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MAXIMUM AND ANTIMAXIMUM PRINCIPLES: BEYOND THE FIRST EIGENVALUE

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(A Jean et à Patrick, avec toute notre amitié)

Abstract. We consider the Dirichlet problem (*)$- \triangle u = \mu u + f$ in $\Omega, u = 0$ on $\partial\Omega$. Let $\widehat{\lambda}$ be an eigenvalue, with $\widehat{\varphi}$ an associated eigenfunction. Under suitable assumptions on f and on the nodal domains of $\widehat{\varphi}$, we show that, if μ is sufficiently close to $\widehat{\lambda}$, then the solution u of (*) has the same number of nodal domains as $\widehat{\varphi}$, and moreover the nodal domains of u appear as small deformations of those of $\widehat{\varphi}$.

1. Introduction

Let us consider the problem

$$-\triangle u = \mu u + f \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

with Ω a smooth bounded domain in $\mathbb{R}^N$ and $f \in L^q(\Omega)$, $q > N$. Let $\lambda_1 > 0$ denote the first eigenvalue of $-\triangle$ on $H_0^1(\Omega)$, with φ_1 an associated eigenfunction, $\varphi_1 > 0$ in Ω. Assume first $f \geqslant 0, f \not\equiv 0$. Then it is well known that the solution u of (1.1) is > 0 in Ω if $\mu < \lambda_1$ (maximum principle) and is < 0 in Ω if $\mu > \lambda_1$ with μ sufficiently close to λ_1 (antimaximum principle, cf. [5]). A conclusion of a similar nature still holds under the weaker assumption $\int_\Omega f\varphi_1 > 0 : u$ is > 0 in Ω if $\mu < \lambda_1$ with μ sufficiently close to λ_1 and u is < 0 in Ω if $\mu > \lambda_1$ with μ sufficiently close to λ_1 (cf. [3]).

These results exhibit a change of sign of the solution u when the parameter μ crosses λ_1. They also show that the number of nodal domains of u remains

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equal to that of φ_1 (here 1!) when μ varies close to λ_1. It is our purpose in this paper to show that the same type of conclusion still holds near a higher eigenvalue.

More precisely let $\widehat{\lambda}$ be a higher eigenvalue, with $\widehat{\varphi}$ an associated eigenfunction. Under suitable assumptions on f and on the nodal domains of $\widehat{\varphi}$, we show that, if μ is sufficiently close to $\widehat{\lambda}$, then the solution u of (1.1) has the same number of nodal domains as $\widehat{\varphi}$ and that the nodal domains of u appear as small deformations of those of $\widehat{\varphi}$; moreover if $\widehat{\varphi}$ is positive in one of its nodal domains, then u is positive (respectively negative) in the corresponding perturbed nodal domain when $\mu < \widehat{\lambda}$ (respectively $\mu > \widehat{\lambda}$), and similarly for a negative nodal domain of $\widehat{\varphi}$. The general idea of the proof is rather simple, although its implementation requires some technicalities. By using a Fourier expansion of f and standard regularity theory, one can write u as $K(\mu)\widehat{\varphi} + u^*(\mu)$, where $K(\mu) \to +\infty$ (respectively $-\infty$) as $\mu \to \widehat{\lambda}$ with $\mu < \widehat{\lambda}$ (respectively $\mu > \widehat{\lambda}$) and where $u^*(\mu)$ remains bounded in $\mathcal{C}^1(\bar{\Omega})$. The dominating term is thus $K(\mu)\widehat{\varphi}$, and it is this term which imposes its nodal structure to the solution u.

Our results are rather satisfactory in dimension $N = 1$ (cf. Theorems 2.1 and 2.2). In higher dimension one has to face two difficulties, even in the simple example of a disk in $\mathbb{R}^2$: the presence of multiple eigenvalues and the lack of regularity of the nodal domains of the eigenfunctions. To deal with multiplicity, we will impose a mild restriction on the Fourier expansion of f (cf. (H_3) in section 3). As far as the nodal domains are concerned, we will require their regularity (cf. (H_1) in section 3). This is a more restrictive assumption and we will comment on it in Examples 3.4 and 3.6.

2. DIMENSION $N = 1$

This section is concerned with the problem

$$-u'' = \mu u + f \text{ in } (0,1), \quad u(0) = u(1) = 0. \tag{2.1}$$

The eigenvalues of $-u''$ in $H_0^1((0,1))$ are $\lambda_k = k^2\pi^2, k = 1, 2, \ldots$; they are simple and the associated eigenfunctions are $\varphi_k(x) = \sin(k\pi x)$.

We first consider the case where μ stays near λ_2 in the sense that $\lambda_1 < \mu < \lambda_2$ or $\lambda_2 < \mu < \lambda_3$. Given $f \in L^2((0,1))$, we write

$$f = f_1 + f_2 + f_3, \tag{2.2}$$

where $f_1 = \alpha_1\varphi_1, f_2 = \alpha_2\varphi_2$ and f_3 is orthogonal to φ_1 and φ_2. Let $u = u(f, \mu)$ be the (unique) solution of (2.1).

Theorem 2.1. *Write f as in (2.2) and assume $\alpha_2 > 0$. Then there exists $\delta = \delta(f) > 0$ with the property that*

(i) *if $\lambda_2 - \delta < \mu < \lambda_2$, then there exists $x_1 \in (0,1)$ such that $u > 0$ on $(0, x_1)$, $u(x_1) = 0$ and $u < 0$ on $(x_1, 1)$;*
(ii) *if $\lambda_2 < \mu < \lambda_2 + \delta$, then there exists $x_1 \in (0,1)$ such that $u < 0$ on $(0, x_1)$, $u(x_1) = 0$ and $u > 0$ on $(x_1, 1)$.*

We, thus, see that for μ close to λ_2, the solution u presents the same type of oscillation as φ_2.

