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DEGENERATE PARABOLIC EQUATION WITH CRITICAL EXPONENT DERIVED FROM THE KINETIC THEORY, II, BLOWUP THRESHOLD

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Abstract. We study a degenerate parabolic equation derived from the kinetic theory using Rényi-Tsallis entropy with the critical exponent. In this paper, we show the existence of the threshold mass for a solution blowing up in finite time.

1. Introduction

We continue the study of the degenerate parabolic equation

$$u_t = \frac{m-1}{m}\Delta u^m - \nabla\cdot(u\nabla\Gamma * u),\quad u\geq 0 \qquad \text{in } \mathbf{R}^n\times(0,T), \tag{1.1}$$

where

$$\Gamma(x) = \frac{1}{\omega_{n-1}(n-2)\,|x|^{n-2}}, \tag{1.2}$$

with ω_{n-1} standing for the area of the boundary of the unit ball in $\mathbf{R}^n$, $n\geq 3$, and $m = 2-\frac{2}{n}$. The solution which we deal with is the weak solution obtained in the previous work [9]. Given the initial value

$$0\leq u_0\in L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n) \qquad \text{with} \qquad u_0^m\in H^1(\mathbf{R}^n), \tag{1.3}$$

we take the approximate solution $u_\varepsilon = u_\varepsilon(x,t)$ satisfying

$$\begin{aligned} u_{\varepsilon t} &= \frac{m-1}{m}\Delta(u_\varepsilon+\varepsilon)^m - \nabla\cdot(u_\varepsilon\nabla\Gamma * u_\varepsilon) && \text{in } \mathbf{R}^n\times(0,T)\\ u_\varepsilon|_{t=0} &= u_{0\varepsilon} && \text{in } \mathbf{R}^n, \end{aligned} \tag{1.4}$$

for $0<\varepsilon\ll 1$, where

$$\begin{aligned} &0\leq u_{0\varepsilon}\in L^1\cap W^{2,p}(\mathbf{R}^n) && \text{for any } p\in[\tfrac{n}{n-1}, n+2]\\ &\|u_{0\varepsilon}\|_p\leq \|u_0\|_p, && \text{for any } p\in[1,\infty] \end{aligned}$$

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$$\|\nabla u_{0\varepsilon}^m\|_2 \leq \|\nabla u_0^m\|_2$$
$$u_{0\varepsilon} \to u_0 \qquad \text{strongly in } L^p(\mathbf{R}^n) \text{ as } \varepsilon \downarrow 0 \text{ for all } p \in [1,\infty). \quad (1.5)$$

Then the weak solution $u = u(\cdot, t)$ local-in-time is obtained by the passage to the limit of the approximate solution u_ε. We also have the total mass conservation

$$\|u(t)\|_1 = \|u_0\|_1 = \lambda \qquad \text{for all } t \in [0,T), \quad (1.6)$$

and the blowup criterion

$$T_{\max} < +\infty \quad \Rightarrow \quad \lim_{t \uparrow T_{\max}} \|u(t)\|_\infty = +\infty, \quad (1.7)$$

where $T_{\max} \in (0, +\infty]$ denotes the maximal existence time of u. Henceforth, $u = u(x,t)$ and $T = T_{\max} \in (0, +\infty]$ denote this weak solution and its maximal existence time, respectively.

In this paper, we shall show the existence of the threshold mass for $T = +\infty$. This phenomenon has its origin in the variational structure of (1.1) and for the moment we shall develop a formal argument to describe this property. First, equation (1.1) is a *model B equation*, see [6], associated with the *free energy*

$$\mathcal{F}(u) = \int_{\mathbf{R}^n} \frac{u^m}{m} dx - \frac{1}{2} \langle \Gamma * u, u \rangle . \quad (1.8)$$

In fact we have

$$\delta\mathcal{F}(u)[v] = \left. \frac{d}{ds}\mathcal{F}(u+sv) \right|_{s=0} = \left\langle v, u^{m-1} - \Gamma * u \right\rangle ,$$

where $\langle \ , \ \rangle$ denotes the L^2-inner product. Identifying $\delta\mathcal{F}(u)$ with $u^{m-1} - \Gamma * u$, we can write (1.1) as

$$u_t = \nabla \cdot \left(\frac{m-1}{m} \nabla u^m - u \nabla \Gamma * u \right) = \nabla \cdot u \nabla \delta \mathcal{F}(u) \qquad \text{in } \mathbf{R}^n \times (0,T). \quad (1.9)$$

Several formulas are derived from (1.9) formally such as the total mass conservation (1.6), the decrease of the free energy

$$\frac{d}{dt}\mathcal{F}(u) = -\int_{\mathbf{R}^n} u \left| \nabla \delta \mathcal{F}(u) \right|^2 dx = -\int_{\mathbf{R}^n} u \left| \nabla (u^{m-1} - \Gamma * u) \right|^2 dx \leq 0, \quad (1.10)$$

and the relation between the second moment and the free energy

$$\frac{d}{dt} \int_{\mathbf{R}^n} |x|^2 u dx = 2(n-2)\mathcal{F}(u), \quad (1.11)$$

derived from $m = 2 - \frac{2}{n}$.

We have already justified (1.6) for the weak solution, see [9]. As for (1.10), the weaker property

$$\mathcal{F}(u(t)) \leq \mathcal{F}(u_0) \tag{1.12}$$

will be proven for all $t \in [0,T)$ in this paper. Furthermore, in the case of

$$\int_{\mathbf{R}^n} |x|^2 u_0(x)dx < +\infty, \tag{1.13}$$

the mapping $t \in [0,T) \mapsto \int_{\mathbf{R}^n} |x|^2 udx$ is locally absolutely continuous, and (1.11) holds for almost every $t \in [0,T)$. As a by-product of the proof of this fact, we obtain the important property

$$u \in C_*([0,T), L^p(\mathbf{R}^n)), \qquad 1 < p \leq \infty, \tag{1.14}$$

regarding $L^p(\mathbf{R}^n) = L^{p'}(\mathbf{R}^n)'$, $\frac{1}{p'} + \frac{1}{p} = 1$. This property is proven in the next section, see Corollary 1, and will be used in our third paper.

Surprisingly, these variational backgrounds of (1.1) induce a blowup mechanism of u, that is, the existence of the threshold of $\lambda = \|u_0\|_1$ for $T = T_{\max} < +\infty$. In fact, the blowup criterion is reduced to $\mathcal{F}(u_0) < 0$ and we obtain the following theorem.

Theorem 1. *There is a constant $\lambda_* > 0$ determined by the dimension $n \geq 3$ such that if $u_0 = u_0(x)$ is the initial value satisfying* (1.3), (1.13), *and $\|u_0\|_1 < \lambda_*$, then $T = +\infty$ holds in* (1.1) *for $m = 2 - \frac{2}{n}$. Each $\lambda > \lambda_*$, on the other hand, takes $u_0 = u_0(x)$ such that* (1.3), (1.13), *$\|u_0\|_1 = \lambda$, and $T < +\infty$.*

The threshold above mass λ_* is also associated with the blowup family of stationary solutions shown in [10, 8]. This profile is quite similar to the Smoluchowski-Poisson equation in two space dimensions, and here we shall spend several lines on stationary solutions of (1.1). First, from (1.6) and (1.10) the stationary state is described by

$$u^{m-1} - \Gamma * u = \text{constant} \ \text{ in } \{u > 0\}, \qquad \int_{\mathbf{R}^n} udx = \lambda. \tag{1.15}$$

If the constant in the right-hand side is denoted by c, then $v = \Gamma * u + c$ satisfies

$$-\Delta v = v_+^q \quad \text{in } \mathbf{R}^n, \qquad \int_{\mathbf{R}^n} v_+^q dx = \lambda, \tag{1.16}$$

for $q > 1$ determined by $m = 1 + \frac{1}{q}$. Since $q = \frac{1}{m-1} = \frac{n}{n-2}$, problem (1.16) is invariant under the scaling transformation

$$v(x) \mapsto v_\mu(x) = \mu^\gamma v(\mu x), \tag{1.17}$$

where $\gamma = n-2$ and $\mu > 0$ is a constant, and consequently, problem (1.16) admits a family of solutions. In fact, the solution is necessarily radially symmetric and $u = v_+^q$ has a compact support, see [10]. The quantized mass $\lambda_* > 0$ is thus defined by

$$\lambda_* = \int_{\mathbf{R}^n} v_{*+}^q dx, \tag{1.18}$$

using the normalized solution $v_* = v_*(x)$:

$$-\Delta v_* = v_{*+}^q, \qquad v_* \leq v_*(0) = 1 \quad \text{in } \mathbf{R}^n. \tag{1.19}$$

