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FRACTIONAL STOCHASTIC EVOLUTION EQUATIONS WITH LÉVY NOISE

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Abstract. In this paper, we consider the problem of existence and uniqueness for a class of abstract integral Volterra equations in a Hilbert space H perturbed by an additive noise of Lévy type, with nonLipschitz nonlinearities. We find the solution in the space of mean square continuous and predictable processes from $\mathbb{R}_+$ into H.

1. Introductory remarks

In recent years, considerable interest has been shown in Volterra integro-differential equations with respect to the fractional kernel $g_\rho(t) = \frac{1}{\Gamma(\rho)} t^{\rho-1}$, as they represent a good link between the heat ($\rho = 1$) and the wave ($\rho = 2$) equation: see [11, 12]. Such models naturally appear in different applications in mathematical physics; in the stochastic case, similar equations – perturbed by cylindrical Wiener processes – were introduced in [8] in connection with the heat equation in materials with memory and in [9] for equations of linear parabolic viscoelasticity. For some recent developments and further bibliographic remarks see the monographs [13], [16] and [1] and, in the stochastic case, [5] or [3, 4]. Notice also that linear Volterra equations with Lévy noise are considered in [14].

Our aim is to obtain a result of existence and uniqueness of solutions for the following equation:

$$u(t) = u_0 + \int_0^t g_\rho(t-s)[Au(s) + f(s,u(s))]\,\mathrm{d}s + L(t), \qquad t \in [0,T], \quad (1.1)$$

where we assume that A is a closed, densely defined linear operator on H and is the generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ of negative type; $\{L(t)\}_{t\geq 0}$ is a Lévy process (see Section 1.2 for details), taking values in H, defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the

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natural filtration generated by the process $\{L(t)\}$ completed with the $\mathbb{P}$-null sets. The initial condition u_0 is taken in $L^2(\Omega, \mathcal{F}_0; H)$.

Let us consider the nonlinear coefficient in (1.1), $f : [0,+\infty)\times\Omega\times H \to H$. We assume that f is measurable with respect to the predictable σ-field $\mathcal{P}_T$ on $\Omega_T = \Omega \times [0, T]$, for arbitrary $T > 0$. In this paper, we do not assume a Lipschitz continuity as in [3] nor a dissipativity condition as in [4]. Rather, we use the class of nonLipschitz conditions of *Taniguchi* type, first proposed in [17]; see Hypothesis 1.6 below.

1.1. **The associated linear Volterra equation.**

Hypothesis 1.1. A is a self-adjoint operator on the separable Hilbert space H. We may and do assume that there exists a complete orthonormal system $\{e_n\}_{n\geq 1}$ of H and a sequence $\{\mu_n\}_{n\geq 1}$, with $\mu_n > 0$, $\mu_n \uparrow \infty$ as $n \to \infty$ and

$$Ae_n = -\mu_n e_n, \qquad n \geq 1.$$

The kernel $\{g_\rho(t)\}$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}_+)$ and we require that the following connection holds between the operator A and the kernel g_ρ:

$$\sum_{n=1}^{\infty} \mu_n^{-1/\rho} < +\infty. \tag{1.2}$$

Remark 1.2. Consider the following toy model. Let us be given a rigid, isotropic, homogeneous material with memory, which occupies a fixed region in space, denoted by $D \subset \mathbb{R}^d$, $d = 1, 2, 3$; i.e., D is a bounded domain with smooth boundary. Assume that A denotes the realization of the Laplace operator with Dirichlet boundary conditions on D. Then equation (1.1) represents the heat conduction law in D.

Notice that condition (1.2) is equivalent to the following condition on ρ and d:

$$\rho < \frac{2}{d}.$$

In our assumptions, the Volterra integral equation

$$u(t) = x + \int_0^t g_\rho(t-\vartheta) Au(\vartheta)\,\mathrm{d}\vartheta, \qquad t \in [0, T], \tag{1.3}$$

is well posed in H; we denote by $\{S(t)\}_{t\geq 0}$ the resolvent associated to (1.3).

