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ENERGY DECAY FOR SOLUTIONS OF THE WAVE EQUATION WITH GENERAL MEMORY BOUNDARY CONDITIONS

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Abstract. We consider the wave equation in a smooth domain subject to Dirichlet boundary conditions on one part of the boundary and dissipative boundary conditions of memory-delay type on the remainder of the boundary, where a general Borel measure is involved. Under quite weak assumptions on this measure, using the multiplier method and a standard integral inequality, we show the exponential stability of the system. Some examples of measures satisfying our hypotheses are given, recovering and extending some of the results from the literature.

Introduction

We consider the wave equation subject to Dirichlet boundary conditions on one part of the boundary and dissipative boundary conditions of memory-delay type on the remainder of the boundary. More precisely, let Ω be a bounded open connected subset of $\mathbb{R}^n (n \geq 2)$ such that, in the sense of Nečas ([8]), its boundary $\partial\Omega$ is of class $\mathcal{C}^2$. Throughout the paper, I denotes the $n \times n$ identity matrix, while A^s denotes the symmetric part of a matrix A. Let m be a $\mathcal{C}^1$ vector field on $\bar{\Omega}$ such that

$$\inf_{\bar{\Omega}} \operatorname{div}(m) > \sup_{\bar{\Omega}} (\operatorname{div}(m) - 2\lambda_m), \tag{0.1}$$

where $\lambda_m(x)$ is the smallest eigenvalue function of the real symmetric matrix $\nabla m(x)^s$.

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Remark 1. The set of all $\mathcal{C}^1$ vector fields on $\bar{\Omega}$ such that (0.1) holds is an open cone. If m is in this set, we denote

$$c(m) = \frac{1}{2}\Big(\inf_{\bar{\Omega}} \operatorname{div}(m) - \sup_{\bar{\Omega}}(\operatorname{div}(m) - 2\lambda_m)\Big).$$

Example 1. 1. An affine example is given by

$$m(x) = (A_1 + A_2)(x - x_0),$$

where A_1 is a positive definite matrix, A_2 a skew-symmetric matrix and x_0 any point in $\mathbb{R}^n$.

2. A nonlinear example is

$$m(x) = (dI + A)(x - x_0) + F(x),$$

where $d > 0$, A is a skew-symmetric matrix, x_0 any point in $\mathbb{R}^n$ and F is a $\mathcal{C}^1$ vector field on $\bar{\Omega}$ such that

$$\sup_{x\in\bar{\Omega}} \|(\nabla F(x))^s\| < \frac{d}{n}$$

($\|\cdot\|$ stands for the usual 2-norm of matrices).

We define a partition of $\partial\Omega$ in the following way. Denoting by $\nu(x)$ the normal unit vector pointing outward from Ω at a point $x \in \partial\Omega$, we consider a partition $(\partial\Omega_N, \partial\Omega_D)$ of the boundary such that the measure of $\partial\Omega_D$ is positive and

$$\begin{aligned} \partial\Omega_N &\subset \{x \in \partial\Omega : m(x)\cdot\nu(x) \geqslant 0\},\\ \partial\Omega_D &\subset \{x \in \partial\Omega : m(x)\cdot\nu(x) \leqslant 0\}. \end{aligned} \tag{0.2}$$

Furthermore, we assume

$$\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} = \varnothing \text{ or } \quad m\cdot n \le 0 \quad \text{on } \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}, \tag{0.3}$$

where n stands for the normal unit vector pointing outward at $\partial\Omega_N$ when considering $\partial\Omega_N$ as a sub-manifold of $\partial\Omega$.

On this domain, we consider the following delayed wave problem:

$$(S)\begin{cases} u'' - \Delta u = 0 & \text{in } \mathbb{R}_+^* \times \Omega\,,\\ u = 0 & \text{on } \mathbb{R}_+^* \times \partial\Omega_D\,,\\ \partial_\nu u + m\cdot\nu \left(\mu_0 u'(t) + \int_0^t u'(t-s)d\mu(s)\right) = 0 & \text{on } \mathbb{R}_+^* \times \partial\Omega_N\,,\\ u(0) = u_0 & \text{in } \Omega\,,\\ u'(0) = u_1 & \text{in } \Omega\,, \end{cases}$$

where u' (respectively u'') is the first (respectively second) time-derivative of u, $\partial_\nu u = \nabla u \cdot \nu$ is the normal outward derivative of u on $\partial\Omega$. Moreover, μ_0 is some positive constant and μ is a Borel measure on $\mathbb{R}^+$.

The above problem covers the case of a problem with memory type as studied for instance in [1, 3, 5, 9], when the measure μ is given by

$$d\mu(s) = k(s)ds, \tag{0.4}$$

where ds stands for Lebesgue measure and k is a nonnegative kernel. But it also covers the case of a problem with a delay as studied for instance in [10, 11, 12], when the measure μ is given by

$$\mu = \mu_1 \delta_\tau, \tag{0.5}$$

where μ_1 is a nonnegative constant and $\tau > 0$ represents the delay. An intermediate case treated in [11] is the case when

$$d\mu(s) = k(s)\chi_{[\tau_1,\tau_2]}(s)ds, \tag{0.6}$$

where $0 \leq \tau_1 < \tau_2$, $\chi_{[\tau_1,\tau_2]}$ is the characteristic equation of the interval $[\tau_1, \tau_2]$ and k is a nonnegative function in $L^\infty([\tau_1, \tau_2])$.

A closer look at the decay results obtained in these references shows that there are different ways to quantify the energy of (S). More precisely, for the measure of the form (0.4), the exponential or polynomial decay of an appropriate energy is proved in [1, 3, 5, 9], by combining the multiplier method (or differential geometry arguments) with the use of suitable Lyapounov functionals (or integral inequalities) under the assumptions that the kernel k is sufficiently smooth and has a certain decay at infinity. On the other hand, for a measure of delay type like (0.5) or (0.6), the exponential stability of the system was proved in [10, 11, 12] by proving an observability estimate obtained by assuming that the term $\int_0^t u'(t-s)d\mu(s)$ is sufficiently small with respect to $\mu_0 u'(t)$. Consequently, our goal here is to obtain some uniform decay results in the general context described above with similar assumptions as in [10, 11, 12]. More precisely, we will show in this paper that if there exists $\alpha > 0$ such that

$$\mu_{\text{tot}} := \int_0^{+\infty} e^{\alpha s} d|\mu|(s) < \mu_0, \tag{0.7}$$

where $|\mu|$ is the absolute value of the measure μ, then the above problem (S) is exponentially stable.

