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GLOBAL EXISTENCE OF SOLUTIONS FOR THE COUPLED VLASOV AND NAVIER-STOKES EQUATIONS

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Abstract. In this work, we obtain a result of global existence for weak solutions of the three-dimensional incompressible Vlasov-Navier-Stokes equations, the coupling being done through a drag force which is linear with respect to the relative velocity of the fluid and particles.

1. Introduction

When we consider a spray, that is, a dispersed phase of particles (or droplets) moving inside a continuous fluid, we can choose between several models. One of them, first introduced by Williams in the framework of combustion theory [16] (see also [3]), consists in coupling a kinetic equation with fluid mechanics equations. In such models, the fluid is described through macroscopic quantities (density and velocity, for instance), whereas the particles are represented by a probability density function. We assume that the spray is thin [15]; i.e., the kinetic and fluid equations are coupled through a drag term (which depends on both the fluid macroscopic quantities, and the

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probability density function and its variables), but not through the volume fraction occupied by the fluid, as in the case of the so-called thick sprays.

In this work, we consider the coupling of the three-dimensional incompressible Navier-Stokes equations with a Vlasov-type equation. This system appears in various situations, for instance, when one wants to model the transport, with no diffusive effects, of an aerosol in a Newtonian, viscous, incompressible airflow, inside the human upper airways [5, 1]. We, here, aim to prove global existence of weak solutions for the three-dimensional Vlasov-Navier-Stokes equations, assuming periodicity with respect to the space variable.

The same kind of model has been studied in [10] where the author considered, in all dimensions, the unsteady Stokes equations coupled with the Vlasov equation in a bounded domain with reflection boundary conditions. The main differences between the present work and [10] are the following:

- We do not neglect the convection term and consider the Navier-Stokes equations.
- Our proof only works in dimension smaller than or equal to 3 (because of the use of Sobolev inequalities).
- Our proof is constructive and does not rely on a fixed-point argument of Schauder type: we build a sequence of approximate solutions that converges towards a solution of the coupled problem.

Note that Goudon, Jabin and Vasseur [8, 9] studied the hydrodynamic limit of the Vlasov-Navier-Stokes system, for some particular regimes of the dispersed phase. In [13], Mellet and Vasseur proved the existence of global weak solutions to the coupled Vlasov-Fokker-Planck and compressible Navier-Stokes equations (hence, including diffusive effects for the particles and compressibility effects for the fluid), for absorption or reflection boundary conditions, and in [14], they investigate some asymptotic regimes of this system.

Local existence of solutions in the case of the compressible Vlasov-Euler equation was proved by Baranger and Desvillettes [2]. One can find, in that same article, references to previous works about fluid-kinetic coupling, in particular, when the fluid is modelled by one single equation [4, 7]. The extension of this result to moderately thick sprays (that is, sprays in which the collisions between droplets has been taken into account) has been done by Mathiaud [12].

We present our model in the next section, along with our main results. Those are proven in Section 3.

2. PRESENTATION OF THE MODEL

Let us consider a finite constant $T > 0$ and set $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. We investigate the interaction of a (monodispersed) spray of particles with a Newtonian, viscous, and incompressible fluid. The fluid is commonly described by macroscopic quantities, such as its density, here assumed to be constant (equal to 1), its velocity $\mathbf{u}(t, \mathbf{x})$ and pressure $p(t, \mathbf{x})$ at time $t \in [0, T]$ and position $\mathbf{x} \in \mathbb{T}^3$. The evolution of $(\mathbf{u}, p)$ is here governed by the incompressible Navier-Stokes equations. The particles are described by a density function f in the phase space which solves a Vlasov-like equation. The quantity $f(t, \mathbf{x}, \mathbf{v})$ is the density of particles located in $\mathbf{x} \in \mathbb{T}^3$ at time t which have the velocity $\mathbf{v} \in \mathbb{R}^3$. Equations for the fluid and the spray are coupled through a drag force, which depends on the relative velocity of the fluid and the particles, namely $\mathbf{u} - \mathbf{v}$, and on the density function f. More precisely, f, $\mathbf{u}$ and p satisfy the following system:

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot [(\mathbf{u} - \mathbf{v}) f] = 0 \qquad \text{in } (0, T) \times \mathbb{T}^3 \times \mathbb{R}^3, \tag{2.1}$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}} p - \Delta_{\mathbf{x}} \mathbf{u} = -\int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v})(\mathbf{u}(t, \mathbf{x}) - \mathbf{v}) \, \mathrm{d}\mathbf{v} + \mathbf{F_{ext}} \qquad \text{in } (0, T) \times \mathbb{T}^3, \tag{2.2}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{u} = 0 \qquad \text{in } (0, T) \times \mathbb{T}^3, \tag{2.3}$$

where $\mathbf{F_{ext}}$ is an external force applied to the fluid that is periodic in space. The system has to be completed by initial conditions for $\mathbf{u}$ and f, which are

$$f(0, \mathbf{x}, \mathbf{v}) = f_{\mathrm{in}}(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in \mathbb{T}^3, \quad \mathbf{v} \in \mathbb{R}^3, \tag{2.4}$$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u_{in}}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{T}^3, \tag{2.5}$$

where $f_{\mathrm{in}} \geq 0$ and $\mathbf{u_{in}}$ are given.

Remark 1. Note that we here make the thin spray assumption. First, there is no interaction (collisions or coalescences) between particles, which implies that, when $\mathbf{u}$ is given, (2.1) is linear with respect to f. Secondly, the drag force is the only coupling phenomenon between the spray and the fluid and appears in both (2.1) and (2.2): the volume fraction of the spray remains negligible with respect to the fluid volume fraction, and is not taken into account explicitly.

If we assume that all quantities are smooth enough, system (2.1)–(2.5) satisfies, at least formally, an energy equality. It is obtained in the following way. We first multiply the Navier-Stokes equations (2.2) by $\mathbf{u}$ and integrate over $\mathbb{T}^3$. Then we multiply the Vlasov equation (2.1) by $|\mathbf{v}|^2/2$ and integrate over $\mathbb{T}^3 \times \mathbb{R}^3$. Finally, we add both previous equalities and obtain, after integrations by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2_{L^2(\mathbb{T}^3)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{T}^3\times\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})|\mathbf{v}|^2 + \|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\mathbb{T}^3)} + \iint_{\mathbb{T}^3\times\mathbb{R}^3} f(\mathbf{u}-\mathbf{v})^2 = \int_{\mathbb{T}^3} \mathbf{F_{ext}}\cdot\mathbf{u}.$$

In the previous equation, the first two terms represent the total kinetic energy of the coupled system, whereas the next two ones represent the energy dissipation coming from, respectively, the fluid itself and the drag force. Note that, since f_{in} is nonnegative, the density function f and consequently $\iint_{\mathbb{T}^3\times\mathbb{R}^3} f(\mathbf{u}-\mathbf{v})^2$ also remain positive.

Remark 2. When no external force is applied, the global energy

$$\frac{1}{2}\,\|\mathbf{u}(t)\|^2_{L^2(\mathbb{T}^3)} + \frac{1}{2}\iint_{\mathbb{T}^3\times\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})|\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}$$

of the system is decreasing. We can also point out that, in the same case, the global momentum of the system is conserved.

In the sequel, we shall use the following.

Assumption 1. *The initial data and the source term satisfy*

$$f_{\mathrm{in}} \in L^\infty(\mathbb{T}^3\times\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |\mathbf{v}|^2 f_{\mathrm{in}}\mathrm{d}\mathbf{v} \in L^\infty(\mathbb{T}^3), \quad \mathbf{u_{in}} \in L^2(\mathbb{T}^3), \quad \mathbf{F_{ext}} \in L^2(0,T;L^2(\mathbb{T}^3)).$$

Under this assumption, the energy equality gives our functional framework. More precisely, we set

$$\mathcal{H} = \left\{\mathbf{w}\in L^2(\mathbb{T}^3):\ \nabla_{\mathbf{x}}\cdot\mathbf{w}=0\right\}, \quad \mathcal{V} = \left\{\mathbf{w}\in H^1(\mathbb{T}^3):\ \nabla_{\mathbf{x}}\cdot\mathbf{w}=0\right\}.$$

Definition 1. *We say that* $(\mathbf{u}, f)$ *is a weak solution of* (2.1)–(2.5) *if the following conditions are satisfied:*

- $\mathbf{u} \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \cap \mathcal{C}^0([0,T];\mathcal{V}')$,
- $f(t,\mathbf{x},\mathbf{v}) \geq 0$, *for any* $(t,\mathbf{x},\mathbf{v}) \in (0,T)\times\mathbb{T}^3\times\mathbb{R}^3$,
- $f \in L^\infty(0,T;L^\infty(\mathbb{T}^3\times\mathbb{R}^3)\cap L^1(\mathbb{T}^3\times\mathbb{R}^3))$,
- $f|\mathbf{v}|^2 \in L^\infty(0,T;L^1(\mathbb{T}^3\times\mathbb{R}^3))$,
- *for all* $\phi \in \mathcal{C}^1([0,T]\times\mathbb{T}^3\times\mathbb{R}^3)$ *with compact support in* $\mathbf{v}$, *such that* $\phi(T,\cdot,\cdot) = 0$,

$$-\int_0^T \iint_{\mathbb{T}^3\times\mathbb{R}^3} f\,[\partial_t\phi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\phi + (\mathbf{u}-\mathbf{v})\cdot\nabla_{\mathbf{v}}\phi]\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_{\mathrm{in}}\,\phi(0,\cdot,\cdot)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}, \quad (2.6)$$

- *for all* $\boldsymbol{\psi}\in\mathcal{C}^1([0,T]\times\mathbb{T}^3)$ *such that* $\nabla_{\mathbf{x}}\cdot\boldsymbol{\psi}=0$, *for a.e.* t,

$$\int_{\mathbb{T}^3}\mathbf{u}(t)\cdot\boldsymbol{\psi}(t)\,\mathrm{d}\mathbf{x} - \int_0^t\int_{\mathbb{T}^3}\mathbf{u}\cdot\partial_t\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}(\mathbf{u}\cdot\nabla_{\mathbf{x}})\,\mathbf{u}\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}\nabla_{\mathbf{x}}\mathbf{u} : \nabla_{\mathbf{x}}\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s$$
$$= -\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3} f(\mathbf{u}-\mathbf{v})\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}\mathbf{F_{ext}}\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \int_{\mathbb{T}^3}\mathbf{u_{in}}\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}\mathbf{x}. \quad (2.7)$$

In the following, the moments of f are denoted, for any $\alpha\geq 0$,

$$m_\alpha f(t,\mathbf{x}) = \int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})|\mathbf{v}|^\alpha\ \mathrm{d}\mathbf{v}, \qquad t\in[0,T],\ \mathbf{x}\in\mathbb{T}^3,$$
$$M_\alpha f(t) = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})|\mathbf{v}|^\alpha\ \mathrm{d}\mathbf{x}\ \mathrm{d}\mathbf{v} = \int_{\mathbb{T}^3} m_\alpha f(t,\mathbf{x})\ \mathrm{d}\mathbf{x}, \quad t\in[0,T].$$

The main result of this work is hereby stated.