Proof of Theorem 2.1. The solution u can be written as $u = u_1 + u_2 + u_3$ where

$$u_1 = \frac{\alpha_1}{\lambda_1 - \mu}\varphi_1, \quad u_2 = \frac{\alpha_2}{\lambda_2 - \mu}\varphi_2,$$

and u_3 solves

$$-u_3'' = \mu u_3 + f_3 \text{ in } (0,1),\ u_3(0) = u_3(1) = 0. \tag{2.3}$$

By the variational characterization of λ_3, one deduces from (2.3) that

$$\lambda_3 \int_0^1 u_3^2 \leqslant \int_0^1 (u_3')^2 = \mu \int_0^1 u_3^2 + \int_0^1 f_3 u_3.$$

Restricting from now on μ to vary with $(\lambda_1 + \lambda_2)/2 < \mu < (\lambda_2 + \lambda_3)/2$, it follows that

$$\|u_3\|_2 \leqslant \frac{2}{\lambda_3 - \lambda_2}\|f_3\|_2.$$

Using again (2.3) and the Sobolev imbedding theorem, one obtains

$$\|u_3\|_{C^1([0,1])} \leqslant C_1 \|u_3\|_{W^{2,2}((0,1))} \leqslant C_2,$$

where the constants C_1, C_2 are independent of μ. Since a similar estimate clearly also holds for u_1, one finally has

$$||u_1 + u_3||_{C^1([0,1])} \leqslant C_3, \tag{2.4}$$

with C_3 independent of μ.

We now consider case (i) where in addition to the restriction on μ indicated above, one has $\mu < \lambda_2$. Since $\varphi_2'(1/2) = -2\pi$, one has, for some $\eta > 0$, $\varphi_2'(x) \leqslant -\pi$ for $x \in (\frac{1}{2} - \eta, \frac{1}{2} + \eta)$, and so, since $\alpha_2 > 0$,

$$u_2'(x) \leqslant -\frac{\pi\alpha_2}{\lambda_2 - \mu} \text{ for } x \in (\tfrac{1}{2} - \eta, \tfrac{1}{2} + \eta).$$

Using (2.4) in the form $(u_1 + u_3)'(x) \leqslant C_3$, one deduces that if μ is further restricted so as to satisfy

$$\lambda_2 - \frac{\pi \alpha_2}{C_3} < \mu < \lambda_2,$$

then

$$u'(x) < 0 \quad \text{for } x \in (1/2 - \eta, 1/2 + \eta). \tag{2.5}$$

Similarly, one has

$$u'(x) > 0 \quad \text{for } x \in (0, \eta) \cup (1 - \eta, 1). \tag{2.6}$$

On the other hand, there exists $\varepsilon > 0$ such that $\varphi_2(x) \geqslant \varepsilon$ for $x \in [\eta, \frac{1}{2} - \eta]$, and so, since $\alpha_2 > 0$,

$$u_2(x) \geqslant \frac{\varepsilon \alpha_2}{\lambda_2 - \mu} \text{ for } x \in [\eta, \tfrac{1}{2} - \eta].$$

Using (2.4) in the form $(u_1 + u_3)(x) \geqslant -C_3$, one deduces that if μ is further restricted so as to satisfy

$$\lambda_2 - \frac{\varepsilon \alpha_2}{C_3} < \mu < \lambda_2,$$

then

$$u(x) > 0 \text{ for } x \in [\eta, \tfrac{1}{2} - \eta]. \tag{2.7}$$

Similarly one obtains

$$u(x) < 0 \text{ for } x \in [\tfrac{1}{2} + \eta, 1 - \eta]. \tag{2.8}$$

The relations (2.5)-(2.8) imply the existence of a unique $x_1 \in (\frac{1}{2} - \eta, \frac{1}{2} + \eta)$ such that $u > 0$ on $(0, x_1), u(x_1) = 0$ and $u < 0$ on $(x_1, 1)$. The conclusion of case (i) is, thus, proved.

In case (ii), $\lambda_2 - \mu < 0$, and so the sign of $u_2 = \alpha_2/(\lambda_2 - \mu)\varphi_2$ is reversed with respect to that of φ_2. The above argument is easily adapted: it is again u_2 which is dominant when μ comes close to λ_2, because of the uniform estimate (2.4). □

Theorem 2.1 can easily be extended to the case of an eigenvalue λ_k with $k \geqslant 3$. Let μ remain close to λ_k in the sense that $\lambda_{k-1} < \mu < \lambda_k$ or $\lambda_k < \mu < \lambda_{k+1}$, and write $f \in L^2((0,1))$ as

$$f = \sum_{i=1}^{k-1} \alpha_i \varphi_i + \alpha_k \varphi_k + f^{\perp},$$

where $f^{\perp}$ is orthogonal to $\varphi_1, \varphi_2, \ldots, \varphi_k$.

Theorem 2.2. *Write f as above and assume $\alpha_k > 0$. Then there exists $\delta = \delta(f) > 0$ with the property that*

(i) *if $\lambda_k - \delta < \mu < \lambda_k$, then there exist $0 < x_1 < x_2 < \ldots < x_{k-1} < 1$ such that $u > 0$ on $(0, x_1), u(x_1) = 0$, $u < 0$ on $(x_1, x_2), u(x_2) = 0, u > 0$ on $(x_2, x_3), \ldots$;*

(ii) *if $\lambda_k < \mu < \lambda_k + \delta$, then there exist $0 < x_1 < x_2 < \ldots < x_{k-1} < 1$ such that $u < 0$ on $(0, x_1), u(x_1) = 0$, $u > 0$ on $(x_1, x_2), u(x_2) = 0$, $u < 0$ on $(x_2, x_3), \ldots$*

Example 2.3. Let us take $f(x) = \sin(\pi x) + \frac{1}{4}\sin(2\pi x) + \frac{1}{4}\sin(3\pi x)$. One has $f > 0$ on $(0,1)$. For $\lambda_1 < \mu < \lambda_1 + \frac{3}{2}\pi^2$, the solution u of (2.1) satisfies $u < 0$ on $(0,1)$, as expected from the antimaximum principle. For μ close to λ_2 (respectively λ_3), u changes sign once (respectively twice), as a consequence of Theorems 2.1 and 2.2. For μ close to λ_4, u has a constant sign, which does not contradict Theorem 2.2 since the coefficient of φ_4 in the expansion of f is zero. All these assertions can easily be observed numerically using, e.g., Maple.