The paper is organized as follows: in the first two sections, we explain how to define an energy using some basic measure theory. Using well-known results, we obtain the existence of energy solutions. In this setting, we present and prove our stabilization result in the third section. Examples of measures μ satisfying our hypotheses are given at the end of the paper, where we show that we recover and extend some of the results from the references cited above.

Finally, in the whole paper we use the notation $A \lesssim B$ for the estimate $A \leq CB$ with some constant C that only depends on Ω, m or μ.

1. First results

In this section, we show that the assumption (0.7) implies the existence of some Borel finite measure λ such that

$$\lambda(\mathbb{R}^+) < \mu_0, \quad |\mu| \leq \lambda \tag{1.1}$$

(in the sense that, for every measurable set $\mathcal{B}$, $|\mu|(\mathcal{B}) \leq \lambda(\mathcal{B})$) and

$$\text{for all measurable set } \mathcal{B}, \int_{\mathcal{B}} \lambda([s, +\infty))ds \leq \alpha^{-1}\lambda(\mathcal{B}). \tag{1.2}$$

Indeed, we show the following equivalence.

Proposition 1. *Let μ be a Borel positive measure on $\mathbb{R}^+$ and μ_0 some positive constant. The following properties are equivalent* :

- $\exists\alpha > 0$ *such that*

$$\int_0^{+\infty} e^{\alpha s} d\mu(s) < \mu_0.$$

- *There exists a Borel measure λ on $\mathbb{R}^+$ such that*

$$\lambda(\mathbb{R}^+) < \mu_0, \quad \mu \leq \lambda,$$

and, for some constant $\beta > 0$,

$$\text{for all measurable set } \mathcal{B}, \int_{\mathcal{B}} \lambda([s, +\infty))ds \leq \beta^{-1}\lambda(\mathcal{B}).$$

Proof. We introduce the application T from the set of positive Borel measures into itself as follows: if μ is some positive Borel measure, we define a positive Borel measure $T(\mu)$ by

$$T(\mu)(\mathcal{B}) = \int_{\mathcal{B}} \mu([s, +\infty))ds,$$

if $\mathcal{B}$ is any measurable set.

($\Leftarrow$) If λ fulfills the second property, then it immediately follows that for all $n \in \mathbb{N}, \beta^n T^n(\mu) \leq \lambda$, where as usual T^n is the composition $T \circ T \cdots \circ T$ n-times. A summation consequently gives, for any $r \in (0,1)$,

$$\sum_{n=0}^{\infty} (r\beta)^n T^n(\mu) \leq \sum_{n=0}^{\infty} r^n \lambda = (1-r)^{-1}\lambda.$$

Using Fubini's theorem, we can now compute

$$\begin{aligned} T^n(\mu)(\mathbb{R}^+) &= \int_0^{+\infty} \Big(\int_{s_{n+1}}^{+\infty} \cdots \int_{s_3}^{+\infty} \Big(\int_{s_2}^{+\infty} d\mu(s_1) \Big) ds_2 \cdots ds_n \Big) ds_{n+1} \\ &= \int_0^{+\infty} \Big(\int_0^{s_1} \cdots \Big(\int_0^{s_n} ds_{n+1} \Big) \cdots ds_2 \Big) d\mu(s_1) \\ &= \int_0^{+\infty} \frac{s_1^n}{n!} d\mu(s_1), \end{aligned}$$

so that, using the monotone convergence theorem, one can obtain

$$\int_0^{+\infty} e^{r\beta s} d\mu(s) \leq (1-r)^{-1} \lambda(\mathbb{R}^+),$$

and our proof ends using the fact that $(1-r)^{-1}\lambda(\mathbb{R}^+) < \mu_0$ for sufficiently small r.

($\Rightarrow$) For any measurable set $\mathcal{B}$, we define

$$\lambda(\mathcal{B}) = \sum_{n=0}^{\infty} \alpha^n T^n(\mu)(\mathcal{B}).$$

It is clear that λ is a Borel measure such that $\mu \leq \lambda$. Moreover, if $\mathcal{B}$ is a measurable set, one has, thanks to the monotone convergence theorem,

$$\begin{aligned} T(\lambda)(\mathcal{B}) = \int_{\mathcal{B}} \lambda([s,+\infty))ds &= \sum_{n=0}^{\infty} \alpha^n \int_{\mathcal{B}} T^n(\mu)([s,+\infty))ds \\ &\leq \alpha^{-1} \sum_{n=0}^{\infty} \alpha^{n+1} T^{n+1}(\mu)(\mathcal{B}); \end{aligned}$$

that is, $T(\lambda) \leq \alpha^{-1}\lambda$.

Finally, another use of the monotone convergence theorem gives

$$\lambda(\mathbb{R}^+) = \int_0^{+\infty} e^{\alpha s} d\mu(s) < \mu_0. \qquad \square$$

Remark 2. 1) If μ satisfies our first property, one can choose $\beta = \alpha$ in our second property.

2) If μ is supported in $(0, \tau]$, it is straightforward to see that, for some small enough constant c,

$$d\lambda(s) = d\mu(s) + c\ \chi_{[0,\tau]}(s)ds$$

fulfills (0.7). This observation allows us to recover the choices of energy in [10, 11].

In the sequel, we can thus consider the measure λ obtained by the application of Proposition 1 to $|\mu|$.

2. Well posedness

2.1. **General results.** Defining $H^1_D(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega_D\}$ and $H^1_0(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}$, we here present an application of Theorem 4.4 of the Propst and Prüss paper (see [14]) in the framework of hypothesis (0.3).