The blowup threshold is also studied in [3], although the potential used there is different from ours. She presents two distinct constants $\lambda_1^* > \lambda_2^* > 0$ associated with the Sobolev constant such that if $u_0 \in L^1 \cap L^\infty(\mathbf{R}^n)$ with $u_0^m \in H^1(\mathbf{R}^n)$ satisfies $\|u_0\|_1 \leq \lambda_2^*$ then $T_{\max} = \infty$, and for each $\lambda > \lambda_1^*$ there exists initial data $u_0 \in L^1 \cap L^\infty(\mathbf{R}^n)$, $u_0^m \in H^1(\mathbf{R}^n)$ with $\|u_0\|_1 = \lambda$ satisfying $T_{\max} < \infty$.

The above described properties of the stationary solution, variational functionals, and the scaling invariance appear to have a similarity to those of the Smoluchowski-Poisson equation in two space dimensions, see [5, 6] and the references therein. It is natural to raise the questions of the finiteness of the blowup points and the non-existence of type I blowup rate to (1.1), $m = 2 - \frac{2}{n}$. Actually, they will be clarified in our forthcoming papers.

Recently, we learned that Theorem 1 is also shown in [1]. There, it is also shown that a solution u to (1.1) exists globally in time if the mass of the initial data u_0 is equal to the quantized mass λ_*, which is described by the Hardy-Littlewood-Sobolev constant. In contrast, we shall use the Trudinger-Moser inequality in this paper. This argument is shown in [7] heuristically.

This paper is composed of three sections. Section 2 is devoted to preliminaries and Theorem 1 is proven in Section 3. Henceforth, C_i $(i = 1, 2, \cdots)$ denotes positive constants whose subscripts are renewed in each section.

2. PRELIMINARIES

Henceforth, u_{ε_j} shall be the approximate solution that converges to the weak solution u to (1.1) as $\varepsilon_j \downarrow 0$.

We recall the following estimate, see Lemma 2.1 of [9].

Lemma 2.1. *We have*

$$\|\Gamma * u\|_\infty \leq C_1(n,p)(\|u\|_1 + \|u\|_p), \quad u \in L^1 \cap L^p(\mathbf{R}^n) \tag{2.1}$$

$$\|\nabla\Gamma * u\|_\infty \leq C_2(n,q)(\|u\|_1 + \|u\|_q), \quad u \in L^1 \cap L^q(\mathbf{R}^n) \tag{2.2}$$

for $p \in (n/2, \infty]$ *and* $q \in (n, \infty]$.

We start with the following lemma, complying with [2].

Lemma 2.2. *Inequality* (1.12) *holds for the weak solution* $u = u(x,t)$.

Proof. For every $T_0 \in (0,T)$, we have

$$u_\varepsilon \in L^\infty(0,T_0;L^1(\mathbf{R}^n)) \cap L^\infty(0,T_0;L^\infty(\mathbf{R}^n)),$$

and

$$M_0 = \sup_{t\in[0,T_0]} \|u_\varepsilon(t)\|_{L^1\cap L^\infty(\mathbf{R}^n)} \le C_3(T_0), \tag{2.3}$$

with C_3 independent of ε, see [9], where

$$\|u\|_{L^1\cap L^\infty(\mathbf{R}^n)} = \|u\|_1 + \|u\|_\infty.$$

We put

$$\mathcal{F}_{\varepsilon,l}(t) = \int_{\mathbf{R}^n} \frac{(u_\varepsilon+\varepsilon)^m}{m}\psi_l dx - \frac{1}{2}\left\langle \Gamma * u_\varepsilon, u_\varepsilon\right\rangle, \tag{2.4}$$

where ψ_l, $l = 1, 2, \cdots$, is the cut-off function, see [2, 9], satisfying

$$\begin{gathered} 0 \le \psi_l \le 1, \quad \psi_l \equiv 1 \text{ in } B(0,l), \quad \text{supp}\psi_l \subset B(0,2l) \\ x\cdot\nabla\psi_l \le 0, \quad |\nabla\psi_l| \le C_4 l^{-1}\psi_l^{1/2}, \quad |\nabla^2\psi_l| \le C_5 l^{-2}, \end{gathered} \tag{2.5}$$

and $\nabla^2\psi_l$ denotes the Hessian matrix of ψ_l. It holds that

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_{\varepsilon,l} = &-\int_{\mathbf{R}^n} \nabla\{(u_\varepsilon+\varepsilon)^{m-1}\psi_l - v_\varepsilon\}\cdot(u_\varepsilon+\varepsilon)\nabla\{(u_\varepsilon+\varepsilon)^{m-1} - v_\varepsilon\}dx \\ &- \varepsilon\int_{\mathbf{R}^n} \nabla\{(u_\varepsilon+\varepsilon)^{m-1}\psi_l - v_\varepsilon\}\cdot\nabla v_\varepsilon dx, \end{aligned} \tag{2.6}$$

for $v_\varepsilon = \Gamma * u_\varepsilon$. Putting $W_\varepsilon = (u_\varepsilon+\varepsilon)^{m-1} - v_\varepsilon$, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_{\varepsilon,l} &= -\int_{\mathbf{R}^n} \nabla\{W_\varepsilon\psi_l + (\psi_l - 1)v_\varepsilon\}\cdot(u_\varepsilon+\varepsilon)\nabla W_\varepsilon dx \\ &\quad -\varepsilon\int_{\mathbf{R}^n} \nabla\{W_\varepsilon\psi_l + (\psi_l-1)v_\varepsilon\}\cdot\nabla v_\varepsilon dx \\ &= -J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{2.7}$$

with

$$\begin{aligned} J_1 &= \int_{\mathbf{R}^n} (u_\varepsilon+\varepsilon)|\nabla W_\varepsilon|^2\psi_l dx \\ J_2 &= -\int_{\mathbf{R}^n} (u_\varepsilon+\varepsilon)(W_\varepsilon+v_\varepsilon)\nabla W_\varepsilon\cdot\nabla\psi_l dx \end{aligned}$$

$$\begin{aligned} J_3 &= -\int_{\mathbf{R}^n} (u_\varepsilon+\varepsilon)\nabla W_\varepsilon \cdot (\psi_l - 1)\nabla v_\varepsilon dx \\ J_4 &= -\varepsilon \int_{\mathbf{R}^n} \nabla\{W_\varepsilon \psi_l + (\psi_l - 1)v_\varepsilon\}\cdot \nabla v_\varepsilon dx. \end{aligned}$$

Using $-\Delta v_\varepsilon = u_\varepsilon$, $u_\varepsilon \geq 0$, and $\psi_l \geq 0$, we have

$$\begin{aligned} J_4 &= \varepsilon \int_{\mathbf{R}^n} \Delta v_\varepsilon \{W_\varepsilon \psi_l + (\psi_l - 1)v_\varepsilon\} dx \\ &= -\varepsilon \int_{\mathbf{R}^n} u_\varepsilon (W_\varepsilon \psi_l + (\psi_l - 1)v_\varepsilon) dx \\ &= -\varepsilon \int_{\mathbf{R}^n} u_\varepsilon (u_\varepsilon+\varepsilon)^{m-1}\psi_l - u_\varepsilon v_\varepsilon \psi_l dx \leq \varepsilon \|u_\varepsilon\|_1 \|v_\varepsilon\|_\infty. \end{aligned} \tag{2.8}$$