3. DIMENSION $N \geqslant 2$

We consider in this section problem (1.1), where Ω is a bounded domain in $\mathbb{R}^N$ of class C^2. The eigenvalues of $-\triangle$ on $H_0^1(\Omega)$ are denoted by $(0 <)\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots$, with repetition according to multiplicity, and the associated orthonormal set of eigenfunctions is denoted by $\varphi_1, \varphi_2, \varphi_3, \ldots$.

Let us consider an eigenvalue $\widehat{\lambda} \geqslant \lambda_2$ and an associated eigenfunction $\widehat{\varphi}$. We will assume that $\widehat{\varphi}$ has n nodal domains $\Omega_1, \ldots, \Omega_n$ and that these nodal domains enjoy the following two properties:

(H_1) each Ω_i satisfies at each $x \in \partial\Omega_i$ the interior ball condition;

(H_2) for σ sufficiently small, say $0 < \sigma < \sigma_0$, each $\Omega_{i,\sigma}$ is arcwise connected, where $\Omega_{i,\sigma} := \{x \in \Omega_i : d(x, \partial\Omega_i) > \sigma\}$.

Examples of eigenpairs $(\widehat{\lambda}, \widehat{\varphi})$ satisfying (or not) (H_1) can be found in [6] (see also Remark 3.3 and Examples 3.4 and 3.6 below).

Denote by r the largest integer k such that $\lambda_k < \widehat{\lambda}$ and by s the smallest k such that $\lambda_k > \widehat{\lambda}$. We also introduce $\sigma_1 > 0$ such that

$$\begin{cases} d(\bar{\Omega}_i, \bar{\Omega}_j) \geqslant 4\sigma_1 \text{ if } \bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset, \\ d(\bar{\Omega}_i, \partial\Omega) \geqslant 4\sigma_1 \text{ if } \bar{\Omega}_i \cap \partial\Omega = \emptyset. \end{cases} \tag{3.1}$$

Given $f \in L^q(\Omega)$ with $q > N$, we suppose that f can be written as

(H_3) $f = \sum_{t=1}^{r} \alpha_t \varphi_t + \alpha \widehat{\varphi} + f^{\perp}$, where $f^{\perp}$ is orthogonal to the eigenspaces associated to $\lambda_1, \dots, \lambda_r, \widehat{\lambda}$.

Note that (H_3) is not a restriction on f when $\widehat{\lambda}$ is simple.

We will study the situation where μ stays near $\widehat{\lambda}$ in the sense that $\lambda_r < \mu < \widehat{\lambda}$ or $\widehat{\lambda} < \mu < \lambda_s$. Let $u = u(f, \mu) \in H_0^1(\Omega)$ be the (unique) solution of (1.1), which belongs to $W^{2,q}(\Omega) \subset C^1(\bar{\Omega})$.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Assume (H_1), (H_2), and let f satisfy (H_3) with $\alpha > 0$. Take $\sigma > 0$ with $\sigma < \sigma_0$ and σ_1. Then there exists $\delta = \delta(f, \sigma) > 0$ such that*

- *(i) if $\widehat{\lambda} - \delta < \mu < \widehat{\lambda}$, then u has exactly n nodal domains $\mathcal{O}_1, \dots, \mathcal{O}_n$, with the following three properties:*
  - *(i_1) $\Omega_{i,\sigma} \subset \mathcal{O}_i \subset \tilde{\Omega}_{i,\sigma}$ for $i = 1, \dots, n$, where $\tilde{\Omega}_{i,\sigma} := \{x \in \Omega : d(x, \Omega_i) < \sigma\}$,*
  - *(i_2) $u(x)\hat{\varphi}(x) > 0$ for all $x \in \mathcal{O}_i \cap \Omega_i$ and any $i = 1, \dots, n$,*
  - *(i_3) if $\bar{\mathcal{O}}_i \cap \bar{\mathcal{O}}_j \neq \emptyset$ with $i \neq j$, then $u(x)u(y) < 0$ for all $x \in \mathcal{O}_i$, $y \in \mathcal{O}_j$;*
- *(ii) if $\widehat{\lambda} < \mu < \widehat{\lambda} + \delta$, then the same conclusion as in (i) above holds, with the only change that in (i_2) one now has $u(x)\widehat{\varphi}(x) < 0$ for all $x \in \mathcal{O}_i \cap \Omega_i$.*

We, thus, see that for μ close to $\widehat{\lambda}$ with $\mu < \widehat{\lambda}$ (respectively $\mu > \widehat{\lambda}$), u looks like $\widehat{\varphi}$ (respectively $-\widehat{\varphi}$) in the sense that they have the same number of nodal domains, that each $\mathcal{O}_i$ appears as a small perturbation of the corresponding Ω_i (cf. (i_1)), with the same (respectively opposite) signs for u and $\hat{\varphi}$ on the intersection (cf. (i_2)). Moreover, the $\mathcal{O}_i$'s enjoy the property that a change of sign occurs when going from one $\mathcal{O}_i$ to a neighboring one (cf.(i_3)); this latter property should be looked at as a regularity property (since for the Ω_i's, (3.7) below follows from (H_1)).