Theorem 1. *Suppose $u_0 \in H^1_D(\Omega), u_1 \in L^2(\Omega)$. Then (S) admits a unique solution $u \in \mathcal{C}(\mathbb{R}^+, H^1(\Omega)) \cap \mathcal{C}^1(\mathbb{R}^+, L^2(\Omega))$ in the weak sense of Propst and Prüss. Moreover, if $u_0 \in H^2(\Omega) \cap H^1_D(\Omega)$ and $u_1 \in H^1_0(\Omega)$, then $u \in \mathcal{C}^1(\mathbb{R}^+, H^1(\Omega)) \cap \mathcal{C}^2(\mathbb{R}^+, L^2(\Omega))$ and in addition*

$$\forall t \geq 0,\ \Delta u(t) \in L^2(\Omega) \quad \partial_\nu u(t)_{|\partial\Omega_N} \in H^{1/2}(\partial\Omega_N).$$

Proof. The proof is the one proposed in [14], Theorem 4.4 except that, for smoother data, we cannot use the elliptic result in the general context of (0.3) to get more regularity. □

Inspired by [10, 11], we now define the energy of the solution of (S) at any positive time t by the following formula:

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega (u'(t,x))^2 + |\nabla u(t,x)|^2 dx \\ &+ \frac{1}{2}\int_{\partial\Omega_N} m\cdot\nu \int_0^t \Big(\int_0^s (u'(t-r,x))^2 dr\Big) d\lambda(s) d\sigma \\ &+ \frac{1}{2}\int_{\partial\Omega_N} m\cdot\nu \int_t^\infty \Big(\int_0^s (u'(s-r,x))^2 dr\Big) d\lambda(s) d\sigma. \end{aligned}$$

Remark 3. 1) In the definition of energy, the measure λ can be replaced by any positive Borel measure ν such that $\nu \leq \lambda$ such as, for instance, $|\mu|$. In fact, we will see later that conditions (1.1) and (1.2) are only here to ensure that the corresponding energy E_λ is nonincreasing, but the decay of another energy E_ν is implied by the decay of E_λ.

2) If μ is compactly supported in $[0,\tau]$, for times greater than τ, one can recover the energies from [10, 11] by choosing the measure λ supported in $[0,\tau]$ given by Remark 2. Indeed, the last term in the energy is null for $t>\tau$, and the second term is reduced to

$$\frac{1}{2}\int_{\partial\Omega_N} m\cdot\nu\int_0^\tau\Big(\int_0^s (u'(t-r,x))^2dr\Big)d\lambda(s)d\sigma.$$

We now identify our energy space.

Proposition 2. *If $u_0\in H^2(\Omega)\cap H^1_D(\Omega)$ and $u_1\in H^1_0(\Omega)$, then*

$$u'\in L^\infty(\mathbb{R}^+,H^1(\Omega)).$$

Consequently, for such initial conditions, the energy $E(t)$ is well defined for any $t>0$ and it uniformly depends continuously on the initial data.

Proof. Let us first pick some solution of (S) with $u_0\in H^2(\Omega)\cap H^1_D(\Omega)$ and $u_1\in H^1_D(\Omega)$. We define the standard energy as

$$E_0(t)=\frac{1}{2}\int_\Omega (u'(t,x))^2+|\nabla u(t,x)|^2dx.$$

As in [6], it is classical that

$$E_0(0)-E_0(T)=-\int_0^T\int_{\partial\Omega_N}\partial_\nu u u' d\sigma dt.$$

Using the form of our boundary condition and Young's inequality, one gets, for any $\epsilon>0$,

$$\begin{aligned} E_0(0)-E_0(T)&=\int_0^T\int_{\partial\Omega_N}(m\cdot\nu)\Big(\mu_0u'(t)^2+u'(t)\int_0^t u'(t-s)d\mu(s)\Big)d\sigma dt\\ &\geq\int_0^T\int_{\partial\Omega_N}(m\cdot\nu)\Big(\Big(\mu_0-\frac{\epsilon}{2}\Big)u'(t)^2-\frac{1}{2\epsilon}\Big(\int_0^t u'(t-s)d\mu(s)\Big)^2\Big)d\sigma. \end{aligned}$$

Using the fact that $\mu\leq|\mu|$ and the Cauchy-Schwarz inequality consequently implies

$$\begin{aligned} E_0(0)-E_0(T)\geq\int_{\partial\Omega_N}(m\cdot\nu)\Big(\Big(\mu_0-\frac{\epsilon}{2}\Big)\int_0^T u'(t)^2dt\\ -\frac{\mu_{\text{tot}}}{2\epsilon}\int_0^t(u'(t-s))^2d|\mu|(s)\Big)d\sigma. \end{aligned}$$

Using now Fubini's theorem two times, one can obtain the following identities:

$$\int_0^T \int_0^t (u'(t-s))^2 d|\mu|(s)dt = \int_0^T \Big(\int_s^T u'(t-s)^2 dt \Big) d|\mu|(s)$$
$$= \int_0^T \Big(\int_0^{T-s} u'(t)^2 dt \Big) d|\mu|(s) = \int_0^T \Big(\int_0^{T-t} d|\mu|(s) \Big) u'(t)^2 dt$$

so that, using the fact that $|\mu|([0, T-t]) \leq \mu_0$,

$$E_0(0) - E_0(T) \geq \int_{\partial\Omega_N} (m \cdot \nu) \Big(\Big(\mu_0 - \frac{\epsilon}{2} - \frac{\mu_{\rm tot}^2}{2\epsilon} \Big) \int_0^T u'(t)^2 dt \Big) d\sigma.$$

The choice of $\epsilon = \mu_{\rm tot}$ finally implies that $E_0(T)$ is bounded. Using the density of $H^2(\Omega) \cap H_D^1(\Omega) \times H_D^1(\Omega)$ in $H_D^1(\Omega) \times L^2(\Omega)$, we get the boundedness of E_0 for solutions with initial data $u_0 \in H_D^1(\Omega), u_1 \in L^2(\Omega)$. In particular, if $u_0 \in H_D^1(\Omega)$ and $u_1 \in L^2(\Omega)$, we obtain that $u \in L^\infty(\mathbb{R}^+, H^1(\Omega))$.