Since $W_\varepsilon + v_\varepsilon = (u_\varepsilon+\varepsilon)^{m-1}$, it holds that

$$\begin{aligned} J_2 &\leq \delta J_1 + \frac{1}{4\delta}\int_{\mathbf{R}^n} (u_\varepsilon+\varepsilon)^{2m-1}\psi_l^{-1}|\nabla\psi_l|^2 dx \\ &\leq \delta J_1 + \delta^{-1} l^{-2} C_6 \int_{\mathbf{R}^n} (u_\varepsilon+\varepsilon)^{2m-1}\chi_{D_l} dx \\ &\leq \delta J_1 + \delta^{-1} l^{-2} 2^{2m-1} C_6 (\|u_\varepsilon\|_{2m-1}^{2m-1} + \varepsilon^{2m-1}|D_l|), \end{aligned} \tag{2.9}$$

for any $\delta > 0$, where $D_l = \mathrm{supp}|\nabla\psi_l|$ and $|A|$ denotes Lebesgue's measure of A. We have $0 \leq \psi_l \leq 1$ and hence

$$\begin{aligned} J_3 =& -\int_{\mathbf{R}^n} (u_\varepsilon+\varepsilon)\nabla(u_\varepsilon+\varepsilon)^{m-1}\cdot(\psi_l - 1)\nabla v_\varepsilon dx \\ &- \int_{\mathbf{R}^n} (u_\varepsilon+\varepsilon)|\nabla v_\varepsilon|^2 (1-\psi_l) dx \\ &\leq -\int_{\mathbf{R}^n} (u_\varepsilon+\varepsilon)\nabla(u_\varepsilon+\varepsilon)^{m-1}\cdot(\psi_l - 1)\nabla v_\varepsilon dx \\ &= -\frac{m-1}{m}\int_{\mathbf{R}^n} \nabla(u_\varepsilon+\varepsilon)^m \cdot (\psi_l - 1)\nabla v_\varepsilon dx \qquad (2.10) \\ &= \frac{m-1}{m}\int_{\mathbf{R}^n} u_\varepsilon(u_\varepsilon+\varepsilon)^m (1-\psi_l) dx + \frac{m-1}{m}\int_{\mathbf{R}^n} (u_\varepsilon+\varepsilon)^m \nabla v_\varepsilon \cdot \nabla\psi_l dx \\ &\leq \int_{\mathbf{R}^n} u_\varepsilon (u_\varepsilon+\varepsilon)^m (1-\psi_l) dx + l^{-1} 2^m C_7 \|\nabla v_\varepsilon\|_\infty (\|u_\varepsilon\|_m^m + \varepsilon^m |D_l|). \end{aligned}$$

Combining (2.7)-(2.10) with $\delta = 1$ and Lemma 2.1, we obtain

$$\mathcal{F}_{\varepsilon,l}(t) - \mathcal{F}_{\varepsilon,l}(0) \leq C_8(n, M_0)\cdot T_0\big\{\varepsilon + l^{-2}(1+\varepsilon^{2m-1}|D_l|)$$

$$+l^{-1}(1+\varepsilon^m|D_l|)\big\}+C_8\int_0^t\int_{\mathbf{R}^n}u_\varepsilon(1-\psi_l)dxds=K_{\varepsilon,l}(t);$$

that is,

$$\int_{\mathbf{R}^n}\frac{(u_\varepsilon(t)+\varepsilon)^m}{m}\psi_l dx-\frac{1}{2}\{\langle\Gamma*u_\varepsilon(t),\psi_l u_\varepsilon(t)\rangle+\langle\Gamma*u_\varepsilon(t),(1-\psi_l)u_\varepsilon(t)\rangle\}$$
$$\leq\int_{\mathbf{R}^n}\frac{(u_{0\varepsilon}+\varepsilon)^m}{m}\psi_l dx-\frac{1}{2}\langle\Gamma*u_{0\varepsilon},u_{0\varepsilon}\rangle+K_{\varepsilon,l}(t),$$

for all $t\in[0,T_0)$. Since we have

$$\|u_\varepsilon(t)\|_1=\|u_{0\varepsilon}\|_1 \qquad \text{for any } t\in[0,T_0) \tag{2.11}$$
$$u_\varepsilon\to u \qquad \text{in } L^\infty(0,T_0;L^p(\mathbf{R}^n)) \text{ for all } p\in[1,\infty), \tag{2.12}$$

see [9], besides (1.5)-(1.6), it holds that

$$\lim_{l\uparrow\infty}\lim_{\varepsilon_j\downarrow 0}K_{\varepsilon_j,l}(t)=0$$
$$\lim_{l\uparrow\infty}\lim_{\varepsilon_j\downarrow 0}\left\langle\Gamma*u_{\varepsilon_j}(t),(1-\psi_l)u_{\varepsilon_j}(t)\right\rangle=0$$
$$\lim_{l\uparrow\infty}\lim_{\varepsilon_j\downarrow 0}\int_{\mathbf{R}^n}\frac{(u_{\varepsilon_j}(t)+\varepsilon_j)^m}{m}\psi_l dx-\frac{1}{2}\left\langle\Gamma*u_{\varepsilon_j}(t),\psi_l u_{\varepsilon_j}(t)\right\rangle$$
$$=\int_{\mathbf{R}^n}\frac{u^m(t)}{m}dx-\frac{1}{2}\langle\Gamma*u(t),u(t)\rangle$$
$$\lim_{l\uparrow\infty}\lim_{\varepsilon_j\downarrow 0}\int_{\mathbf{R}^n}\frac{(u_{0\varepsilon_j}+\varepsilon_j)^m}{m}\psi_l dx-\frac{1}{2}\left\langle\Gamma*u_{0\varepsilon_j},\psi_l u_{0\varepsilon_j}\right\rangle$$
$$=\int_{\mathbf{R}^n}\frac{u_0^m}{m}dx-\frac{1}{2}\langle\Gamma*u_0,u_0\rangle,$$

and hence (1.12) holds. □

Next, we examine the scaling property of the variational functional $\mathcal{F}(u)$.

Lemma 2.3. It holds that

$$j_*=\inf\{\mathcal{F}(u)\mid 0\leq u\in L^m(\mathbf{R}^n),\ \int_{\mathbf{R}^n}u=\lambda_*\}=0,$$

if $m=2-\frac{2}{n}$, where $\lambda_*=\lambda_*(n)$ is the quantized mass determined by (1.18)-(1.19).

Proof. The higher-dimensional Trudinger-Moser inequality holds on the bounded domain, and in particular, we obtain, see [11, 10],

$$j_R=\inf\{\mathcal{F}(u)\mid u\geq 0,\ \text{supp}\ u\subset B_R,\ \int_{\mathbf{R}^n}u=\lambda_*\}>-\infty, \tag{2.13}$$

where $B_R = B(0,R)$. Here, it follows that

$$\mathcal{F}(u_\mu) = \mu^{n-2}\mathcal{F}(u), \tag{2.14}$$

for $u_\mu(x) = \mu^n u(\mu x)$ since $m = 2 - \frac{2}{n}$. Since supp $u_\mu \subset B_{\mu^{-1}R}$ if and only if supp $u \subset B_R$, therefore, we obtain

$$j_{\mu^{-1}R} = \mu^{n-2} j_R \geq j_R,$$

for $\mu > 1$. This implies $j_R \geq 0$ and hence $j_* \geq 0$ because $R > 0$ is arbitrary. We have $j_* = \mu^{n-2} j_*$ again by the above scaling, where $\mu > 0$ is arbitrary. Then, it follows that $j_* = 0$. □

Now, we show the key lemma.

Lemma 2.4. *If* (1.13) *holds, the mapping*

$$t \in [0,T) \mapsto \int_{\mathbf{R}^n} |x|^2 u(x,t)dx$$

is locally absolutely continuous and it follows that (1.11) *holds for almost every* $t \in [0,T)$.

We shall use additional lemmas for our purpose.

Lemma 2.5. *Given* $\varphi \in C_0^\infty(\mathbf{R}^n)$, *the mapping*

$$t \in [0,T) \mapsto \int_{\mathbf{R}^n} \varphi(x) u(x,t)dx$$

is locally absolutely continuous.