Let x be an eigenvector of A with eigenvalue $-\mu$; then $S(t)x$ can be expressed in terms of the scalar resolvent function $\{s(\mu, t)\}_{t\geq 0}$

$$S(t)x = s(\mu; t)x, \qquad t \geq 0,$$

where $\{s(\mu,t)\}_{t\ge 0}$ is the solution of the one-dimensional Volterra equation

$$s(\mu;t) + \mu \int_0^t g_\rho(t-\vartheta) s(\mu;\vartheta)\, \mathrm{d}\vartheta = 1, \qquad t \ge 0.$$

We recall that the Mittag-Leffler's function $\{\mathcal{E}_{\rho,\theta}(z)\}$ is defined as

$$\mathcal{E}_{\rho,\theta}(z) = \sum_{n=0}^{\infty} \frac{(-1)+z^k}{\Gamma(\rho k+\theta)}, \qquad z \in \mathbb{C},\ \rho,\theta > 0$$

(notice that our convention on the sign differs from [10]). For short, we write $\mathcal{E}_\rho(z) = \mathcal{E}_{\rho,1}(z)$. This function is bounded and continuous, $\mathcal{E}_\rho(0) = 1$ and $\mathcal{E}_\rho(t) = O(\frac{1}{t})$ as $t \to \infty$ – see [1, 5] for details.

It is known that $\{s(\mu,t)\}_{t\ge 0}$ can be expressed in terms of the Mittag-Leffler's function:

$$s(\mu;t) = \mathcal{E}_\rho(\mu t^\rho), \qquad t \ge 0;$$

in future sections we use this identification in order to prove some relevant estimates for the resolvent operator family and, in turn, properties of the solution of (1.1).

1.2. **The stochastic convolution process.** We are given a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ satisfying the usual hypotheses

(i) $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$, for all $t \ge 0$;
(ii) $\mathcal{F}_t = \mathcal{F}_t^+$ is right-continuous for all $t \ge 0$, where $\mathcal{F}_t^+ = \cap_{u>t} \mathcal{F}_u$.

Our main reference space will be the space $L^2_{\mathcal{P}}(\mathbb{R}_+; L^2(\Omega;H))$ of predictable processes $u : [0,\infty) \times \Omega \to H$ such that

$$\sup_{t\in[0,T]} \mathbb{E}|u(t)|^2 < +\infty \qquad \forall T > 0. \tag{1.4}$$

We shall consider a Lévy process $\{L(t)\}_{t\ge 0}$ on $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ with values in the real, separable Hilbert space H; i.e.,

(i) $\{L(t)\}_{t\ge 0}$ is stochastically continuous,
(ii) $\{L(t)\}_{t\ge 0}$ is adapted to $\{\mathcal{F}_t\}_{t\ge 0}$,
(iii) $L(0) = 0$ a.s.,
(iv) $\{L(t)\}_{t\ge 0}$ has increments independent from the past; i.e., $L(t)-L(s)$ is independent of $\mathcal{F}_s$ if $0 \le s < t$,
(v) $\{L(t)\}_{t\ge 0}$ has càdlàg trajectories, and
(vi) $\{L(t)\}_{t\ge 0}$ has stationary increments; i.e., $L(t) - L(s)$ has the same distribution as $L(t-s)$ for $0 \le s < t$.

Hypothesis 1.3. $\{L(t)\}_{t\ge 0}$ is a Lévy process with reproducing kernel Hilbert space (RKHS for short) $\mathcal{H}$ contained in H.

The RKHS of the process $\{L(t)\}$ (which equals that of the random variable $L(1)$), taking values in H, is the space $\mathcal{H} = Q^{1/2}(H)$, where Q is the covariance operator of $L(1)$. The previous assumption is, roughly speaking, equivalent to the requirement that the covariance operator is bounded in H.