Let now u be a solution of (S) with $u_0 \in H^2(\Omega) \cap H_D^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$. Using Theorem 1, one can define the limit in $L^2(\Omega)$ u_2 of $u''(t)$ as $t \to 0$ and in this situation, as in [14], it is easy to see that u' is a solution of (S) with initial data $u_1 \in H_D^1(\Omega)$ and $u_2 \in L^2(\Omega)$.

Indeed, one can see that Fubini's theorem gives, on $\partial\Omega_N$,

$$\int_0^t u'(t-s) d\mu(s) = \int_0^t \Big(\int_0^s u''(s-r) d\mu(r) \Big) ds,$$

provided $u_1 = 0$ on $\partial\Omega_N$, so that

$$\frac{d}{dt} \Big(\int_0^t u'(t-s) d\mu(s) \Big) = \int_0^t u''(t-s) d\mu(s).$$

Using the proof above, one concludes that $u' \in L^\infty(\mathbb{R}^+, H^1(\Omega))$ which, thanks to a classical trace result, implies that $u' \in L^\infty(\mathbb{R}^+, L^2(\partial\Omega))$.

The first three terms of the energy $E(t)$ are consequently defined for any time $t > 0$. We only need to take a look at the last one to achieve our result. Using Fubini's theorem again, one has

$$\int_t^{+\infty} \Big(\int_0^s (u'(t-r))^2 dr \Big) d\lambda(s) = \int_0^{+\infty} u'(r)^2 \Big(\int_{\max(r,t)}^{+\infty} d\lambda(s) \Big) dr,$$

so that, using (1.2),

$$\int_{\partial\Omega_N} m \cdot \nu \int_t^\infty \Big(\int_0^s (u'(s-r,x))^2 dr \Big) d\lambda(s) d\sigma$$

$$\leq \|m\|_\infty \|u'\|_{L^\infty(L^2(\partial\Omega))} \int_0^{+\infty} \lambda([r,+\infty))dr$$
$$\leq \alpha^{-1}\|m\|_\infty \lambda(\mathbb{R}^+)\|u'\|_{L^\infty(L^2(\partial\Omega))}. \qquad \square$$

2.2. **Compactly supported measure and semigroup approach.** In this second approach, we assume that μ is supported in $[0,\tau]$ and that $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} = \varnothing$. We here simply follow the result obtained by Nicaise-Pignotti ([11]).

First, observe that, for $t > \tau$, (S) is reduced to

$$\begin{cases} u'' - \Delta u = 0 & \text{in } (\tau,+\infty)\times\Omega\,, \\ u = 0 & \text{on } (\tau,+\infty)\times\partial\Omega_D\,, \\ \partial_\nu u + m\cdot\nu\left(\mu_0 u'(t) + \int_0^\tau u'(t-s)d\mu(s)\right) = 0 & \text{on } (\tau,+\infty)\times\partial\Omega_N\,, \\ u(0) = u_0 & \text{in } \Omega\,, \\ u'(0) = u_1 & \text{in } \Omega\,. \end{cases}$$

We define $X_\tau = L^2(\partial\Omega_N \times (0,1)\times(0,\tau), d\sigma\, d\rho\, s\, d\mu(s)))$ and $Y_\tau = L^2(\partial\Omega_N \times (0,\tau); H^1(0,1),\, d\sigma\, s\, d\mu(s))$.

One can use the same strategy as in the proof of Theorem 2.1 in [11] to get the following.

Theorem 2. 1) *If $u(\tau) \in H^1_D(\Omega)$, $u'(\tau) \in L^2(\Omega)$ and $u'(\tau-\rho s,x) \in X_\tau$, (S) has a unique solution $u \in \mathcal{C}([\tau,+\infty), H^1_D(\Omega)) \cap \mathcal{C}^1([\tau,+\infty), L^2(\Omega))$. Moreover, if $u(\tau) \in H^2(\Omega)\cap H^1_D(\Omega)$, $u'(\tau) \in H^1(\Omega)$ and $u'(\tau-\rho s,x) \in Y_\tau$, then*

$$\begin{cases} u \in \mathcal{C}^1([\tau,+\infty), H^1_D(\Omega)) \cap \mathcal{C}([\tau,+\infty), H^2(\Omega)); \\ t \mapsto s u''(t-\rho s, x) \in \mathcal{C}([\tau,+\infty), X_\tau). \end{cases}$$

2) *If $(u^n_\tau(x), v^n_\tau(x), g^n(s,\rho,x)) \to (u(\tau,x), u'(\tau,x), u'(\tau-\rho s,x))$ in $H^1_D(\Omega) \times L^2(\Omega)\times X_\tau$, then the solution u^n of*

$$\begin{cases} u'' - \Delta u = 0 & \textit{in } (\tau,+\infty)\times\Omega\,, \\ u = 0 & \textit{on } (\tau,+\infty)\times\partial\Omega_D\,, \\ \partial_\nu u + m\cdot\nu\left(\mu_0 u'(t) + \int_0^\tau u'(t-s)d\mu(s)\right) = 0 & \textit{on } (\tau,+\infty)\times\partial\Omega_N\,, \\ u(\tau) = u^n_\tau & \textit{in } \Omega\,, \\ u'(\tau) = u^n_\tau & \textit{in } \Omega\,, \\ u'(x,\tau-\rho s) = g^n(x,s,\rho) & \textit{in } \Omega_N\times(0,\tau)\times(0,1) \end{cases}$$

is such that $E(u^n)$ converges uniformly with respect to time towards $E(u)$.