Proof of Theorem 3.1. We start as in the proof of Theorem 2.1. The solution u can be written as $u = \tilde{u} + \widehat{u} + u^{\perp}$ where

$$\tilde{u} = \sum_{t=1}^{r} \frac{\alpha_t}{\lambda_t - \mu} \varphi_t, \quad \widehat{u} = \frac{\alpha}{\widehat{\lambda} - \mu} \widehat{\varphi},$$

and $u^{\perp}$ solves

$$-\triangle u^{\perp} = \mu u^{\perp} + f^{\perp} \text{ in } \Omega, \quad u^{\perp} = 0 \text{ on } \partial\Omega. \tag{3.2}$$

By the variational characterization of λ_s, one deduces from (3.2) that

$$\lambda_s \int_\Omega |u^\perp|^2 \leqslant \int_\Omega |\bigtriangledown u^\perp|^2 = \mu \int_\Omega |u^\perp|^2 + \int_\Omega f^\perp u^\perp.$$

Restricting from now on μ to vary with $(\widehat{\lambda} + \lambda_r)/2 < \mu < (\widehat{\lambda} + \lambda_s)/2$, it follows that

$$\|u^\perp\|_2 \leqslant \frac{2}{\lambda_s - \widehat{\lambda}} \|f^\perp\|_2. \tag{3.3}$$

Using a bootstrap argument in the equation (3.2) and the Sobolev imbedding theorem, one obtains

$$\|u^\perp\|_{C^1(\bar{\Omega})} \leqslant C_1 \|u^\perp\|_{W^{2,q}(\Omega)} \leqslant C_2,$$

where the constants C_1 and C_2 are independent of μ. Since a similar bound clearly also holds for $\tilde{u}$, one finally has

$$\|\tilde{u} + u^\perp\|_{C_1(\bar{\Omega})} \leqslant C_3, \tag{3.4}$$

with C_3 independent of μ.

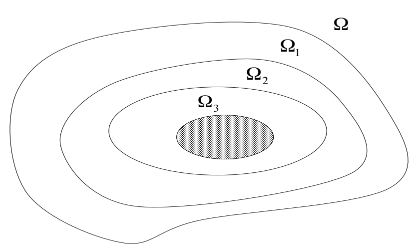
We now consider some consequences of hypothesis (H_1) and of the regularity of Ω. Simple geometric considerations on intersections of balls imply

$$\begin{cases} \bar{\Omega}_i \cap \bar{\Omega}_j \cap \bar{\Omega}_k = \emptyset \text{ if } i, j, k \text{ are different,} \\ \bar{\Omega}_i \cap \bar{\Omega}_j \cap \partial\Omega = \emptyset \text{ if } i, j \text{ are different.} \end{cases} \tag{3.5}$$

For a fixed nodal domain Ω_i, denote $J_i := \{j \neq i : \bar{\Omega}_i \cap \bar{\Omega}_j \neq \emptyset\}$. Any $x \in \partial\Omega_i$ belongs either to $\partial\Omega$ or to some unique $\partial\Omega_j$ with $j \in J_i$ (by (3.5)). This implies that $\partial\Omega_i$ can be written as the finite disjoint union

$$\partial\Omega_i = \cup_{l \in L_i} \gamma_l, \tag{3.6}$$

where γ_l for $l \in L_i$ is either of the form $\partial\Omega_i \cap \partial\Omega_j$ for some $j \epsilon J_i$ or of the form $\partial\Omega_i \cap \partial\Omega$. The following picture illustrates the situation:



Another consequence of (H_1), using the Hopf lemma, is that

$$\text{if } \bar{\Omega}_i \cap \bar{\Omega}_j \neq \emptyset \text{ with } i \neq j, \text{ then } \widehat{\varphi}(x)\widehat{\varphi}(y) < 0 \text{ for all } x \in \Omega_i, y \in \Omega_j. \quad (3.7)$$

Let us now concentrate for a moment on the case of a γ_l in (3.6) which is the common boundary between one Ω_i and one Ω_j, with say $\widehat{\varphi} < 0$ on Ω_i. By (H$_1$) and (3.5), for any $x \in \gamma_l$, there exists a ball $B(\xi, r) \subset \Omega_i$, containing x on its boundary and such that $B(\xi, 2r) \subset \bar{\Omega}_i \cup \Omega_j$. For $y \in B(\xi, 2r), y \neq \xi$, and any C^1 function g, we define the directional derivative

$$\frac{\partial g}{\partial \nu}(y) := \nabla g(y) \frac{y - \xi}{|y - \xi|}.$$

Since, for $y = x$, the direction ν is outward to the ball $B(\xi, r)$ at x, one has by the Hopf lemma $\frac{\partial \widehat{\varphi}}{\partial \nu}(x) > 0$, and consequently one can find $\tilde{\sigma} > 0$ such that $\tilde{\sigma} < \sigma, 4\tilde{\sigma} < r$ and

$$\tilde{m} := \inf \left\{ \frac{\partial \widehat{\varphi}}{\partial \nu}(y) : y \in B(x, 3\tilde{\sigma}) \right\} > 0. \quad (3.8)$$

Note that everything here depends on the point x: $\xi, r, \tilde{\sigma}, \tilde{m}$; in particular $\tilde{\sigma} = \tilde{\sigma}_{x,l,i}$ and $\tilde{m} = \tilde{m}_{x,l,i}$. When x varies in γ_l, the balls $B(x, \tilde{\sigma})$ cover γ_l, and so a finite number of them still cover γ_l, say $B(x_p, \tilde{\sigma}_p)$ for $p \in P_l$.