Theorem 1. *Let* $T>0$. *Under Assumption* 1, *there exists at least one weak solution* $(f,\mathbf{u})$ *to* (2.1)–(2.5) *on* $(0,T)$ *in the sense of Definition* 1. *Moreover, this solution satisfies the following estimates:*

$$\frac{1}{2}M_2 f(t) + \frac{1}{2}\|\mathbf{u}(t)\|^2_{L^2(\mathbb{T}^3)} + \int_0^t\|\nabla_{\mathbf{x}}\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s + \int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3} f|\mathbf{u}-\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s$$
$$\leq \frac{1}{2}M_2 f_{\mathrm{in}} + \frac{1}{2}\|\mathbf{u_{in}}\|^2_{L^2(\mathbb{T}^3)} + \int_0^t\int_{\mathbb{T}^3}\mathbf{F_{ext}}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s, \quad (2.8)$$
$$\|f\|_{L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)} \leq e^{3T}\|f_{\mathrm{in}}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}.$$

Proof of Theorem 1. The proof of Theorem 1 is split into three main steps.

In the first one, we build a sequence of approximate solutions where the convection velocity of the Navier-Stokes equations and the fluid velocity appearing in the Vlasov equation are regularized. We also cut off the high particle velocities in the drag force applied by the spray on the flow. In order to prove the existence of a solution to this regularized but still coupled system, we build a sequence of uncoupled problems where we first solve the kinetic part, then the fluid part. We prove that the sequence of solutions of these regularized uncoupled systems converges towards the solution of the regularized system. This is the object of Proposition 2 below.

In a second step, we prove that the approximate solutions satisfy uniform bounds (with respect to the regularization) on a certain time interval allowing us to pass to the limit when the parameter of regularization tends to 0. This is the object of Proposition 1 below.

We finally prove that the local (in time) solution obtained in the previous step can in fact be extended to the whole interval $[0, T]$, thanks to the energy estimate.

2.1. **The regularized system.** First, we define the regularized system. Let $\varepsilon > 0$ and define θ_ε to be a mollifier such that $\theta_\varepsilon = \varepsilon^3\theta(\mathbf{x}/\varepsilon)$ with $\theta \in \mathcal{C}^\infty(\mathbb{R}^3)$, $\theta \geq 0$, and $\int_{\mathbb{R}^3}\theta = 1$. Moreover, we introduce $\gamma_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^3)$ whose support is included in the ball $B(0, 1/\varepsilon)$, $0 \leq \gamma_\varepsilon \leq 1$, $\gamma_\varepsilon = 1$ on $B(0, 1/2\varepsilon)$ and such that $\gamma_\varepsilon(\mathbf{v}) \longrightarrow 1$ as ε goes to zero. We then consider the following regularization of the initial problem:

$$\partial_t f_\varepsilon + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_\varepsilon + \nabla_{\mathbf{v}}\cdot[(\theta_\varepsilon \star \mathbf{u}_\varepsilon - \mathbf{v})f_\varepsilon] = 0, \quad \text{in } (0,T)\times\mathbb{T}^3\times\mathbb{R}^3, \tag{2.9}$$

$$\partial_t f_\varepsilon + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_\varepsilon + \nabla_{\mathbf{v}}\cdot[(\theta_\varepsilon \star \mathbf{u}_\varepsilon - \mathbf{v})f_\varepsilon] = 0 \text{ in } (0,T)\times\mathbb{T}^3\times\mathbb{R}^3, \tag{2.10}$$

$$\begin{aligned}&\partial_t \mathbf{u}_\varepsilon + ((\theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot\nabla_{\mathbf{x}})\mathbf{u}_\varepsilon + \nabla_{\mathbf{x}} p_\varepsilon - \Delta_{\mathbf{x}}\mathbf{u}_\varepsilon \qquad (2.11)\\ &\qquad = -\int_{\mathbb{R}^3} f_\varepsilon(t,\mathbf{x},\mathbf{v})(\mathbf{u}_\varepsilon(t,\mathbf{x}) - \mathbf{v})\gamma_\varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{v} + \mathbf{F}^\varepsilon_{\mathbf{ext}} \qquad \text{in } (0,T)\times\mathbb{T}^3,\end{aligned}$$

$$\nabla_{\mathbf{x}}\cdot\mathbf{u}_\varepsilon = 0 \qquad \text{in } (0,T)\times\mathbb{T}^3, \tag{2.12}$$

where $\mathbf{F}^\varepsilon_{\mathbf{ext}}$ is a space regularization of the force $\mathbf{F}_{\mathbf{ext}}$.

We complete the system with initial conditions for $\mathbf{u}_\varepsilon$ and f_ε:

$$f_\varepsilon(0,\mathbf{x},\mathbf{v}) = f^\varepsilon_{\text{in}}(\mathbf{x},\mathbf{v}), \qquad \mathbf{x}\in\mathbb{T}^3, \quad \mathbf{v}\in\mathbb{R}^3, \tag{2.13}$$

$$\mathbf{u}_\varepsilon(0,\mathbf{x}) = \mathbf{u}^\varepsilon_{\mathbf{in}}(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{T}^3, \tag{2.14}$$

where $f^\varepsilon_{\text{in}}$ and $\mathbf{u}^\varepsilon_{\mathbf{in}}$ are $\mathcal{C}^\infty$ approximations of f_{in} and $\mathbf{u}_{\mathbf{in}}$ such that $(f^\varepsilon_{\text{in}})$ converges to f_{in} strongly in $L^p(\mathbb{T}^3\times\mathbb{R}^3)$, for all $p < \infty$, and weakly-$*$ in $L^\infty(\mathbb{T}^3\times\mathbb{R}^3)$, $(M_2 f^\varepsilon_{\text{in}})$ is uniformly bounded with respect to ε and converges strongly towards $(M_2 f_{\text{in}})$ in $L^\infty(\mathbb{T}^3)$, and $(\mathbf{u}^\varepsilon_{\mathbf{in}})$ strongly converges to $\mathbf{u}_{\mathbf{in}}$ in

$L^2(\mathbb{T}^3)$. We can also assume, without any loss of generality, that $f_{\rm in}^\varepsilon$ has compact support in $\mathbf{v}$.

We prove that there exists a solution $(\mathbf{u}_\varepsilon, f_\varepsilon)$ of (2.9)–(2.14) in the functional space $\big(L^2(0,T;H^1(\mathbb{T}^3)) \cap L^\infty(0,T;L^2(\mathbb{T}^3))\big) \times L^\infty(0,T;L^\infty(\mathbb{T}^3\times\mathbb{R}^3))$, such that $\partial_t \mathbf{u}_\varepsilon \in L^2(0,T;L^2(\mathbb{T}^3))$, satisfying the following variational formulation:

- for any $\phi \in \mathcal{C}^1([0,T]\times\mathbb{T}^3\times\mathbb{R}^3)$, for a.e. t,

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t)\,\phi(t,\cdot,\cdot)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v} - \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon\left[\partial_t\phi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\phi + (\theta_\varepsilon\star\mathbf{u}_\varepsilon - \mathbf{v})\cdot\nabla_{\mathbf{v}}\phi\right]\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_{\rm in}^\varepsilon\,\phi(0,\cdot,\cdot)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}, \quad (2.15)$$

- for all $\boldsymbol{\psi} \in \mathcal{C}^1([0,T]\times\mathbb{T}^3)$ such that $\nabla_{\mathbf{x}}\cdot\boldsymbol{\psi} = 0$, for a.e. t,

$$\int_0^t \int_{\mathbb{T}^3} \partial_t\mathbf{u}_\varepsilon\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} ((\theta_\varepsilon\star\mathbf{u}_\varepsilon)\cdot\nabla_{\mathbf{x}})\,\mathbf{u}_\varepsilon\,\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon : \nabla_{\mathbf{x}}\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s = -\int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{v})\,\gamma_\varepsilon(\mathbf{v})\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}\mathbf{F}^\varepsilon_{\mathbf{ext}}\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s. \quad (2.16)$$

We also obtain the following proposition, which is crucial to get the asymptotics $\varepsilon \to 0$ and obtain the existence of at least one weak solution to the original problem (2.1)–(2.3).

Proposition 1. *There exists a time $T^* \in (0,T]$, only depending on T and both quantities $\|\mathbf{u_{in}}\|^2_{L^2(\mathbb{T}^3)} + |M_2 f_{\rm in}| + \|\mathbf{F_{ext}}\|^2_{L^2(0,T;L^2(\mathbb{T}^3))}$ and $\|f_{\rm in}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}$, such that*

$$\|\mathbf{u}_\varepsilon\|^2_{L^\infty(0,T^*;L^2(\mathbb{T}^3))} + \|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon\|^2_{L^2(0,T^*;L^2(\mathbb{T}^3))} + \|M_2 f_\varepsilon\|_{L^\infty(0,T^*)} \le C(T,T^*,\|\mathbf{u_{in}}\|^2_{L^2(\mathbb{T}^3)} + |M_2 f_{\rm in}| + \|\mathbf{F_{ext}}\|^2_{L^2(0,T;L^2(\mathbb{T}^3))}, \|f_{\rm in}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}).$$

Here, $C(A,B,\dots)$ denotes a nonnegative constant depending on A, B, ...