Proof. We define $z(x,\rho,s,t)=u'(t-\rho s,x)$ for $x\in\partial\Omega_N$, $t>\tau$, $s\in(0,\tau)$, $\rho\in(0,1)$. Problem (S) is then equivalent to

$$\begin{array}{ll}
u''-\Delta u=0 & \text{in } (\tau,+\infty)\times\Omega\,,\\
sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0 & \text{in}\partial\Omega_N\times(0,1)\times(0,\tau)\times(\tau,+\infty),\\
u=0 & \text{on } (\tau,+\infty)\times\partial\Omega_D\,,\\
\partial_\nu u+m\cdot\nu(\mu_0u'(t)+\int_0^\tau u'(t-s)d\mu(s))=0 & \text{on } (\tau,+\infty)\times\partial\Omega_N\,,\\
u(\tau)=u(\tau) & \text{in } \Omega\,,\\
u'(\tau)=u(\tau) & \text{in } \Omega\,,\\
z(x,0,t,s)=u'(t,x) & \text{on } \partial\Omega_N\times(\tau,+\infty)\times(0,\tau),\\
z(x,\rho,\tau,s)=f_0(x,\rho,s) & \text{on } \partial\Omega_N\times(0,1)\times(0,\tau),
\end{array}$$

where $f_0(x,\rho,s)=u'(\tau-\rho s,x)$.

Consequently, (S) can be rewritten as

$$U'=\mathcal{A}U,\qquad U(\tau)=(u(\tau),u'(\tau),f_0)^T,$$

where the operator is defined by $\mathcal{A}\begin{pmatrix}u\\v\\z\end{pmatrix}=\begin{pmatrix}v\\\Delta u\\-s^{-1}z_\rho\end{pmatrix}$ with domain

$$\begin{aligned}\mathcal{D}(\mathcal{A})=\ \Big\{\ &(u,v,z)^T\in H_D^1(\Omega)\times L^2(\Omega)\times Y_\tau:\Delta u\in L^2(\Omega),\\
&\partial_\nu u(x)=-(m\cdot\nu)\Big(\mu_0v(t)+\int_0^\tau z(x,1,s)d\mu(s)\Big)\text{ on }\partial\Omega_N,\\
&v(x)=z(x,0,s)\text{ on }\partial\Omega_N\times(0,\tau)\Big\}\,.\end{aligned}$$

The proof of Theorem 2.1 in [11] shows us that $\mathcal{A}$ is a maximal monotone operator on the Hilbert space $\mathcal{H}:=H_D^1(\Omega)\times L^2(\Omega)\times X_\tau$ endowed with the product topology. It consequently generates a contraction semigroup on $\mathcal{H}$. Moreover, if $(u(\tau,x),u'(\tau,x),u'(\tau-\rho s,x))\in\mathcal{D}(\mathcal{A})$, one gets that

$$\begin{cases}u\in\mathcal{C}^1([\tau,+\infty),H_D^1(\Omega))\cap\mathcal{C}([\tau,+\infty),H^2(\Omega));\\ t\mapsto su''(t-\rho s,x)\in\mathcal{C}([\tau,+\infty),X_\tau).\end{cases}$$

This ends the proof. □

We can consequently deduce another way to obtain solutions.

Corollary 1. *Suppose that $u_0\in H^2(\Omega)\cap H_D^1(\Omega)$ and $u_1\in H_0^1(\Omega)$; then (S) has a unique solution $u\in\mathcal{C}([\tau,+\infty),H_D^1(\Omega))\cap\mathcal{C}^1([\tau,+\infty),L^2(\Omega))$.*

Proof. Thanks to Theorem 1, one only needs to check that if u belongs to $\mathcal{C}^1([0,\tau],H_D^1(\Omega))$, then $u'(x,\tau-\rho s)\in X_\tau$; this is straightforward using Fubini's theorem. □

3. Linear stabilization

We begin with a classical elementary result due to Komornik [6].

Lemma 1. *Let $E : [0,+\infty) \to \mathbb{R}_+$ be a nondecreasing function that fulfils the following:*

$$\forall t \geq 0, \int_t^\infty E(s)ds \leq TE(t),$$

for some $T > 0$. Then, one has

$$\forall t \geqslant T, E(t) \leq E(0) \exp\big(1 - \frac{t}{T}\big).$$

We will now show the following stabilization result.

Theorem 3. *Assume* (0.1)-(0.7). *Then, if $u_0 \in H^2(\Omega) \cap H^1_D(\Omega)$ and $u_1 \in H^1_0(\Omega)$, there exists $T > 0$ such that the energy $E(t)$ of the solution u of (S) satisfies*

$$\forall t \geqslant T, \quad E(t) \leq E(0) \exp\big(1 - \frac{t}{T}\big).$$

Proof. Our goal is to perform the multiplier method and deal with the delay terms to show that one can apply Lemma 1 to the energy.

Lemma 2. *There exists $C > 0$, such that, for any solution u of (S) and any $S \leq T$,*

$$E(S) - E(T) \geqslant C \int_S^T \int_{\partial\Omega_N} (m \cdot \nu)\Big((u'(t))^2 + \int_0^t (u'(t-s))^2 d\lambda(s)\Big) d\sigma dt.$$

In particular, the energy is a nonincreasing function of time.

Proof. We start from the classical result that

$$E_0(S) - E_0(T) = -\int_S^T \int_{\partial\Omega} \partial_\nu u u' d\sigma dt.$$

As above, one gets, for any $\epsilon > 0$,

$$E_0(S) - E_0(T) \geq \int_S^T \int_{\partial\Omega_N} (m \cdot \nu)\Big(\Big(\mu_0 - \frac{\epsilon}{2}\Big)u'(t)^2 - \frac{\mu_{\text{tot}}}{2\epsilon} \int_0^t (u'(t-s))^2 d|\mu|(s)\Big) d\sigma dt.$$

We will now split $E - E_0$ into two terms:

$$[E - E_0]_S^T = -\frac{1}{2}\Big(\int_{\partial\Omega_N} (m \cdot \nu)[f(t,x) - g(t,x)]_S^T d\sigma\Big),$$

where

$$f(t,x) = \int_0^t \Big(\int_0^s (u'(t-r))^2 dr\Big) d\lambda(s), \quad g(t,x) = \int_0^t \Big(\int_0^s u'(r)^2 dr\Big) d\lambda(s).$$