Proof. Put

$$F(t) = \int_{\mathbf{R}^n} \varphi(x) u(x,t)dx,$$

and fix $T_0 \in (0,T)$. We shall show the existence of $f = f(t) \in L^1(0,T_0)$ such that

$$F(t) - F(0) = \int_0^t f(s)ds \qquad \text{for } t \in [0,T_0]. \tag{2.15}$$

Such an f is actually given by

$$f(t) = \frac{m-1}{m}\int_{\mathbf{R}^n} u^m \Delta\varphi dx + \frac{1}{2}\iint_{\mathbf{R}^n\times\mathbf{R}^n} \rho_\varphi u \otimes u dx dx',$$

where $u \otimes u = u(x,t)u(x',t)$ and

$$\rho_\varphi = \rho_\varphi(x,x') = (\nabla\varphi(x) - \nabla\varphi(x')) \cdot \nabla\Gamma(x-x').$$

First, we have

$$\frac{d}{dt}\int_{\mathbf{R}^n}\varphi u_\varepsilon dx = \frac{m-1}{m}\int_{\mathbf{R}^n}(u_\varepsilon+\varepsilon)^m\Delta\varphi dx + \frac{1}{2}\iint_{\mathbf{R}^n\times\mathbf{R}^n}\rho_\varphi u_\varepsilon \otimes u_\varepsilon dx dx', \tag{2.16}$$

for all $t\in[0,T_0)$. In particular,

$$F_\varepsilon(t) = \int_{\mathbf{R}^n}\varphi(x)u_\varepsilon(x,t)dx$$

is absolutely continuous on $[0,T_0]$ because the right-hand side of (2.16) is integrable on $(0,T_0)$, and therefore, it holds that

$$\begin{aligned} F_\varepsilon(t)-F_\varepsilon(0) &= \frac{m-1}{m}\int_0^t ds\int_{\mathbf{R}^n}(u_\varepsilon+\varepsilon)^m\Delta\varphi dx \\ &+\frac{1}{2}\int_0^t ds\iint_{\mathbf{R}^n\times\mathbf{R}^n}\rho_\varphi u_\varepsilon\otimes u_\varepsilon dx dx', \end{aligned} \tag{2.17}$$

for $t\in[0,T_0]$. Here, we have

$$\begin{aligned} &F_{\varepsilon_j}(t)-F_{\varepsilon_j}(0)\to F(t)-F(0) \\ &\iint_{Q_t}(u_{\varepsilon_j}+\varepsilon_j)^m\Delta\varphi dx ds \to \iint_{Q_t}u^m\Delta\varphi dx ds, \end{aligned} \tag{2.18}$$

as $\varepsilon_j\downarrow 0$ by (2.12), where $Q_t=\mathbf{R}^n\times(0,t)$. Let $1<p,q<\infty$ be as in $\frac{1}{p}+\frac{1}{q}+\frac{n-2}{n}=2$. We have

$$\sup_{t\in[0,T_0]}\|u(t)\|_{L^1\cap L^\infty(\mathbf{R}^n)}\le C_9(T_0) \tag{2.19}$$

$$u_{\varepsilon_j}\to u \quad \text{for a.e. } x\in\mathbf{R}^n \text{ and for all } t\in[0,T), \tag{2.20}$$

see [9], and observe

$$|\rho_\varphi(x,x')|\le(n-2)\|\nabla^2\varphi\|_\infty\Gamma(x-x') = \omega_{n-1}^{-1}\|\nabla^2\varphi\|_\infty|x-x'|^{2-n}. \tag{2.21}$$

By virtue of (2.3) and (2.19) -(2.21), we can apply the HLS inequality and the (2.12) to obtain

$$\begin{aligned} &\int_0^t ds\iint_{\mathbf{R}^n\times\mathbf{R}^n}|\rho_\varphi(x,x')|\cdot|u_\varepsilon(x,s)u_\varepsilon(x',s)-u(x,s)u(x',s)|dx dx' \\ &\le \omega_{n-1}^{-1}\|\nabla^2\varphi\|_\infty \\ &\quad\times\int_0^t dx\iint_{\mathbf{R}^n\times\mathbf{R}^n}\frac{(|u_\varepsilon(x,s)|+|u(x,s)|)\cdot|u_\varepsilon(x',s)-u(x's)|}{|x-x'|^{n-2}}dx dx' \\ &\le C_{10}(n,p)\|\nabla^2\varphi\|_\infty\int_0^t(\|u(s)\|_p+\|u_\varepsilon(s)\|_p)\cdot\|u_\varepsilon(s)-u(s)\|_q dx \end{aligned}$$

$$\leq C_{10}(n,p)\|\nabla^2\varphi\|_\infty(C_9(T_0)+C_3(T_0))$$
$$\times\int_0^t\left(\int_{\mathbf{R}^n}|u_\varepsilon(x,s)-u(x,s)|^q\right)^{1/q}ds \to\ 0, \tag{2.22}$$

as $\varepsilon\downarrow 0$ for all $t\in[0,T_0]$. Combining (2.16)-(2.18) and (2.22), we have the desired result. □

Corollary 1. *We have* (1.14) *for our weak solution* u.

Proof. Fix $T_0\in(0,T)$. The uniform estimate (2.19) gives

$$\sup_{t\in[0,T_0]}\left|\int_{\mathbf{R}^n}\varphi(x)u(x,t)dx\right|\leq C_9\|\varphi\|_{p'},$$

for all $\varphi\in C_0^\infty(\mathbf{R}^n)$ and $p\in(1,\infty]$ with $1/p+1/p'=1$. The mapping

$$[0,T_0]\ni t\ \mapsto\ \int_{\mathbf{R}^n}\varphi(x)u(x,t)dx$$

is continuous thanks to Lemma 2.5, and thus $u(\cdot,t)\in L^p(\mathbf{R}^n)$ for all $t\in[0,T_0]$ and $p\in(1,\infty]$. The continuity follows from the density of the embedding $C_0^\infty(\mathbf{R}^n)\subset L^{p'}(\mathbf{R}^n)$, and therefore (1.14) holds. □

Lemma 2.6. *If* (1.13) *holds, then*

$$\int_{\mathbf{R}^n}|x|^2u(x,t)dx\in L^\infty_{loc}[0,T).$$

Proof. We fix $T_0\in(0,T)$ and set

$$E_{\varepsilon,l}(t)=\int_{\mathbf{R}^n}|x|^2\psi_l(x)u_\varepsilon(x,t)dx,$$

for $0<\varepsilon\ll 1$ and $l=1,2,\cdots$, where $\psi_l\in C_0^\infty(\mathbf{R}^n)$ is the cut-off function satisfying (2.5). Then, it holds that

$$\frac{d}{dt}E_{\varepsilon,l}=\frac{m-1}{m}\int_{\mathbf{R}^n}(u_\varepsilon+\varepsilon)^m\Delta(|x|^2\psi_l)dx$$
$$+\int_{\mathbf{R}^n}u_\varepsilon\nabla\Gamma*u_\varepsilon\cdot\nabla(|x|^2\psi_l)dx=I+II, \tag{2.23}$$

for all $t\in[0,T_0]$. Henceforth, we write C_4 and C_5 in (2.5) as M_∇ and M_Δ, respectively. The estimate (2.19), combined with (2.5) and Lemma 2.1, gives

$$II\ \ \leq\ \ 2\int_{\mathbf{R}^n}u_\varepsilon|\nabla\Gamma*u_\varepsilon|\cdot|x|\psi_l dx+\int_{\mathbf{R}^n}u_\varepsilon\nabla\Gamma*u_\varepsilon\cdot|x|^2\nabla\psi_l dx$$

$$\begin{aligned}
&\leq \int_{\mathbf{R}^n}|x|^2\psi_l u_\varepsilon dx+\int_{\mathbf{R}^n}\psi_l u_\varepsilon|\nabla\Gamma*u_\varepsilon|^2dx\\
&\quad+\frac{M_\nabla}{l}\cdot 2l\int_{\mathbf{R}^n}|x|\psi_l^{1/2}u_\varepsilon|\nabla\Gamma*u_\varepsilon|dx\\
&\leq (1+M_\nabla)\int_{\mathbf{R}^n}|x|^2\psi_l u_\varepsilon dx+(1+M_\nabla)\int_{\mathbf{R}^n}u_\varepsilon|\nabla\Gamma*u_\varepsilon|^2dx\\
&\leq (1+M_\nabla)E_{\varepsilon,l}+C_{11}(n,T_0), \qquad (2.24)
\end{aligned}$$