Recall that $\{e_n\}$ is an orthonormal complete system in H. Let us consider the process $L(t) = \{L_n(t)\}_{n\geq 1}$ where the $\{L_n(t)\}$ are independent, real-valued Lévy processes with characteristic function

$$\mathbb{E}[e^{ixL_n(t)}] = e^{-t\psi_n(x)},$$

where

$$\psi_n(t) = \frac{1}{2}q_n x^2 + \int_{\mathbb{R}} (1 - e^{ixy})\, \nu_n(\mathrm{d}x).$$

Our construction follows the idea in [15], page 144. In order to give a meaning to the stochastic convolution term

$$Z(t) = \int_0^t S(t-r)\,\mathrm{d}L(r), \qquad t \geq 0, \tag{1.5}$$

we consider each component separately. Namely, we can write

$$Z_n(t) = \int_0^t s(\mu_n, t-r)\,\mathrm{d}L_n(r), \qquad t \geq 0,$$

and we obtain

$$\mathbb{E}[e^{i\langle x, Z(t)\rangle}] = \sum_{n=1}^\infty \exp\Big\{ - \int_0^t \psi_n(\langle x, e_n\rangle s(\mu_n, r))\,\mathrm{d}r \Big\}.$$

The next result provides a necessary and sufficient condition in order to have that Z takes values in H.

Proposition 1.4. *The process $\{Z(t)\}$ takes values in H if and only if*

$$\begin{aligned}\mathbb{E}|Z(t)|^2 &= \sum_{n=1}^\infty q_n \int_0^t |s(\mu_n, r)|^2\,\mathrm{d}r \\ &+ \sum_{n=1}^\infty \Big[\int_0^t \Big(|s(\mu_n, r)|^2 \int_{-\frac{1}{|s(\mu_n,r)|}}^{\frac{1}{|s(\mu_n,r)|}} |y|^2 \nu_n(\mathrm{d}y) + \nu_n(y : |y| > \frac{1}{|s(\mu_n, r)|}) \Big)\mathrm{d}r \Big] \\ &< +\infty. \end{aligned} \tag{1.6}$$

As a consequence of the above proposition, we find the following regularity result for the stochastic convolution process $Z(t)$.

Corollary 1.5. *$\{Z(t)\}_{t\geq 0}$ is a predictable process with values in H and satisfies*

$$\sup_{t\in[0,T]} \mathbb{E}|Z(t)|^2 \leq C$$

for arbitrary $T > 0$.

Proof. The proof is based on the discussion concerning Example 7.5 in [15], which shows that Hypothesis 1.3 is equivalent to the boundedness of the sequence $\left\{q_n + \int_{\mathbb{R}} |y|^2 \,\nu_n(\mathrm{d}y)\right\}_{n\geq 1}$ and the estimate

$$\mathbb{E}|Z(t)|^2 \leq \sum_{n=1}^{\infty} \frac{q_n + \int_{\mathbb{R}} |y|^2 \,\nu_n(\mathrm{d}y)}{\mu_n^{1/\rho}},$$

which implies (1.6); then the claim follows. □

1.3. **The nonlinear term.** In the applications, it is often found that (global) Lipschitz conditions are too strong assumptions on the coefficients in (1.1). Following [17] we may consider the following conditions, that are already known in the literature, compare [7, 6, 2].

Hypothesis 1.6. $f : [0,T] \times \Omega \times H \to H$ is measurable with respect to the predictable σ-field $\mathcal{P}_T$ on $\Omega_T = \Omega \times [0,T]$ and continuous in x for any fixed $(t,\omega) \in \Omega_T$.

(H1) There exists a function $H(t,x) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, locally integrable in $t \geq 0$ for any fixed $x \geq 0$ and continuous, monotone nondecreasing and concave in x for any fixed $t \in [0,T]$, such that the following hold.

(a) For any $t \geq 0$ and $X \in L^2(\Omega; H)$

$$\mathbb{E}|f(t,X)|^2 \leq H(t, \mathbb{E}|X|^2).$$

(b) For any constant $\gamma > 0$ and $x \in H$, the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = \gamma H(t, X(t)), \\ X(0) = x, \end{cases}$$

has a solution on $[0,T]$.

(H2) There exists a function $G(t,x) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, locally integrable in $t \geq 0$ for any fixed $x \geq 0$ and continuous, monotone nondecreasing and concave in x for any fixed $t \geq 0$, and $G(t,0) = 0$.

