak solution u in $L^\infty(\Omega)$ such that $\|u\|_\infty \in (a_1, a_2]$, then*

$$\int_{a_1}^{a_2} f(s)ds > 0.$$

We remark again that the condition $f(0) > 0$ will be removed later.

Proof. For the given Ω we choose an open ball B as above and for the given solution u we define the function α on $\bar{B}$ as before. Further define $\beta(x) = a_2$ for all x in B. Since β and α are a supersolution and subsolution, respectively, see Lemma 3.2, (3.1) has a maximum solution $\bar{u}$ such that

$\alpha(x) \le \bar{u}(x) \le a_2$ for all x in B (as we shall see below, it is important here that $\bar{u}$ is a maximum weak solution, whose existence is guaranteed by Remark 1.5 in [5]; earlier sub-supersolution results for the $p-$Laplacian, like those in [2], do not guarantee the existence of such maximal solutions). This means, for all solution v of (3.1) with $\alpha(x) \le v(x) \le a_2$, $v \le \bar{u}$. We claim that this implies that $\bar{u}$ is radially symmetric; i.e., $\bar{u}(x_1) = \bar{u}(x_2)$ for all x_1, x_2 in B such that $|x_1| = |x_2|$. Assuming this is not the case, there exist x_1 and x_2 in B with $|x_1| = |x_2|$ such that $\bar{u}(x_1) < \bar{u}(x_2)$. Let P be an $N \times N$ matrix in $SO(N, \mathbb{R})$, the special orthogonal group, such that $x_2 = Px_1$. Note that the transpose matrix P^T of P is also its inverse matrix. Let $u_1(x) = \bar{u}(Px)$. Since for all x in Ω

$$\nabla u_1(x) = P\nabla\bar{u}(Px),$$

and the map $x \mapsto Px$ is an isometry, it follows that

$$|\nabla u_1(x)| = |P\nabla\bar{u}(Px)| = |\nabla\bar{u}(Px)|.$$

We next show that u_1 is a weak solution of (3.1). That is, we need to verify for all φ in $W_0^{1,p}(B)$

$$\int_B |\nabla u_1(x)|^{p-2}\nabla u_1(x) \cdot \nabla\varphi(x)dx = \int_B f(u_1(x))\varphi(x)dx. \tag{3.2}$$

Let $\psi(x) = \varphi(P^T x) \in W_0^{1,p}(B)$. The left-hand side of (3.2) becomes

$$\begin{aligned}
&\int_B |\nabla\bar{u}(Px)|^{p-2}P\nabla\bar{u}(Px) \cdot \nabla\varphi(x)dx \\
&= \int_B |\nabla\bar{u}(Px)|^{p-2}P\nabla\bar{u}(Px) \cdot (P\nabla\psi(Px))\, dx \\
&= \int_B |\nabla\bar{u}(y)|^{p-2}\nabla\bar{u}(y) \cdot \nabla\psi(y) \det P dy = \int_B f(\bar{u}(y))\psi(y)dy \\
&= \int_B f(\bar{u}(PP^T y))\psi(PP^T y)dy = \int_B f(\bar{u}(Px))\psi(Px) \det P^T dx \\
&= \int_B f(u_1(x))\varphi(x)dx.
\end{aligned}$$

Hence, (3.2) holds.

Now, (3.1) has two subsolutions, α and u_1. It follows from Theorem 1.4 in [5] that (3.1) has another solution u_2 such that $\max\{\alpha, u_1\} \le u_2 \le \beta$. Since $\bar{u}$ is the maximum solution with respect to the pair of sub-supersolutions (α, β), we have $\bar{u}(x_1) \ge u_2(x_1) \ge u_1(x_1) = \bar{u}(x_2) > \bar{u}(x_1)$. This contradiction shows that $\bar{u}$ must be radially symmetric.