Moreover, we have

$$\|f_\varepsilon\|_{L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)} \le e^{3T}\|f_{\rm in}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}. \tag{2.17}$$

Finally, there exists $(f,\mathbf{u}) \in L^\infty((0,T^*)\times\mathbb{T}^3\times\mathbb{R}^3)\times(L^\infty(0,T^*;L^2(\mathbb{T}^3))\cap L^2(0,T^*;H^1(\mathbb{T}^3))\cap\mathcal{C}^0([0,T^*];\mathcal{V}'))$ *such that, up to a subsequence,* (f_ε) *converges weakly-$*$ towards f in* $L^\infty((0,T^*)\times\mathbb{T}^3\times\mathbb{R}^3)$ *and* $(\mathbf{u}_\varepsilon)$ *weakly converges to* $\mathbf{u}$ *in* $L^2(0,T^*;H^1(\mathbb{T}^3))$ *and weakly$*$ in* $L^\infty(0,T^*;L^2(\mathbb{T}^3))$*. Moreover,* $(\mathbf{u}_\varepsilon)$ *strongly converges towards* $\mathbf{u}$ *in* $L^2(0,T^*;L^2(\mathbb{T}^3))$*.*

Finally, in order to prove the existence of a solution to the regularized system (2.9)–(2.14), we have to use another system, which is defined in the next subsection.

2.2. **The regularized and uncoupled system.** For a fixed $\varepsilon>0$, for any $n\in\mathbb{N}$ and a given function $\mathbf{u}_\varepsilon^n$ in $H^1(0,T;L^2(\mathbb{T}^3))\cap L^2(0,T;H^1(\mathbb{T}^3))$, let us consider the following problem defined by $\mathbf{u}_\varepsilon^0=\mathbf{u}_{\mathbf{in}}^\varepsilon$ and

$$\partial_t f_\varepsilon^n + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_\varepsilon^n + \nabla_{\mathbf{v}}\cdot[(\theta_\varepsilon\star\mathbf{u}_\varepsilon^n-\mathbf{v})f_\varepsilon^n]=0 \text{ in } (0,T)\times\mathbb{T}^3\times\mathbb{R}^3, \tag{2.18}$$

$$\begin{aligned}&\partial_t\mathbf{u}_\varepsilon^{n+1}+((\theta_\varepsilon\star\mathbf{u}_\varepsilon^{n+1})\cdot\nabla_{\mathbf{x}})\mathbf{u}_\varepsilon^{n+1}+\nabla_{\mathbf{x}}p_\varepsilon^{n+1}-\Delta_{\mathbf{x}}\mathbf{u}_\varepsilon^{n+1} \qquad (2.19)\\ &\quad=-\int_{\mathbb{R}^3} f_\varepsilon^n(t,\mathbf{x},\mathbf{v})(\mathbf{u}_\varepsilon^n(t,\mathbf{x})-\mathbf{v})\gamma_\varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{v}+\mathbf{F}_{\mathbf{ext}}^\varepsilon \qquad \text{in } (0,T)\times\mathbb{T}^3,\end{aligned}$$

$$\nabla_{\mathbf{x}}\cdot\mathbf{u}_\varepsilon^{n+1}=0 \qquad \text{in } (0,T)\times\mathbb{T}^3, \tag{2.20}$$

supplemented with initial conditions (2.13)–(2.14).

Proposition 2. *For any $n\in\mathbb{N}$, there exists a unique solution* $(f_\varepsilon^n,\mathbf{u}_\varepsilon^{n+1})$ *of* (2.18)–(2.20)*, supplemented with initial conditions* (2.13)–(2.14)*, with the regularity* $\mathcal{C}^1([0,T];\mathcal{C}^k(\mathbb{T}^3\times\mathbb{R}^3))\times\big(H^1(0,T;L^2(\mathbb{T}^3))\cap L^2(0,T;H^1(\mathbb{T}^3))\big)$*, for any $k\in\mathbb{N}$. Moreover, it satisfies the following estimates:*

$$\|f_\varepsilon^{n+1}\|_{L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)} \le e^{3T}\|f_{\rm in}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}, \tag{2.21}$$

$$\|\mathbf{u}_\varepsilon^{n+1}\|_{L^\infty(0,T;L^2(\mathbb{T}^3))\cap L^2(0,T;H^1(\mathbb{T}^3))} \le C(\varepsilon), \tag{2.22}$$

$$\|\partial_t\mathbf{u}_\varepsilon^{n+1}\|_{L^2(0,T;L^2(\mathbb{T}^3))} \le C(\varepsilon), \tag{2.23}$$

where $C(\varepsilon)$ may depend on ε but does not depend on n. Furthermore, the sequence $(f_\varepsilon^n,\mathbf{u}_\varepsilon^n)$ *strongly converges in* $L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)\times L^\infty(0,T;L^2(\mathbb{T}^3))$ *towards a variational solution of* (2.9)–(2.14)*, i.e., satisfying* (2.15)–(2.16).

For the sake of simplicity, we shall drop the subscript ε, and denote, for instance, f_ε^n or $\mathbf{u}_\varepsilon^n$ by f^n or $\mathbf{u}^n$.

3. Proof of the main results

We first study the properties of the sequence $(f^n, \mathbf{u}^n)_{n\in\mathbb{N}}$ defined by (2.18)–(2.20), and derive bounds needed for the asymptotics $n \to +\infty$. We can then obtain the existence of a solution $(f_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ of the regularized problems (2.9)–(2.14); that is the assertion of Proposition 2.

3.1. **Proof of Proposition 2.**

3.1.1. *Existence and uniqueness of* $(f^n, \mathbf{u}^n)_{n\in\mathbb{N}}$. First, we focus on the kinetic equation (2.18). For the sake of clarity, in this subsection, we will omit the subscript ε in $(f^n_\varepsilon, \mathbf{u}^n_\varepsilon)_{n\in\mathbb{N}}$. For any n, let us assume that $\mathbf{u}^n$ is given and belongs to $H^1(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;H^1(\mathbb{T}^3))$. Since we have regularized the fluid velocity appearing in the kinetic equation, the unique solution f^n of (2.18) can be defined thanks to the method of characteristics.

Indeed, $g^n = e^{-3t} f^n$ satisfies the transport equation

$$\partial_t g^n + \mathbf{v}\cdot\nabla_{\mathbf{x}} g^n + (\theta_\varepsilon \star \mathbf{u}^n - \mathbf{v})\cdot\nabla_{\mathbf{v}} g^n = 0.$$

The solution of the previous equation can be written in terms of characteristics. As a matter of fact, let us consider the solutions of the following system:

$$\frac{\mathrm{d}\mathbf{x}^n}{\mathrm{d}t}(t) = \mathbf{v}^n(t), \tag{3.1}$$

$$\frac{\mathrm{d}\mathbf{v}^n}{\mathrm{d}t}(t) = (\theta_\varepsilon \star \mathbf{u}^n)(t, \mathbf{x}^n(t)) - \mathbf{v}^n(t), \tag{3.2}$$

with initial conditions $\mathbf{x}^n(0) = \mathbf{x}$ and $\mathbf{v}^n(0) = \mathbf{v}$, and set $\boldsymbol{\chi}^n(t,\mathbf{x},\mathbf{v}) = (\mathbf{x}^n(t), \mathbf{v}^n(t))$ for any $(t,\mathbf{x},\mathbf{v})$. Then we have

$$g^n(t,\mathbf{x},\mathbf{v}) = f^\varepsilon_{\mathrm{in}}(\boldsymbol{\chi}^n(t,\mathbf{x},\mathbf{v})), \qquad \forall\, t,\mathbf{x},\mathbf{v}. \tag{3.3}$$

Since $\mathbf{u}^n \in H^1(0,T;L^2(\mathbb{T}^3))$, $\theta_\varepsilon \star \mathbf{u}^n \in \mathcal{C}^0([0,T];\mathcal{C}^k(\mathbb{T}^3\times\mathbb{R}^3))$ for any $k\in\mathbb{N}$, and consequently f^n lies in $\mathcal{C}^1([0,T];\mathcal{C}^k(\mathbb{T}^3\times\mathbb{R}^3))$ for any $k\in\mathbb{N}$.

Next, we focus on the modified Navier-Stokes equation (2.19).

Assume that $(f^n, \mathbf{u}^n)$ is given in $L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)\times L^\infty(0,T;L^2(\mathbb{T}^3))$. Since we cut off the high particle velocities, the drag force $\int_{\mathbb{R}^3} f^n\,(\mathbf{v} - \mathbf{u}^n)\,\gamma(\mathbf{v})\,\mathrm{d}\mathbf{v}$ at least belongs to $L^2(0,T;L^2(\mathbb{T}^3))$. Applying standard results about the Navier-Stokes system [6] with a mollified convection term, one can prove the existence and uniqueness of $\mathbf{u}^{n+1} \in L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;H^1(\mathbb{T}^3))$ solving (2.19).

Note that the uniqueness relies on the fact that the convection velocity has been regularized. Moreover, since the convection velocity has been regularized and the initial velocity is smooth, one can check that $\partial_t \mathbf{u}^n \in L^2(0,T;L^2(\mathbb{T}^3))$. By a simple induction argument, these properties hold for all $n \in \mathbb{N}$.

In the next subsection, we derive uniform estimates in these spaces with respect to n satisfied by the sequence $(f^n, \mathbf{u}^n)_{n\in\mathbb{N}}$.

3.1.2. *Uniform estimates for the sequence* $(f^n, \mathbf{u}^n)_{n\in\mathbb{N}}$. Using (3.3), we immediately obtain (2.21). Note that this estimate does not depend on the fluid velocity $\mathbf{u}^n$.