A change of variables allows us to get

$$f(t,x) = \int_0^t \int_0^t u'(r)^2 dr d\lambda(s) - \int_0^t \int_0^{t-s} u'(r)^2 dr d\lambda(s).$$

An application of Fubini's theorem consequently gives us

$$f(t,x) - g(t,x) = \int_0^t u'(r)^2 \lambda([0,r]) dr - \int_0^t \int_0^{t-s} u'(r)^2 dr d\lambda(s),$$

and, as above, one can use Fubini's theorem to deduce that

$$\int_S^T \int_0^t (u'(t-s))^2 d\lambda(s) dt = \Big[\int_0^t \int_0^{t-s} u'(r)^2 dr d\lambda(s)\Big]_S^T.$$

One now uses the fact that $\lambda([0,r]) \leqslant \lambda(\mathbb{R}^+)$ to conclude that

$$[E - E_0]_S^T \geq \frac{1}{2} \int_S^T \int_{\partial\Omega_N} m \cdot \nu \Big(\int_0^t (u'(t-s))^2 d\lambda(s) - \lambda(\mathbb{R}^+) u'(t)^2\Big) d\sigma dt.$$

Summing up and using the fact that $|\mu| \leq \lambda$, we have obtained

$$\begin{aligned} E(S) - E(T) \ &\geqslant \ \int_S^T \int_{\partial\Omega_N} (m \cdot \nu) \Big(\Big(\mu_0 - \frac{\lambda(\mathbb{R}^+) + \epsilon}{2}\Big) u'(t)^2 \\ &+ \ \frac{1}{2}\Big(1 - \frac{\mu_{\text{tot}}}{\epsilon}\Big) \int_0^t (u'(t-s))^2 d\lambda(s)\Big) d\sigma dt. \end{aligned}$$

We finally chose $\epsilon = \mu_0$ which gives us our result since $\lambda(\mathbb{R}^+) < \mu_0$. □

In the multiplier method, one may use Rellich's relation, especially in the context of singularities. In our framework (0.3), the following Rellich inequality (see the proof of Theorem 4 in [4] or Proposition 4 in [2]) is useful.

Proposition 3. *Suppose $u \in H^1(\Omega)$ is such that*

$$\Delta u \in L^2(\Omega), u_{|\partial\Omega_D} \in H^{\frac{3}{2}}(\partial\Omega_D) \text{ and } \partial_\nu u_{|\partial\Omega_N} \in H^{\frac{1}{2}}(\partial\Omega_N).$$

Then u satisfies $2\partial_\nu u(m.\nabla u) - (m \cdot \nu)|\nabla u|^2 \in L^1(\partial\Omega)$ and we have the following inequality:

$$2\int_\Omega \triangle u(m.\nabla u) dx \ \leq \ \int_\Omega (\text{div}(m) I - 2(\nabla m)^s)(\nabla u, \nabla u) dx$$

$$+ \quad \int_{\partial\Omega} (2\partial_\nu u(m.\nabla u) - (m\cdot\nu)|\nabla u|^2)d\sigma.$$

With this result, we can prove the following multiplier estimate.

Lemma 3. *Let $Mu = 2m.\nabla u + a_0 u$, where*

$$a_0 := \frac{1}{2}\Big(\inf_{\bar\Omega} \operatorname{div}(m) + \sup_{\bar\Omega} \left(\operatorname{div}(m) - 2\lambda_m\right) \Big).$$

Then, under the assumptions of Theorem 3, the following inequality holds true:

$$\begin{aligned} & c(m)\int_S^T \int_\Omega (u')^2 + |\nabla u|^2 dxdt \\ & \leq -\Big[\int_\Omega u' Mu \Big]_S^T + \int_S^T \int_{\partial\Omega_N} Mu\partial_\nu u + (m\cdot\nu)((u')^2 - |\nabla u|^2) d\sigma dt. \end{aligned}$$

Proof. Firstly, we consider $M = 2m\cdot\nabla u + au$ where a will be fixed later. Using the fact that u is a regular solution of (S) and noting that $u''Mu = (u'Mu)' - u'Mu'$, an integration by parts gives

$$\begin{aligned} 0 &= \int_S^T \int_\Omega (u'' - \triangle u) Mudxdt \\ &= \Big[\int_\Omega u' Mudx \Big]_S^T - \int_S^T \int_\Omega (u'Mu' + \triangle u Mu) dxdt. \end{aligned}$$

Now, thanks to Proposition 3, we have

$$\begin{aligned} \int_\Omega \triangle u Mudx &\leq a\int_\Omega \triangle u udx + \int_\Omega (\operatorname{div}(m) I - 2(\nabla m)^s)(\nabla u, \nabla u)dx \\ &\quad + \int_{\partial\Omega} (2\partial_\nu u(m.\nabla u) - (m.\nu)|\nabla u|^2)d\sigma. \end{aligned}$$

Consequently, the Green-Riemann formula leads to the following:

$$\begin{aligned} \int_\Omega \triangle u Mudx &= \int_\Omega ((\operatorname{div}(m) - a)I - 2(\nabla m)^s)(\nabla u, \nabla u)dx \\ &\quad + \int_{\partial\Omega} (\partial_\nu u Mu - (m\cdot\nu)|\nabla u|^2)d\sigma. \end{aligned}$$

Using the fact that $\nabla u = \partial_\nu u \nu$ on $\partial\Omega_D$ and $m\cdot\nu \leqslant 0$ on $\partial\Omega_D$, we have then

$$\int_\Omega \triangle u Mudx \leq \int_\Omega ((\operatorname{div}(m) - a)I - 2(\nabla m)^s)(\nabla u, \nabla u)dx$$

$$+ \int_{\partial\Omega_N} (\partial_\nu u M u - (m\cdot\nu)|\nabla u|^2) d\sigma.$$

On the other hand, another use of Green's formula gives us

$$\int_\Omega u' M u' dx = \int_\Omega (a - \operatorname{div}(m))(u')^2 dx + \int_{\partial\Omega_N} (m\cdot\nu)|u'|^2 d\sigma.$$