Proceeding in this way first for each γ_l from (3.6) (including the possible case where γ_l is part of $\partial\Omega$, which can be treated in a similar manner), then for each "negative" Ω_i, and finally for each "positive" Ω_i (where here $\partial\widehat{\varphi}/\partial\nu(x) < 0$ and so (3.8) should be replaced by $\tilde{m} := \inf\{-\partial\widehat{\varphi}/\partial\nu(y) : y \in B(x, 3\tilde{\sigma})\} > 0$), we define

$$\bar{m} := \min \tilde{m}_{x_p,l,i} > 0, \ \bar{\sigma} := \min \tilde{\sigma}_{x_p,l,i} > 0. \quad (3.9)$$

Note that these quantities now only depend on the spectral data $\Omega, \widehat{\lambda}, \widehat{\varphi}$ and on the given σ; they are in particular independent of u, μ.

We now consider case (i) in Theorem 3.1, i.e., $\mu < \widehat{\lambda}$, and concentrate again for a moment on the case of a γ_l in (3.6) which is the common boundary between a "negative" Ω_i and a "positive" Ω_j. Restricting μ from below by

$$\mu > \hat{\lambda} - \frac{\alpha \bar{m}}{C_3}, \quad (3.10)$$

and using (3.4), one obtains

$$\frac{\partial u}{\partial \nu}(y) \geqslant \frac{\alpha}{\widehat{\lambda} - \mu} \frac{\partial \widehat{\varphi}}{\partial \nu}(y) - C_3 \geqslant \frac{\alpha \bar{m}}{\widehat{\lambda} - \mu} - C_3 > 0,$$

for $y \in B(x_p, 3\tilde{\sigma}_p)$. We define $\gamma_{l,\bar{\sigma}} := \{y \in \bar{\Omega}_i \cup \bar{\Omega}_j : d(y, \gamma_l) < \bar{\sigma}\}$. Since clearly

$$\gamma_{l,2\bar{\sigma}} \subset \cup_{p \in P_l} B(x_p, 3\tilde{\sigma}_p), \tag{3.11}$$

one has

$$\frac{\partial u}{\partial \nu}(y) > 0 \text{ for all } y \in \gamma_{l,2\bar{\sigma}}. \tag{3.12}$$

Note that there is a priori a little ambiguity in (3.12) because $y \in \gamma_{l,2\bar{\sigma}}$ may belong to several balls $B(x_p, 3\tilde{\sigma}_p)$ and so several directions ν can be used; in fact inequality (3.12) holds for any of these choices.

In case γ_l is of the form $\partial\Omega_i \cap \partial\Omega$ with $\widehat{\varphi} < 0$ on Ω_i, one proceeds in a similar way: one defines $\gamma_{l,\bar{\sigma}} := \{y \in \bar{\Omega} : d(y, \gamma_l) < \bar{\sigma}\}$ and one again has (3.12) in $\gamma_{l,2\bar{\sigma}}$. One also works similarly for a "positive" Ω_j and again reaches (3.12) with now $\partial u/\partial\nu(y) < 0$ instead of > 0.

Inequality (3.12) (respectively its reverse) is the information on u that we need near the boundary of the "negative" (respectively "positive") Ω_i's.

We now look at what happens away from these boundaries. Recalling the definition of $\Omega_{i,\bar{\sigma}}$ (cf. (H_2)) and considering one "negative" Ω_i and one "positive" Ω_j, there clearly exist $\varepsilon_i, \varepsilon_j > 0$ such that

$$\sup\{\widehat{\varphi}(y) : y \in \Omega_{i,\bar{\sigma}}\} \leqslant -\varepsilon_i, \ \inf\{\widehat{\varphi}(y) : y \in \Omega_{j,\bar{\sigma}}\} \geqslant \varepsilon_j,$$

and consequently, by (3.4),

$$\begin{aligned} \sup\{u(y) : y \in \Omega_{i,\bar{\sigma}}\} &\leqslant -\tfrac{\alpha\varepsilon_i}{\widehat{\lambda}-\mu} + C_3, \\ \inf\{u(y) : y \in \Omega_{j,\bar{\sigma}}\} &\geqslant \tfrac{\alpha\varepsilon_j}{\widehat{\lambda}-\mu} - C_3. \end{aligned} \tag{3.13}$$

We define $\varepsilon := \min\{\bar{m}, \varepsilon_1, \ldots, \varepsilon_n\}, \delta := \frac{\alpha\varepsilon}{C_3}$, and finally restrict μ by $\widehat{\lambda} - \delta < \mu < \widehat{\lambda}$. Clearly (3.10) holds and so also (3.12) (or its reverse in case of a "positive" Ω_j). Moreover, by (3.13), for all "negative" Ω_i's and "positive" Ω_j's,

$$u(y) < 0 \text{ for } y \in \Omega_{i,\bar{\sigma}} \text{ and } u(y) > 0 \text{ for } y \in \Omega_{j,\bar{\sigma}}. \tag{3.14}$$

This is the information on u that we need away from the boundaries of the Ω_i's or Ω_j's.

We now define

$$\mathcal{O}_i := \Omega_{i,\bar{\sigma}} \cup \cup_{l \in L_i} \{y \in \gamma_{l,2\bar{\sigma}} : u(y) < 0\},$$

for a "negative" Ω_i, and

$$\mathcal{O}_j := \Omega_{j,\bar{\sigma}} \cup \cup_{l \in L_j} \{y \in \gamma_{l,2\bar{\sigma}} : u(y) > 0\},$$

for a "positive" Ω_j.