To obtain the uniform bounds on $(\mathbf{u}^n)_{n\in\mathbb{N}}$, let us multiply (2.19) by $\mathbf{u}^{n+1}$ and integrate over $\mathbb{T}^3$. Since the fluid is incompressible and the problem periodic in the space variable, we have

$$\int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}^{n+1}) \cdot \nabla_{\mathbf{x}})\mathbf{u}^{n+1} \cdot \mathbf{u}^{n+1} \,\mathrm{d}\mathbf{x} = 0.$$

Moreover, thanks to (2.21) and the use of the cutoff function γ_ε,

$$\Big| \iint_{\mathbb{T}^3\times\mathbb{R}^3} f^n(\mathbf{v} - \mathbf{u}^n)\cdot \mathbf{u}^{n+1}\, \gamma_\varepsilon(\mathbf{v}) \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\Big| \\ \leq C(\varepsilon)\left(1 + \|\mathbf{u}^n\|^2_{L^2(\mathbb{T}^3)} + \|\mathbf{u}^{n+1}\|^2_{L^2(\mathbb{T}^3)}\right).$$

Consequently, using the periodicity, the incompressibility and the fact that

$$\Big| \int_{\mathbb{T}^3} \mathbf{F}^\varepsilon_{\mathbf{ext}} \cdot \mathbf{u}^{n+1} \,\mathrm{d}\mathbf{x}\Big| \leq \frac{1}{2}\|\mathbf{u}^{n+1}\|^2_{L^2(\mathbb{T}^3)} + \frac{1}{2}\|\mathbf{F}^\varepsilon_{\mathbf{ext}}\|^2_{L^2(\mathbb{T}^3)},$$

we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}^{n+1}\|^2_{L^2(\mathbb{T}^3)} + \|\nabla_{\mathbf{x}}\mathbf{u}^{n+1}\|^2_{L^2(\mathbb{T}^3)} \\ \leq C(\varepsilon)\left(1 + \|\mathbf{u}^n\|^2_{L^2(\mathbb{T}^3)} + \|\mathbf{u}^{n+1}\|^2_{L^2(\mathbb{T}^3)}\right).$$

Hence, $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{T}^3))\cap L^2(0,T;H^1(\mathbb{T}^3))$ (see Lemma 3 in Appendix A).

Estimate (2.23) is obtained by multiplying (2.19) by $\partial_t \mathbf{u}^{n+1}$. Indeed, integrating over $\mathbb{T}^3$, using estimate (2.22) on $(\mathbf{u}^n)$, the fact that $\int_{\mathbb{R}^3} f^n(\mathbf{v} - \mathbf{u}^n)\,\gamma_\varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{v}$ is uniformly bounded in $L^2(0,T;L^2(\mathbb{T}^3))$ with respect to n and the fact that the convection velocity has been regularized, we deduce the uniform bound of $(\partial_t\mathbf{u}^n)_{n\in\mathbb{N}}$ in $L^2(0,T;L^2(\mathbb{T}^3))$.

Remark 3. One could also obtain a uniform (with respect to n) bound of $(f^n)_{n\in\mathbb{N}}$, for instance, in $W^{1,\infty}((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)$.

3.1.3. *Convergence.* We now prove that the sequence $(f^n, \mathbf{u}^n)_{n\in\mathbb{N}}$ is convergent in $L^\infty(((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)\times L^\infty(0,T;L^2(\mathbb{T}^3))$. To that purpose, we study the differences $\mathbf{w}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}^n$ and $f^{n+1} - f^n$, for any $n \in \mathbb{N}$.

The function $\mathbf{w}^{n+1}$ satisfies the following weak formulation:

$$\int_{\mathbb{T}^3} \partial_t \mathbf{w}^{n+1}\cdot\boldsymbol{\psi} + \int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{w}^{n+1})\cdot\nabla_{\mathbf{x}})\,\mathbf{u}^{n+1}\cdot\boldsymbol{\psi} + \int_{\mathbb{T}^3} ((\theta_\varepsilon\star\mathbf{u}^n)\cdot\nabla_{\mathbf{x}})\,\mathbf{w}^{n+1}\cdot\boldsymbol{\psi} + \int_{T^3}\nabla_{\mathbf{x}}\mathbf{w}^{n+1} : \nabla_{\mathbf{x}}\boldsymbol{\psi}$$
$$= \iint_{\mathbb{T}^3\times\mathbb{R}^3} f^n(\mathbf{u}^n - \mathbf{v})\,\gamma_\varepsilon(\mathbf{v})\cdot\boldsymbol{\psi} - \iint_{\mathbb{T}^3\times\mathbb{R}^3} f^{n-1}(\mathbf{u}^{n-1}-\mathbf{v})\,\gamma_\varepsilon(\mathbf{v})\cdot\boldsymbol{\psi},$$

for any $\boldsymbol{\psi}\in L^2(0,T;H^1(\mathbb{T}^3))$ such that $\nabla_{\mathbf{x}}\cdot\boldsymbol{\psi} = 0$.

The function $\boldsymbol{\psi} = \mathbf{w}^{n+1}$ is then an admissible test function. The only terms which we must take care of are the nonlinear ones: the convection one

$$T_1 = \int_{\mathbb{T}^3} ((\theta_\varepsilon\star\mathbf{w}^{n+1})\cdot\nabla_{\mathbf{x}})\,\mathbf{u}^{n+1}\cdot\mathbf{w}^{n+1} + \int_{\mathbb{T}^3}((\theta_\varepsilon\star\mathbf{u}^n)\cdot\nabla_{\mathbf{x}})\,\mathbf{w}^{n+1}\cdot\mathbf{w}^{n+1},$$

and the drag force one

$$T_2 = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f^n(\mathbf{u}^n-\mathbf{v})\,\gamma_\varepsilon(\mathbf{v})\cdot\mathbf{w}^{n+1} - \iint_{\mathbb{T}^3\times\mathbb{R}^3} f^{n-1}(\mathbf{u}^{n-1}-\mathbf{v})\,\gamma_\varepsilon(\mathbf{v})\cdot\mathbf{w}^{n+1}.$$

Thanks to the space periodicity and the fluid incompressibility, we have

$$\int_{\mathbb{T}^3}((\theta_\varepsilon\star\mathbf{w}^{n+1})\cdot\nabla_{\mathbf{x}})\,\mathbf{u}^{n+1}\cdot\mathbf{w}^{n+1} = -\int_{\mathbb{T}^3}((\theta_\varepsilon\star\mathbf{w}^{n+1})\cdot\nabla_{\mathbf{x}})\,\mathbf{w}^{n+1}\cdot\mathbf{u}^{n+1},$$

and

$$\int_{\mathbb{T}^3}((\theta_\varepsilon\star\mathbf{u}^n)\cdot\nabla_{\mathbf{x}})\,\mathbf{w}^{n+1}\cdot\mathbf{w}^{n+1} = 0.$$

Consequently, using (2.22), we successively get

$$\begin{aligned}|T_1| &\le \|\theta_\varepsilon\star\mathbf{w}^{n+1}\|_{L^\infty(\mathbb{T}^3)}\|\mathbf{u}^{n+1}\|_{L^2(\mathbb{T}^3)}\|\nabla_{\mathbf{x}}\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)},\\ &\le C(\varepsilon)\,\|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}\|\nabla_{\mathbf{x}}\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}.\end{aligned}$$

Finally, we have

$$|T_1| \le C(\varepsilon)\,\|\mathbf{w}^{n+1}\|^2_{L^2(\mathbb{T}^3)} + \frac{1}{2}\|\nabla_{\mathbf{x}}\mathbf{w}^{n+1}\|^2_{L^2(\mathbb{T}^3)}.$$

We now take care of T_2:

$$T_2 = \iint_{\mathbb{T}^3\times\mathbb{R}^3}(f^n - f^{n-1})(\mathbf{u}^n-\mathbf{v})\,\gamma_\varepsilon(\mathbf{v})\cdot\mathbf{w}^{n+1}$$

$$- \iint_{\mathbb{T}^3\times\mathbb{R}^3} f^{n-1}(\mathbf{u}^n - \mathbf{u}^{n-1})\,\gamma_\varepsilon(\mathbf{v})\cdot\mathbf{w}^{n+1}.$$

Taking into account the cutoff function and (2.21)–(2.22), we get

$$\begin{aligned} |T_2| \;&\leq\; C(\varepsilon)\,\big(\|f^n - f^{n-1}\|_{L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)}\|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)} \\ &\qquad\qquad + \;\|\mathbf{w}^{n+1}\|_{L^2(\mathbb{T}^3)}\|\mathbf{w}^n\|_{L^2(\mathbb{T}^3)}\big) \\ &\leq\; C(\varepsilon)\,\Big(\|f^n - f^{n-1}\|^2_{L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)} \\ &\qquad\qquad + \;\|\mathbf{w}^{n+1}\|^2_{L^2(\mathbb{T}^3)} + \|\mathbf{w}^n\|^2_{L^2(\mathbb{T}^3)}\Big). \end{aligned} \tag{3.4}$$

The quantity $\|f^n - f^{n-1}\|^2_{L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)}$ has now to be estimated. We recall that, for all $n\in\mathbb{N}$ and $(t,\mathbf{x},\mathbf{v})$, $f^n(t,\mathbf{x},\mathbf{v}) = e^{3t} f^\varepsilon_{\mathrm{in}}(\boldsymbol{\chi}^n(t,\mathbf{x},\mathbf{v}))$ and that $f^\varepsilon_{\mathrm{in}}$ has been mollified. Consequently, we have

$$\|f^n - f^{n-1}\|_{L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)} \leq C(\varepsilon,T)\,\|\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1}\|_{L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)}. \tag{3.5}$$