(a) For any $t \geq 0$ and $X, Y \in L^2(H)$ it holds that

$$\mathbb{E}|f(t,X) - f(t,Y)|^2 \leq G(t, \mathbb{E}|X-Y|^2).$$

(b) For any constant $\gamma > 0$, if a nonnegative function $\{x(t)\}$ on $[t_0, T]$ satisfies

$$x(t) \le \gamma \int_{t_0}^{t} G(s, x(s))\, \mathrm{d}s,$$

on $[t_0, T]$, then $x(t) \equiv 0$.

We now briefly discuss the applicability of the above assumptions. In particular, it seems interesting to give some examples of functions G that satisfy **(H2)**. A complete presentation is given in [6].

Let $G(t, x) = \lambda(t)\gamma(x)$, where λ is a nonnegative, locally integrable function,

$$0 \le \int_0^T \lambda(t)\, \mathrm{d}t < \infty, \qquad T > 0,$$

and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function such that

$$\int_{0+} \frac{1}{\gamma(x)}\, \mathrm{d}x = +\infty.$$

If, in addition, $\gamma(x)$ or $\frac{\gamma(x)^2}{x}$ is a concave function, then in order to have **(H2)** it is sufficient to require

$$|f(t, X) - f(t, Y)|^2 \le \lambda(t)\gamma(|X - Y|^2), \qquad a.s. \tag{1.7}$$

For instance one can take

$$\gamma(x) = \begin{cases} x[\log(x^{-1/2})]^\alpha, & x \le \delta, \\ \delta[\log(\delta^{-1/2})]^\alpha, & x \ge \delta, \end{cases}$$

for arbitrary $\alpha \le 1$ and sufficiently small δ. As an example of a function f that satisfies condition (1.7), in the setting of Hypothesis 1.1, with $\{e_n\}$ an orthonormal system in H and $|x|$ the norm in this space, we can choose

$$f(x) = \sum_{n=1}^{\infty} n^{-3/2}[\cos(n|x|) + \sin(n|x|)]\, e_n.$$

Compare for instance [6, Section 4].

2. Existence and uniqueness of the solution

Using the family of resolvent operators, our approach to the Volterra equation is quite similar to the functional approach that is developed, for instance, in [15] for SPDEs. The *mild solution* to (1.1) in H is a predictable process $u \in L^2_{\mathcal{P}}(\mathbb{R}_+; L^2(\Omega; H))$ such that

$$\sup_{t\in[0,T]} \mathbb{E}|u(t)|^2 < +\infty \qquad \forall T > 0,$$

that satisfies $\mathbb{P}$-a.s. the stochastic integral equation

$$u(t) = S(t)x + \int_0^t \int_0^{t-r} S(t-r-\sigma)g_{\rho-1}(\sigma)\,\mathrm{d}\sigma\, f(r,u(r))\,\mathrm{d}r + \int_0^t S(t-r)\,\mathrm{d}L(r), \qquad t > 0. \tag{2.1}$$

2.1. **Picard approximation scheme.** Fix an arbitrary $T > 0$ and denote by $D_T = L^2_{\mathcal{P}}([0,T]; L^2(\Omega; H))$ the space of predictable processes from $[0,T]$ into H endowed with the norm in (1.4).

Define $v(t) = u(t) - Z(t)$, $t \in [0,T]$, where $Z(t)$ is the stochastic convolution process introduced in (1.5), and note that (2.1) can be written as

$$v(t) = S(t)x + \int_0^t \int_0^{t-r} S(t-r-\sigma)g_{\rho-1}(\sigma)\,\mathrm{d}\sigma\, f(r, v(r)+Z(r))\,\mathrm{d}r. \tag{2.2}$$

We are going to solve (2.2) pathwise, since $\{Z(t)\}_{t\geq 0}$ belongs to D_T.

We introduce the following map in D_T:

$$\Phi X(t) = S(t)x + \int_0^t \int_0^{t-r} S(t-r-u)g_{\rho-1}(u)\,\mathrm{d}u\, f(r, X(r)+Z(r))\,\mathrm{d}r. \tag{2.3}$$

The first lemma provides useful estimates on the D_T-norm of a convolution integral. We consider the case of the fractional derivative kernel and we show that the estimate holds independently of T; this allows us, in Theorem 2.3, to study the behavior of the solution on the whole positive half-line.