for all $t\in[0,T_0]$. Since

$$\Delta(|x|^2\psi_l)\leq 2n\psi_l+|x|^2\Delta\psi_l$$

holds by (2.5), we obtain

$$\begin{aligned}
I &\leq 2n\int_{\mathbf{R}^n}(u_\varepsilon+\varepsilon)^m\psi_l dx+\int_{\mathbf{R}^n}(u_\varepsilon+\varepsilon)^m|x|^2\Delta\psi_l dx\\
&\leq 2^m n\int_{\mathbf{R}^n}(u_\varepsilon^m+\varepsilon^m)\psi_l dx+2^m M_\Delta\left(\frac{2l}{l}\right)^2\int_{\mathrm{supp}\Delta\psi_l}(u_\varepsilon^m+\varepsilon^m)dx\\
&\leq C_{12}(n)\sup_{t\in[0,T_0]}\|u_\varepsilon(t)\|_m^m+\varepsilon^m C_{13}(n,l)\\
&\leq C_{14}(n,T_0)+\varepsilon^m C_{15}(n,l), \qquad (2.25)
\end{aligned}$$

for all $t\in[0,T_0]$. Then, (2.23) -(2.25) imply

$$E_{\varepsilon,l}(t)\leq e^{(1+M_\nabla)T_0}\left\{E_{\varepsilon,l}(0)+(1+C_{16}(n,l)\varepsilon^m)C_{17}(n,T_0)\right\},$$

for all $t\in[0,T_0]$. By Fatou's lemma, we have, for all $t\in[0,T_0]$,

$$\begin{aligned}
\int_{\mathbf{R}^n}|x|^2u(x,t)dx &\leq \lim_{l\uparrow\infty}\liminf_{\varepsilon\downarrow 0}E_{\varepsilon,l}(t)\\
&\leq e^{(1+M_\nabla)T_0}\Big\{\int_{\mathbf{R}^n}|x|^2u_0(x)dx+C_{18}(n,T_0)\Big\},
\end{aligned}$$

and hence the desired inequality by the assumption (1.13). □

Proof of Lemma 2.4. We fix $T_0\in(0,T)$ and put

$$E(t)=\int_{\mathbf{R}^n}|x|^2u(x,t)dx,\qquad E_l(t)=\int_{\mathbf{R}^n}|x|^2\psi_l(x)u(x,t)dx,$$

$l=1,2,\cdots$. From Lemma 2.5 with $\varphi=|x|^2\psi_l$, $E_l=E_l(t)$ is absolutely continuous on $[0,T_0]$ and it holds that

$$\frac{d}{dt}E_l(t)=\frac{m-1}{m}\int_{\mathbf{R}^n}u^m\Delta(|x|^2\psi_l)dx+\frac{1}{2}\iint_{\mathbf{R}^n\times\mathbf{R}^n}\rho_\varphi u\otimes u dx dx',$$

for almost every $t \in (0, T_0)$. Using

$$\Delta(|x|^2\psi_l) = 2n\psi_l + 4x \cdot \nabla\psi_l + |x|^2\Delta\psi_l,$$

$$\rho_\varphi(x, x') = 2\psi_l(x)(x - x') \cdot \nabla\Gamma(x - x') + 2\{\psi_l(x) - \psi_l(x')\}x' \cdot \nabla\Gamma(x - x') + \{|x|^2\nabla\psi_l(x) - |x'|^2\nabla\psi_l(x')\} \cdot \nabla\Gamma(x - x'),$$

$$2(x - x') \cdot \nabla\Gamma(x - x') = -2(n - 2)\Gamma(x - x'),$$

we obtain

$$\frac{d}{dt}E_l(t) = I_1 + I_2 + II_1 + II_2 + II_3 + II_4, \tag{2.26}$$

with

$$I_1 = \frac{m-1}{m} \cdot 2n \int_{\mathbf{R}^n} \psi_l u^m dx$$

$$I_2 = -(n - 2) \iint_{\mathbf{R}^n \times \mathbf{R}^n} \psi_l(x)\Gamma(x - x')u \otimes u dx dx'$$

$$II_1 = \frac{4(m-1)}{m} \int_{\mathbf{R}^n} u^m (x \cdot \nabla\psi_l) dx$$

$$II_2 = \frac{m-1}{m} \int_{\mathbf{R}^n} u^m |x|^2 \Delta\psi_l dx$$

$$II_3 = \iint_{\mathbf{R}^n \times \mathbf{R}^n} \{\psi_l(x) - \psi_l(x')\}x' \cdot \nabla\Gamma(x - x')u \otimes u dx dx'$$

$$II_4 = \frac{1}{2} \iint_{\mathbf{R}^n \times \mathbf{R}^n} \{|x|^2\nabla\psi_l(x) - |x'|^2\nabla\psi_l(x')\} \cdot \nabla\Gamma(x - x')u \otimes u dx dx'.$$

By (2.19) and the dominated convergence theorem, first, we have

$$I_1 \to \frac{m-1}{m} \cdot 2n \int_{\mathbf{R}^n} u^m dx \tag{2.27}$$

$$I_2 \to -(n - 2) \iint_{\mathbf{R}^n \times \mathbf{R}^n} \Gamma(x - x')u \otimes u dx dx', \tag{2.28}$$

as $l \uparrow +\infty$ for almost every $t \in (0, T_0)$. Using (2.5), (2.19) and Lemma 2.6, next, we have

$$|II_1| \le 4 \int_{\mathbf{R}^n} |x|u^m \cdot l^{-1} M_\nabla dx \le l^{-1} \cdot 4M_\nabla \left(\int_{\mathbf{R}^n} |x|^2 u dx \right)^{1/2} \left(\int_{\mathbf{R}^n} u^{2m-1} dx \right)^{1/2} \to 0, \tag{2.29}$$

and

$$|II_2| \le 2l^{-1} M_\Delta \int_{\mathbf{R}^n} |x| u^m dx$$
$$\le l^{-1} \cdot 2M_\Delta \left(\int_{\mathbf{R}^n} |x|^2 u dx \right)^{1/2} \left(\int_{\mathbf{R}^n} u^{2m-1} dx \right)^{1/2} \to 0, \qquad (2.30)$$

as $l \uparrow \infty$ for almost every $t \in (0, T_0)$. We also note

$$\frac{|\psi_l(x) - \psi_l(x')|}{|x-x'|} |(x-x') \cdot \nabla\Gamma(x-x')| \le \|\nabla \psi_l\|_\infty \cdot \frac{1}{\omega_{n-1}|x-x'|^{n-2}}$$
$$\le l^{-1} \cdot \omega_{n-1}^{-1} M_\nabla |x-x'|^{2-n},$$

and use the HLS inequality, (2.19) and Lemma 2.6 to obtain

$$|II_3| \le \; l^{-1} \cdot \omega_{n-1}^{-1} M_\nabla \iint_{\mathbf{R}^n \times \mathbf{R}^n} |x-x'|^{2-n} \cdot u(x,t) \cdot |x'| u(x',t) dx dx'$$
$$\le l^{-1} \cdot C_{19}(n) \|u(t)\|_{\frac{2n}{n+2}} \cdot \left\{ \int_{\mathbf{R}^n} |x|^{\frac{2n}{n+2}} u^{\frac{n}{n+2}} \cdot u^{\frac{n}{n+2}} dx \right\}^{\frac{n+2}{2n}}$$
$$\le l^{-1} \cdot C_{20}(n, T_0) \|u(t)\|_{n/2}^{1/2} \left\{ \int_{\mathbf{R}^n} |x|^2 u(x,t) dx \right\}^{1/2}$$
$$\le l^{-1} \cdot C_{21}(n, T_0) \to 0, \qquad (2.31)$$

as $l \uparrow \infty$ for almost every $t \in (0, T_0)$. Similarly, we use Lemma 2.1,