Definition (3.1)–(3.2) of the characteristics $\boldsymbol{\chi}^n = (\mathbf{x}^n, \mathbf{v}^n)$ and estimate (2.22) imply that, for any t,

$$\begin{aligned} &(\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1})(t)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \\ &\quad\leq C(\varepsilon)\left(\int_0^t \|(\theta_\varepsilon\star\mathbf{u}^n - \theta_\varepsilon\star\mathbf{u}^{n-1})(s)\|_{L^\infty(\mathbb{T}^3)}\,\mathrm{d}s\right. \\ &\qquad\left. + \int_0^t (1 + \|(\theta_\varepsilon\star\mathbf{u}^n)(s)\|_{W^{1,\infty}(\mathbb{T}^3)})\,\|(\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1})(s)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}\,\mathrm{d}s\right). \end{aligned}$$

Thus, for any t, we can write

$$\begin{aligned} &\|(\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1})(t)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \\ &\quad\leq C(\varepsilon)\,\left(\int_0^t \|\mathbf{w}^n(s)\|_{L^2(\mathbb{T}^3)}\,\mathrm{d}s + \int_0^t \|(\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1})(s)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}\,\mathrm{d}s\right). \end{aligned}$$

Using Gronwall's lemma, we obtain, for any $t\leq T$,

$$\|(\boldsymbol{\chi}^n - \boldsymbol{\chi}^{n-1})(t)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \leq C(\varepsilon,T)\,\int_0^t \|\mathbf{w}^n(s)\|_{L^2(\mathbb{T}^3)}\,\mathrm{d}s. \tag{3.6}$$

Consequently, with (3.5) and (3.6), (3.4) becomes

$$|T_2| \leq C(\varepsilon,T)\,\left(\int_0^t \|\mathbf{w}^n(s)\|^2_{L^2(\mathbb{T}^3)}\mathrm{d}s + \|\mathbf{w}^{n+1}\|^2_{L^2(\mathbb{T}^3)} + \|\mathbf{w}^n\|^2_{L^2(\mathbb{T}^3)}\right).$$

We finally obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}^{n+1}\|^2_{L^2(\mathbb{T}^3)} + \frac{1}{2}\|\nabla_{\mathbf{x}}\mathbf{w}^{n+1}\|^2_{L^2(\mathbb{T}^3)} \leq C(\varepsilon,T)\left(\int_0^t \|\mathbf{w}^n(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s + \|\mathbf{w}^{n+1}\|^2_{L^2(\mathbb{T}^3)} + \|\mathbf{w}^n\|^2_{L^2(\mathbb{T}^3)}\right).$$

Therefore, integrating the previous inequality with respect to t, and using

$$\int_0^t\int_0^s \|\mathbf{w}^n(r)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}r\,\mathrm{d}s \leq T\int_0^t \|\mathbf{w}^n(r)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}r,$$

we obtain, for any t, $0 \leq t \leq T$,

$$\|\mathbf{w}^{n+1}(t)\|^2_{L^2(\mathbb{T}^3)} \leq C(\varepsilon,T)\left(\int_0^t \|\mathbf{w}^n(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s + \int_0^t \|\mathbf{w}^{n+1}(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s\right).$$

Lemma 3 then implies that there exists a constant $K(\varepsilon,T) > 0$, which does not depend on n, such that

$$\|\mathbf{w}^n\|^2_{L^\infty(0,T;L^2(\mathbb{T}^3))} \leq \frac{K^nT^n}{n!}.$$

Consequently, the sequence $(\mathbf{u}^n)_{n\in\mathbb{N}}$ converges in $L^\infty(0,T;L^2(\mathbb{T}^3))$. That also implies, thanks to (3.5) and (3.6), that the sequence $(f^n)_{n\in\mathbb{N}}$ converges in $L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)$.

Remark 4. Note that, at this stage, to study the regularized and uncoupled system, we used the high particle velocity cutoff function to control the force applied by the spray on the fluid. Moreover, the space regularization of the fluid velocity in the Vlasov equation enables us to apply the method of characteristics and obtain a strong solution (this will also be true at the limit as $n \to \infty$ for any fixed $\varepsilon > 0$). Finally, the space regularization of the convection fluid velocity simplifies the study of the fluid equations by making the convection term easier to deal with.

3.2. **Asymptotics with respect to n.** As $\varepsilon > 0$ is still fixed, we can pass to the limit when n goes to infinity. We already know that $(\mathbf{u}^n_\varepsilon)_n$ converges to $\mathbf{u}_\varepsilon$ strongly in $L^\infty(0,T;L^2(\mathbb{T}^3))$ and, thanks to (2.22), weakly in $L^2(0,T;H^1(\mathbb{T}^3))$, and that $(f^n_\varepsilon)_n$ strongly converges to f_ε in $L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)$. Thus, since f^n_ε belongs to $\mathcal{C}^0([0,T]\times\mathbb{T}^3\times\mathbb{R}^3)$, so does f_ε.

These convergences enable us to pass to the limit with respect to n in the following weak formulation, for all t, for all $\phi \in \mathcal{C}^1([0,T]\times\mathbb{T}^3\times\mathbb{R}^3)$:

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} f^n_\varepsilon(t)\,\phi(t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}$$

$$- \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^n \left[\partial_t\phi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\phi + (\theta_\varepsilon \star \mathbf{u}_\varepsilon^n - \mathbf{v})\cdot\nabla_{\mathbf{v}}\phi\right] \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_{\mathrm{in}}^\varepsilon\,\phi(0)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}. \quad (3.7)$$

Remark 5. Note that here the test functions are not necessarily with compact support in $\mathbf{v}$. Indeed, thanks to (3.6), the sequence $(\boldsymbol{\chi}_\varepsilon^n)_n$ converges to $\boldsymbol{\chi}_\varepsilon$ in $L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)$ when n goes to $+\infty$. Then we check

$$f_\varepsilon(t,\mathbf{x},\mathbf{v}) = e^{3t} f_{\mathrm{in}}^\varepsilon(\boldsymbol{\chi}_\varepsilon(t,\mathbf{x},\mathbf{v})). \quad (3.8)$$

Since we assumed that $f_{\mathrm{in}}^\varepsilon$ has been chosen with compact support in $\mathbf{v}$, it is also the case for $f_\varepsilon(t,\mathbf{x},\cdot)$ for all $(t,\mathbf{x})\in[0,T]\times\mathbb{T}^3$.

Note that here the solution f_ε is in fact a strong solution. Furthermore, for all $\boldsymbol{\psi}\in\mathcal{C}^1([0,T]\times\mathbb{T}^3)$ such that $\nabla_{\mathbf{x}}\cdot\boldsymbol{\psi}=0$, for almost every t,

$$\begin{aligned} &\int_0^t \int_{\mathbb{T}^3} \partial_t \mathbf{u}_\varepsilon^{n+1}\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{T}^3} \left((\theta_\varepsilon\star\mathbf{u}_\varepsilon^{n+1})\cdot\nabla_{\mathbf{x}}\right)\,\mathbf{u}_\varepsilon^{n+1}\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \nabla_{\mathbf{x}}\mathbf{u}_\varepsilon^{n+1} : \nabla_{\mathbf{x}}\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\ &= -\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^n(\mathbf{u}_\varepsilon^n - \mathbf{v})\,\gamma_\varepsilon(\mathbf{v})\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \mathbf{F}_{\mathbf{ext}}^\varepsilon\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s. \end{aligned}$$

This yields (2.15) and (2.16). The limit function $(f_\varepsilon, \mathbf{u}_\varepsilon)$ thus solves the (variational formulation of the) regularized system (2.9)–(2.14).

This ends the proof of Proposition 2.

We now derive uniform bounds in ε for $(f_\varepsilon,\mathbf{u}_\varepsilon)$. In a first step, these estimates only hold locally in time (but they only depend on T, $\|\mathbf{u}_{\mathbf{in}}\|_{L^2(\mathbb{T}^3)}^2 + |M_2 f_{\mathrm{in}}| + \|\mathbf{F}_{\mathbf{ext}}\|_{L^2(0,T;L^2(\mathbb{T}^3))}^2$ and $\|f_{\mathrm{in}}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}$; this will be crucial in the last part of the proof of Theorem 1). This constitutes the following.

3.3. **Proof of Proposition 1.** We here obtain some uniform estimates on the sequence $(f_\varepsilon,\mathbf{u}_\varepsilon)_{\varepsilon>0}$. The fact that $\|f_{\mathrm{in}}^\varepsilon\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \le \|f_{\mathrm{in}}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}$, together with equality (3.8), implies (2.17). Therefore, $(f_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty((0,T)\times\mathbb{T}^3\times\mathbb{R}^3)$. Consequently, and thanks to Remark 5, $|\mathbf{v}|^2$ is an admissible test function in the variational formulation (2.15).

A similar argument to the one used in the proof of Proposition 2 ensures that f_ε at least belongs to $\mathcal{C}^1([0,T]\times\mathbb{T}^3\times\mathbb{R}^3)$. That allows us to write

$$\frac{\mathrm{d}}{\mathrm{d}t}M_2 f_\varepsilon + 2M_2 f_\varepsilon \le 2\int_{\mathbb{T}^3} |\theta_\varepsilon\star\mathbf{u}_\varepsilon(t,\mathbf{x})|\,m_1 f_\varepsilon(t,\mathbf{x})\,\mathrm{d}\mathbf{x}. \quad (3.9)$$

To estimate $m_1 f_\varepsilon$ in terms of second-order moments of f_ε, we need the following.

Lemma 1. *Let $\beta > 0$ and g be a nonnegative function in $L^\infty((0,T)\times \mathbb{T}^3\times\mathbb{R}^3)$, such that $m_\beta g(t,\mathbf{x}) < +\infty$, for almost every $(t,\mathbf{x})$. The following estimate holds for any $\alpha < \beta$:*

$$m_\alpha g(t,\mathbf{x}) \le \big(\frac{4}{3}\pi\|g(t,\mathbf{x},\cdot)\|_{L^\infty(\mathbb{R}^3)} + 1\big) m_\beta g(t,\mathbf{x})^{\frac{\alpha+3}{\beta+3}}, \qquad a.e.\ (t,\mathbf{x}).$$

The same kind of result can be found in [13] or [10].