Lemma 2.1. *In our assumptions, there exists a constant $C > 0$ such that, for every $\varphi \in D_T$,*

$$\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t \int_0^{t-r} S(t-r-u)g_{\rho-1}(u)\,\mathrm{d}u\, \varphi(r)\,\mathrm{d}r \right|^2 \leq C \int_0^T \|\varphi\|^2_{D_s}\,\mathrm{d}s. \tag{2.4}$$

Proof. Recall that $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system in H of eigenvectors of A; we compute

$$\left| \int_0^t \int_0^{t-r} S(t-r-u)g_{\rho-1}(u)\,\mathrm{d}u\, \varphi(r)\,\mathrm{d}r \right|^2$$

$$\le \sum_{k=1}^{\infty} \Big| \int_0^t \int_0^{t-r} s(\mu_k, t-r-\sigma) g_{\rho-1}(\sigma)\, \mathrm{d}\sigma \, \langle e_k, \varphi(r)\rangle \, \mathrm{d}r \Big|^2$$
$$\le \sum_{k=1}^{\infty} \Big| \int_0^t r(\mu_k, t-r) \, \langle e_k, \varphi(r)\rangle \, \mathrm{d}r \Big|^2$$
$$\le \sum_{k=1}^{\infty} \|r(\mu_k; \cdot)\|^2_{L^2(\mathbb{R}_+)} \int_0^t \langle e_k, \varphi(r)\rangle^2 \, \mathrm{d}r$$
$$\le \sum_{k=1}^{\infty} C \frac{1}{\mu_k^{(2\rho-1)/\rho}} \int_0^t \langle e_k, \varphi(r)\rangle^2 \, \mathrm{d}r \le C \frac{1}{\omega_A^{(2\rho-1)/\rho}} \int_0^t |\varphi(r)|^2 \, \mathrm{d}r,$$

where $r(\mu; z) = z^{\rho-1}\mathcal{E}_{\rho,\rho}(\mu z^\rho)$ is defined in terms of Mittag-Leffler's function (see Section 1.1) and $\omega_A = \mu_1$ is the (absolute value of the) first eigenvalue of A; by Hypothesis 1.1, $\omega_A > 0$ and the claim follows by taking the expectation on both sides in the last formula. □

We now introduce the necessary tools we need to prove the existence and uniqueness theorem. At first, we prove that the mapping Φ is well defined and continuous.

Lemma 2.2. *Φ maps D_T in D_T. Further, given $u, v \in D_T$,*

$$\|\Phi(u) - \Phi(v)\|^2_{D_T} \le C_T \int_0^T G(s, \|u-v\|^2_{D_s}) \, \mathrm{d}s. \tag{2.5}$$

Proof. By definition of D_T, we shall prove that $\sup_{t\in[0,T]} \big(\mathbb{E}|(\Phi u)(t)|^2\big) < \infty$ for given $u \in D_T$. We have

$$\mathbb{E}|(\Phi u)(t)|^2 \le 2\mathbb{E} \sup_{t\in[0,T]} |S(t)x|^2 + 2\mathbb{E}\Big| \int_0^t \int_0^{t-r} S(t-r-\sigma) g_{\rho-1}(\sigma)\, \mathrm{d}\sigma \, f(r, u(r) + Z(r)) \, \mathrm{d}r \Big|^2,$$

and Lemma 2.1 and Hypothesis 1.6-**(H1)** imply

$$\sup_{t\in[0,T]} \mathbb{E}|(\Phi u)(t)|^2 \le 2M\mathbb{E}|x|^2 + 2C \int_0^T H(s, \mathbb{E}|u(s) + Z(s)|^2) \, \mathrm{d}s$$
$$\le 2M\mathbb{E}|x|^2 + 2C \int_0^T H(s, \sup_{r\in[0,T]} \mathbb{E}|u(r) + Z(r)|^2) \, \mathrm{d}s < +\infty, \tag{2.6}$$

since for fixed

$$x = 2\Big(\sup_{r\in[0,T]} \mathbb{E}|u(r)|^2 + \sup_{r\in[0,T]} \mathbb{E}|Z(r)|^2\Big),$$

Hypothesis 1.6-**(H1)** implies that $\int_0^T H(r,x)\,\mathrm{d}r$ is finite.