Consequently,

$$\int_S^T \int_\Omega (\operatorname{div}(m) - a)(u')^2 + ((a - \operatorname{div}(m))I + 2(\nabla m)^s)(\nabla u, \nabla u) dx dt$$

$$\leq \; -\Big[\int_\Omega u' M u dx\Big]_S^T + \int_S^T \int_{\partial\Omega_N} \partial_\nu u M u + (m\cdot\nu)((u')^2 - |\nabla u|^2) d\sigma dt.$$

Our goal is now to find a such that $\operatorname{div}(m) - a$ and $(a - \operatorname{div}(m))I + 2(\nabla m)^s$ are uniformly minorized on Ω. One has to find a such that, uniformly on Ω,

$$\begin{cases} \operatorname{div}(m) - a \geq c \\ 2\lambda_m + (a - \operatorname{div}(m)) \geq c, \end{cases} \tag{3.1}$$

for some positive constant c. The latter condition is then equivalent to finding a which fulfills

$$\inf_{\bar\Omega} \operatorname{div}(m) > a > \sup_{\bar\Omega} (\operatorname{div}(m) - 2\lambda_m),$$

and its existence is now guaranteed by (0.1). Moreover, it is straightforward to see that the greatest value of c such that (3.1) holds is

$$c(m) = \frac{1}{2}\Big(\inf_{\bar\Omega} \operatorname{div}(m) - \sup_{\bar\Omega}(\operatorname{div}(m) - 2\lambda_m)\Big),$$

and is obtained for $a = a_0$. This ends the proof. □

Consequently, the following result holds.

Lemma 4. *For every $\tau \leq S < T < \infty$, the following inequality holds true:*

$$\int_S^T \int_\Omega (u')^2 + |\nabla u|^2 dx dt \lesssim E(S).$$

Proof. We start from Lemma 3. First of all, the Young and Poincaré inequalities give

$$\Big|\int_\Omega u' M u dx\Big| \lesssim E(t),$$

so that

$$-\Big[\int_\Omega u' Mu dx\Big]_S^T \lesssim E(S) + E(T) \leqslant CE(S).$$

Now, from the boundary condition, one has

$$\begin{aligned}&Mu\partial_\nu u + (m\cdot\nu)((u')^2 - |\nabla u|^2)\\ &= (m\cdot\nu)\Big(\Big(\mu_0 u' + \int_0^t u'(t-s)d\mu(s)\Big)Mu + (u')^2 - |\nabla u|^2\Big).\end{aligned}$$

Using the definition of Mu and Young's inequality, we get for any $\epsilon > 0$

$$\begin{aligned}&Mu\partial_\nu u + (m\cdot\nu)((u')^2 - |\nabla u|^2)\\ &\leqslant (m\cdot\nu)\Big(\Big(1 + \|m\|_\infty^2 + \mu_0^2\frac{a_0^2}{2\epsilon}\Big)(u')^2 + \frac{a_0^2}{2\epsilon}\Big(\int_0^t u'(t-s)d\mu(s)\Big)^2 + \epsilon u^2\Big).\end{aligned}$$

Another use of the Poincaré inequality consequently allows us to choose $\epsilon > 0$ such that

$$\epsilon\int_{\partial\Omega_N}(m\cdot\nu)u^2 d\sigma \leqslant \frac{c(m)}{2}\int_\Omega |\nabla u|^2 dx.$$

The Cauchy-Schwarz inequality consequently leads to

$$\begin{aligned}&\frac{c(m)}{2}\int_S^T\int_\Omega (u')^2 + |\nabla u|^2 dxdt\\ &\lesssim E(S) + \int_S^T\int_{\partial\Omega_N}(m\cdot\nu)\Big(u'(t)^2 + \int_0^t (u'(t-s))^2 d|\mu|(s)\Big)d\sigma dt,\end{aligned}$$

and, since $|\mu| \leq \lambda$, Lemma 2 gives us the desired result

$$c(m)\int_S^T\int_\Omega (u')^2 + |\nabla u|^2 dxdt \lesssim E(S). \qquad \square$$

To conclude we need to absorb the two last integral terms for which we use the following result.

Lemma 5. 1) *For any solution u and any $S < T$,*

$$\begin{aligned}&\int_S^T\int_{\partial\Omega_N} m\cdot\nu\int_0^t\Big(\int_0^s (u'(t-r,x))^2 dr\Big)d\lambda(s)d\sigma dt\\ &\lesssim \int_S^T\int_{\partial\Omega_N} m\cdot\nu\int_0^t (u'(t-s,x))^2 d\lambda(s)d\sigma dt.\end{aligned}$$

2) *For any solution* u *and any* $S < T$,

$$\int_S^T \int_{\partial\Omega_N} m\cdot\nu \int_t^{+\infty} \Big(\int_0^s (u'(s-r,x))^2 dr\Big) d\lambda(s) d\sigma dt$$
$$\lesssim \int_S^T \int_{\partial\Omega_N} m\cdot\nu \int_0^t (u'(t-s,x))^2 d\lambda(s) d\sigma dt + \int_S^{+\infty} \int_{\partial\Omega_N} m\cdot\nu \; u'^2 d\sigma dt.$$

Proof. 1) We start from the left-hand side term. We fix $x \in \partial\Omega_N$ and $t \in [S,T]$ and we use Fubini's theorem to estimate integrals with respect to time:

$$\int_0^t \Big(\int_0^s (u'(t-r,x))^2 dr\Big) d\lambda(s) = \int_0^t (u'(t-r,x))^2 \lambda([r,t]) dr$$
$$\le \int_0^t (u'(t-r,x))^2 \lambda([r,+\infty)) dr \le \alpha^{-1} \int_0^t (u'(t-r,x))^2 d\lambda(r),$$

which gives the required result after an integration with respect to t and x.