Proof of Lemma 1. We write, for almost every $(t,\mathbf{x})$,

$$\begin{aligned} m_\alpha g(t,\mathbf{x}) &= \int_{|\mathbf{v}|\le R} |\mathbf{v}|^\alpha g + \int_{|\mathbf{v}|\ge R} |\mathbf{v}|^\alpha g \\ &\le \frac{4}{3}\pi R^{\alpha+3}\|g(t,\mathbf{x},\cdot)\|_{L^\infty(\mathbb{R}^3)} + \frac{1}{R^{\beta-\alpha}}\, m_\beta g(t,\mathbf{x}). \end{aligned}$$

By choosing $R = m_\beta g(t,\mathbf{x})^{\frac{1}{\beta+3}}$, we find the required result. This ends the proof of Lemma 1.

Applying Lemma 1, we obtain, thanks to (2.17),

$$0 \le m_1 f_\varepsilon(t,\mathbf{x}) \le C\, m_2 f_\varepsilon(t,\mathbf{x})^{4/5},$$

which yields

$$\|m_1 f_\varepsilon(t)\|_{L^{5/4}(\mathbb{T}^3)} \le C\, M_2 f_\varepsilon(t)^{4/5}, \tag{3.10}$$

where C only depends on T and $\|f_{\rm in}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}$. This estimate, together with (3.9), gives

$$\frac{\mathrm{d}}{\mathrm{d}t} M_2 f_\varepsilon + 2M_2 f_\varepsilon \le \|\theta_\varepsilon\star\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)}\|m_1 f_\varepsilon\|_{L^{5/4}(\mathbb{T}^3)} \le C\|\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)}(M_2 f_\varepsilon)^{4/5}.$$

Setting $H(t) = M_2 f_\varepsilon(t)^{1/5}$, we obtain

$$H' + H \le C\|\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)}.$$

Since $M_2 f_\varepsilon \ge 0$, and assuming that $M_2 f^\varepsilon_{\rm in} \le C\, M_2 f_{\rm in}$, we have, for almost every t,

$$H(t) \le C\Big(1 + \int_0^t \|\mathbf{u}_\varepsilon(s)\|_{L^5(\mathbb{T}^3)}\,\mathrm{d}s\Big),$$

and thus

$$M_2 f_\varepsilon(t) \le C\Big(1 + \int_0^t \|\mathbf{u}_\varepsilon(s)\|_{L^5(\mathbb{T}^3)}\,\mathrm{d}s\Big)^5, \tag{3.11}$$

where C only depends on the initial data.

Next, we take care of the fluid velocity. The function $\mathbf{u}_\varepsilon$ is an admissible test function for the variational formulation (2.16). Indeed, by density, (2.16) remains valid for any $\boldsymbol{\psi} \in L^2(0,T;H^1(\mathbb{T}^3))$. After integration by parts, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)} + \|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)} = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon \mathbf{u}_\varepsilon \cdot (\mathbf{v}-\mathbf{u}_\varepsilon)\,\gamma_\varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v} + \int_{\mathbb{T}^3} \mathbf{F}_{\mathbf{ext}}\cdot \mathbf{u}_\varepsilon.$$

Since $\gamma_\varepsilon \geq 0$, that implies

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)} + \|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)}\\
&\leq \int_{\mathbb{T}^3} |\mathbf{u}_\varepsilon|\, m_1 f_\varepsilon \mathrm{d}\mathbf{x} + \|\mathbf{F}_{\mathbf{ext}}\|_{L^2(\mathbb{T}^3)}\|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}\\
&\leq \|\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)}\,\|m_1 f_\varepsilon\|_{L^{5/4}(\mathbb{T}^3)} + \|\mathbf{F}_{\mathbf{ext}}\|_{L^2(\mathbb{T}^3)}\|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}.
\end{aligned}$$

Applying Young's inequality and remembering (3.10) and (3.11), we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)} + \|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)}\\
&\quad\leq C\|\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)}\left(1+\int_0^t \|\mathbf{u}_\varepsilon\|_{L^5(\mathbb{T}^3)}\right)^4 + C\|\mathbf{F}_{\mathbf{ext}}\|^2_{L^2(\mathbb{T}^3)} + C\|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)}.
\end{aligned}$$

Next, using the continuous injection of $H^1(\mathbb{T}^3)$ in $L^5(\mathbb{T}^3)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)} + \|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)}\\
&\qquad\leq \frac{1}{2}\|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)} + C\|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)} + C\\
&\qquad\quad + C\left(\int_0^t \left(\|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)} + \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{T}^3)}\right)\right)^8 + C\|\mathbf{F}_{\mathbf{ext}}\|^2_{L^2(\mathbb{T}^3)}. \qquad (3.12)
\end{aligned}$$

Using

$$\int_0^t\int_0^s \|\mathbf{u}_\varepsilon(r)\|^8_{L^2(\mathbb{T}^3)}\,\mathrm{d}r\,\mathrm{d}s \leq T\int_0^t \|\mathbf{u}_\varepsilon(r)\|^8_{L^2(\mathbb{T}^3)}\,\mathrm{d}r,$$

and Hölder's inequality, we integrate (3.12) to obtain

$$\frac{1}{2}\|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)}(t) + \frac{1}{4}\int_0^t \|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)}$$

$$\leq \frac{1}{4}\int_0^t \|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)} + \frac{1}{2}\|\mathbf{u}^\varepsilon_{\mathbf{in}}\|^2_{L^2(\mathbb{T}^3)} + C + C\int_0^t \left(\int_0^s \|\nabla_\mathbf{x}\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)}\right)^4 + C\int_0^t \|\mathbf{u}_\varepsilon\|^8_{L^2(\mathbb{T}^3)} + C\int_0^t \|\mathbf{F_{ext}}\|^2_{L^2(\mathbb{T}^3)}.$$

A nonlinear Gronwall lemma thus implies that there exist a time $T^* > 0$ and a constant C such that, for almost every $t \leq T^*$,

$$\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\mathbb{T}^3)} + \int_0^t \|\nabla_\mathbf{x}\mathbf{u}_\varepsilon(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s \leq C. \tag{3.13}$$

We recall here that

$$C := C(T, T^*, \|\mathbf{u_{in}}\|^2_{L^2(\mathbb{T}^3)} + |M_2 f_{\mathrm{in}}| + \|\mathbf{F_{ext}}\|^2_{L^2(0,T;L^2(\mathbb{T}^3))}, \|f_{\mathrm{in}}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}).$$

Finally, estimates (3.11) and (3.13) imply that

$$M_2 f_\varepsilon(t) \leq C, \qquad \text{a.e. } t \leq T^*.$$

3.4. **Asymptotics with respect to $\boldsymbol{\varepsilon}$.** In order to pass to the limit in the weak formulations (2.15)–(2.16), we need to have some compactness properties for the sequence $(f_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$. In fact, proving that $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is compact in $L^p(0,T;L^2(\mathbb{T}^3))$, for some p, is enough for the asymptotics $\varepsilon \to 0$. From (2.16), it is easy to check that $(\partial_t \mathbf{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^{3/2}(0,T^*;\mathcal{V}')$. As a matter of fact, once again, the only two terms we have to take care of are the convection and drag force ones.

For the convection term, we use standard arguments:

$$\int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot\nabla_\mathbf{x})\,\mathbf{u}_\varepsilon \cdot \boldsymbol{\psi} = -\int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot\nabla_\mathbf{x})\,\boldsymbol{\psi}\cdot\mathbf{u}_\varepsilon.$$

But $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(0,T^*;L^2(\mathbb{T}^3)) \cap L^2(0,T^*;H^1(\mathbb{T}^3))$, which is continuously embedded in $L^3(0,T^*;L^4(\mathbb{T}^3))$. Thus,

$$\left|\int_0^{T^*}\int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot\nabla_\mathbf{x})\,\mathbf{u}_\varepsilon\cdot\boldsymbol{\psi}\right| \leq C\|\nabla_\mathbf{x}\boldsymbol{\psi}\|_{L^3(0,T^*;L^2(\mathbb{T}^3))}.$$

Thus, the form

$$\boldsymbol{\psi} \mapsto \int_0^{T^*}\int_{\mathbb{T}^3} ((\theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot\nabla_\mathbf{x})\,\mathbf{u}_\varepsilon\cdot\boldsymbol{\psi}$$

is bounded in $L^{3/2}(0,T^*;\mathcal{V}')$.

The term coming from the drag force can be estimated as follows:

$$\left|\int_0^{T^*}\iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{v})\gamma_\varepsilon(\mathbf{v})\boldsymbol{\psi}\right| \\ \le C\|\mathbf{u}_\varepsilon\|_{L^2(0,T^*;L^6(\mathbb{T}^3))}\|\boldsymbol{\psi}\|_{L^2(0,T^*;L^6(\mathbb{T}^3))}\|m_0 f_\varepsilon\|_{L^\infty(0,T^*;L^{3/2}(\mathbb{T}^3))} \\ + \|\boldsymbol{\psi}\|_{L^2(0,T^*;L^5(\mathbb{T}^3))}\|m_1 f_\varepsilon\|_{L^2(0,T^*;L^{5/4}(\mathbb{T}^3))}.$$

But, thanks to Lemma 1, we have

$$m_0 f_\varepsilon \le C(m_2 f_\varepsilon)^{3/5}, \qquad m_1 f_\varepsilon \le C(m_2 f_\varepsilon)^{4/5},$$

and we proved that $\|M_2 f_\varepsilon\|_{L^\infty(0,T^*)} \le C$.

Therefore, $(m_0 f_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0,T^*;L^{5/3}(\mathbb{T}^3))$ and $(m_1 f_\varepsilon)_{\varepsilon>0}$ in $L^\infty(0,T^*;L^{5/4}(\mathbb{T}^3))$.

Moreover, the sequence $(\mathbf{u}_\varepsilon)$ is bounded in $L^2(0,T^*;H^1(\mathbb{T}^3))$ and thus in $L^2(0,T^*;L^6(\mathbb{T}^3))$.