Claim 1. *For each $i = 1, \ldots, n, \mathcal{O}_i$ is arcwise connected.*

Proof of Claim 1. We deal with a $\mathcal{O}_i$ associated to a "negative" Ω_i; a similar argument holds in the "positive" case. Since, by (H_2), $\Omega_{i,\bar{\sigma}}$ is arcwise connected, in order to prove that $\mathcal{O}_i$ is arcwise connected, it suffices to show that any given $\bar{y} \in \gamma_{l,2\bar{\sigma}}$ with $l \in L_i$ and $u(\bar{y}) < 0$ can be joined within $\mathcal{O}_i$ to a point of $\Omega_{i,\bar{\sigma}}$. We keep concentrating on the case where γ_l is the common boundary of Ω_i and one "positive" Ω_j; the case where γ_l is part of $\partial\Omega$ is similar but simpler (because γ_l is then a fixed nodal set). By (3.11), $\bar{y} \in B(x_p, 3\tilde{\sigma}_p)$ for some $x_p \in \gamma_l$. Consider the ball $B(\xi_p, r_p)$ associated to x_p by the interior ball condition. Since $\tilde{\sigma}_p < r_p$, one has, using (3.9), $\xi_p \in \Omega_{i,\bar{\sigma}}$. Consider the segment $[\bar{y}, \xi_p]$. Claim 1 will be proved if we show that $[\bar{y}, \xi_p] \subset \mathcal{O}_i$.

For that purpose we first observe that $[\bar{y}, \xi_p] \subset \Omega_{i,\bar{\sigma}} \cup \gamma_{l,2\bar{\sigma}}$. This inclusion can be verified as follows: since $[\tilde{y}, \xi_p] \subset B(\xi_p, 2r_p) \subset \bar{\Omega}_i \cup \Omega_j$, it suffices to deal separately with $[\bar{y}, \xi_p] \cap \bar{\Omega}_i$ and $[\bar{y}, \xi_p] \cap \bar{\Omega}_j$; the inclusion of $[\bar{y}, \xi_p] \cap \bar{\Omega}_i$ into $\Omega_{i,\bar{\sigma}} \bigcup \gamma_{l,2\bar{\sigma}}$ is obvious, while that of $[\bar{y}, \xi_p] \bigcap \bar{\Omega}_j$ into $\Omega_{i,\bar{\sigma}} \cup \gamma_{l,2\bar{\sigma}}$ follows from the fact that for $z \in [\bar{y}, \xi_p] \cap \bar{\Omega}_j$, one has $d(z, x_p) \leqslant d(\bar{y}, x_p) < 2\bar{\sigma}$. We thus see that, when y varies along $[\bar{y}, \xi_p]$ from ξ_p towards $\bar{y}$, one first has, by (3.14), $u(y) < 0$ as long as $y \in \Omega_{i,\sigma}$, then $\frac{\partial u}{\partial \nu}(y) > 0$ by (3.12), to finally reach $u(\bar{y})$ which is < 0 by assumption. This implies that u remains < 0 all over the segment $[\bar{y}, \xi_p]$, and the conclusion $[\bar{y}, \xi_p] \subset \mathcal{O}_i$ follows. Claim 1 is proved.

Claim 2. *For each $i = 1, \ldots, n, \mathcal{O}_i$ is a nodal domain for u.*

Proof of Claim 2. Again we limit ourselves to the case of a "negative" $\mathcal{O}_i$. One has to show the maximality. Suppose by contradiction the existence of $D_i \supset \mathcal{O}_i, D_i \neq \mathcal{O}_i, D_i$ open and arcwise connected such that $u < 0$ on D_i. Take $\bar{z} \in D_i \backslash \mathcal{O}_i$. There are three possibilities. (i) $\bar{z} \in \tilde{\Omega}_{i,\bar{\sigma}}$ but then $u(\bar{z}) < 0$ implies, by definition of $\mathcal{O}_i$, that $\bar{z} \in \mathcal{O}_i$, a contradiction. (ii) $\bar{z} \notin \tilde{\Omega}_{i,\bar{\sigma}}$ and there exists $j \in J_i$ such that $\bar{z} \in \Omega_j$. Since $u(\bar{z}) < 0, \bar{z} \notin \Omega_{j,\bar{\sigma}}$, and so the distance from $\bar{z}$ to another Ω_k or to $\partial\Omega$ must be $\leqslant \bar{\sigma}$. Since $\bar{\sigma} < \sigma_1$, one then deduces from (3.1) that $d(\bar{z}, \Omega_i) \geqslant 3\bar{\sigma}$. (iii) $\bar{z} \notin \tilde{\Omega}_{i,\bar{\sigma}}$ and there does not exist $j \in J_i$ with $\bar{z} \in \Omega_j$. This means that $\bar{z} \in \bar{\Omega}_k$ for some $k \notin J_i$, and (3.1) implies $d(\bar{z}, \Omega_i) \geqslant 4\bar{\sigma}$. We thus conclude from (ii) and (iii) that any $z \in D_i \backslash \mathcal{O}_i$ satisfies $d(\bar{z}, \mathcal{O}_i) \geqslant 3\bar{\sigma}$. This contradicts the connectedness of D_i, and Claim 2 is proved.

To conclude the proof of Theorem 3.1 in case (i), it remains to verify $(i_1), (i_2), (i_3)$. One has $\Omega_{i,\bar{\sigma}} \subset \mathcal{O}_i \subset \tilde{\Omega}_{i,\bar{\sigma}}$, where the last inclusion follows from the definition of $\mathcal{O}_i$ and (3.14); since $\bar{\sigma} < \sigma$, property (i_1) follows. Property (i_2) is clear from the construction of $\mathcal{O}_i$. To prove (i_3) we first observe that if $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$, then since $\bar{\sigma} < \sigma_1$ and σ_1 satisfies (3.1), $\bar{\mathcal{O}}_i \cap \bar{\mathcal{O}}_j = \emptyset$. In other words, if $\bar{\mathcal{O}}_i \bigcap \bar{\mathcal{O}}_j \neq \emptyset$, then $\bar{\Omega}_i \cap \bar{\Omega}_j \neq \emptyset$, and the construction of $\mathcal{O}_i, \mathcal{O}_j$ yields $u(x)u(y) < 0$ for $x \in \mathcal{O}_i, y \in \mathcal{O}_j$.