Now, we turn to the second part of the claim. Let $u, v \in D_T$, then

$$|\Phi(u) - \Phi(v)|^2 \leq \Big| \int_0^t \int_0^{t-r} S(t-r-s) g_{\rho-1}(s) \mathrm{d}s \, [f(r, u(r) + Z(r)) - f(r, v(r) + Z(r))] \mathrm{d}r \Big|^2. \tag{2.7}$$

Setting $\varphi(r) = f(r, u(r) + Z(r)) - f(r, v(r) + Z(r))$, it follows from Lemma 2.1 that

$$|\Phi(u) - \Phi(v)|^2 \leq C_T \int_0^T \mathbb{E}|f(r, u(r) + Z(r)) - f(r, v(r) + Z(r))|^2 \, \mathrm{d}s,$$

and Hypothesis 1.6-**(H2)** leads to

$$|\Phi(u) - \Phi(v)|^2 \leq C \int_0^T G(r, \mathbb{E}|u(r) - v(r)|^2) \, \mathrm{d}s.$$

Then claim (2.5) follows by the monotonicity of $G(r,\cdot)$ for each $r \in [0,T]$ as required in Hypothesis 1.6-**(H2)**. □

Theorem 2.3. *Assume Hypotheses* 1.1, 1.3, *and* 1.6. *Then there exists a unique solution* $v = \{v(t)\} \in L^2_{\mathcal{P}}(\mathbb{R}_+; L^2(\Omega; H))$ *of equation* (2.2). *Further, setting* $u(t) = v(t) + Z(t)$, $\{u(t)\} \in L^2_{\mathcal{P}}(\mathbb{R}_+; L^2(\Omega; H))$ *is the unique mild solution of equation* (2.1).

Proof. We consider the Picard iteration scheme

$$\begin{aligned} v_0(t) &= S(t)x \\ v_{n+1}(t) &= \Phi(v_n)(t) = S(t)x \\ &\qquad + \int_0^t \int_0^{t-r} S(t-r-u) g_{\rho-1}(u) \, \mathrm{d}u \, f(r, v_n(r) + Z(r)) \, \mathrm{d}r. \end{aligned} \tag{2.8}$$

First step. We prove that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded in D_T. Proceeding as in the proof of (2.6), by the concavity of $H(t,\cdot)$ we obtain

$$\begin{aligned} \|v_{n+1}\|^2_{D_T} &\leq 2M^2 \mathbb{E}|x|^2 + 4C \int_0^T H(s, \|Z\|^2_{D_s}) \, \mathrm{d}s + 4C \int_0^T H(s, \|v_n\|^2_{D_s}) \, \mathrm{d}s \\ &\leq k_1 + k_2 \int_0^T H(s, \|v_n\|^2_{D_s}) \, \mathrm{d}s, \end{aligned}$$

where

$$k_1 = \max\left\{M^2, 2M^2\mathbb{E}|x|^2 + 4C\int_0^T H(s, \|Z\|^2_{D_s})\,\mathrm{d}s\right\}.$$

We shall compare this quantity with the solution $\{y(t)\}$ of the Cauchy problem

$$y(t) = k_1 + k_2\int_0^t H(s, y(s))\,\mathrm{d}s, \qquad t \in [0,T].$$

This problem has a unique solution on $[0,T]$ thanks to Hypothesis 1.6. Notice that our choice of k_1 implies

$$\mathbb{E}|v_0(t)|^2 \le y(t), \qquad t \in [0,T].$$

We shall prove by induction that the estimate

$$\sup_{s\in[0,t]} \mathbb{E}|v_n(s)|^2 \le y(t), \qquad t \in [0,T],$$

holds for any $n \ge 0$. In fact, if it holds for v_n, then since H is nondecreasing we have

$$y(t) - \|v_{n+1}\|^2_{D_t} \ge k_2\int_0^T \left[H(s, y(s)) - H(s, \|v_n\|^2_{D_s})\right]\,\mathrm{d}s \ge 0,$$

and the claim follows.