$$\frac{|\nabla\psi_l(x) - \nabla\psi_l(x')|}{|x-x'|} |(x-x') \cdot \nabla\Gamma(x-x')|$$
$$\le \|\nabla^2 \psi_l\|_\infty \cdot \frac{1}{\omega_{n-1}|x-x'|^{n-2}} \le l^{-2} \cdot (n-2) M_\Delta \cdot \Gamma(x-x'),$$

and

$$\big| |x|^2 - |x'|^2 \big| \, |\nabla\psi_l(x')| \le l^{-1} \cdot M_\nabla |x-x'| (|x| + |x'|),$$

to get

$$|II_4| \le l^{-2} \cdot \frac{(n-2)M_\Delta}{2} \int_{\mathbf{R}^n} |x|^2 u(x,t) \cdot \Gamma * u(x,t) dx$$
$$+ l^{-1} \cdot \omega_{n-1}^{-1} M_\nabla \iint_{\mathbf{R}^n \times \mathbf{R}^n} |x-x'|^{2-n} \cdot u(x,t) \cdot |x'| u(x',t) dx dx'$$
$$\le l^{-2} \cdot C_{22}(n) \|\Gamma * u\|_{L^\infty(Q_{T_0})} \int_{\mathbf{R}^n} |x|^2 u(x,t) dx + l^{-1} \cdot C_{23}(n, T_0)$$
$$\le (l^{-2} + l^{-1}) \cdot C_{24}(n, T_0) \to 0, \qquad (2.32)$$

as $l \uparrow \infty$ for almost every $t \in (0, T_0)$. Now, (2.26) reads

$$E_l(t) - E_l(0) = \int_0^t (I_1 + I_2 + II_1 + II_2 + II_3 + II_4)(s)ds,$$

and it is clear that $E_l(t) - E_l(0) \to E(t) - E(0)$ for all $t \in [0, T_0]$. Thus, (2.27)-(2.32) imply

$$E(t) = E(0) + \int_0^t e(s)ds,$$

for all $t \in [0, T_0]$ with

$$e(t) = \frac{m-1}{m} \cdot 2n \int_{\mathbf{R}^n} u^m(x,t)dx - (n-2) \iint_{\mathbf{R}^n \times \mathbf{R}^n} \Gamma(x-x')u(x,t)u(x',t)dxdx',$$

summable on $(0, T_0)$. The proof is complete. □

3. PROOF OF THEOREM 1

At this stage, the following lemma is immediate.

Lemma 3.1. *If the initial value u_0 satisfies $\mathcal{F}(u_0) < 0$ and* (1.13), *then $T < +\infty$ arises.*

Proof. From (1.12) and (1.11), it follows that

$$\int_{\mathbf{R}^n} |x|^2 u(x,t)dx < 0 \qquad \text{for } t \gg 1,$$

if both $\mathcal{F}(u_0) < 0$ and $T = +\infty$ occur, a contradiction. Thus, $\mathcal{F}(u_0) < 0$ implies $T < +\infty$. □

We use the following lemma to treat the sub-critical mass case.

Lemma 3.2. *If $\lambda < \lambda_*$, we have*

$$\|u(t)\|_m + \langle \Gamma * u(t), u(t) \rangle \le C_1, \qquad t \in [0, T). \tag{3.1}$$

Proof. We have $\|v\|_1 = \lambda_*$ for $v = \frac{\lambda_*}{\lambda} u$, and this implies

$$\begin{aligned} \mathcal{F}(u_0) \quad &\geq \quad \mathcal{F}(u) = \int_{\mathbf{R}^n} \frac{u^m}{m} dx - \frac{1}{2} \langle \Gamma * u, u \rangle \\ &= \quad \left(\frac{\lambda}{\lambda_*}\right)^m \int_{\mathbf{R}^n} \frac{v^m}{m} dx - \frac{1}{2} \left(\frac{\lambda}{\lambda_*}\right)^2 \langle \Gamma * v, v \rangle \end{aligned}$$

$$\geq \begin{cases} \frac{1}{2}\left\{\left(\frac{\lambda}{\lambda_*}\right)^m - \left(\frac{\lambda}{\lambda_*}\right)^2\right\}\langle \Gamma * v, v\rangle \\ \left\{\left(\frac{\lambda}{\lambda_*}\right)^m - \left(\frac{\lambda}{\lambda_*}\right)^2\right\}\int_{\mathbf{R}^n} \frac{v^m}{m} dx, \end{cases}$$

by Lemma 2.3. Then, (3.1) follows from $0 < \lambda < \lambda_*$ and $1 < m < 2$. □

Proof of Theorem 1. Since the Trudinger-Moser inequality (2.13) is sharp, it holds that

$$\inf\{\mathcal{F}(u) : 0 \leq u \in L^m(\mathbf{R}^n),\ \mathrm{supp}\ u \subset B_R,\ \int_{\mathbf{R}^n} u = \lambda\} = -\infty,$$

for any $R > 0$ and $\lambda > \lambda_*$. Each $\lambda > \lambda_*$, in particular, admits an initial value $u_0 = u_0(x) \geq 0$ with compact support such that $\|u_0\|_1 = \lambda$ and $\mathcal{F}(u_0) < 0$. Then, $T < +\infty$ follows from Lemma 3.1.

If $\lambda = \|u_0\|_1 < \lambda_*$ is the case, Lemma 3.2 implies

$$\sup_{t\in[0,T)} \|u(t)\|_p \leq C_2, \tag{3.2}$$

for $p = m = 2 - \frac{2}{n} > 1$. We shall show that inequality (3.2) with some $p > 1$ implies

$$\limsup_{t\uparrow T} \|u(t)\|_\infty < +\infty. \tag{3.3}$$

Then, $T = +\infty$ holds by (1.7). For this purpose, we have only to derive estimates of u_ε uniform in ε. We shall write $u = u_\varepsilon$ and $U = u_\varepsilon + \varepsilon$ again.

First, (1.4) implies

$$\begin{aligned} \frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1} &= -\frac{m-1}{m}(\nabla U^m, \nabla u^r) + (u\nabla\Gamma * u, \nabla u^r) \\ &= -\int_{\mathbf{R}^n} (m-1)r \cdot U^{m-1}u^{r-1}|\nabla u|^2 - \frac{r}{r+1}\nabla u^{r+1} \cdot \nabla\Gamma * u\ dx \\ &\leq -\int_{\mathbf{R}^n} (m-1)r \cdot u^{m+r-2}|\nabla u|^2 - \frac{r}{r+1}u^{r+2}\ dx \\ &= -\frac{4r(m-1)}{(m+r)^2}\int_{\mathbf{R}^n} |\nabla u^{\frac{m+r}{2}}|^2 + \frac{r}{r+1}\int_{\mathbf{R}^n} u^{r+2}\ dx, \end{aligned} \tag{3.4}$$

for $r \geq 1$. Next, we have

$$q_1 = 1 < \frac{r+m}{2} \leq q_2 = r + 2 \leq \frac{n}{n-2}(r+m),$$

since $m = 2 - \frac{2}{n}$, and, therefore, Gagliardo-Nirenberg's inequality is applicable. We obtain, see [2],

$$\|w\|_{q_2} \leq C^{\frac{2}{r+m}} \|w\|_{q_1}^{1-\theta} \|\nabla w^{\frac{m+r}{2}}\|_2^{\frac{2\theta}{m+r}}, \tag{3.5}$$

where $C = C_0^{1/\beta}$ with $C_0 > 0$ independent of $r \geq 1$,

$$\begin{aligned} \beta &= \frac{q_2 - \frac{r+m}{2}}{q_2 - q_1} \Big[\frac{2q_1}{r+m} + (1 - \frac{2q_1}{r+m}) \cdot \frac{2n}{n+2} \Big] \\ &= \frac{\frac{r+1}{2} + \frac{1}{n}}{r+1} \Big[\frac{2n}{n+2} - \frac{n-2}{n+2} \cdot \frac{2}{r+m} \Big] \sim \frac{n}{n+2}, \qquad r \uparrow \infty, \end{aligned}$$

and

$$\begin{aligned} \theta &= \frac{r+m}{2} (\frac{1}{q_1} - \frac{1}{q_2})(\frac{1}{n} - \frac{1}{2} + \frac{r+m}{2q_1})^{-1} \\ &= \frac{m+r}{2} (1 - \frac{1}{r+2})(\frac{1}{n} - \frac{1}{2} + \frac{r+m}{2})^{-1} = \frac{r+m}{r+2} \sim 1, \qquad r \uparrow \infty. \end{aligned}$$

Since

$$\begin{aligned} (1-\theta) q_2 &= \frac{2-m}{r+2} \cdot (r+2) = \frac{2}{n} \\ \frac{2\theta}{m+r} \cdot q_2 &= \frac{1}{r+m} \cdot 2 \cdot \frac{r+m}{r+2} (r+2) = 2, \end{aligned}$$

inequality (3.5) reads

$$\|w\|_{r+2}^{r+2} \leq C_3 \|w\|_1^{2/n} \|\nabla w^{\frac{m+r}{2}}\|_2^2, \tag{3.6}$$

with a constant $C_3 > 0$ independent of $r \geq 1$.