Finally, case (ii) of Theorem 3.1 where $\mu > \widehat{\lambda}$ can of course be treated in a similar way; the main difference is a change of sign in (3.12), (3.14). This completes the proof of Theorem 3.1. □

Remark 3.2. The above proof shows that if the part γ_l of $\partial\Omega_i$ is included in $\partial\Omega$, then $\mathcal{O}_i = \Omega_i$ in a neighborhood of γ_l. More precisely, if $\partial\Omega_i \cap \partial\Omega \neq \emptyset$, then $\partial\mathcal{O}_i \cap \partial\Omega = \partial\Omega_i \cap \partial\Omega$, and moreover the product $\partial u/\partial\nu . \partial\widehat{\varphi}/\partial\nu$ is > 0 (respectively < 0) on $\partial\Omega_i \cap \partial\Omega$ when $\mu < \widehat{\lambda}$ (respectively $\mu > \widehat{\lambda}$). Here, ν denotes the exterior normal to $\partial\Omega$.

Remark 3.3. If Ω_i is of class C^2, then (H$_1$) and (H$_2$) hold. This can be verified by a construction similar to that described on page 355 of [10].

Example 3.4. Take for Ω the unit disk in $\mathbb{R}^2$. The eigenfunctions in polar coordinates (ρ, θ) are of the form

$$J_n(k_{n,m}\rho)(\alpha \cos n\theta + \beta \sin n\theta),$$

where $k_{n,m}$ denotes the m^{th} root of the n^{th} Bessel function J_n and where α, β are arbitrary constants (cf. [6]). The corresponding eigenvalues are $k_{n,m}^2$. When $n \neq 0$, the nodal curves include radial lines $\theta =$ const., and Theorem 3.1 does not apply. This is the case for $\lambda_2 = k_{1,1}^2$ (where the nodal domains are half-disks and so (H$_1$) does not hold), for $\lambda_4 = k_{2,1}^2$ (where the nodal domains are quarter-disks and so (H$_1$) as well as (3.5) and (3.7) do not hold), ... On the contrary, Theorem 3.1 applies for the (simple) eigenvalues $k_{0,m}^2$, for which the nodal curves are circles included in Ω. This is the case for $\lambda_6 = k_{0,2}^2, \lambda_{15} = k_{0,3}^2$,...

The proof of Theorem 3.1 can be adapted to yield the following result which concerns only one nodal domain Ω_{i_0} but in an arbitrary domain Ω.

Theorem 3.5. *Let Ω be a bounded domain in $\mathbb{R}^N$. Let $\widehat{\lambda} \geqslant \lambda_2$ be an eigenvalue of $-\triangle$ on $H_0^1(\Omega)$ and $\widehat{\varphi}$ an associated eigenfunction. Assume that one nodal domain of $\widehat{\varphi}$, say Ω_{i_0}, enjoys the following properties:*

(H_4) $\bar{\Omega}_{i_0} \subset \Omega$ and Ω_{i_0} satisfies at any $x \in \partial\Omega_{i_0}$ the interior ball condition;

(H_5) for σ sufficiently small, say $0 < \sigma < \sigma_0, \Omega_{i_0,\sigma}$ is arcwise connected.

Take $f \in L^q(\Omega)$ with $q > N$ and satisfying (H_3). Finally, for μ near $\widehat{\lambda}$ (in the sense that $\lambda_r < \mu < \widehat{\lambda}$ or $\widehat{\lambda} < \mu < \lambda_s$), let u be the (unique) solution of

(1.1). *Take $\sigma > 0$ with $\sigma < \sigma_0$ and $\sigma < \sigma_1$. Then there exists $\delta = \delta(f, \sigma) > 0$ such that*

(*i*) *if $\widehat{\lambda} - \delta < \mu < \widehat{\lambda}$, then u admits a nodal domain $\mathcal{O}_{i_0}$ with the following three properties:*

(i_1) $\Omega_{i_0,\sigma} \subset \mathcal{O}_{i_0} \subset \tilde{\Omega}_{i_0,\sigma}$,

(i_2) $u(x)\widehat{\varphi}(x) > 0$ *for* $x \epsilon \mathcal{O}_{i_0} \cap \Omega_{i_o}$,

(i_3) $u(x)u(y) < 0$ *for* $x \epsilon \mathcal{O}_{i_0}$ *and* $y \epsilon A$ *where* A *is the union of* $\tilde{\Omega}_{i_o,\sigma} \backslash \bar{\mathcal{O}}_{i_0}$ *and of the* $\Omega_{j,\sigma}$*'s for* j *such that* $\bar{\Omega}_{i_0} \cap \bar{\Omega}_j \neq \emptyset$;

(*ii*) *if $\widehat{\lambda} < \mu < \widehat{\lambda} + \delta$, the same conclusion as in* (i) *holds, with only one change of sign in* (i_2).

In the statement of Theorem 3.5, $\sigma_1 > 0$ is defined in an analogous way as in (3.1): $d(\bar{\Omega}_{i_0}, \bar{\Omega}_j) \geqslant 4\sigma_1$ for any nodal domain Ω_j with $\bar{\Omega}_{i_0} \cap \bar{\Omega}_j = \emptyset$, and $d(\bar{\Omega}_{i_0}, \partial\Omega) \geqslant 4\sigma_1$. It is worth observing here that the Courant nodal domain theorem remains valid in an arbitrary bounded domain (the classical proof of [6], [11], [2] can be adapted by using Lemma 5.6 from [7]).