Second step. We aim to prove that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in D_T. Let

$$r(t) = \limsup_{n,m\to\infty} \|v_n - v_m\|^2_{D_t}, \qquad t \in [0,T].$$

Notice that $\{r(t)\}_{t\in[0,T]}$ is a monotone nondecreasing function with $r(t) \le 2y(t)$. Also, by the second part of Lemma 2.2 and Fatou's lemma, it follows that

$$r(t) \le C_T\int_0^T G(s, r(s))\,\mathrm{d}s,$$

and Hypothesis 1.6-**(H2)** implies that

$$0 = r(t) = \limsup_{n,m\to\infty} \|v_n - v_m\|^2_{D_t} = \lim_{n,m\to\infty} \|v_n - v_m\|^2_{D_t}, \qquad t \in [0,T].$$

This proves that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in D_T, as claimed; denote by v its limit. Letting $n \to \infty$ in (2.8) we immediately get that v is a solution of (2.2).

Third step. It remains to prove that the solution to (2.2) is unique. Assume by contradiction that v and w are two fixed points of the mapping Φ. Then, with the aid of the estimate in the second part of Lemma 2.2 we obtain

$$\|v-w\|_{D_T}^2 \le C_T \int_0^T G(s, \|v-w\|_{D_s}^2)\,\mathrm{d}s,$$

and Hypothesis 1.6-**(H2)** leads to the thesis. □

References

[1] Emilia G. Bajlekova, *Fractional evolution equations in Banach spaces*, Eindhoven University of Technology, Eindhoven, 2001, Dissertation.

[2] Dorel Barbu and Gheorghe Bocşan, *Successive approximations to solutions of stochastic semilinear functional differential equations in hilbert spaces,* Technical Report 162, University of Timisoara, Faculty of Mathematics, 2004, STPA - Seminar of Probability Theory and Applications.

[3] Stefano Bonaccorsi and Marco Fantozzi, *Large deviation principle for semilinear stochastic Volterra equations,* Dynam. Systems Appl., 13 (2004), 203–219.

[4] Stefano Bonaccorsi and Marco Fantozzi, *Infinite dimensional stochastic Volterra equations with dissipative nonlinearity,* Dynam. Systems Appl., 15 (2006), 465–478.

[5] Stefano Bonaccorsi and Luciano Tubaro, *Mittag-Leffler's function and stochastic linear Volterra equations of convolution type,* Stochastic Anal. Appl., 21 (2003), 61–78.

[6] Guilan Cao and Kai He, *Successive approximation of infinite dimensional semilinear backward stochastic evolution equations with jumps,* Stochastic Process. Appl., 117 (2007), 1251–1264.

[7] Guilan Cao, Kai He, and Xicheng Zhang, *Successive approximations of infinite dimensional SDEs with jump,* Stoch. Dyn., 5 (2005), 609–619.

[8] Philippe Clément and Giuseppe Da Prato, *White noise perturbation of the heat equation in materials with memory,* Dynam. Systems Appl., 6 (1997), 441–460.

[9] Philippe Clément, Giuseppe Da Prato, and Jan Prüss, *White noise perturbation of the equations of linear parabolic viscoelasticity,* Rend. Istit. Mat. Univ. Trieste, 29 (1998), 207–220.

[10] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi, "Higher Transcendental Functions," Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981.

[11] Yasuhiro Fujita, *Integrodifferential equation which interpolates the heat equation and the wave equation,* Osaka J. Math., 27 (1990), 309–321.

[12] Yasuhiro Fujita, *Integrodifferential equation which interpolates the heat equation and the wave equation, II,* Osaka J. Math., 27 (1990), 797–804.

[13] G. Gripenberg, S.-O. Londen, and O. Staffans, "Volterra Integral and Functional Equations," volume 34 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1990.

[14] Anna Karczewska, *Stochastic Volterra convolution with Lévy process,* Int. J. Pure Appl. Math., 18 (2005), 109–120.

[15] S. Peszat and J. Zabczyk, "Stochastic Partial Differential Equations with Lévy Noise," volume 113 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2007.

[16] Jan Prüss, "Evolutionary Integral Equations and Applications," volume 87 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1993.

[17] Takeshi Taniguchi, *Successive approximations to solutions of stochastic differential equations,* J. Differential Equations, 96 (1992), 152–169.