Next, define a C^1-function $u : [0, R) \to \mathbb{R}^+$ by $u(|x|) = \bar{u}(x)$ for all x in B, where R is the radius of B. Using the chain rule for classical differentiation, we have for all r in $(0, R)$

$$\begin{aligned} \frac{\partial \bar{u}}{\partial x_i} &= \frac{du}{dr}\frac{\partial r}{\partial x_i} = u'\frac{x_i}{r}, \ i = 1, ..., N \\ |\nabla \bar{u}| &= |u'|\Big(\sum_{i=1}^{N} \frac{x_i^2}{r^2}\Big)^{\frac{1}{2}} = |u'|. \end{aligned}$$

For any v in $C_0^\infty(0, R)$, put

$$w(r) = \frac{v(r)}{r^{N-1}}, \qquad r \in (0, R), \ w(0) = 0,$$

and

$$\bar{v}(x) = v(|x|), \ \bar{w}(x) = w(|x|), \qquad x \in B.$$

Now, as a weak solution of (3.1), $\bar{u}$ satisfies

$$\int_B |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \bar{w} dx = \int_B f(\bar{u}) \bar{w} dx.$$

But, $\frac{\partial \bar{w}(x)}{\partial x_i} = w' \frac{x_i}{|x|}$, and thus,

$$\int_0^R |u'|^{p-2} u' w' r^{N-1} dr = \int_0^R f(u) w r^{N-1} dr.$$

Substituting $w = \frac{v}{r^{N-1}}$ and $w' = \frac{v'}{r^{N-1}} - \frac{N-1}{r^N} v$ into this equation, we obtain

$$\int_0^R |u'|^{p-2} u' \Big(\frac{v'}{r^{N-1}} - \frac{N-1}{r^N} v\Big) r^{N-1} dr = \int_0^R f(u) \frac{v}{r^{N-1}} r^{N-1} dr,$$

or

$$\int_0^R |u'|^{p-2} u' v' dr - \int_0^R \frac{N-1}{r} |u'|^{p-2} u' v dr = \int_0^R f(u) v dr,$$

for all v in $C_0^\infty(0, R)$. This implies that u is a C^1 weak solution of the equation

$$-\partial(|u'|^{p-2} u') = \frac{N-1}{r} |u'|^{p-2} u' + f(u), \tag{3.3}$$

and by the continuity of the right-hand side, the distributional derivative ∂ above becomes a classical derivative and hence u is a classical solution of (3.3).

Since $\bar{u}$ is radially symmetric, $u'(0) = 0$. Hence, u is a solution of (3.3) subject to the condition $u'(0) = 0 = u(R)$. Let $r_0 \in [0, R)$ such that $u_{\max} =$

$u(r_0) = \max\{u(r) : r \in [0, R)\}$. Multiplying both sides of (3.3) by u' and integrating it, we obtain

$$-\Big(\int_{r_0}^{r} (p-1)|u'|^{p-2} u' u'' d\tau + (N-1) \int_{r_0}^{r} \frac{|u'|^p}{\tau} d\tau \Big) = \int_{r_0}^{r} f(u) u' d\tau,$$

for all $0 < r < R$. Now, since $u_{\max} = u(r_0)$ is greater than a_1, we can choose $r \in (0, R)$ such that $u(r) = a_1$. The above equality becomes

$$\int_{u_{\max}}^{a_1} f(s) ds = -(p-1) \int_0^{u'(r)} |s|^{p-2} s ds - (N-1) \int_{r_0}^{r} \frac{|u'|^p}{\tau} d\tau < 0.$$

This equation shows $\int_{a_1}^{u_{\max}} f(s) ds > 0$. Because $f \leq 0$ in $(a_1, b_1]$, $u_{\max} \in (b_1, a_2]$ and f is nonnegative in $[u_{\max}, a_2]$, we get the desired result because

$$\int_{a_1}^{a_2} f(s) ds \geq \int_{s_0}^{u_{\max}} f(s) ds > 0. \qquad \square$$