Proof of Theorem 3.5. We start as in the proof of Theorem 3.1 and reach (3.3) through the variational argument described there (which is valid without any regularity assumption on Ω, cf. e.g. [9]). Using a bootstrap argument based on Theorem 7.1 from [1] and the Sobolev imbedding theorem, one obtains that for any subdomain D with $\bar{\Omega}_{i_0} \subset D \subset \bar{D} \subset \Omega$,

$$\|u^{\perp}\|_{C^1(\bar{D})} \leqslant C,$$

where the constant $C = C(D)$ is independent of μ. As before this leads to the estimate

$$\|\tilde{u} + u^{\perp}\|_{C^1(\bar{D})} \leqslant C', \tag{3.15}$$

with $C' = C'(D)$ independent of μ.

We now consider some consequences of (H_4):

$$\bar{\Omega}_{i_0} \cap \bar{\Omega}_j \cap \bar{\Omega}_k = \emptyset \text{ if } i_0, j, k \text{ are different,} \tag{3.16}$$

$$\text{if } \bar{\Omega}_{i_0} \cap \bar{\Omega}_j \neq \emptyset, \text{then } \widehat{\varphi}(x)\widehat{\varphi}(y) < 0 \text{ for all } x \in \Omega_{i_0},\ y \in \Omega_j. \tag{3.17}$$

The proof of these properties is a little more involved than before because here no regularity is assumed on Ω_j with $j \neq i_0$. By using the Hopf lemma and the reciprocal function theorem as on pages 672-673 of [8], one can see that any $z \in \partial\Omega_{i_0}$ has an open neighborhood $\mathcal{U}$ such that $\{x \in \mathcal{U} : \widehat{\varphi}(x) < 0\}$ and $\{x \in \mathcal{U} : \widehat{\varphi}(x) > 0\}$ are connected. This easily leads to (3.16), and (3.17) then follows from another application of the Hopf lemma.

From this stage the proof can be pursued by obvious adaptations of that of Theorem 3.1. One looks at Ω_{i_0} and only considers those Ω_j's with $j \epsilon J_{i_0}$.

The numbers $\bar{m}$ and $\bar{\sigma}$ are introduced in a similar way as before, and one then makes use of (3.15) with $D = \Omega_{\bar{\sigma}}$. This gives a constant C' which will replace the constant C_3 in (3.10) and later, in the final choice of δ. $\square$

Example 3.6. Take $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. Then the eigenfunctions are of the form

$$\psi_{k,l}(x,y) := \sin(k\pi x)\sin(l\pi y),$$

and the corresponding eigenvalues are $\nu_{k,l} := \pi^2(k^2 + l^2)$ (cf.([6]). The first of these eigenvalues are

$$\nu_{1,1} = 2\pi^2 < \nu_{1,2} = \nu_{2,1} = 5\pi^2 < \nu_{2,2} = 8\pi^2 < \nu_{1,3} = \nu_{3,1} = 10\pi^2 < \cdots .$$

In particular, the eigenfunction

$$\varphi_5(x,y) = \psi_{1,3}(x,y) + \psi_{3,1}(x,y) = \sin(\pi x)\sin(\pi y)(6 - 4\sin^2(\pi x) - 4\sin^2(\pi y)),$$

associated to $\widehat{\lambda} = \lambda_5 = \nu_{1,3} = \nu_{3,1} = 10\pi^2$, has two nodal domains, a negative one Ω_1 with $\bar{\Omega}_1 \subset \Omega$ and a positive one Ω_2 touching $\partial\Omega$. The nodal line γ is a smooth closed curve contained in Ω whose equation is $\sin^2(\pi x) + \sin^2(\pi y) = 3/2$. The assumptions of Theorem 3.5 are satisfied by Ω_1. It follows that, for f satisfying (H_3) and for $\mu < \widehat{\lambda}$, μ close to $\widehat{\lambda}$, the solution u admits a negative nodal domain $\mathcal{O}_1$ which satisfies $\bar{\mathcal{O}}_1 \subset \Omega$ and which appears as a perturbation of Ω_1; the new eigencurve separates $\mathcal{O}_1$ from a nodal domain $\mathcal{O}_2$ where u is > 0, which appears near γ as a perturbation of Ω_2, and which contains $A = [\tilde{\Omega}_{1,\sigma} \backslash \mathcal{O}_1] \cup \Omega_{2,\sigma}$.

Remark 3.7. A more detailed analysis of the preceding example allows us to prove the following result concerning $\mathcal{O}_2$: If T is a given union of closed segments included in $\partial\Omega$ but avoiding the corners of Ω, then $\bar{\mathcal{O}}_2 \supset T$ when μ is sufficiently close to $\widehat{\lambda}$. The proof is based on Lemma 3.8 below. Using this lemma, one sees that estimate (3.15) remains valid for any subdomain D of Ω with $\bar{D} \subset (\bar{\Omega}\backslash$ its four corners). This is the key point which allows the adaptation of the proof of Theorem 3.1 near T. So if one removes from Ω four small disks centered at the corners, the solution u has exactly two nodal domains in the remaining set when μ is sufficiently close to $\widehat{\lambda}$. There is, however, little hope to derive any information on the sign of u near the corners of Ω, as the counterexample of [4] seems to indicate.

Lemma 3.8. *Let Ω be a bounded domain with a C^2 boundary portion $T \subset \partial\Omega$. Let $u \in H_0^1(\Omega)$ be a solution of $-\triangle u = g$ in Ω with $g \in L^2(\Omega)$. If $g \in L^p_{loc}(\Omega \cup T)$ for some $p > 1$, then $u \in W^{2,p}_{loc}(\Omega \cup T)$ and for any*

$\Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2 \cup T \subset \bar{\Omega}_2 \cup T \subset \Omega \cup T$,

$$\|u\|_{W^{2,p}(\Omega_1)} \leqslant C(\|u\|_{L^p(\Omega_2)} + \|g\|_{L^p(\Omega_2)}),$$

where the constant C only depends on p and the domains.

Proof. This is easily adapted from that of Theorems 7.1 and 8.1 from [1]. See also Theorem 9.13 in [10] for a related result. □

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