Given $k \geq 1$ and $l \geq 1$, we take the cut-off function ψ_l satisfying (2.5) and the truncation $\overline{u}(x,t) = \min\{u(x,t), k\}$. It is clear that

$$\|u\|_{r+2}^{r+2} \leq 2^{r+1} (\|u - \overline{u}\|_{r+2}^{r+2} + \|\overline{u}\|_{r+2}^{r+2}). \tag{3.7}$$

We find

$$\begin{aligned} \|u - \overline{u}\|_{r+2}^{r+2} &\leq C_3 \|u - \overline{u}\|_1^{2/n} \|\nabla (u - \overline{u})^{\frac{r+m}{2}}\|_2^2 \\ &\leq C_3 \|u - \overline{u}\|_1^{2/n} \|\nabla u^{\frac{r+m}{2}}\|_2^2, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\|\overline{u}\|_{r+2}^{r+2} \leq 2^{r+1} (\|\overline{u}\psi_l\|_{r+2}^{r+2} + \|\overline{u}(1-\psi_l)\|_{r+2}^{r+2}) \\ &\leq 2^{r+1} k^{r+2} |B(0,2l)| + 2^{r+1} C_3 \|\overline{u}(1-\psi_l)\|_1^{2/n} \|\nabla \{u(1-\psi_l)\}^{\frac{r+m}{2}}\|_2^2 \\ &\leq 2^{r+1} k^{r+2} |B(0,2l)| + 2^{r+2} C_3 \|\overline{u}(1-\psi_l)\|_1^{2/n} (\|\nabla u^{\frac{r+m}{2}}\|_2^2 + \|u\|_{r+m}^{r+m}), \end{aligned} \tag{3.9}$$

by using (3.6) and (2.5). It follows from (3.6) and (1.11)-(1.13) that

$$\|u-\overline{u}\|_1 \le \|u\chi_{\{u>k\}}\|_1 \le \int_{\{u>k\}} k^{-p+1}u^p dx \le k^{-p+1}C_2^p, \tag{3.10}$$

$$\|\overline{u}(1-\psi_l)\|_1 \le \int_{|x|\ge l} udx \le l^{-2}\int_{\mathbf{R^n}} |x|^2 udx \le \frac{C_4}{l^2}(1+T). \tag{3.11}$$

It holds that

$$\|u\|_{r+m}^{r+m} \le \lambda^{(r+m)\mu}\|u\|_{r+2}^{(r+m)(1-\mu)} \le \|u\|_{r+2}^{r+2} + \lambda^{\frac{(r+2)(r+m)\mu}{(r+2)-(r+m)(1-\mu)}}, \tag{3.12}$$

since $1<m<2$, where $\frac{1}{r+m} = \frac{\mu}{1} + \frac{1-\mu}{r+2}$. We collect inequalities (3.7)-(3.12) and conclude that

$$\begin{aligned}&\left\{1 - l^{-4/n}\cdot 2^{2r+3}C_3C_4^{2/n}(1+T)^{2/n}\right\}\|u\|_{r+2}^{r+2} \\ &\le 4^{r+1}k^{r+2}|B(0,2l)| + l^{-4/n}\cdot 2^{2r+3}C_3C_4^{2/n}(1+T)^{2/n}\lambda^{\frac{(r+2)(r+m)\mu}{(r+2)-(r+m)(1-\mu)}} \\ &\quad + \{k^{-\frac{2(p-1)}{n}}\cdot 2^{r+1}C_2^{\frac{2p}{n}}C_3 + l^{-4/n}\cdot 2^{2r+3}C_3C_4^{2/n}(1+T)^{2/n}\}\|\nabla u^{\frac{r+m}{2}}\|_2^2.\end{aligned} \tag{3.13}$$

Combining (3.4) with (3.13) and making $k\gg 1$ and $l \gg 1$, we obtain

$$\sup_{t\in[0,T)} \|u(t)\|_q \le C_5(q,T), \tag{3.14}$$

for each $q\ge 1$ and, therefore,

$$\begin{aligned}\sup_{t\in[0,T)} \|\nabla^2\Gamma * u(t)\|_q &\le C_6(q,T) \\ \sup_{t\in[0,T)} \|\nabla\Gamma * u(t)\|_\infty &\le C_7(n,\lambda,T),\end{aligned} \tag{3.15}$$

by the Calderón-Zygmund estimate and Lemma 2.1.

Let $v=\Gamma * u$. From (3.4) we deduce

$$\begin{aligned}&\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1} \\ &\le -\frac{4r(m-1)}{(r+m)^2}\int_{\mathbf{R^n}} |\nabla u^{\frac{r+m}{2}}|^2 dx + \frac{2r}{r+m}\int_{\mathbf{R^n}} u^{\frac{r-m+2}{2}}\nabla u^{\frac{r+m}{2}}\cdot\nabla v dx \\ &\le -\frac{3r(m-1)}{(r+m)^2}\int_{\mathbf{R^n}} |\nabla u^{\frac{r+m}{2}}|^2 dx + \frac{r}{m-1}\int_{\mathbf{R^n}} u^{r-m+2}|\nabla v|^2 dx \\ &\le -\frac{3r(m-1)}{(r+m)^2}\|\nabla u^{\frac{r+m}{2}}\|_2^2 + \frac{r}{m-1}\|\nabla v\|_\infty^2\|u\|_{r-m+2}^{r-m+2}.\end{aligned} \tag{3.16}$$

By the interpolation inequality for L^p-norms and (1.6), it holds that

$$\|u\|_{r-m+2} \le \|u\|_1^\alpha\|u\|_{r+1}^{1-\alpha} = \lambda^\alpha\|u\|_{r+1}^{1-\alpha}, \tag{3.17}$$

with

$$\alpha = \frac{m-1}{r(r-m+2)}. \tag{3.18}$$

We use Gagliardo-Nirenberg's inequality (3.5) to obtain

$$\|u\|_{r+1} \le C_0^{\frac{2}{\beta(r+m)}} \|u\|_{\frac{r+1}{4}}^{1-\theta} \|\nabla u^{\frac{r+m}{2}}\|_2^{\frac{2\theta}{r+m}}, \tag{3.19}$$

with

$$\begin{aligned}\theta &= \frac{r+m}{2}\Big(\frac{4}{r+1} - \frac{1}{r+1}\Big)\Big(\frac{1}{n} - \frac{1}{2} + \frac{2(r+m)}{r+1}\Big)^{-1} \\ &= \frac{3nr+6n-6}{(3n+2)r+7n-6} \to \frac{3n}{3n+2}, \qquad r\uparrow\infty,\end{aligned} \tag{3.20}$$

and

$$\beta = \frac{r+1-\frac{r+m}{2}}{r+1-\frac{r+1}{4}}\Big(\frac{2}{r+m} + \Big(1 - \frac{r+1}{2(r+m)}\Big)\cdot\frac{2n}{n+2}\Big) \to \frac{2n}{3(n+2)}, \quad r\uparrow\infty. \tag{3.21}$$