Finally, we can write

$$\left|\int_0^{T^*}\iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{v})\gamma_\varepsilon(\mathbf{v})\boldsymbol{\psi}\right| \le C\|\boldsymbol{\psi}\|_{L^2(0,T^*;L^6(\mathbb{T}^3))} \le C\|\boldsymbol{\psi}\|_{L^2(0,T^*;H^1(\mathbb{T}^3))},$$

and the drag force $\left(\int_{\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{v})\,\gamma_\varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{v}\right)_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T^*;\mathcal{V}')$.

That ensures that $(\partial_t\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^{3/2}(0,T^*;\mathcal{V}')$.

Thanks to those bounds, there exists $(f,\mathbf{u}) \in L^\infty((0,T^*)\times\mathbb{T}^3\times\mathbb{R}^3)\times (L^\infty(0,T^*;L^2(\mathbb{T}^3))\cap L^2(0,T^*;H^1(\mathbb{T}^3))\cap\mathcal{C}^0([0,T^*];\mathcal{V}'))$ such that, up to a subsequence,

$$f_\varepsilon \rightharpoonup f, \qquad \text{weakly } -* \text{ in } L^\infty((0,T^*)\times\mathbb{T}^3\times\mathbb{R}^3),$$

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}, \qquad \nabla_{\mathbf{x}}\mathbf{u}_\varepsilon \rightharpoonup \nabla_{\mathbf{x}}\mathbf{u}, \quad \text{weakly in} \qquad L^2((0,T)\times\mathbb{T}^3),$$

and

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}, \qquad \text{weakly}-* \text{ in} \quad L^\infty((0,T);L^2(\mathbb{T}^3)).$$

Thanks to the Aubin compactness lemma (respectively the Ascoli theorem), since $(\partial_t\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^{3/2}(0,T^*;\mathcal{V}')$, we can state that $(\mathbf{u}_\varepsilon)$ strongly converges to $\mathbf{u}$ in $L^2(0,T^*;L^2(\mathbb{T}^3))$ (respectively in $\mathcal{C}^0([0,T^*];\mathcal{V}')$).

Remark 6. The fact that $\mathbf{u}$ belongs to $\mathcal{C}^0(0,T^*;\mathcal{V}')$ and $L^\infty(0,T^*;\mathcal{H})$ implies that $\mathbf{u}$ is continuous with respect to time, with values in $L^2(\mathbb{T}^3)$ endowed with its weak topology (see [11], Lemma 8.1).

These convergences allow us to pass to the limit in the following weak formulations:

$$- \int_0^{T^*} \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon \left[\partial_t\phi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\phi + (\theta_\varepsilon \star \mathbf{u}_\varepsilon - \mathbf{v})\cdot\nabla_{\mathbf{v}}\phi\right] \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_{\mathrm{in}}^\varepsilon\,\phi(0)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v},$$

for all $\phi \in \mathcal{C}^1([0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3)$ with compact support in $\mathbf{v}$ and such that $\phi(T^*)=0$, and, for almost every $t \le T^*$,

$$\begin{aligned}
&- \int_{\mathbb{T}^3} \mathbf{u}_\varepsilon(t)\cdot\boldsymbol{\psi}(t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{T}^3} ((\theta_\varepsilon\star\mathbf{u}_\varepsilon)\cdot\nabla_{\mathbf{x}})\ \mathbf{u}_\varepsilon\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \nabla_{\mathbf{x}}\mathbf{u}_\varepsilon : \nabla_{\mathbf{x}}\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&= - \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon\,(\mathbf{u}_\varepsilon - \mathbf{v})\,\gamma_\varepsilon(\mathbf{v})\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \mathbf{F}^\varepsilon_{\mathbf{ext}}\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s,
\end{aligned}$$

for all $\boldsymbol{\psi}\in\mathcal{C}^1([0,T^*]\times\mathbb{T}^3)$ such that $\nabla_{\mathbf{x}}\cdot\boldsymbol{\psi}=0$. We then obtain a solution satisfying (2.6)–(2.7). This ends the proof of Proposition 1.

3.5. **Global existence.** We just derived the existence locally in time of a weak solution $(f,\mathbf{u})$. In this section, we prove that this solution is, in fact, global in time, which constitutes the last step of the proof of Theorem 1. We start with the following.

Lemma 2. *The solution $(f,\mathbf{u})$ obtained above satisfies the pointwise estimate, for almost every $t \le T^*$,*

$$\begin{aligned}
&\frac{1}{2}M_2 f(t) + \frac{1}{2}\|\mathbf{u}(t)\|^2_{L^2(\mathbb{T}^3)} \\
&+ \int_0^t \|\nabla_{\mathbf{x}}\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s + \int_0^t\iint_{\mathbb{T}^3\times\mathbb{R}^3} f|\mathbf{u}-\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s \qquad (3.14)\\
&\le (1+e^T+T\,e^T)\left(\frac{1}{2}\|\mathbf{u_{in}}\|^2_{L^2(\mathbb{T}^3)} + \frac{1}{2}|M_2 f_{\mathrm{in}}| + \frac{1}{2}\,\|\mathbf{F_{ext}}\|^2_{L^2(0,T;L^2(\mathbb{T}^3))}\right),
\end{aligned}$$

and the maximum principle

$$\|f\|_{L^\infty((0,T^*)\times\mathbb{T}^3\times\mathbb{R}^3)} \le e^{3T}\|f_{\mathrm{in}}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)}.$$

Proof of Lemma 2. We first note that the second inequality is directly obtained from the definition of f, as the weak-$*$ limit of (f_ε) in $L^\infty((0,T^*)\times\mathbb{T}^3\times\mathbb{R}^3)$. For the first inequality, we shall study the energy estimate satisfied by $(f_\varepsilon,\mathbf{u}_\varepsilon)$ obtained by choosing once again $\phi=|\mathbf{v}|^2/2$ as a test function in (2.15), and $\mathbf{u}_\varepsilon$ as a test function in (2.16):

$$\frac{1}{2}M_2 f_\varepsilon(t) + \frac{1}{2}\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\mathbb{T}^3)} + \int_0^t \|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s + \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s = \frac{1}{2}M_2 f^\varepsilon_{\mathrm{in}} + \frac{1}{2}\|\mathbf{u}^\varepsilon_{\mathbf{in}}\|^2_{L^2(\mathbb{T}^3)} + \int_0^t \int_{\mathbb{T}^3} \mathbf{F}^\varepsilon_{\mathbf{ext}}\cdot\mathbf{u}_\varepsilon\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + R_\varepsilon(t), \quad (3.15)$$

where

$$\begin{aligned} R_\varepsilon(t) \quad = \quad & \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon|\mathbf{u}_\varepsilon|^2(1-\gamma_\varepsilon(\mathbf{v}))\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s \\ & - \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon\mathbf{u}_\varepsilon\cdot\mathbf{v}(1-\gamma_\varepsilon(\mathbf{v}))\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s \\ & + \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon\star\mathbf{u}_\varepsilon)\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s. \end{aligned}$$

Let $R^i_\varepsilon, i = 1,2,3$, denote each of the three terms appearing in R_ε and $h_\varepsilon = f_\varepsilon(1-\gamma_\varepsilon)$.

We first have, for almost every t,

$$0 \le R^1_\varepsilon(t) \le \|m_0 h^\varepsilon\|_{L^\infty(0,T^*;L^{3/2}(\mathbb{T}^3))}\,\|\mathbf{u}_\varepsilon\|^2_{L^2(0,T^*;L^6(\mathbb{T}^3))}.$$

Lemma 1 implies that

$$\|m_0 h_\varepsilon(t)\|_{L^{3/2}(\mathbb{T}^3)} \le C\,M_{3/2}h_\varepsilon(t)^{3/2},$$

and

$$M_{3/2}h_\varepsilon(t) = \iint_{\mathbb{T}^3\times\mathbb{R}^3} |\mathbf{v}|^{3/2} f_\varepsilon(1-\gamma_\varepsilon) \le \iint_{\mathbb{T}^3\times\{|\mathbf{v}|\ge\frac{1}{2}\varepsilon\}} |\mathbf{v}|^{3/2} f_\varepsilon \le \sqrt{2\varepsilon}\,M_2 f_\varepsilon(t).$$

Consequently, from the uniform bound obtained on $(M_2 f_\varepsilon)_{\varepsilon>0}$ in $L^\infty(0,T^*)$, we deduce that the sequence $(\|m_0 h^\varepsilon\|_{L^\infty(0,T^*;L^{3/2}(\mathbb{T}^3))})$ converges to 0 when ε goes to 0.

Finally, since the sequence $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ has a uniform bound in the functional spaces $L^2(0,T^*;H^1(\mathbb{T}^3)) \hookrightarrow L^2(0,T^*;L^6(\mathbb{T}^3))$, we have proven that (R^1_ε) goes to 0 when $\varepsilon \to 0$.

To treat the second term R^2_ε, we proceed in the same way. For almost every t, we have

$$0 \le R^2_\varepsilon(t) \le \|m_1 h^\varepsilon\|_{L^\infty(0,T^*;L^{6/5}(\mathbb{T}^3))}\,\|\mathbf{u}_\varepsilon\|_{L^2(0,T^*;L^6(\mathbb{T}^3))}.$$

Lemma 1 implies that

$$\|m_1 h_\varepsilon\|_{L^{5/6}(\mathbb{T}^3)}(t) \le C\,M_{9/5}h_\varepsilon(t)^{5/6},$$

and

$$M_{9/5}h_\varepsilon(t) = \iint_{\mathbb{T}^3\times\mathbb{R}^3} |\mathbf{v}|^{9/5} f_\varepsilon(1-\gamma_\varepsilon) \le \iint_{\mathbb{T}^3\times\{|\mathbf{v}|\ge\frac{1}{2}\varepsilon\}} |\mathbf{v}|^{9/5} f_\varepsilon \le (2\varepsilon)^{\frac{1}{5}} M_2 f_\varepsilon(t).$$

Consequently, from the uniform bound obtained on $(M_2 f_\varepsilon)_{\varepsilon>0}$ in $L^\infty(0,T^*)$, we deduce that the sequence $(\|m_1 h^\varepsilon\|_{L^\infty(0,T^*;L^{5/6}(\mathbb{T}^3))})$ converges to 0 when ε goes to 0.