2) As above, fixing $x \in \partial\Omega_N$, we obtain

$$\int_S^T \int_t^{+\infty} \Big(\int_0^s (u'(s-r,x)^2 dr\Big) d\lambda(s) dt$$
$$= \int_S^T \int_0^{+\infty} (u'(r,x))^2 \lambda([\max(r,t),+\infty]) dr dt$$
$$= \int_S^T \int_0^t (u'(t-r,x))^2 dr \lambda([t,+\infty)) dt$$
$$+ \int_S^T \Big(\int_t^{+\infty} (u'(r,x))^2 \lambda([r,+\infty)) dr\Big) dt.$$

Since for all $r \le t$, $\lambda([t,+\infty) \le \lambda([r,+\infty))$, we first have

$$\int_S^T \int_0^t (u'(t-r,x))^2 dr \lambda([t,+\infty)) dt$$
$$\le \int_S^T \int_0^t (u'(t-r,x))^2 \lambda([r,+\infty)) dr dt$$
$$\le \alpha^{-1} \int_S^T \int_0^t (u'(t-r,x))^2 d\lambda(r) dt.$$

On the other hand, Fubini's theorem gives us

$$\int_S^T \Big(\int_t^{+\infty} (u'(r,x))^2 \lambda([r,+\infty)) dr\Big) dt$$

$$= \int_S^{+\infty} u'(r)^2 \lambda([r,+\infty))(\min(T,r) - S)dr.$$

We now note that

$$r\lambda([r,+\infty)) \leq \int_r^{+\infty} s d\lambda(s) \leq \int_0^{+\infty} s d\lambda(s),$$

and

$$\int_0^{+\infty} s d\lambda(s) = \int_0^{+\infty} \lambda([t,+\infty))dt \leq \alpha^{-1}\lambda(\mathbb{R}^+).$$

We consequently obtain

$$\int_S^T \left(\int_t^{+\infty} (u'(r,x))^2 \lambda([r,+\infty))dr \right) dt \lesssim \int_S^{+\infty} u'^2,$$

which gives the required result after an integration over $\partial\Omega_N$. □

Up to now, we have proven that

$$\int_S^T E(t)dt \lesssim E(S) + \int_S^{+\infty} \int_{\partial\Omega_N} m \cdot \nu \ u'^2 d\sigma dt$$
$$+ \int_S^T \int_{\partial\Omega_N} m \cdot \nu \int_0^t (u'(t-s,x))^2 d\lambda(s) d\sigma dt.$$

Lemma 2 allows us to conclude the proof, since it gives

$$\int_S^{+\infty} \int_{\partial\Omega_N} m \cdot \nu \ u'^2 d\sigma dt \lesssim E(S),$$

and

$$\int_S^T \int_{\partial\Omega_N} m \cdot \nu \int_0^t (u'(t-s,x))^2 d\lambda(s) d\sigma dt \lesssim E(S).$$

□

Remark 4. In the case of some compactly supported measure μ, one can also obtain exponential decay results for the following problem:

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \mathbb{R}_+^* \times \mathbb{R}_+^* , \\ u = 0 & \text{on } \mathbb{R}_+^* \times \partial\Omega_D , \\ \partial_\nu u + \mu_0 u'(t) + \int_0^t u'(t-s) d\mu(s) = 0 & \text{on } \mathbb{R}_+^* \times \partial\Omega_N , \\ u(0) = u_0 & \text{in } \Omega , \\ u'(0) = u_1 & \text{in } \Omega , \end{cases}$$

as was done in [11] using the work of Lasiecka-Triggiani-Yao [7] since the system is time invariant for $t \gg 1$.

Moreover, a careful attention shows that our proof allows us to obtain decay for this system without assumptions on the support of μ provided that $\inf_{\partial\Omega_N} m \cdot \nu > 0$.

4. EXAMPLES

We start with two general results and then particularize them to recover results from the literature.

Example 2. If μ is some Borel measure such that

$$|\mu|(\mathbb{R}^+) < \mu_0 \text{ and } \int_0^{+\infty} e^{\beta s} d|\mu|(s) < +\infty,$$

for some $\beta > 0$, then μ fulfils the assumption (0.7) for an appropriate α. Indeed, for any $0 \leq \alpha \leq \beta$, the expression $\int_0^{+\infty} e^{\alpha s} d|\mu|(s)$ is finite and by the dominated convergence theorem of Lebesgue we have

$$\int_0^{+\infty} e^{\alpha s} d|\mu|(s) \to |\mu|(\mathbb{R}^+) \text{ as } \alpha \to 0.$$

Consequently, by the assumption $|\mu|(\mathbb{R}^+) < \mu_0$, we get (0.7) for α small enough.

Example 3. One can choose

$$\mu = \sum_{i=1}^{\infty} \mu_i \delta_{\tau_i},$$

where $(\tau_i)_{i=1}^{\infty}$, $(\mu_i)_{i=1}^{\infty}$ are some families such that $\tau_i > 0$ and are two by two disjoint, and

$$\sum_{i=1}^{\infty} |\mu_i| e^{\alpha \tau_i} < \mu_0$$

for some $\alpha > 0$.

Example 4. Choose $d\mu(s) = k(s)ds$ where k is a kernel satisfying

$$\int_0^{+\infty} |k(s)| ds < \mu_0 \text{ and } \int_0^{+\infty} |k(s)| e^{\beta s} ds < \infty,$$

for some $\beta > 0$. Then, as a consequence of Example 2, we get an exponential decay rate for the system (S) under the (very weak) condition above; in particular we do not need any differentiability assumptions on k, nor uniform exponential decay of k at infinity as in [1, 3, 5, 9].

Example 5. Choose $d\mu(s) = k(s)\chi_{[\tau_1,\tau_2]}(s)ds$, where k is an integrable function in $[\tau_1, \tau_2]$ such that

$$\int_{\tau_1}^{\tau_2} |k(s)|ds < \mu_0;$$

then we get an exponential decay for the system (S) as a consequence of Example 2, because the second assumption trivially holds. In that case, we extend the results of [11] to a larger class of kernels k, for instance, in the class of functions of bounded variation.

Example 6. Take $\mu(s) = \mu_1\delta_\tau(s)$, where μ_1 is a constant and $\tau > 0$ represents the delay satisfying $|\mu_1| < \mu_0$; then we recover the decay results from [10, 12].

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