From (3.18), (3.20) and (3.21), it is clear that there exist $r_0' = r_0'(n) \gg 1$ and $r_0'' = r_0''(n) \gg 1$ such that

$$\frac{2(1-\alpha)(r-m+2)}{\beta(r+m)} \le \frac{3(n+2)+1}{n} < 6 \quad \text{for all } r \ge r_0', \tag{3.22}$$

$$\frac{\theta(1-\alpha)(r-m+2)}{r+m} \le \frac{3n}{3n+2} \quad \text{for all } r \ge r_0'', \tag{3.23}$$

and

$$\begin{aligned}(1-\theta)(1-\alpha)(r-m+2) &= \frac{h_1(r)}{h_2(r)} \\ := \frac{2r^3 + (n+4-2m)r^2 + \{2(n+1)-(n+2)m\}r - n(m-1)}{(3n+2)r^2+(7n-6)r}.\end{aligned}$$

Given $K > 0$ in

$$2K + \frac{2(7n-6)}{3n+2} - (n+4-2m) > 0, \tag{3.24}$$

we see that there exists $r_K = r_K(n,K) > 0$ such that

$$\frac{h_1(r)}{h_2(r)} \le \frac{2}{3n+2}(r+K) \quad \text{for all } r \ge r_K,$$

since

$$\frac{2}{3n+2}(r+K)h_2(r) - h_1(r) = \Big\{2K + \frac{2(7n-6)}{3n+2} - (n+4-2m)\Big\}r^2$$

$$+\left\{\frac{2(7n-6)K}{3n+2}+(n+2)m-2(n+1)\right\}r+n(m-1).$$

Since $K=n+1$ satisfies (3.24), there exists $r_0'''=r_0'''(n)>0$ such that

$$(1-\theta)(1-\alpha)(r-m+2)=\frac{h_1(r)}{h_2(r)}\le\frac{2(r+1+n)}{3n+2}\quad\text{for all } r\ge r_0'''. \tag{3.25}$$

Combining (3.17), (3.19), and (3.22)-(3.25), we obtain

$$\frac{r}{m-1}\|\nabla v\|_\infty^2\|u\|_{r-m+2}^{r-m+2}\le\frac{r(m-1)}{(r+m)^2}\|\nabla u^{\frac{r+m}{2}}\|_2^2+C_8(n)r^{3n+1}C_0^{3(3n+2)}$$
$$\cdot\,(1+\lambda^{\frac{(3n+2)(m-1)}{2r}})(1+\|\nabla v\|_\infty^{3n+2})\cdot\max\left\{1,\|u\|_{\frac{r+1}{4}}^{r+1+n}\right\}, \tag{3.26}$$

for all $r\ge r_0^*=r_0^*(n)=\max\{r_0',r_0'',r_0'''\}$, where we used $\frac{1}{1-\mu}\le\frac{3n+2}{2}$ for all $r\ge r_0^*$ by (3.23) for $\mu=\frac{\theta(1-\alpha)(r-m+2)}{r+m}$.

Inequalities (3.16) and (3.26) imply

$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1}\le r^{3n+1}C_9(n,\lambda)(1+\|\nabla v\|_\infty^{3n+2})\cdot\max\left\{1,\|u\|_{\frac{r+1}{4}}^{r+1+n}\right\},$$

and hence

$$\begin{aligned}\|u(t)\|_{r+1}^{r+1}&\le\|u_0\|_{r+1}^{r+1}+(r+1)^{3n+2}C_{10}(n,\lambda)\\&\quad\times\int_0^t(1+\|\nabla v(s)\|_\infty^{3n+2})\cdot\max\left\{1,\|u(s)\|_{\frac{r+1}{4}}^{r+1+n}\right\}dx\\&\le(r+1)^{4n}C_{11}\cdot\max\left\{\|u_0\|_{r+1}^{r+1},\sup_{t\in[0,T)}\|\nabla v(t)\|_\infty^{4n},\sup_{t\in[0,T)}\|u(t)\|_{\frac{r+1}{4}}^{r+1+n},1\right\}.\end{aligned}$$

Thus it holds that

$$\begin{aligned}\|u(t)\|_{r+1}&\le\{(r+1)C_{12}\}^{\frac{4n}{r+1}}\\&\quad\times\max\left\{\|u_0\|_{r+1},\sup_{t\in[0,T)}\|\nabla v(t)\|_\infty,\sup_{t\in[0,T)}\|u(t)\|_{\frac{r+1}{4}}^{1+\frac{n}{r+1}},1\right\},\end{aligned} \tag{3.27}$$

for all $r\ge\max\{r_0^*,4n-1\}$ and $t\in[0,T)$ with $C_{13}=C_{13}(n,\lambda,T)$.

Using inequalities (3.27) with (3.14)-(3.15), we develop Moser's iteration scheme, taking k_0 in $4^{k_0}\ge\max\{r_0^*,4n\}$. Putting

$$\Phi_k=\max\left\{\sup_{t\in[0,T)}\|u(t)\|_{4^k},\sup_{t\in[0,T)}\|\nabla v(t)\|_\infty,d\right\},\quad d=\max\{\|u_0\|_1,\|u_0\|_\infty,1\},$$

for $k\ge k_0$, we obtain

$$\Phi_k\le C_{13}^{4^{-k}}\cdot 4^{C_{14}(n)\cdot k4^{-k}}\cdot\Phi_{k-1}^{1+n\cdot 4^{-k}}$$

$$\leq C_{13}^{4^{-k}+\sum_{j=2}^{k-k_0}\left\{4^{-(k+j-1)}\prod_{p=k-j+2}^{k}(1+n\cdot 4^{-p})\right\}}$$
$$\times\ 4^{C_{14}(n)\cdot 4^{-k}+C_{14}(n)\sum_{j=2}^{k-k_0}\left\{(k+1-j)4^{-(k+j-1)}\prod_{p=k-j+2}^{k}(1+n\cdot 4^{-p})\right\}}$$
$$\times\ \Phi_{k_0}^{\prod_{p=k_0+1}^{k}(1+n\cdot 4^{-p})} \leq (4C_{13}\Phi_{k_0})^{C_{15}(n)},$$

by (3.27), and therefore

$$\sup_{t\in[0,T)}\|u(t)\|_\infty \leq C_{16}(n,\lambda,\|u_0\|_\infty,T),$$

by (3.14)-(3.15). This implies $T=+\infty$ and the proof is complete. □

REFERENCES

[1] A. Blanchet, J. A. Carrillo, and P. Laurençot, *Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimension*, Calc. Var., 35 (2009), 133–168.

[2] Y. Sugiyama, *Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems*, Differential and Integral Equations, 19 (2006), 841–876.

[3] Y. Sugiyama, *Application of the best constant of the Sobolev inequality to degenerate Keller-Segel models*, Adv. Differential Equations, 12 (2007), 124–144.

[4] Y. Sugiyama and H. Kunii, *Global existence and decay properties for a degenerate Keller-Segel model with a power factor in drift term*, J. Differential Equations, 227 (2006), 333–364.

[5] T. Suzuki, "Free Energy and Self-Interacting Particles," Birkhäuser, Boston, 2005.

[6] T. Suzuki, "Mean Field Theories and Dual Variation," Atlantis Press, Amsterdam, 2008.

[7] T. Suzuki, *Degenerate parabolic equation derived from the kinetic theory*, Kokyuroku RIMS, 1591 (2008), 164–188.

[8] T. Suzuki and F. Takahashi, *Nonlinear eigenvalue problem with quantization*, Handbook of Differential Equations, Stationary Partial Differential Equations, 5 (M. Chipot ed.), pp. 277–370, Elsevier, Amsterdom, 2008.

[9] T. Suzuki and R. Takahashi, *Degenerate parabolic equation with critical exponent derived from the kinetic theory, I, generation of the weak solution*, Adv. Deifferential Equations, 14 (2009), 433–476.

[10] G. Wang and D. Ye, *On a nonlinear elliptic equation arising in a free boundary problem*, Math. Z., 244 (2003), 531–548.

[11] G. Wolansky, *Critical behaviour of semi-linear elliptic equations with sub-critical exponents*, Nonlinear Anal., 26 (1996), 971–995.