Finally, since the sequence $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ has a uniform bound in the functional spaces $L^2(0,T^*;H^1(\mathbb{T}^3)) \hookrightarrow L^2(0,T^*;L^6(\mathbb{T}^3))$, we have proven that (R^2_ε) goes to 0 when $\varepsilon \to 0$.

We now take care of the last term R^3_ε. Let us write, for any t,

$$\begin{aligned} R^3_\varepsilon(t) &= \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot \mathbf{v}\gamma_\mu(\mathbf{v})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s \\ &\quad + \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot \mathbf{v}(1-\gamma_\mu(\mathbf{v}))\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s, \end{aligned}$$

where $\mu > 0$. From arguments similar to those used for the second term, we can check that

$$0 \le \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot \mathbf{v}(1-\gamma_\mu(\mathbf{v}))\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s \le C\,(2\mu)^{1/5},$$

where C depends neither on ε nor on μ.

Furthermore, since $(f_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty((0,T^*)\times\mathbb{T}^3\times\mathbb{R}^3)$, we simultaneously have, for almost every t,

$$\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot \mathbf{v}\,\gamma_\mu(\mathbf{v})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s \ge 0,$$

and

$$\begin{aligned} &\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot \mathbf{v}\,\gamma_\mu(\mathbf{v})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s \\ &\qquad\qquad \le C(\mu)\|\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon\|_{L^1(0,T^*;L^1(\mathbb{T}^3))}. \end{aligned}$$

Consequently, since $\mathbf{u}_\varepsilon$ strongly converges in $L^2(0,T^*;L^2(\mathbb{T}^3))$, the sequence

$$\left(\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(\mathbf{u}_\varepsilon - \theta_\varepsilon \star \mathbf{u}_\varepsilon)\cdot \mathbf{v}\,\gamma_\mu(\mathbf{v})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s\right)_{\varepsilon>0}$$

goes to 0 for all μ when $\varepsilon \to 0$. That ensures that $(R^3_\varepsilon)_{\varepsilon>0}$ goes to 0 when $\varepsilon \to 0$. We have finally proven that $(R_\varepsilon)_{\varepsilon>0}$ converges to 0 as ε goes to 0.

For the force term, we note that, by Young's inequality,

$$\left|\int_0^t \int_{\mathbb{T}^3} \mathbf{F}^\varepsilon_{\mathbf{ext}} \cdot \mathbf{u}_\varepsilon\right| \leq \frac{1}{2}\int_0^t \|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)} + \frac{1}{2}\int_0^t \|\mathbf{F}^\varepsilon_{\mathbf{ext}}\|^2_{L^2(\mathbb{T}^3)}.$$

Equality (3.15) implies that

$$\begin{aligned}
&\frac{1}{2}M_2 f_\varepsilon(t) + \frac{1}{2}\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\mathbb{T}^3)} \\
&\quad + \int_0^t \|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s + \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s \\
&\qquad \leq \frac{1}{2}M_2 f^\varepsilon_{\mathrm{in}} + \frac{1}{2}\|\mathbf{u}^\varepsilon_{\mathbf{in}}\|^2_{L^2(\mathbb{T}^3)} \\
&\qquad\quad + \frac{1}{2}\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\mathbf{F}^\varepsilon_{\mathbf{ext}}(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s.
\end{aligned}$$

We then apply Gronwall's lemma to get

$$\begin{aligned}
&\frac{1}{2}M_2 f_\varepsilon(t) + \frac{1}{2}\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\mathbb{T}^3)} \\
&\quad + \int_0^t \|\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s + \int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s \\
&\qquad \leq (1+e^t)\left(\frac{1}{2}\|\mathbf{u}^\varepsilon_{\mathbf{in}}\|^2_{L^2(\mathbb{T}^3)} + \frac{1}{2}|M_2 f^\varepsilon_{\mathrm{in}}|\right) \\
&\qquad\quad + \frac{1}{2}(1+t\,e^t)\int_0^t \|\mathbf{F}^\varepsilon_{\mathbf{ext}}\|^2_{L^2(0,T;L^2(\mathbb{T}^3))}\mathrm{d}s,
\end{aligned}$$

and let $\varepsilon \to 0$ in the equation.

Using the weak$-*$ convergence of f_ε in $L^\infty((0,T^*)\times\mathbb{T}^3\times\mathbb{R}^3)$ it is easy to check that

$$M_2 f(t) \leq \liminf_{\mu\to 0} M_2(f\gamma_\mu)(t),$$

and

$$\|M_2(f\gamma_\mu)\|_{L^\infty(0,T^*)} \leq \liminf_{\varepsilon\to 0}\|M_2(f_\varepsilon\gamma_\mu)\|_{L^\infty(0,T^*)} \leq \liminf_{\varepsilon\to 0}\|M_2 f_\varepsilon\|_{L^\infty(0,T^*)}.$$

We treat the term $\int_0^t \iint_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}s$ in the same way to obtain a similar result, using this time also the strong convergence of $\mathbf{u}_\varepsilon$ in $L^2((0,T^*)\times\mathbb{T}^3)$.

Finally, the two previous convergences and the weak convergence of $\nabla_{\mathbf{x}}\mathbf{u}_\varepsilon$ in $L^2((0,T^*)\times\mathbb{T}^3)$ enable us to obtain (3.14). This ends the proof of Lemma 2. □

Thanks to this lemma, we can now end the proof of Theorem 1.

3.6. **Extension of the solution.** Let $T > 0$ be fixed, and consider a set of admissible initial data and an external force. Let us choose $\mathcal{K}$ such that

$$\begin{aligned} \frac{1+e^T+T\,e^T}{2}\Big(\|\mathbf{u_{in}}\|^2_{L^2(\mathbb{T}^3)}+|M_2 f_{\rm in}|+\|\mathbf{F_{ext}}\|^2_{L^2(0,T;L^2(\mathbb{T}^3))}\Big) &\le \mathcal{K},\\ e^{3T}\|f_{\rm in}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} &\le \mathcal{K}. \end{aligned}$$

Then all the constants appearing in the proof of the local estimates can be bounded by a constant $C_{T,\mathcal{K}}$ only depending on T and $\mathcal{K}$. Moreover, because of the choice of $\mathcal{K}$, the external force obviously satisfies

$$\frac{1}{2}\,\|\mathbf{F_{ext}}\|^2_{L^2(0,T;L^2(\mathbb{T}^3))} \le \mathcal{K}$$

and the initial data satisfy

$$\frac{1}{2}\|\mathbf{u_{in}}\|^2_{L^2(\mathbb{T}^3)} \le \mathcal{K},\quad \frac{1}{2}|M_2 f_{\rm in}| \le \mathcal{K},\quad \|f_{\rm in}\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \le \mathcal{K}.$$

Consequently, T^* (defined in Proposition 1) can be bounded below by a nonnegative time $\tau_{T,\mathcal{K}} > 0$, only depending on T and $\mathcal{K}$. From Lemma 2, we can then check that, for almost every $t \le T^*$,

$$\frac{1}{2}\|\mathbf{u}(t)\|^2_{L^2(\mathbb{T}^3)} \le \mathcal{K},\quad \frac{1}{2}|M_2 f(t)| \le \mathcal{K},\quad \|f(t)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \le \mathcal{K}.$$

Moreover, we have

$$\frac{1}{2}\int_t^T \|\mathbf{F_{ext}}(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s \le \mathcal{K},\qquad \text{a.e. } t.$$

That ensures that, starting again from time $T^*-\sigma$, $\sigma > 0$ small enough, for which $\mathbf{u}(T^*-\sigma) \in L^2(\mathbb{T}^3)$, $f(T^*-\sigma) \in L^\infty(\mathbb{T}^3\times\mathbb{R}^3)$ and $M_2 f(T^*-\sigma) \in L^\infty(\mathbb{T}^3)$, and applying once again Proposition 1, we can extend the solution $(f,\mathbf{u})$ on $(T^*-\sigma, T^*+\tau_{T,\mathcal{K}}-\sigma)$. We can then iterate this process until we reach T.

Estimate (2.8) is obtained by letting $\varepsilon \to 0$ in estimate (3.15). The estimates are similar to those used in order to get (3.14)

That ends the proof of Theorem 1.

APPENDIX A. GRONWALL LEMMA

In this section, we state and prove the Gronwall lemma we used several times in this work.

Lemma 3. *Let $T > 0$ and consider a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative continuous functions on $[0,T]$. Assume that (a_n) satisfies, for any n,*

$$a_{n+1}(t) \le A + B\int_0^t a_n(s)\,\mathrm{d}s + C\int_0^t a_{n+1}(s)\,\mathrm{d}s, \qquad 0 \le t \le T,$$

where A, B and C are nonnegative constants.

If $A = 0$, there exists a constant $K \ge 0$ such that

$$a_n(t) \le \frac{K^{n+1}t^n}{n!}, \qquad 0 \le t \le T, \qquad n \in \mathbb{N}. \tag{A.1}$$

If $A > 0$, there exists a constant $K \ge 0$ depending on A, B, C such that

$$a_n(t) \le K\exp(Kt), \qquad 0 \le t \le T, \qquad n \in \mathbb{N}. \tag{A.2}$$

Proof of Lemma 3: If both $A = C = 0$, we choose

$$K = \max\Big(B, \max_{[0,T]} a_0\Big),$$

and the induction is immediate to get (A.1).

If $A > 0$ and $C = 0$, we choose

$$K = \max\Big(A, B, \max_{[0,T]} a_0\Big),$$

and the induction is once again immediate to get (A.2).

If $C > 0$, we can apply one of the two previous cases to the sequence (α_n) defined, for any n, by

$$\alpha_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big[\int_0^t a_n(s)\,\mathrm{d}s\Big]\exp(-Ct)\Big),$$

and obtain (A.1) or (A.2).

That ends the proof of Lemma 3.

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