4. SOME REMARKS

Remark 1. We can remove the condition $f(0) > 0$ in Theorem 3.1. In fact, assume that $f(0) \leq 0$ and again assume that (1.1) has a nonnegative solution u in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying $\|u\|_\infty \in (a_1, a_2]$. Let $\tilde{f}$ be a continuous function so that $\tilde{f}(0) > 0$, $\tilde{f}(s) \geq f(s)$ when $0 \leq s \leq a_1$ and $\tilde{f}(s) = f(s)$ on $[a_1, \infty)$. Then u is a subsolution of

$$\begin{cases} -\triangle_p u = \tilde{f}(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{4.1}$$

and as before, we may use $\beta(x) \equiv a_2$ as a supersolution for (4.1). Hence, (4.1) has a solution $\tilde{u}$ satisfying $u \leq \tilde{u} \leq a_2$. We now proceed as in the first part of the proof with $\tilde{f}$ in place of f and obtain

$$\int_{a_1}^{a_2} f(s) ds = \int_{a_1}^{a_2} \tilde{f}(s) ds > 0.$$

The previous remark and Theorem 3.1 consider the case $0 < a_1 < b_1 < a_2$. Now, we study the case $0 = a_1 < b_1 < a_2$.

Remark 2. Let $a < b$ be two positive numbers and let f be a continuous function on $[0, b]$ such that $f(0) = 0$, $f < 0$ on $(0, a)$, $f > 0$ on (a, b) and $f(a) = f(b) = 0$. Assume the problem

$$\begin{cases} -\triangle_p u = f(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{4.2}$$

has a nonnegative weak solution u in $E \cap L^\infty(\Omega)$ such that $\|u\|_\infty$ is in $(a, b]$, then

$$\int_0^b f(s)ds \geq 0.$$

Proof. Let $\epsilon < a$ be an arbitrarily small positive number. For each n in $\mathbb{N}$, define a continuous function g_n satisfying $g_n(0) = 1$, $g_n > 0$ on $(0, \epsilon)$, $f \leq g_n < 0$ on (ϵ, a), $g_n = f$ on $[a, b]$ and

$$\int_\epsilon^a g_n(s)ds < \int_0^a f(s)ds + \frac{1}{n}$$

(see e.g. figure 2). Since u is a solution of (4.2) and $g_n \geq f$, u is a subsolution

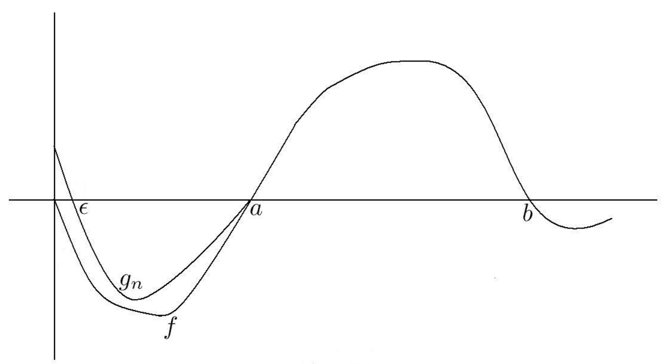

FIGURE 2

of

$$\begin{cases} -\triangle_p u = g_n(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{4.3}$$

Now, (4.3) has a pair of sub-supersolutions (u, b), and, as in the proof of Theorem 3.1, (4.3) has a weak solution u_1 such that $u \leq u_1 \leq b$ almost everywhere in Ω. Using Theorem 3.1 and the assumption on g_n, we have

$$\int_0^b f(s)ds + \frac{1}{n} > \int_\epsilon^b g_n(s)ds > 0,$$

from which the assertion follows. □

Let us next consider the problem (1.3) under the assumption that $f(0)$ is $-\infty$. Such problems have been studied extensively in recent years, see, e.g. [3, 7, 9]. To give a particular example, we show that one such result, Theorem 1.1 in [3], may be deduced from Theorem 3.1 and Remark 2 when all functions in the problem in [3] are independent of x. Of course, our result is somewhat more general than Theorem 1.1 in [3] since we are considering a problem of $p-$Laplacian type and also remove some smoothness and growth conditions required in that paper. This idea will be given in Remark 3.

Let g be continuous on $(0,\infty)$ and $\lim_{s\to 0^+} g(s)=\infty$. Let $h:[0,\infty)\to[0,\infty)$ be continuous. Assume there exists $s_0>0$ such that $f=-g+h$ is negative on $(0,s_0)$ and positive on (s_0,∞). The following holds.

Remark 3. If $\int_0^{s_0} g(s)ds=\infty$, then the problem

$$\begin{cases} -\Delta_p u + g(u) &= h(u) \quad \text{in } \Omega,\\ u &> 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{4.4}$$

has no nonnegative weak solution in $L^\infty(\Omega)$.

Proof. Assume that u is a nonnegative weak solution of (4.4) such that $M=\|u\|_\infty<\infty$. It follows that $M>s_0$. For, if $M\le s_0$, then

$$0\le\int_\Omega |\nabla u|^p dx=\int_\Omega(-g(u)+h(u))udx\le 0,$$

and, thus, $u=0$.

Define a continuous function f on $(0,\infty)$ and a sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ on $[0,\infty)$ such that

$$f(s)=\begin{cases} -g(s)+h(s) & s\in(0,M),\\ \ge 0 & s\in(M,M+1),\\ 0 & s>M+1,\end{cases}$$

and

$$\begin{cases} f_n(0)=0, & \\ f(s)\le f_n(s)\le 0 & s\in(0,\frac{1}{n}],\\ f_n(s)=f(s) & s\in(\frac{1}{n},\infty).\end{cases}$$

(See e.g. figure 3.)

Since $u\le M$ almost everywhere in Ω and $f_n\ge f$ for all $n\in\mathbb{N}$, u is a subsolution of

$$\begin{cases} -\Delta_p u &= f_n(u) \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

for all $n\in\mathbb{N}$. Applying Theorem 1.4 in [5], we deduce that (4.5) has a weak solution u_n such that $u\le u_n\le M+1$ almost everywhere in Ω. Now,

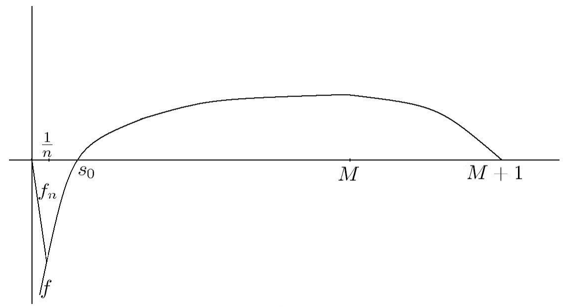

Figure 3

Remark 2 implies that

$$\int_0^{M+1} f_n(s)ds \geq 0,$$

and

$$\int_0^{s_0} -f_n(s)ds \leq \int_{s_0}^{M+1} f_n(s)ds = \int_{s_0}^{M+1} f(s)ds,$$

for all $n \in \mathbb{N}$.

On the other hand, $-f_n(s)$ converges to $-f(s) = g(s) - h(s)$ for every s in $(0, s_0)$ as n approaches ∞. Noting that $-f_n \geq 0$ on $(0, s_0)$, we can use Fatou's lemma to get

$$\int_0^{s_0} (g(s) - h(s))ds \leq \liminf_{n\to\infty} \int_0^{s_0} -f_n(s)ds \leq C < \infty,$$

where $C = \int_{s_0}^{M+1} f(s)ds$ is a constant. It follows that $\int_0^{s_0} g(s)ds < \infty$ which contradicts the hypothesis. □

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